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### Linear operations in Saks spaces (II)

**W. Orlicz** (Poznań)

3.1. We shall prove now some general theorems concerning the Saks spaces $^1$.

Let $X$ denote the fundamental Banach space or an incomplete Banach space with the norm $\| \cdot \|$, and let $\| \cdot \|$ be a $B$- or $F$-norm, defining together with the former norm the Saks space $X_q$.

(A) Let us denote by $X_q$ the space composed of the elements of $X$, with the norm

$$
\|x\| = \|x\| + \|\eta\|;
$$

then $X_q$ is a Fréchet space.

(A') If we apply, in the definition of a Saks space, the norm $\|x\|_s$ as the “starred” norm, then the sphere $\|x\|_s \leq 1$ forms a Saks space $X_q$, satisfying conditions (1), (2), and (3).

(A'') If the set $X'$ is dense with respect to the norm $\| \cdot \|$, in the sphere $\|x\|_s \leq 1$, then it is dense in $X_q$.

To prove (A'), let us suppose that $\|x_n\|_s \leq 1$, $\|x_n - x_m\|_s \to 0$ as $n,m \to \infty$.

The space $X'$ being complete, there exists an element $x_{\cdot\cdot}$ of $X'$ such that

$$
\lim_{n \to \infty} \|x_n - x_{\cdot\cdot}\| = 0.
$$

Since (as we have noticed in 2.2, p. 266) the space $X$ is such that

$$
\lim_{n \to \infty} \|y_n\|_s = 0,
$$

we see that $x_n - x_{\cdot\cdot} \to x_{\cdot\cdot} - x_{\cdot\cdot}$ implies

$$
\lim_{n \to \infty} \|x_n - x_{\cdot\cdot}\| \geq \|x_{\cdot\cdot} - x_{\cdot\cdot}\|,
$$

$^1$ This is the second part of the paper [8] which will be denoted by the abbreviation LO(II). Subsequently some definitions and notations of LO(II) will be used.

The results of this paper, as those of LO(II) were presented at 26th September 1944 to the VI Congress of Polish Mathematicians.
and since \( \|x_n - x_m\| \to 0 \) as \( n, m \to \infty \), we see that \( \|x_n - x_m\| \to 0 \) and, since \( \|x_n - x_m\| \to 0 \), we get \( \|x_n - x_m\| \to 0 \). Thus the space \( X_n \) is complete.

To prove that the space \( (X_n) \), satisfies conditions \( (\Sigma_1) \) and \( (\Sigma_2) \), it is sufficient in virtue of 1.32 to state that condition \( (\Sigma) \) is satisfied at every point of a dense set in \( X_n \). For this purpose, let us consider the sphere \( \|x - x_0\| \leq \epsilon \), \( |\epsilon| \leq 1 \), \( |\epsilon| \leq 1 \). The element \( y = (1 - \alpha)x \), \( 0 < \alpha < 1 \), lies in this sphere for sufficiently small \( \alpha \). Condition \( (\Sigma) \) is satisfied at the point \( y \), because \( \|x - x_0\| < \delta = 1 - |\epsilon| \) implies \( y - x \in X_n \).

Statement (A) immediately results from the completeness of the space \( (X_n) \).

To prove \( (A^{'}) \) let us notice that if \( \|x_n\| < 1 \), then there exists \( x \in X \) such that \( \|x - x_n\| < (1 - \|x_n\|) \) for \( 0 < \|x\| < 1 \). Hence \( \|x\| < 1 \), \( \|x - x_n\| < \epsilon \), and since the set of the elements \( x_n \) satisfying \( \|x_n\| < 1 \) is obviously dense in \( X_n \), the set \( X_n \) is dense in \( X_n \).

(B) If \( x_n \in X \) and \( \|x_n\| \to 0 \) implies \( \|x_n\| \to 0 \), then \( X \) is a Banach space with respect to the norm \( \|\| \), and conversely.

Only completeness with respect to the norm \( \|\| \) is to be proved. By hypothesis, \( \|x_n - x_m\| \to 0 \) as \( n, m \to \infty \) implies \( \|x_n - x_m\| \to 0 \), whence also \( \|x_n - x_m\| \to 0 \) (where \( \|\| \) is the norm defined by (1)). It suffices to notice that the space \( X \) is complete with respect to the norm \( \|\| \) in virtue of (A).

Suppose now \( X_n \), with the norm \( \|\| \), is a Banach space. From (A) it follows that \( \|x_n\| \) is also complete with respect to the norm \( \|\| \); moreover, \( \|x_n\| \to 0 \), whence, by a well-known theorem of Banach, \( \|x_n\| \to 0 \) implies \( \|x_n\| \to 0 \), which in turn implies \( \|x_n\| \to 0 \).

(C) Each of the following conditions is necessary and sufficient in order that the norm \( \|\| \) be equivalent to the B-norm \( \|\| \):

(a) \( X \) is a Banach space with respect to the norm \( \|\| \) and \( \|x_n\| \to 0 \) implies \( \|x_n\| \to 0 \),

(b) \( X \) is a Fréchet space with respect to the norm \( \|\| \) and \( \|x_n\| \to 0 \) implies \( \|x_n\| \to 0 \),

(c) \( X \) is a Banach space with respect to the norm \( \|\| \) and the set \( X^n = \{ \|x_n\| \leq 1 \} \) is of the second category in \( X \).

Ad (a) and (b). If the norm \( \|\| \) is equivalent to \( \|\| \) then the norm \( \|\| \) is equivalent to \( \|\| \) and \( \|\| \); by 3.1 (A) \( X \) is a Banach space with the norm \( \|\| \), whence (a) and (b) are necessary.

If (a) is satisfied, then \( \|x_n\| \to 0 \) implies \( \|x_n\| \to 0 \) and in virtue of 3.1 (B) \( \|x_n\| \to 0 \) implies \( \|x_n\| \to 0 \).

If (b) is satisfied, then by 3.1 (B) \( X_n \), provided with the norm \( \|\| \), is a Banach space, and since this space is also an F-space with respect to the norm \( \|\| \) and since \( \|x_n\| \to 0 \) implies \( \|x_n\| \to 0 \), therefore also \( \|x_n\| \to 0 \) implies \( \|x_n\| \to 0 \), by the theorem of Banach cited above.

Ad (c). The necessity is obvious (a). To prove the sufficiency, it is enough, in virtue of (B), to prove that \( \|x_n\| \to 0 \) implies \( \|x_n\| \to 0 \). Suppose that it is not so. Then there exist \( \theta_n \in X_n \) such that \( \|x_n\| \to 0 \), \( \|x_n\| \to 1 \).

Write \( X_n = \{ \|x_n\| \leq 1 - 1/n \} \), \( n = 2, 3, \ldots \),

and let \( K(x_n, q) \) be an arbitrary sphere in \( X_n \). We shall prove that there are points \( y \) in \( K(x_n, q) \) such that \( y = 1 \). If \( \|y\| = 1 \), then \( y = x_n \) gives the desired result; if \( \|y\| < 1 \), let us choose a sequence \( \theta_n \) of numbers such that \( \|y_n + \theta_n x_n\| = 1 \) for \( n = 1, 2, \ldots \). Then

\[
\lim_{n \to \infty} \theta_n < \infty
\]

for \( 1 = \|y + \theta_n x_n\| \geq \|y_n + \theta_n x_n\| - \|\theta_n x_n\| - \|y_n - x_n\| \geq \|y_n - x_n\| \) (I). It follows that \( \|y_n + \theta_n x_n\| \to 0 \), whence \( y = x_n \). It follows that \( \|\| \) is of the first category, in contradiction to the hypothesis.

(D) Every \( (X_n, Y) \)-linear operation may be extended in a unique way to an operation \( V(x) \) defined in the whole of \( X \) and \( (X_n, Y) \)-linear.

For \( x \in X \) let us set \( V(x) = U(x, y) \) where \( |y| \geq |x| \).

Since \( (y)/Q \) is a norm and since \( yU(x, y) = U(x, y) = V(x) \) is an extension of \( U(x, y) \), Let us choose \( y \) such that \( \max \{ |y|, |y| |x| + |x| \} \), then

\[
V(x + y) = yU(x + y) = yU(x) + yU(y) = V(x) + V(y).
\]

Let \( \|x_n - x_m\| \to 0 \); then there exists \( y \|y\| \leq \|x_m\| \), whence \( \|x_n - x_m\| \to 0 \),

\[
V(x_n) = U(x_n, y) \to yU(x_m, y) = V(x).
\]

Finally, it is obvious that the requirement that the extended operation be \( (X_n, Y) \)-linear determines uniquely the operation \( V(x) \).

3.11. Sometimes, if the norm \( \|\| \) is equal to the norm \( \|\| \), we shall say that \( X \) is a Banach space.
3.2. In the sequel $U_n(x)$ will stand for $(X, Y)$-linear operations into a Banach or Fréchet space $Y$. We shall denote by $\omega(x)$ the oscillation of the sequence $\{U_n(x)\}$ at $x_i$, i.e. the number defined by the formula

$$\omega(x) = \lim \sup_{n \to +\infty} \|U_n(x) - U_n(x)\|.$$ 

We shall denote by $E$ and $S_\omega$ the sets of the points $x$ at which $\omega(x) > \sigma$ and $\omega(x) < \sigma$ respectively (the number $\sigma$ may be equal to $\infty$). Then $S_\omega$ is a $F_\sigma$-set, since the sets

$$E \bigcup_{n \in \mathbb{N}} \{x \in X : \|U_n(x) - U_n(x)\| < \varepsilon\}$$

are closed for each finite $\varepsilon$.

By $C$ we shall denote the set of those points of $X$ for which $\omega(x) = \sigma$, i.e. the sets of points of convergence of the sequence $\{U_n(x)\}$; $D$ will stand for the set of divergence of this sequence, i.e. $D = E \bigcup \{x \in X : \omega(x) = 0\}$.

3.21. The functional $\omega(x)$ is, in general, of Baire's second class and therefore may be discontinuous everywhere; this functional has the following properties:

(a) $U_n(x)$ converges to $x_i$ if and only if $\omega(x_i) = 0$; $\omega(0) = 0$.

(b) $|\omega(x) - \omega(y)| < 2\omega(|x - y|)$ if $\omega(x)$ and $\omega(y)$ are finite.

(c) If $x_0, x_1, \ldots, x_n \in X$, then $\omega(x_0 + \ldots + x_n) < \omega(x_0) + \ldots + \omega(x_n)$.

(d) The sequence $\{U_n(x)\}$ is bounded at $x$ if and only if

$$\lim_{n \to +\infty} \omega(\varepsilon) = 0.$$ 

If $X$ is a Banach space, this condition is equivalent to $\omega(x) < \infty$. In general Fréchet spaces the inequality $\omega(x) < \infty$ is only necessary for the boundedness of the sequence.

(e) If the oscillation $\omega(x)$ is continuous at $x_0$, then the sequence $\{U_n(x)\}$ is bounded everywhere in $X$.

(f) If the oscillation $\omega(x)$ is continuous at $x_0$, $\|x_0\| < 1$ and $\omega(x_0) = 0$, then $\omega(x)$ is continuous everywhere.

(g) Let the space $X$ satisfy condition (\Sigma). If $\omega(x)$ is continuous at a point $x_i$ belonging to the closure of the set $C$, then it is continuous in the whole of $X$.

(h) Suppose that the space $X$ satisfies condition (\Sigma) and $\omega(x)$ vanishes in a set dense in $X$. Then either $\omega(x) = 0$ and hence it is continuous in the entire space, or $\omega(x)$ is discontinuous at any point, whence it is of Baire's second class.

The proofs of (a) and (c) are trivial; (b) follows from the inequality

$$\|U_n(x) - U_m(x)\| - \|U_n(y) - U_m(y)\| \leq \|U_n(x) - U_m(x)\| - \|U_n(y) - U_m(y)\| < 2\|U_n(x) - U_m(x)\|.$$ 

To prove the sufficiency of (d), choose an $\varepsilon_1 > 0$ such that $|y| < \varepsilon_1$, $|\delta| < \delta_1$, then $|\delta y| < \varepsilon_1$. This is possible by 1.22. There exists a $\delta_1$ such that $0 < \delta_1 < 1$ and such that $\omega(\delta x) < \varepsilon_1$ if $|\delta| < \delta_1$. Thus we can find a $k$ such that $\|U_n(x_0) - U_n(x_0)\| = \|U_n(x_0) - U_n(x_0)\| = \varepsilon_1$ for $m, n > k$. Hence we get $\|U_n(x_0) - U_n(x_0)\| < \varepsilon_1$. The boundedness of the sequence $\{U_n(x)\}$ follows from the inequality

$$\|\theta U_n(x)\| < \|U_n(\theta x)\| + \|U_n(x) - U_n(x)\| < 2\varepsilon$$

valid for sufficiently small $\theta$ and $m,n > k$.

Suppose now that the sequence $\{U_n(x)\}$ is bounded. Given an $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\theta| < \delta_1$, $n = 1, 2, \ldots,$ implies $\|U_n(x)\| < \varepsilon/2$. Hence $\|U_n(\theta x) - U_n(x)\| < \varepsilon$ for $|\theta| < \delta_1$, $n = 1, 2, \ldots,$ it follows that $\omega(\delta x) < \varepsilon$.

(e) follows immediately from (d).

We prove (f) only for $x_i = 0$. By (o) and (d) we get $\omega(x) < \infty$ everywhere; then we apply (b).

Now we prove (g). By the continuity of $\omega(x)$ at $x_i$ we get $\omega(x_i) = 0$. Given any $\varepsilon > 0$ we have $\omega(x) < \varepsilon/2$ in a sphere $K(x_i, \varepsilon)$. By (\Sigma) there exists a $\delta > 0$ such that $\|x_i\| < \delta$ implies $x = x_i - a_0, a_0 < x_i \in K(x_i, \varepsilon)$, whence $\omega(x) = \omega(x_i) + \omega(a_0) < \varepsilon/2$. Thus $\omega(x)$ is continuous at $x = 0$ and it is sufficient to apply (f).

(h) follows directly from (g).

3.3. For a more detailed study of the convergence and divergence of the sequence $\{U_n(x)\}$ we shall introduce some numerical constants connected with this sequence.

The supremum of the numbers $l$ for which the set $R_l \neq 0$ will be called the index of oscillation or shortly $\omega(x)$-index of the sequence $\{U_n(x)\}$.

Thus $l = \sup_{x \in X} \omega(x)$. The greatest number $\sigma$ for which the set $R_\sigma$ is residual will be called the index of residual oscillation or shortly $\omega(x)$-index of the sequence $\{U_n(x)\}$. Its existence follows from the fact that $\omega(x), \omega(x) < \varepsilon$, implies $R_\varepsilon = \prod_{n=1}^\sigma R_n$. 


The number
\[ \omega_0 = \lim_{t \to 0} \sup_{x \in (0,t)} \omega(x) \]
will be called the zero-oscillation or shortly the \( \omega_0 \)-oscillation of the sequence \( \{U_n(x)\} \).

The number
\[ \mu_0 = \lim_{t \to 0} \sup_{x \in (0,t)} \|U_n(x)\| \]
will be called the zero-modulus of continuity or shortly \( \mu_0 \)-modulus of the sequence \( \{U_n(x)\} \).

These constants may assume infinite values.

**3.4. Theorem 1.** If the space \( X \) satisfies condition (\( \Sigma \)), then

(i) \[ \omega_0 \leq 2\sigma_1, \]

(ii) \[ \mu_0 \leq 2\sigma_1. \]

**Proof.** Let \( \sigma_1 < \infty \), \( \sigma_2 < \sigma \). Then \( S_2 \) is an \( F_2 \)-set of the second category, whence \( S_2 \) contains a sphere \( K(a_1, \theta) \). By (\( \Sigma_1 \)) there exists a \( \delta > 0 \) such that \( \|x\| < \delta \) implies \( x = x_0 + w \) with \( x_0, w \in K(a_1, \theta) \). Hence
\[ \omega(x) \leq \omega(x_0) + \omega(\theta) < 2\sigma, \]
and this implies (i).

Now let us prove (ii). It is sufficient to consider only the case \( \sigma_2 < \infty \). Let \( \varepsilon > 0 \) be chosen freely, then let \( \sigma_1 < \sigma_1 + \frac{\varepsilon}{2} \) and set
\[ S_2^\varepsilon = \left\{ x \in S_2 : \sup_{n \in \mathbb{N}} \|U_n(x) - U_n(x_0)\| < \sigma_1 + \frac{\varepsilon}{2} \right\}, \]

these sets are closed, \( S_2 = \bigcup_{n=1}^\infty S_2^\varepsilon \), and since \( S_2 \) is of the second category, one of the sets \( S_2^\varepsilon \), say \( S_2^\varepsilon_1 \), contains a sphere. By (\( \Sigma \)) we get for \( \|x\| < \delta \)
\[ \sup_{n \in \mathbb{N}} \|U_n(x) - U_n(x_0)\| \leq 2\sigma_2 + 2\varepsilon, \]
\[ \sup_{m \in \mathbb{N}} \|U_m(x)\| \leq 2\sigma_1 + 2\varepsilon + \|U_1(x)\|, \]
whence for sufficiently small \( \|x\| < \delta \)
\[ \mu_0 - \varepsilon \leq \sup_{n \in \mathbb{N}} \|U_n(x)\| \leq 2\sigma_2 + 3\varepsilon, \]
and therefore \( \mu_0 \leq 2\sigma_2 + 3\varepsilon. \)

**Theorem 2.** In every space \( X \) the following inequalities are satisfied:

(i) \[ \omega_0 \geq \sigma_1, \]

(ii) \[ \omega_0 \leq 3\mu_0. \]

If \( X \) satisfies condition (\( \Sigma \)) or (\( \Sigma_2 \)), then the inequalities

(iii) \[ \mu_0 \leq 2\omega_0 \]

are satisfied respectively.

**Proof.** Since \( \omega(x) \geq \sigma_1 \) is a residual set, we get
\[ \sup_{x \in S_2 \cap \mathbb{R}^n} \omega(x) \leq \sigma_1, \]
and hence \( \omega_0 \geq \sigma_1 \). The inequalities
\[ \sup_{x \in S_2 \cap \mathbb{R}^n} \|U_n(x) - U_n(x_0)\| \leq 2\sup_{x \in S_2 \cap \mathbb{R}^n} \|U_n(x)\| \]

imply (ii'), without any additional hypotheses about \( X \).

If \( X \) satisfies condition (\( \Sigma \)), condition (ii') directly results from (ii') and (i).

Now let us consider the case where \( X \) satisfies condition (\( \Sigma_2 \)). Choose a sequence \( x_n \to 0 \) and a sequence \( \{t_n\} \) of indices so that
\[ \lim_{n \to \infty} \|U_{t_n}(x_n)\| = \mu_0. \]

By (\( \Sigma_2 \)) there exists a sequence \( \{a_n\} \) satisfying conditions (\( \Sigma_3 \)):
(i)-(iii) of 1.31 and condition 2° of LO(I), p. 270, and such that
\[ \lim_{n \to \infty} \|U_{t_n}(a_n)\| = \mu_0 \]
where \( a_n \to a_0 \). Similarly to 3.4 (proof of Theorem 2), we can prove that
\[ \sum_{n=1}^\infty \frac{1}{a_n} \frac{\tilde{a}_n}{\mu_0} \leq 2 \cdot \mu_0. \]

For any \( a \in (a_n) \) belonging to the space (I) (cf. LO(I), p. 243) consider the sequence of operations \( \{V_n(a)\} \) defined by the formula
\[ V_n(a) = \sum_{k=1}^\infty a_k U_k(\tilde{a}_n/2). \]

Denoting the zero-modulus of continuity and the zero-oscillation of this sequence by \( \mu_0^2 \) and \( \omega_0^2 \) respectively, we get \( 2\omega_0^2 \leq \mu_0^2, \omega_0^2 \leq a_0. \) Hence, space (I) satisfying condition (\( \Sigma_4 \)), we get \( \mu_0^2 \leq 16\omega_0^2, \mu_0 \leq 4a_0 \).

\(^1\) In LO(I), p. 276, line 12, 14 and 17, \( \tilde{a}_n \) is misprinted for \( \tilde{a}_n/2. \)
3.41. We need the following

**Lemma.** Let $K(x_1, x_2)$ be a fixed sphere, and denote by $\gamma$ the infimum of the numbers $\sigma$ for which the set $S_\sigma$ is dense in $K(x_1, x_2)$. Then, given any $x \in K(x_1, x_2)$,

$$\omega(x) \leq 2a_0 + \gamma.$$ (k)

Suppose that $\omega(x) < \infty$; then $\omega(x) < \infty$ everywhere. If $x \in K(x_1, x_2)$ and $y \in S_\sigma, K(x_1, x_2)$, $\sigma > \gamma$ and the distance $d(x, y)$ is sufficiently small, we get by 3.21(b), $\varepsilon > 0$ being arbitrary,

$$\omega(x) - \omega(y) \leq 2a_0 ||x - y|| \leq 2a_0 \varepsilon,$$ (l)

$$\omega(x) \leq 2a_0 + \sigma + \varepsilon.$$ (l')

3.42. **Theorem 3.** Suppose that the sequence $\{u_n(x)\}$ converges in a set dense in $X_n$. Then

(i) $\lambda_n \geq \omega(x)$, (i') $\lambda_n \geq \omega(x)$, (i'') $2\alpha \geq \lambda_n$.

Proof. (i) and (i') trivially result from the definitions. To prove (i''), we apply lemma 3.41 (in this case $\gamma = 0$).

3.43. **Theorem 4.** If the space $X_n$ satisfies condition $\{(\Sigma_1)\}$ or $\{(\Sigma_2)\}$, then the following conditions are equivalent:

(m) $\alpha_n = 0$, \hspace{1cm} (m') $\mu_n = 0$,

(m'') $\omega(x)$ is continuous everywhere.

Proof. The equivalence of (m) and (m'') results from Theorem 1 and 2; $\omega_n = 0$ is equivalent to the continuity of $\omega(x)$ at $0$, and this by 3.21(b), implies the continuity of $\omega(x)$ in the whole of $X_n$.

Remark. By Theorem 2, (m) and (m'') follow from (m') without any additional assumptions on $X_n$.

Theorem 5. If the space $X_n$ satisfies condition $\{(\Sigma_1)\}$, then each of the conditions (m), (m'), (m'') is equivalent to the following:

(n) $\omega_n = 0$.

Proof. This is an immediate consequence of theorems 1, 2 and 4.

We complete these theorems by the following remark. If $X_n$ are Banach spaces, then the $\mu_n$-modulus may assume the values 0 or $\infty$ only; $\mu_n = 0$ if and only if $\lim_{n \to \infty} \|u_n\| < \infty$, where $\|u_n\| = \sup_{x \in X_n} \|u_n(x)\|$.

Indeed, if $\mu_n < \infty$, then there exists a $\varepsilon > 0$ such that $\|u_n(x)\| \leq \mu_n + 1$ for $n = 1, 2, \ldots$ and $\|x\| = \|x\| = 1$, whence

$$\|u_n(x)\| \leq \mu_n + 1 \leq \varepsilon \|x\|$$

for $x \in X_n$;

it follows hence that the $\mu_n$-modulus of the sequence $\{u_n(x)\}$ is equal to 0. The last inequality implies also $\|u_n\| \leq 1/\varepsilon$. If

$$\lim_{n \to \infty} \|u_n\| \leq \infty,$$

then from the inequalities $\|u_n(x)\| \leq \|u_n\| \leq \infty$ it is apparent that $\mu_n = 0$.

Let us notice that every Saks space has the following property:

3.44. If $K(x_1, x_2) \subset C$, then $C = X_n$.

For the proof let us remark first that we may assume $\|x_n\| < 1$. Let $x'_n = \min(1, \|x_n\|)$. Then every element $x$ satisfying the inequalities $\|x - x_n\| \leq \|x'_n - x_n\| \leq \|x - x'_n\|$ is in $K(x_1, x_2)$. Therefore $x_n + 2\alpha x_k x(x_1, x_2)$ for every $x \in X_n$ and sufficiently small $\alpha$, whence $x_n + 2\alpha x \in C$, and $x_n \in C$ implies $x \in C$.

3.45. **Theorem 6.** Let $\mu_n = 0$; then for every Saks space $X_n$ the set $C$ of the points of convergence is either non-dense or identical with the entire space.

1) If $\mu_n = 0$, the set $C$ is of the first category, whence $D$ is residual if $X_n$ satisfies condition $\{(\Sigma_1)\}$; if $X_n$ satisfies condition $\{(\Sigma_2)\}$, then the set $D$ is dense in $X_n$.

Proof. Ad 1\perp. If $\mu_n = 0$ and $X_n$ is an arbitrary Saks space, then by the remark which follows Theorem 4, the set $C$ is closed. It suffices to apply 3.44.

Ad 2\perp. If $X_n$ satisfies condition $\{(\Sigma_2)\}$, it suffices to apply Theorem 1; if $X_n$ satisfies condition $\{(\Sigma_1)\}$, Theorems 2 and 3 and 3.44 lead to the desired result.

The following theorem is now immediately obtained:

**Theorem 8.** Let $X_n$ satisfy condition $\{(\Sigma_1)\}$; then the set $C$ of convergence of the sequence $\{u_n(x)\}$ is either of the first category, or identical with $X_n$.

We complete Theorem 6 by the following remark. Let $X_n$ satisfy condition $\{(\Sigma_2)\}$ and let the sequence $\{u_n(x)\}$ converge everywhere; then this sequence converges uniformly on every compact set.

Suppose that the set $X_n$ is compact and that the sequence $\{u_n(x)\}$ does not converge uniformly on $X_n$. Then there exists an $\varepsilon > 0$ and elements
\[ a_n \in X^* \text{ such that the sequence } [a_n] \text{ converges to an element } a \text{ of } X^* \]
\[ \|U_n T(z_n) - U_n(z_n)\| \geq \varepsilon \]
for \( i = 1, 2, \ldots \). This, however, is impossible for \( \mu = 0 \).

**Theorem 7.** Let \( U_n(x) \to U(x) \) in the whole of \( X_n \). If \( X_n \) satisfies condition \((\Sigma)\) or \((\Sigma')\), then \( U(x) \) is a \((X_n, \Psi)\)-linear operation \(^*\).

Proof. The additivity of \( U(x) \) is obvious. By Theorem 6, \( \mu = 0 \), whence it immediately follows that \( U(x) \) is continuous at 0. Now it suffices to apply 2.2 (A).

**3.6.** Theorems 6 and 2.2 immediately imply the following one:

Let \( X_n \) satisfy condition \((\Sigma)\) or \((\Sigma')\) and let \( X^* \) be a family of linear functionals in \( X \) such that every sequence \( \eta \in Y^* \) contains a subsequence convergent everywhere in \( X \); then the functionals \( \eta \in Y^* \) are equicontinuous.

**3.5.** By \( B \) we shall henceforth denote the set of those \( x \in X \) for which the sequence \( \{U_n(x)\} \) is bounded, by \( U \) the set \( X_n - B \), i.e. the set of those points at which the sequence is unbounded.

**Theorem 8.** Let \( X \) be an arbitrary Saks space. Then \( B = X \), if and only if the following condition is satisfied:

(*) Given any \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that

\[ |\delta| \leq \varepsilon, \quad |\delta|^n \leq |\delta| \quad \text{implies} \quad \|U_n(x)\| < \varepsilon \quad \text{for} \quad n = 1, 2, \ldots \]

Proof. Let \( V_n(x) \) denote the extension of \( U_n(x) \) as in 3.1(1). If \( B = X_n \), then the sequence \( \{V_n(x)\} \) is bounded for every \( x \in X_n \), as can easily be seen. By a known theorem (Mazur and Orlicz [4]) there exists an \( \varepsilon > 0 \) such that

\[ \|V_n(x)\| < \varepsilon \quad \text{for} \quad n = 1, 2, \ldots, |\varepsilon| + |\varepsilon|^n \leq \delta \]

Let us choose \( \delta' \) so that \( |\delta|^n \leq \delta' \) imply \( |\delta|^n \leq \delta' / 2 \) for \( |\delta| \leq 1 \). Let \( \delta = \min(1, \delta', \delta / 2) \). Since \( |\delta| \leq |\delta|, \quad |\delta|^n \leq |\delta| \), \( \delta \in X_n \) and \( |\delta|^n \leq |\delta'| \leq \delta \), it follows from \( U_n(x) = V_n(x) \) and (3) that (2) is satisfied.

To prove the necessity let us notice that condition (*) implies \( \|U_n(x)\| = \|U_n(x)\| < \varepsilon \) for almost all \( n \) if \( \delta = 0 \).

**3.51.** (A) The inequality \( |\delta|^n \leq |\delta| \) in condition (2) may be replaced by \( x \in X_n \) if one of the following conditions is satisfied:

(a) the fundamental space \( X \) is complete with respect to the norm \( \|\| \),
(b) \( \|\| \) is a \( B \)-norm.

\[ (a) \quad \text{Concerning this theorem in the case when the condition } (\Sigma) \text{ is satisfied cf. Alexiewicz [1].} \]

The proofs result from the fact that in case (a) \( |\delta|^n \to 0 \) implies \( \|U_n(x)\| \to 0 \) and in case (b) the norm \( \|\| \) is homogeneous.

Let us denote by \( \mu_\varepsilon(\phi) \) the zero-module of continuity of the sequence \( \{\phi_n(x)\} \). Then it is easily seen that

\[ \text{(B)} \quad \text{Condition (*) is equivalent to the relation} \]

\[ (4) \quad \lim_{\varepsilon \to 0} \mu_\varepsilon(\phi) = 0. \]

For \( |\phi| \leq 1 \) the inequality

\[ (5) \quad \mu_\varepsilon \leq \mu_\varepsilon(\phi) \]

is always satisfied.

**Theorem 9.** 1. If for arbitrary \( X_n \), if \( \lim \mu_\varepsilon(\phi) = 0 \), the set \( B \) of points of boundedness is identical with the whole space.

2. For every \( X_n \), if

\[ \lim_{\varepsilon \to 0} \mu_\varepsilon(\phi) > 0, \]

the set of unboundedness \( U \) is dense in \( X_n \). If \( X_n \) satisfies condition \((\Sigma)\), \( B \) is of the first category, whence the set \( U \) is residual.

Proof. Ad 1. It suffices to apply Theorem 8 and 3.5(II).

Ad 2. As in 3.44, we prove that \( B \) contains a sphere, then \( B = X_n \). If

\[ \varepsilon_\varepsilon = \lim \mu_\varepsilon(\phi) > 0, \]

and \( X_n \) is an arbitrary Saks space, then the set \( U \) is non-empty whence, by the foregoing remark, it is dense in \( X_n \). Suppose that \( X_n \) satisfies condition \((\Sigma)\). Then there exist a sequence \( \phi_n \to 0 \), a sequence of elements \( \{x_n\} \) and a sequence \( \{x_n\} \) of indices such that \( \|\phi_n U_n(x_n)\| > \varepsilon_\varepsilon / 2 \) and \( |x_n|^n \to 0 \). Then the zero-module of the sequence of operations \( \{\phi_n U_n(x_n)\} \) exceeds \( \varepsilon_\varepsilon / 2 \), whence by Theorem 6 this sequence diverges in a residual set \( D \). Obviously \( B \cap U \).

Suppose that the set \( B \) is dense in \( X_n \) and \( B = X_n \). If \( X_n \) is a Banach space, then by 3.51 (A) and (2)

\[ |\phi_n U_n(x_n)| \leq \varepsilon \quad \text{for} \quad n = 1, 2, \ldots, x \in X_n, \]

whence the inequality

\[ \|U_n(x)\| = |\phi_n U_n(x_n)| < \varepsilon, \quad n = 1, 2, \ldots, \]

is satisfied for \( |x| = |x|^n \leq \varepsilon \).

It follows that \( \mu_\varepsilon = 0 \) and, by Theorem 6, \( C = X_n \) in this case.

Let us notice that for arbitrary Saks spaces, even those which satisfy
condition \((\Sigma_i)\) or \((\Sigma_2)\), the condition \(B=X_\psi\) does not imply \(\mu_0=0\); the set \(C\) may in that case be dense in \(X_\psi\) without, however, being identical with the whole space \(X_\psi\). On the other hand, by \((5)\), \(\mu_0=0\) implies \(B=X_\psi\).

3.6. Let \(y_n\) be elements of a Fréchet space \(X\) satisfying the following conditions:

\begin{enumerate}
  \item[(a)] the set \(y_n, i=1,2,\ldots,\) is bounded,
  \item[(b)] \(\sup_{n=1}^{\infty} \|y_n\| \to \eta > 0\) for \(n=1,2,\ldots,\)
  \item[(c)] the series \(\sum_{n=1}^{\infty} \|y_n\|, i=1,2,\ldots,\) are uniformly convergent in
  the interval \(0 \leq \theta \leq 1,\)
  \item[(d)] \(\lim_{i \to \infty} y_n = y_\ast\) for \(n=1,2,\ldots,\)
  \item[(e)] \(\lim_{i \to \infty} y_\ast = 0.\)
\end{enumerate}

Under these hypotheses there exists a nought-or-one sequence \(\{\lambda_n\}\) such that the series

\[ x_i = \sum_{n=1}^{\infty} \lambda_n y_n \]

converges for \(i=1,2,\ldots,\) and the sequence \(\{x_i\}\) is bounded and divergent.

Proof. Suppose, for a moment, that \(y_n=0\) for \(n=1,2,\ldots;\) this, together with the hypotheses \((a)-(e)\), implies the possibility of defining two increasing sequences of indices \(\{k_n\}\) and \(\{l_n\}\) such that

\[ 1^\circ \sum_{n=k_n+1}^{\infty} \|y_n\| < \frac{1}{2^k} \quad \text{for} \quad i=1,2,\ldots, k_n, \quad 0 \leq \theta \leq 1, \]

\[ 2^\circ \sum_{l=1}^{l_n} \|y_l\| < \frac{1}{2^k} \quad \text{for} \quad i=1,2,\ldots, l_n, \quad 0 \leq \theta \leq 1 \]

(we set \(k_0=1\)). For \(i_{n+1} \leq n \leq l_n, \quad k=1,2,\ldots, \quad 0 \leq \theta \leq 1,\) the following inequality is satisfied:

\[ \sum_{n=k_n+1}^{\infty} \|y_n\| \leq \sum_{l=k_n+1}^{\infty} \|y_n\| + \sum_{l=k_n+1}^{\infty} \|y_l\| < 1/2^{k-1} + 1/2^{k+1}. \]

By \((a)\) there exists an index \(j_n (n=1,2,\ldots)\) such that

\[ \|y_{j_n}\| \approx \frac{3}{4} \eta. \]

Let us write

\[ \lambda_n = \begin{cases} 1+(-1)^k & \text{for} \quad n = j_n, k=1,2,\ldots, \\ 2 & \text{for} \quad n = j_n, k=1,2,\ldots, \\ 0 & \text{elsewhere}; \end{cases} \]

then we define the elements \(z_i\) by formula \((6)\).
Given a sequence \( \{ U_n(x) \} \) we shall denote by \( C^u(Y) \) the set of those \( \varepsilon \in X \) for which this sequence is \( Y \)-weakly convergent to an element of \( Y \).

**Theorem 11.** Let \( Y \) be a separable space and let \( X \) satisfy the condition \( (\Sigma) \); then the set \( C^u(Y) \) is either of the first category or identical with the entire space.

Proof. Suppose that the set \( C^u(Y) \) is of the second category. By Theorems 6' and 7 for every \( \varepsilon \in Y \), the functionals \( \eta(U_n(x)) \) are convergent in the whole space \( X \) to a linear functional \( \xi_\varepsilon(x) \); moreover, there exists \( v(\varepsilon) \in Y \) such that \( \xi_\varepsilon(x) = \eta(v(\varepsilon)) \) for \( x \in C^u(Y) \). Let \( \eta \in Y \); since the norms \( \| y \| \) are bounded, the separability of \( Y \) implies that the sequence \( \xi_\varepsilon(x) \) contains a partial sequence, convergent for every \( x \in C^u(Y) \), whence, by Theorem 6', in the whole of \( X \). The functionals \( \xi_\varepsilon(x) \) are thus, in virtue of 3.46, equiconvergent everywhere. This implies

\[
\| \eta(v(x)) - v(x) \| \rightarrow 0
\]

when \( x_1, \varepsilon \in C^u(Y) \), \( x_1, x_2 \). Indeed, it is sufficient to choose \( \eta \in Y \) so that

\[
| \xi_\varepsilon(x_1) - \xi_\varepsilon(x_2) | = | \eta(U_{i}^2 - v(x_1)) | = \| U_{i}^2(x_1) - v(x_1) \| \leq \varepsilon \| U_{i}^2(x_1) - v(x_1) \| / 2,
\]

and then apply equiconvergence at 0 of the functionals \( \xi_\varepsilon(x) \) (in the formula above, \( \varepsilon \) denotes the constant in condition 2.3 (\( f \)). Now we define, for \( x \in C^u(Y) \), the operation

\[
V(x) = \lim_{\varepsilon \rightarrow 0} \eta(v(x))
\]

where \( x_1 \in C^u(Y) \), \( x_2 \rightarrow x \). By (8), this kind of definition is justified and the defined operation is uniquely determined; moreover \( V(x) = v(x) \) for \( x \in C^u(Y) \) and \( V(x) \) is a \( (X, Y) \)-continuous operation in \( C^u(Y) \). Since \( x_1 \rightarrow x_2 \in C^u(Y) \), \( \eta \in Y \), implies

\[
\xi_\varepsilon(x_1) = \eta(v(x_1)) \rightarrow \xi_\varepsilon(x_2)
\]

\[
\eta(v(x)) \rightarrow \eta(V(x))
\]

we get \( \xi_\varepsilon(x_1) = \eta(V(x)) \) in \( C^u(Y) \). There exists a sphere \( K \) contained in \( C^u(Y) \); therefore

\[
\xi_\varepsilon(x) = \eta(V(x))
\]

for every \( x \in K \). Hence, by \( (\Sigma) \), we see that the operation \( V(x) \) may be extended over the whole space \( X \), so that (9) be satisfied.

**Theorem 11**. The assertion of Theorem 11 remains unaffected if we replace the hypothesis of separability of \( Y \) by that of separability of \( X \) and suppose, moreover, that \( Y \rightarrow Y \).
Proof. Let $X_n$ denote the closed linear span of the elements $\{U_n(x)\}$ where $x \in X$, $n=1,2,\ldots$. The space $X_n$ being separable, $(X_n, Y)$-continuity of $U_n(x)$ implies that the set $Y_n$ is separable. Since, for every $x \in G(Y)$ and $y \in Y$, $\eta(U_n(x)) = \eta(y_n(x))$ where $v(x) \in Y$, it follows by a well-known theorem that the elements $v(x)$ are in $Y_n$, and it is sufficient to apply Theorem 11 with $Y$ replaced by $X_n$.

**Theorem 12.** Let the sequence $\{U_n(x)\}$ be $Y$-weakly convergent to $U(x)$ for every $x \in X$. Then the following conditions are sufficient in order that the weak limit $U(x)$ be an $(X_n, Y)$-linear operation:

1. $Y$ is separable; $X_n$ satisfies condition $\left(\Sigma_1\right)$;
2. $Y$ is an arbitrary Banach space, $X_n$ is separable and satisfies condition $\left(\Sigma_1\right)$, and $X_n=Y$;
3. $Y$ is an arbitrary Banach space satisfying condition $\left(\Sigma_1\right)$ (see LO(1), p. 371), $X_n$ satisfies condition $\left(\Sigma_1\right)$;
4. $Y$ is an arbitrary Banach space, $X_n$ satisfies condition $\left(\Sigma_2\right)$, and $X_n=Y$.

Proof. Let us remark first that in each case under consideration the uniqueness of the weak limit implies the additivity of the operation $U(x)$. Moreover, by Theorem 7, $\eta(U_n(x))$ is a linear functional in $X_n$ for every $x \in X_n$. In case (1), (2), and (3) the $(X_n, Y)$-continuity follows by the application of Theorems 1, 1', and 3' of LO(1) respectively. In case (4) the proof is analogous to that of Theorem 1 of LO(1). Only, it is to be noticed that, actually, in the proof on p. 268 in LO(1) we do not use the separability of the whole space $Y$ but only of the closed linear span $X_n$ of the elements $U_n(x)$. The hypothesis $X_n=Y$ ensures the separability of $Y$.

3.8. Let $A$ denote the space of nought-or-one sequences $\lambda=\{\lambda_n\}$, $Y$ be a Fréchet space. The series

$$\sum_{n=1}^{\infty} \lambda_n y_n$$

is called perfectly bounded if for arbitrary $\lambda \in A$ the partial sums of the series $\sum_{n=1}^{\infty} \lambda_n y_n$ compose a bounded set; the series is called perfectly convergent if for every $\lambda \in A$ the series $\sum_{n=1}^{\infty} \lambda_n y_n$ is convergent. It is easily seen that every perfectly convergent series is perfectly bounded. In some spaces perfect boundedness of a series implies its perfect convergency; according to the terminology adopted in LO(1) such spaces are said to have property (Z). If $Y$ is a Banach space, then the series $(*)$ is perfectly bounded if and only if there exists a $K>0$ such that

$$\|\sum_{n=1}^{\infty} \lambda_n y_n\| \leq K$$

for every $\lambda_n \in A$ and $i=1,2,\ldots$.

The set $Z_n$ (the set $Z_n^\prime$) of those sequences $x=\{y_n\}$ for which the series $(*)$ is perfectly convergent (perfectly bounded) forms a linear space under the usual definitions of addition and multiplication by scalars. If we define the norm of the element $x \in Z_n$ by the formula

$$\|x\|_n = \sup_{\|y\|_n \leq 1} \sum_{n=1}^{\infty} \lambda_n y_n,$$

then $Z_n$ becomes a Fréchet or Banach space according to the space $Y$. The same norm may be defined in the space $Z_n^\prime$. The series $(*)$ is called absolutely convergent if the series $\sum_{n=1}^{\infty} |y_n|$ converges. Let $Z_n$ denote the set of those sequences $x=\{y_n\}$ for which the series converges absolutely.

**3.31.** The set $Z_n$ is linear under the usual definitions of addition and multiplication by scalars if and only if the following condition is satisfied:

1. There exists $a_n > 0$ such that $|a| \leq a_n$ implies $|a y| \leq K(a) |y|$, where $K(a) < \infty$, and $\varphi$ may depend on $a$.

If (1) is satisfied, then it is easy to prove that, given real $a, \beta$, there exists a $K < \infty$ such that for sufficiently small $|y|, |y'|$,

$$||y\varphi(y-\beta y')| \leq K(||y|| + ||y'||).$$

Hence $\sum_{n=1}^{\infty} |y_n| < \infty$ and $\sum_{n=1}^{\infty} |y_n| < \infty$, implies $\sum_{n=1}^{\infty} |a y_n + \beta y_n| < \infty$. Suppose now that the set $Z_n$ is linear and condition (1) is satisfied. Then there exist an $a > 0$ and a sequence $\{y_n\}$ such that

$$|\lambda y_n| \geq a_n |y_n|,$$

$|y_n| \leq 1/n^2$, $b_n \geq n$ for $n = 1,2,\ldots$.

Let us choose positive integers $n_n$ so that $1/n^2 \leq r_n < 2/n^2$, set $r_n = 0$ and let

$$y_n = y_n$$

for $r_1 + r_2 + \cdots + r_n < i < r_{n+1} + r_1 + \cdots + r_n$.

Obviously

$$\sum_{n=1}^{\infty} |y_n|^2 \leq \sum_{n=1}^{\infty} \frac{2}{n^2} < \infty.$$

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on the other hand, however,
\[ \sum_{n=1}^{\infty} |y_n|^\alpha \geq \sum_{n=1}^{\infty} \frac{b_n}{n^\alpha} = \infty, \]
which is contradictory.

3.82. In $\mathcal{Z}_a$ we define the norm as
\[ \|x\|_a = \sum_{n=1}^{\infty} |y_n|. \]

The set $\mathcal{Z}_a$ with the norm defined by (10) is a Fréchet space if and only if condition (o) is satisfied.

The necessity being trivial, we shall prove the sufficiency only. We need the following Lemma:

Let $\xi_n(a)$ be continuous functions in $-\infty < a < \infty$, satisfying the conditions:
1. $\xi_n(a') + \xi_n(a) \geq \xi_n(a') \xi_n(a)$ for arbitrary $a'$, $a$,
2. $\xi_n(-a) = \xi_n(a)$,
3. $\lim_{a \to \infty} \xi_n(a) = 0$ for every $a$.

Then there exist a $c > 0$ and $K > 0$ such that
\[ \xi_n(a) \leq K \quad \text{for} \quad |a| \leq c, \quad n=1,2, \ldots \]

The proof may be carried out by the classical category method, similarly to the proof of the well-known theorem on sequences of linear operations.

Now we proceed to the proof of our theorem. If condition (o) is satisfied, then it is obvious that norm (10) satisfies conditions 1.1(a), (b) and the space $\mathcal{Z}_a$ is complete. It remains to prove 1.1(c'). For this purpose it is sufficient to prove that in condition (o) we may always assume that $q$ does not depend on $\alpha$, $K(\alpha) \leq K < \infty$, provided that $q$ is sufficiently small. In the contrary case there would exist sequences $\alpha_n \to 0$, $y_n \to 0$ such that $|\xi_n(y_n)|/|y_n| \to \infty$. This, however, is impossible, for the functionals $\xi_n(a) = |\xi_n(y_n)|/|y_n|$ satisfy the conditions of the lemma (3) follows immediately by (o)), and the assertion of the lemma does not hold for these functionals.

Let $\mathcal{Z}_a^c (\gamma > 1)$ denote the set of all sequences $\mathbf{a} = \{a_n\}$ for which the series
\[ \sum_{n=1}^{\infty} |y_n|^\gamma < \infty, \]

In $\mathcal{Z}_a^c$ we define the norm as
\[ \|\mathbf{a}\|_{\mathcal{Z}_a^c} = \left( \sum_{n=1}^{\infty} |y_n|^\gamma \right)^{1/\gamma}. \]

In the same way as for $\mathcal{Z}_a$, one can prove

3.83. The space $\mathcal{Z}_a^c$ with norm (10') is a Fréchet space if and only if condition (o) is satisfied.

Theorem 3.82 and 3.83 may be completed by the trivial remark that if $Y$ is a Banach space, so are the spaces $\mathcal{Z}_a$ and $\mathcal{Z}_a^c$. Condition (o) is obviously satisfied if the norm is monotonic, i.e. satisfies the inequality $|y_n| \leq |y|$ for $0 < a < 1$, $y \in Y$ (this is the case in the spaces $S, K', \ell_1 (0 < a < 1)$ which are not Banach spaces).

3.84. Theorem 13. Let $X_n$ satisfy condition (Σ); then the set of those points $\mathbf{x}$ where the series
\[ \sum_{n=1}^{\infty} U_n(x) \]
converges perfectly (is perfectly bounded) is either of the first category, or identical with the space $\mathcal{X}_n$.
A similar proposition holds also for the set of the points of absolute convergence and for the set of those points for which $|U_n(x)| \leq \mathcal{E}$, the norm $\|\mathbf{x}\|$ in $Y$ being supposed to satisfy condition (o).

Proof. We shall prove the theorem for the set of points of perfect convergence. Suppose that the series converges perfectly in a set of the second category. Let us define an operation $V_0(x)$ from $X_n$ to $Z_p$, setting
\[ V_0(x) = \{ U_1(x), U_2(x), \ldots, U_n(x), 0, 0, \ldots \}. \]

It is easily seen that if the series (*) converges perfectly, then the series $\sum_{n=1}^{\infty} \lambda_n y_n$ form a sequence convergent to $0$ as $n \to \infty$ uniformly in $A'$

Set $Y = \{ V_0(x) ; \} \|V_0(x)\|_p \to 0$ as $x \to \infty$ in a set of the second category, whence by Theorem 8' the sequence $\{ V_n(x) \}$ converges in $Z_p$ for every $x \in X_n$.

3.9. In this section we shall illustrate the applications of the foregoing theorems on sequences of operations by means of some examples.

Let $A = [a_{ij}]; i,j = 1,2,\ldots$ be a matrix of real numbers. Let us write
\[ A_1 = \sum_{n=1}^{\infty} |a_{nn}|, \quad A_2 = \sum_{n=1}^{\infty} |a_{nn}|. \]

In the sequel we shall deal with matrices satisfying some of the following conditions:
By theorem 1.6 and from the above remarks it follows that:

(a) If the sequence \( A(x) \) converges in a set of the second category, then it converges in the whole of \( X_\sigma \), \( \mu_\sigma = 0 \), and the conditions \( (a) \), \( (b) \), \( (\gamma) \), \( (\eta) \), \( (\theta) \) are satisfied.

(\theta) If \( 0 < \mu_\sigma < \infty \), then the sequence \( A(x) \) is bounded, but in a residual set its oscillation is greater than \( \mu_\sigma^2 \).

(c) If \( \mu_\sigma = \infty \), then the sequence \( A(x) \) is unbounded in a residual set.

(4) If \( \mu_\sigma = 0 \) and condition \( (\gamma) \) is satisfied, then the sequence \( A(x) \) converges in the entire space \( X_\sigma \).

3.9. Let us now show that the matrix \( A \) satisfies condition 3.9, \( (a)-(\gamma) \) with \( \sigma = 1, a_n = 0 \) for \( n = 1, 2, \ldots \) Every matrix of this type defines a permanent (matrix)-method of summability, with the field of summability \( A^{**} \). By \( A(x) \) we shall denote the \( A \)-limit of the sequence \( x \); by \( A^*_x, A^*_x, A_x \) we shall denote the subsets of \( A^* \) composed of bounded sequences, bounded sequences \( A \)-summable to 0, and sequences \( A \)-summable to 0 respectively. The term \( "A \)-summability method \( A = (a_n) \)" will mean the method of summability corresponding to the matrix \( (a_n) \).

Let us now consider the Saks space \( X_\sigma \) (LO(1), p. 243-244) whose elements are sequences of \( A^*_x \) with the following definition of norms:

\[
|\alpha| = \sup_{n} |a_n|, \quad |\alpha|^2 = \sum_{n=1}^{\infty} |a_n|^2 + \sup_{n} |A(x)| ,
\]

where \( A(x) \) denotes the \( n \)-th transform of the sequence \( x \), defined by formula (11). As shown in LO(1), p. 248-249, the space satisfies conditions \( \Sigma \) and \( \Sigma' \).

THEOREM 14. Let \( A = (a_n) \) and \( B = (b_n) \) be permanent methods of summability. If every bounded \( A \)-summable sequence is \( B \)-summable, then \( A(x)B(x) \) for every \( x \in A^*_x \) (Mazur and Orlicz [3] and [3]).

Proof. The transforms \( B(x) \) are linear functionals in the space \( X_\sigma \), and, by hypothesis, \( B(x) \rightarrow B(x) \) in \( X_\sigma \). By theorem 7, \( B(x) \) is a \( X_\sigma \)-linear functional. The set of the sequences converging to 0 and with the terms absolutely less than 1 is dense in \( X_\sigma \) (Mazur and Orlicz [5], p. 137), and since, for every sequence converging to 0, \( B(x) = 0 \), this relation must hold in the whole of \( X_\sigma \) and thus in \( A^*_x \). To complete the proof, it suffices to write every sequence \( x \rightarrow [a_n] e_{\sigma} A^*_x \) in the form \( a_n = a_n + A(x) \), and notice that, by the permanency of method \( A \), \( \{a_n\} e_{\sigma} A^*_x \).
of the method $\Phi_{[\phi]}$, i.e., the set of all $\Phi_{[\phi]}$-summable sequences, will be denoted by $\mathcal{D}_{\phi}$; its subset composed of those sequences for which $\Phi(\phi)=0$, or of those which are bounded, or of those which satisfy both of these conditions will be denoted by $\mathcal{D}_{\phi}^*$, $\mathcal{D}_{\phi}^0$, and $\mathcal{D}_{\phi}^{0+}$ respectively.

**Theorem 16.** Let $\Phi_{[\phi]}$ be a permanent continuous functional method of summability such that the series

$$\sum_{n=1}^{\infty} |\phi_n(t)|$$

converges uniformly in every interval $(t, t')$ where $t < t < \infty$. Let the matrix method $A=(a_{nk})$ be permanent, and let every sequence $x \in \mathcal{D}_{\phi}^*$ be $\Phi_{[\phi]}$-summable. Then $A(x)=\Phi(x)$, for every $x \in \mathcal{D}_{\phi}^*$.

The theorem remains true when the method $A$ is replaced by a permanent continuous functional method $\mathcal{D}_{[\phi]}^{[\phi]}$.

**Proof.** Let us denote by $X$ the Banach space whose elements are the sequences of $\mathcal{D}_{\phi}^*$, the norms being defined as

$$||x||=\sup_{n} |\phi_n(t)|,$$

$$||x||^*=\sum_{n=1}^{\infty} |\phi_n(t)|/2^n + \sup_{t \in [0, \infty)} |\phi(t)|.$$

We shall prove that the sequences converging to 0 form a dense subset of $X$, and that conditions (15) and (16) are satisfied in $X$. Then the proof of our theorem may be carried out in the same way as that of Theorem 14.

The permanency of the method $\Phi_{[\phi]}$ and the hypothesis on the series (15) imply

$$K=\sup_{t \in [0, \infty)} \sum_{n=1}^{\infty} |\phi_n(t)| < \infty.$$  

Let $K_{\phi}(x, \phi)$ be an arbitrary sphere in $X_{\phi}$, let $x_{\phi}=[\phi_n(t)]$. To prove (15) and (16), it is sufficient, in view of 1.32, to prove that there exists an element $x \in X_{\phi}$ for which the condition (15) is satisfied. For $i=1, 2, \ldots$, let us define the sequences

$$d_{\phi}^0=\phi_n(t), \quad d_{\phi}^1=\phi_n(t), \quad \text{for} \quad n=1, 2, \ldots, i,$$

$$d_{\phi}^i=0 \quad \text{elsewhere}.$$

Let $x=[\phi_n(t)]$, $f(t)=\Phi(t, x)$, $f(t)=\Phi(t, x_n)$; by the hypotheses of our theorem and by (15) it follows that the functions $f(t)$ and $f(t)$ satisfy the hypotheses of 1.51 (B). By this Lemma, for every positive integer $s$ there exist non-negative numbers $d_{\phi}^0, d_{\phi}^1, \ldots, d_{\phi}^s$ such that

$$\sum_{n=1}^{\infty} d_{\phi}^s < \varepsilon/4.$$

Under analogous hypotheses, this theorem was first proved by M. Altman [2].
Let us consider the sequence of linear functionals in $X$,

$$T_n(x) = \int \frac{1}{n} x(t) y_n(t) dt.$$  

The following propositions are valid for the sequence $\{T_n(x)\}$:

(3) condition (a) implies $\Phi = \mu_0$,

(e) condition (a) is equivalent to $\mu_0 < \infty$,

(γ) condition (γ) is equivalent to $\mu_0 = 0$.

Hence theorems 1, 6, 9 imply that

(a) The sequence $\{T_n(x)\}$ converges in the whole of $X$, if it converges in a set of the second category. Moreover, in this case, $\mu_0 = \Phi = 0$ and conditions (a), (γ) are satisfied.

(b) If $0 < \mu_0 < \infty$, then the sequence $\{T_n(x)\}$ is bounded; in a residual set, however, its oscillation is not less than $\Phi/2$.

(c) $\mu_0 = \infty$ implies the unboundedness of the sequence $\{T_n(x)\}$ in a residual set.

(d) If condition (β) is satisfied and $\mu_0 = \Phi = 0$, then the sequence $\{T_n(x)\}$ converges in the entire space*. 

*) Theorems (a), (c), and (d) are known.

References


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