The Heisenberg group and the group Fourier transform of regular homogeneous distributions

by

SUSAN ELIZABETH SLOME (New York, NY)

Abstract. We calculate the group Fourier transform of regular homogeneous distributions defined on the Heisenberg group, $\mathbb{H}^n$. All such distributions can be written as an infinite sum of terms of the form $f(\theta)\overline{w}^{-k}P(z)$, where $(z,t) \in \mathbb{C}^n \times \mathbb{R}$, $w = |z|^2 - it$, $\theta = \arg(w/z)$ and $P(z)$ is an element of an orthonormal basis for the spherical harmonics. The formulas derived give the Fourier transform of the distribution in terms of a smooth kernel of the variable $\theta$ and the Weyl correspondent of $P$.

1. Introduction. In this paper we derive formulas for the group Fourier transform of regular homogeneous distributions on the Heisenberg group, $\mathbb{H}^n$. (We use coordinates $(z,t) \in \mathbb{C}^n \times \mathbb{R}$ on $\mathbb{H}^n$.) It can be shown that all such distributions can be expressed as an infinite sum $\sum f_i(\theta)\overline{w}^{-k_i}P_i(z)$. Here, $w = |z|^2 - it$, $\theta = \arg(w/z)$ and the $P_i(z)$ are elements of an orthonormal basis for the spherical harmonics.

The group Fourier transform is a map from $L^1(\mathbb{H}^n)$ into the space of families of bounded operators defined on a Hilbert space. In many applications the Hilbert space is taken to be $L^2(\mathbb{R}^n)$. The domain of definition of the transform can be extended to include tempered distributions on $\mathbb{H}^n$. The group Fourier transform is of interest because it extends to a unitary map from $L^2(\mathbb{H}^n)$ to the space of families of Hilbert–Schmidt operators. Also, the group Fourier transform (which we will denote by $\hat{\mathcal{F}}$) behaves nicely with respect to convolution defined by the group multiplication on $\mathbb{H}^n$. That is, $(f \ast g)\hat{\mathcal{F}} = \hat{f} \cdot \hat{g}$, where the multiplication on the right is composition of operators.

The group Fourier transform is closely related to the Weyl correspondence. In fact, the formulas we present give the group Fourier transform of a regular homogeneous distribution in terms of the Weyl correspondent of $P_i$. This correspondent will be denoted by $\mathcal{W}(P_i)$. The set

$$\{\mathcal{W}(P) \mid P \text{ is a homogeneous harmonic polynomial}\}$$

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is of particular interest in its own right. It was shown by D. Geller [5] that the $W(P)$ constitute operator analogues of spherical harmonics.

Since every regular homogeneous distribution $K(t, z)$ expands to a sum

$$\sum f_i(\theta)\overline{\omega^{-k_i}}P_i(z),$$

when calculating $R_H$ it is enough to consider terms of the form $f(\theta)\overline{\omega^{-k}}P$. We will see, in the case $P = 1$, that the Fourier transform is a diagonal operator with respect to the usual Hermite basis.

There is a classical analogue [9] to the result proved herein. If $K$ is a regular homogeneous distribution on $\mathbb{C}^n$, then $K$ is an infinite sum of terms of the form $\langle z \rangle^{-2k}P(z)$. The result referred to states that

$$F(\Gamma(k)|\langle z \rangle^{-2k}P(z)) = \Gamma(\binom{j}{2}P(\zeta))$$

where $F$ is the usual Fourier transform, and $j = n + 2 - k$. The calculation in our case is complicated by the presence of the function $f(\theta)$ in the expression for $K$.

Sections 2, 3, and 4 contain introductory definitions and results; for more detail see [7], [8], and [9].

In Section 6, we calculate the group Fourier transform of $K$, homogeneous of degree $>-2n-2$ (hence, $K$ defines a distribution). In Section 7 we consider $K$ homogeneous of degree $\leq -2n-2$. In this case $K$ no longer defines a distribution; however, it is possible to define a distribution $A_K$ which agrees with $K$ away from the origin. We compute the Fourier transform of $A_K$.

I wish to thank D. Geller for many enlightening discussions.

2. The Heisenberg group. The Heisenberg group, $H^n$, is a Lie group with underlying manifold $\mathbb{C}^n \times \mathbb{R}$. For $(\xi, t)$ and $(\eta, s)$ in $H^n$, the multiplication is given by

$$(\xi, t) \cdot (\eta, s) = (\xi + \eta, t + s + 2\text{Im}(\xi \cdot \eta)).$$

Let $\zeta = (z_1, \ldots, z_n)$ and $\zeta_j = z_j + iy_j$. Then the left invariant vector fields which agree with $\partial/\partial z_j, \partial/\partial y_j$, and $\partial/\partial t$ at the origin are, respectively, $X_j = \partial/\partial z_j + 2y_j\partial/\partial t, Y_j = \partial/\partial y_j - 2z_j\partial/\partial t$ and $T = \partial/\partial t$. These vector fields form a basis for the Lie algebra of $H^n$ and they satisfy the following commutation relations:

$$[Y_j, X_k] = 4\delta_{jk}T.$$

All other commutators are zero.

We shall be interested in a particular class of unitary representations of the Heisenberg group and the corresponding representations of the Lie algebra. For all real $\lambda$ different from zero, define a mapping $R_\lambda$ from $H^n$ to the group of unitary operators on $L^2(\mathbb{R}^n)$ by

$$(R_\lambda(\zeta, t)f)(x) = e^{2\pi i\lambda(\overline{\omega^{-\lambda}}+u+v/2+i/4)}f(x + v).$$

Here $\zeta = u + iv$ and $f \in L^2(\mathbb{R}^n)$. These representations are irreducible, and up to unitary equivalence these are all the irreducible, infinite-dimensional representations of the Heisenberg group.

We now turn to the connection between the above representation and the Weyl correspondence.

3. The Weyl correspondence. The Weyl correspondence was originally introduced in the development of quantum mechanics. Classical mechanics involves the study of functions dependent on $2n$ variables, $a(p_1, \ldots, p_n, q_1, \ldots, q_n)$. The quantum mechanic approach is to replace the $p_j$ and $q_j$ variables by operators $P_j$ and $Q_j$ acting on a Hilbert space $H$, satisfying the commutation relations

$$[P_j, Q_k] = (\lambda/(2\pi i))\delta_{jk}I.$$

The question then arises: how, in general, is the operator $a(P_1, \ldots, P_n, Q_1, \ldots, Q_n) = a(P, Q)$ defined? Weyl answered the question in the following way.

First consider the function

$$(a, b) = e^{2\pi i(u \cdot q + v \cdot p)}.$$

Since the operator $2\pi i(u \cdot Q + v \cdot P)$ is skew-adjoint, the operator $W(u, v) = e^{2\pi i(u \cdot Q + v \cdot P)}$ is unitary and this is the operator assigned to the exponential function (3). Then by analogy with the Fourier transform and its inverse, to any function $a(p, q)$ assign

$$(4) \quad W(a) = \int_{\mathbb{R}^{2n}} e^{2\pi i(u \cdot Q + v \cdot P)}a(u, v) \, du \, dv.$$

Here $\overline{a(u, v)} = \int_{\mathbb{R}^{2n}} e^{-2\pi i(u \cdot v + v \cdot u)}a(x, y) \, dx \, dy$. The operator $W(a)$ is called the Weyl correspondent to $a(p, q)$. One realization of this scheme is to take $L^2(\mathbb{R}^n)$ as the Hilbert space, fix $\lambda = 1$, and set

$$(a, b)(x) = (Q_j f)(x) = a(x, y) \quad \text{and} \quad (P_j f)(x) = b(x, y),$$

and

$$a(x, y) = \frac{1}{2\pi^2} \frac{\partial f}{\partial x_j}.$$

Given this choice of $H$ and $P$ and $Q$, we can see how $W(u, v)$ operates on a function $f$ in $L^2(\mathbb{R}^n)$. First observe that

$$(5) \quad W(\partial u \cdot v, \partial v \cdot u) = W(u(s + t), v(s + t)).$$
and
\[ \frac{\partial W}{\partial s}(u, v) \bigg|_{s=0} = 2\pi i(u \cdot Q + v \cdot P). \]

Now, \( W \) defined by
\[ [W(u, v)f](x) = e^{2\pi iux \cdot v} f(x + v) \]
satisfies properties (5) and (6). On the other hand, these properties uniquely determine \( W \).

More generally we work with operators which depend on the nonzero real parameter \( \lambda \). That is, we have unbounded operators \( Q_\lambda \) and \( P_\lambda \) and these in turn give rise to operators \( W_\lambda(u, v) \) for \( \lambda \in \mathbb{R}, \lambda \neq 0 \) defined by
\[ W_\lambda(u, v) = e^{2\pi i(u \cdot Q + v \cdot P_\lambda)}, \]
and corresponding Weyl correspondent \( \mathcal{W}_\lambda(a) \). We can view the operators \( P_\lambda \) and \( Q_\lambda \) as operators on Hilbert spaces \( H_\lambda \) also indexed by the nonzero real parameter. However, when convenient we may identify these Hilbert spaces and drop the subscript. Again, if we identify the \( H_\lambda \) with \( L^2(\mathbb{R}^n) \), then the \( P_\lambda \) and \( Q_\lambda \) can be defined by
\[ Q_\lambda f = \lambda x_j f \quad \text{and} \quad P_\lambda f = \frac{1}{2\pi i} \frac{\partial f}{\partial x_j}. \]

Notice that in this case, \( R_\lambda(\zeta, 0) = W_\lambda(u, v) \).

4. The group Fourier transform. The operators defined in (7) allow us to define a mapping from \( L^1(H^n) \) into the space of families of bounded operators on \( H_\lambda \). We denote this space of families by \( B \). The map is defined as follows:
\[ \widehat{f}_H(\lambda) = \int_{H^n} e^{\pi i \lambda \zeta / 2} W_\lambda(u, v) f(\zeta, t) d\zeta dt. \]

The operator \( \widehat{f}_H \) is called the group Fourier transform of \( f \). Observe that the following diagram commutes:
\[ \begin{array}{ccc} L^1(H^n) & \xrightarrow{\mathcal{F}_H} & \mathbb{A} \\ \downarrow W_\lambda & & \uparrow \mathcal{F}_\lambda \\ L^1(\mathbb{R}^{2n}) \end{array} \]

The diagonal arrow is the Fourier transform
\[ \mathcal{F}_\lambda(x, y) = \int_{H^n} e^{2\pi i(u \cdot x + v \cdot y + t\lambda / 4)} f(u, v, t) du dv dt. \]

In the case where \( H_\lambda \) is identified with \( L^2(\mathbb{R}^n) \) and if \( \phi, \psi \in L^2(\mathbb{R}^n) \), the operator \( \widehat{f}_H \) is given by
\[ \langle \widehat{f}_H(\lambda)\phi, \psi \rangle = \int_{\mathbb{H}^n} \langle R_\lambda(\zeta, t)\phi, \psi \rangle f(\zeta, t) d\zeta dt. \]

The group Fourier transform may also be defined in complex notation, and for what follows, complex notation will be more convenient.

Beginning with the Weyl correspondent, we wish to assign an operator to the function \( f(\zeta, \bar{\zeta}, t) \). To do this we start with unbounded operators \( W_{1+}, W_{1-}, \ldots, W_{n+}, W_{n-} \) on a Hilbert space \( H_\lambda \) satisfying
\[ W_{j+} = W_{j-}^*, \quad [W_{j+}^+, W_{k+}] = 2\delta_{jk} I. \]

The connection with the real case is given by the relation
\[ W_{\lambda} = P_{\lambda} + i Q_{\lambda}. \]

Then with the function \( e^{-x \cdot \bar{z} + \bar{x} \cdot z} \) associate the operator
\[ W_{\lambda} = e^{-x \cdot \bar{z} + \bar{x} \cdot z}. \]

The definition of the Weyl correspondent is analogous to (4). That is,
\[ W_{\lambda}(f) = \int_{\mathbb{C}^n} W_{\lambda}(x, \bar{z}) (F^{-1} f)(z) dV. \]

Here \( F^{-1} f \) is the inverse Fourier transform \( \int_{\mathbb{C}^n} \exp(z \cdot \bar{z} - \bar{z} \cdot z) f(z) dV. \) Now if \( f \in L^1(\mathbb{H}^n) \), the group Fourier transform is given by
\[ \mathcal{F}_H(\lambda) = \int_{\mathbb{H}^n} e^{\lambda \zeta} W_{\lambda}(x, \bar{z}) f(z, t) dV. \]

If we again take as our Hilbert space \( L^2(H^n) \), then one realization of the operators \( W_{j+}, W_{j-} \) is
\[ W_{j+} = \iota(2\lambda x_j + (1/2) \partial / \partial x_j), \]
\[ W_{j-} = -\iota(2\lambda x_j + (1/2) \partial / \partial x_j). \]

Associated with each \( H_\lambda \) there is a preferred orthonormal basis \( \{E_{\alpha \lambda}\} \) where \( \alpha \in (\mathbb{Z}^+)^n \). For a fixed value of \( \lambda \) the basis of \( L^2(H^n) \), \( \{E_{\alpha \lambda}\} \), associated with the operators \( W_\lambda \), \( W_\lambda^+ \) is defined as follows. Set
\[ E_{\alpha \lambda}(x) = (|\lambda|/\pi)^{n/4} e^{-2\lambda |x|^2}. \]

Then for all \( \alpha \in (\mathbb{Z}^+)^n \), define
\[ E_{\alpha \lambda} = \begin{cases} (2|\lambda|)^{n/4} e^{-2\lambda |\alpha|^2} E_{\alpha \lambda} & \text{if } \lambda > 0, \\ (2|\lambda|)^{n/4} e^{-2\lambda |\alpha|^2} E_{\alpha \lambda}^* & \text{if } \lambda < 0. \end{cases} \]

In the case \( \lambda = 1/4 \), notice \( \{-i|\alpha| E_{\alpha, 1/4}\} \) are the Hermite functions.
In general, for a fixed \( \lambda \), the operators \( W_{j\lambda} \) and \( W_{j\lambda}^* \) act as weighted shift operators with respect to \( \{ E_{\alpha \lambda} \} \). That is, \[
W_{j\lambda} E_{\alpha \lambda} = (2|\alpha| \lambda)^{1/2} E_{\alpha - e_k}, \quad \text{zero if } \alpha_k = 0,
\]
\[
W_{j\lambda}^* E_{\alpha \lambda} = [2(\alpha_k + 1)|\lambda|^{1/2} E_{\alpha + e_k}
\]
for \( \lambda > 0 \). The right sides are reversed if \( \lambda < 0 \). Here, \( e_k \) denotes \( \{0, 0, \ldots, 1, \ldots, 0 \} \in (\mathbb{Z}^+)^n \) with the 1 in the \( k \)th position.

Many of the nice properties of the usual Fourier transform have parallels for the group Fourier transform. For example,
\[
(f * g)_\lambda = \hat{f}_\lambda \overline{\hat{g}_\lambda}, \quad f, g \in L^1(\mathbb{H}^n).
\]

Here, the convolution is with respect to the group multiplication, and the multiplication on the right is composition of operators. Also, if we set
\[
Z_j = (1/2)(X_j + iY_j), \quad j = 1, 2, \ldots, n
\]
then
\[
(Tf)_\lambda = -i\lambda \hat{f}_\lambda, \quad (Z_j f)_\lambda = \hat{f}_\lambda W_{j\lambda}^*, \quad (Z_j f)_\lambda^* = -\hat{f}_\lambda W_{j\lambda}.
\]

In the last three equations, we assume that \( f \) is in the Schwartz space \( S(\mathbb{H}^n) \).

There is also an analog of the Plancherel theorem. Let \( B = \{ \text{bounded families } R; \text{for each } \lambda, \text{ } R(\lambda) \text{ is a bounded operator on } H_\lambda, \| R \| = sup_{\lambda} \| R(\lambda) \| < \infty \} \) and for all \( \alpha, \beta \) the map \( \lambda \mapsto (R(\lambda)E_{\alpha \lambda}, E_{\beta \lambda}) \) is measurable. Then the group Fourier transform (which henceforth we denote by \( \hat{\cdot} \)) is a map from \( L^2(\mathbb{H}^n) \) into \( B \) and \( \| \hat{f} \| \leq \| f \|_1 \). Further, if we set
\[
R_2 = \left\{ R: \text{for almost every } \lambda, \| R(\lambda) \|_2 < \infty \right\}
\]
then \( \| R \|_2 = \left\{ \| R(\lambda) \|_2(2|\lambda|)\lambda^n \right\} d\lambda < \infty \).

This pairing allows us to extend the definition of the group Fourier transform to the space of tempered distributions \( S'(\mathbb{H}^n) \). Suppose that \( R \) is an operator family and that (13) holds for all \( \lambda = \tilde{\lambda}, \tilde{f} \in S(\mathbb{H}^n) \). Then we say that for \( F \in S'; \tilde{F} = R \) in the sense of tempered distributions if \( F(f) = c_n(R\tilde{f}) \) for all \( f \in S(\mathbb{H}^n) \).

5. Regular homogeneous distributions on \( \mathbb{H}^n \). Consider the following dilations on the Heisenberg group: For \( r > 0 \), set \( D_r(\zeta, t) = (r\zeta, r^2t) \). Notice that \( D_r \) is an automorphism of \( \mathbb{H}^n \) whereas the usual dilation is not.

We wish to calculate the group Fourier transform of regular homogeneous distributions on \( \mathbb{H}^n \). Regular means that the distribution agrees with a \( C^\infty \) function away from the origin. A distribution \( K \) on \( \mathbb{H}^n \) is homogeneous of degree \( l \) if for all \( \phi(\zeta, t) \in S \),
\[
\langle K, r^{-2n-2}\phi(r^{-1}\zeta, r^{-2}t) \rangle = \langle r^l K, \phi \rangle.
\]

If \( K \) is a function, this condition is equivalent to requiring \( K(r\zeta, r^2t) = r^l K(\zeta, t) \).

It can be shown [3] that every regular homogeneous distribution on \( \mathbb{H}^n \) can be expressed in the form
\[
K(t, x) = \sum_i K_i(t, |x|^2)P_i(x)
\]
where \( \{ P_i \} \) form an orthonormal basis for the bigraded spherical harmonics and the \( K_i \) are homogeneous of the appropriate degree. We can write
\[
K_i(t, |x|^2) = f_i(\theta)\overline{\omega}^{-k_i}
\]
with \( w = w(t, x) = |x|^2 - it, \theta = \arg(\omega/w) \) and \(-2k_i + \deg P_i = \deg K(f_i(\theta) \text{ is homogeneous of degree zero})\).

In view of (14) and (15), when calculating the group Fourier transform of a regular homogeneous distribution, it is enough to consider distributions of the form
\[
K(t, x) = \sum_i K_i(t, |x|^2)P_i(x).
\]

This was done in [5] for the case where \( K(t, x) = \overline{\omega}^{-k} \omega^{-\gamma} P(x) \), that is, for \( f(\theta) = e^{it\theta} \). The precise result is stated below in Theorem 5.2, but first we need to consider the Weyl correspondent of a polynomial \( Q(z) \).

The proof of the following theorem is given in [5].

**Theorem 5.1.** Suppose \( P \) is a harmonic polynomial in \( \zeta, \zeta \) where \( \zeta \in \mathbb{C}^n \). If \( P = \sum \alpha_{\gamma} W^\gamma W^\gamma \) then \( \mathcal{W}(P) = \sum \alpha_{\gamma} W^\gamma W^\gamma = \sum \alpha_{\gamma} W^\gamma W^\gamma \).

In equation (14) we stated that the \( P_i \) were elements of an orthonormal basis for the bigraded spherical harmonics. We wish now to clarify that
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Theorem 6.6. Given the regular homogeneous distribution $K(t, x) = f(\theta)\omega^{-k}P(x)$, with $P \in H_{pq}$, $p + q = \kappa$ and $-2n - 2 < \kappa - 2\Re k < 0$, the group Fourier transform of $K$ is $\tilde{K}(\lambda) = J(\lambda)$, where $J$ is defined by

$$J(\lambda)E_{\alpha\lambda} = C\left(\int_{-\pi}^{\pi} f(\theta)K_M(\theta)d\theta\right)W_{\lambda}(P)E_{\alpha\lambda}.$$

Here $M = |\alpha| - p$ if $\lambda > 0$, $M = |\alpha| - q$ if $\lambda < 0$, and

$$C = (-1)^{q-p+1}2^{n-\kappa}|\lambda|^{-j},$$

where $j = n + \kappa + 1 - k$. The function $K_M$ is a smooth function of $\theta$ defined in equation (17) below.

The following theorem was proved in [5] as Proposition 7.1.

Theorem 5.2. Suppose $k, \gamma \in C$, $\kappa = p + q$. Suppose $\gamma$ and $k - \gamma$ are not equal to $0, -1, -2, \ldots$, and that

$$-2n - 2 < \kappa - 2\Re k < 0,$$

or that

$$\kappa \geq 1, \quad \kappa - 2k = -2n - 2,$$

Define

$$G_{k, \gamma}(w) = \Gamma(\gamma)\Gamma(k - \gamma)\omega^{-k}w^{-\gamma}$$

and

$$K_{k, \gamma}(u) = G_{k, \gamma}(w(u))P(x).$$

Let $j = n + \kappa + 1 - k$. Then $\tilde{K}_{k, \gamma}(\lambda) = J_{k, \gamma}(\lambda)P(x)$, defined by

$$J_{k, \gamma}(\lambda)E_{\alpha\lambda} = (-1)^{q-p+1}2^{n-\kappa}|\lambda|^{-j}c_{j\gamma}(\alpha, \lambda)W_{\lambda}(P)E_{\alpha\lambda},$$

where the $c_{j\gamma}$ are given as follows: Let $p' = p$ if $\lambda > 0$, $p' = q$ if $\lambda < 0$, $\gamma' = \gamma$ if $\lambda > 0$, $\gamma' = k - \gamma$ if $\lambda < 0$, and if $M = |\alpha| - p' \geq 0$ then

$$c_{j\gamma}(\alpha, \lambda) = |\lambda|^{-j}\Gamma(M + \gamma')\Gamma(j)\Gamma(M + \gamma' + j)^{-1}.$$
If \( c \neq 0, -1, -2, \ldots \), then define the hypergeometric function \( F \) by
\[
F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n
\]
where \( (a)_n = a(a+1)(a+2)\cdots(a+n-1) \).

If \( a + b = -l \), \( l \in \mathbb{Z}^+ \), then \( (-l)_n = (-1)^n n!/(l-n)! \), \( n \leq l \), so in these cases the sum is finite.

**Proposition 5.5.** If \( c \not\in \mathbb{Z}^- \) and \( \Re(c-a-b) > 0 \), then for \( z = 1 \),
\[
F(a, b; c; 1) = \Gamma(c) \Gamma(c-a-b)/\Gamma(c-a) \Gamma(c-b)^{-1}.
\]

**Theorem 5.6.** Suppose \( \Re(\gamma), \Re(k-\gamma) > 0 \) and \( M \) is a positive integer. Then
\[
\int_{-\pi}^{\pi} \Gamma(k-\gamma) \Gamma(\gamma) e^{i\gamma \theta} K_M(\theta) d\theta = \Gamma(\gamma+M) \Gamma(j)[\Gamma(\gamma+j+M)]^{-1}
\]
where \( K_M \) is defined by
\[
K_M(\theta) = (2\pi)^{-1} e^{i(1-k)\theta} (e^{i\theta} + 1)^{k+j-2} \times \Gamma(j) \Gamma(k+j-1) \Gamma(\gamma+j+M)^{-1} F(-M, j; k+j-1; e^{i\theta} + 1).
\]

**Proof.** By Definition 5.4,
\[
\int_{-\pi}^{\pi} \Gamma(k-\gamma) \Gamma(\gamma) e^{i\gamma \theta} e^{i(1-k)\theta} (e^{i\theta} + 1)^{k+j-2} \Gamma(j) \Gamma(k+j-1) \Gamma(\gamma+j+M)^{-1} \times F(-M, j; k+j-1; e^{i\theta} + 1) d\theta
\]
\[
= \int_{-\pi}^{\pi} (2\pi)^{-1} \Gamma(k-\gamma) \Gamma(\gamma) e^{i\gamma \theta} e^{i(1-k)\theta} (e^{i\theta} + 1)^{k+j-2} \Gamma(j) \times \sum_{m=0}^{M} \frac{1}{m!} (-M)_m(j)_m \Gamma(k+m+j-1) \Gamma(\gamma+j+M)^{-1} d\theta
\]
\[
= (2\pi)^{-1} \Gamma(k-\gamma) \Gamma(\gamma) \sum_{m=0}^{M} \frac{1}{m!} (-M)_m(j)_m \Gamma(k+m+j-1) \Gamma(\gamma+j+M)^{-1} \times \int_{-\pi}^{\pi} e^{i(\gamma+j+m)\theta} e^{i(1-k+m+j)\theta} (e^{i\theta} + 1)^{k+m+j-2} d\theta
\]
\[
= (2\pi)^{-1} \Gamma(k-\gamma) \Gamma(\gamma) \sum_{m=0}^{M} \frac{1}{m!} (-M)_m(j)_m \Gamma(k+m+j-1) \Gamma(\gamma+j+M)^{-1} \times \frac{2\pi \Gamma(k+m+j-1)}{\Gamma(k+m+j)(\gamma+j+M)}
\]

The second to last equality follows from Lemma 5.3.

Now, by Definition 5.4 and using Proposition 5.5 where we take \( a = -M \), \( b = j \) and \( c = \gamma + j \) we obtain
\[
\frac{\Gamma(\gamma) \Gamma(j)}{\Gamma(\gamma+j)} \sum_{m=0}^{M} \frac{1}{m!} (-M)_m(j)_m \Gamma(\gamma+j+M)^{-1} \times \frac{\Gamma(\gamma+j+M)}{\Gamma(\gamma+j)} = \frac{\Gamma(\gamma) \Gamma(j)}{\Gamma(\gamma+j)} \Gamma(\gamma+j) \Gamma(\gamma+j+M)^{-1} \times \frac{\Gamma(\gamma+j+M)}{\Gamma(\gamma+j)}
\]
and the proof is complete.

**Corollary 5.7.** Suppose \( K_{k,\gamma, P} \) is defined as in Theorem 5.2. Then for all \( \gamma, k-\gamma \in \mathbb{C} - \mathbb{Z}^- \),
\[
\hat{K}_{k,\gamma, P} E_\alpha = \left\{ \begin{array}{ll}
C \left( \int_{-\pi}^{\pi} \Gamma(k-\gamma) \Gamma(\gamma) e^{i\gamma \theta} K_M(\theta) d\theta \right) \mathcal{W}_{A}(P) \mathbb{E}_\alpha, & \lambda > 0, \\
C \left( \int_{-\pi}^{\pi} \Gamma(k-\gamma) \Gamma(\gamma) e^{i(\gamma-\gamma) \theta} K_M(\theta) d\theta \right) \mathcal{W}_{A}(P) \mathbb{E}_\alpha, & \lambda < 0,
\end{array} \right.
\]
where \( C = (-1)^{\gamma+n+1} 2^{1-n-\gamma} \lambda^{-1} \). For either \( \gamma \) or \( k-\gamma \) \( \in \mathbb{Z}^- \), \( \Gamma(k-\gamma) \Gamma(\gamma) \) is replaced by \( (-1)^{\gamma} \Gamma(k+1)(\gamma)^{-1} \), where \( l \) is defined in the remarks after 5.2.

**Proof.** The corollary follows directly from the theorem for \( \Re(\gamma) > 0 \) and \( \Re k-\gamma > 0 \) but the result holds for all \( \gamma, k-\gamma \in \mathbb{C} \) by analytic continuation. That is, both sides of the equation
\[
\int_{-\pi}^{\pi} \Gamma(k-\gamma) \Gamma(\gamma) e^{i\gamma \theta} K_M(\theta) d\theta = \Gamma(\gamma+j+M) \Gamma(j)[\Gamma(\gamma+j+M)^{-1}
\]
have analytic extensions as functions of \( \gamma \) to the entire complex plane and since they agree for \( \Re\gamma > 0 \) they are equal for all \( \gamma \). (Here \( \gamma \) is as in Theorem 5.2.)

6. The group Fourier transform of regular homogeneous distributions. The goal of this section is to show that functions \( f(\theta) \in C^1([-\pi, \pi]) \) can be approximated in \( C^1 \) by linear combinations over \( C \) of functions of the form \( e^{i\gamma \theta} \) for \( \gamma \in \mathbb{C} \). We will denote the algebra of all such functions by \( A \). Then the group Fourier transform of distributions of the form given in (16) will be calculated in terms of the kernel \( K_M(\theta) \).

First we need a lemma:
LEMMA 6.1. Every polynomial \( p(\theta) \) can be approximated in \( C^1 \) by elements of the algebra \( A \).

Set \( S^N = \{ f \in \mathbb{C}^N : f^{(k)}(-\pi) = f^{(k)}(\pi) \text{ for } 0 \leq k \leq N \} \). We know that if \( f \in S^N \), the Fourier series for \( f \) converges to \( f \) in \( C^{N-2} \).

So, given \( p(\theta) \), it is enough to show that there exists an \( F \in A \) such that \( p(\theta) - F(\theta) \in S^{N+2} \). That is, we need an \( F \in A \) such that \( F^{(l)}(\pi) = F^{(l)}(-\pi) \) for \( 0 \leq l \leq N+2 \). Denote the constants \( p^{(l)}(-\pi) - p^{(l)}(\pi) \) by \( C_n \). Next, choose constants \( \gamma_0, \gamma_1, \ldots, \gamma_{N+2} \in \mathbb{C} \) distinct and not integers. Set

\[
F(\theta) = \sum_{k=0}^{N+2} c_k e^{i\gamma_k \theta},
\]

the \( c_k \) to be determined. Then

\[
F^{(l)}(\theta) = \sum_{k=0}^{N+2} c_k (i\gamma_k)^l e^{i\gamma_k \theta}.
\]

Hence,

\[
F^{(l)}(\pi) - F^{(l)}(-\pi) = \sum_{k=0}^{N+2} c_k (i\gamma_k)^l e^{-i\gamma_k \pi} + e^{i\gamma_k \pi}.
\]

If we set \( d_k = c_k (e^{-i\gamma_k \pi} - e^{i\gamma_k \pi}) \), we have the system of equations

\[
\sum_{k=0}^{N+2} d_k (i\gamma_k)^n = C_n.
\]

The matrix associated with this system is the Vandermonde matrix so the system has a solution.

PROPOSITION 6.2. For any continuous function \( f(\theta) \in C^1([-\pi, \pi]) \), there exist functions \( f_k(\theta) = \sum_{j=0}^{N_k} c_{j,k} e^{i\gamma_{k,j} \theta} \) such that \( f_k \) converges to \( f \) in \( C^1 \).

P. R. O. O. F. Notice that the algebra, \( A \), of functions consisting of the linear combinations of the \( e^{i\gamma \theta} \), \( \gamma \in [-\pi, \pi] \), separates points and is closed under complex conjugation. Hence, by the complex Stone-Weierstrass theorem every continuous function \( f(\theta) \) can be uniformly approximated by functions in this algebra.

Next, consider a function \( f(\theta) \in C^1 \). Its \( l \)th derivative \( f^{(l)}(\theta) \) is continuous and so there exists a sequence, \( f_k \in A \), which converges uniformly to \( f^{(l)} \). Set \( F_k(\theta) = \int_0^\theta f_k(\phi) d\phi \). Then \( F_k(\theta) \) converge uniformly to \( f^{(l-1)} \) and the convergence is uniform. By repeated integrations, we obtain a sequence of functions in \( A \) which converges to \( f(\theta) - p(\theta) \in C^1 \) so by the lemma there exists \( f_k \in A \) which converges to \( f \in C^1 \).

PROPOSITION 6.3. The distribution \( K(t, z) = f(\theta)\overline{w}^{-k}P(z) \) is contained in \( C^1(\mathbb{H}^n - \{0\}) \) if and only if \( f \) is contained in \( C^1 \).

P. R. O. O. F. First, assume \( f \in C^1 \). Recall that \( \theta = \arg(\overline{w}/w) \). Set \( \theta = -i(\ln \overline{w} - \ln w) \), where the natural logarithm is defined on the principal branch. (That is, \( \ln z \) is analytic away from the negative real axis.)

Now \( w = |z|^2 - it \), hence \( Re \overline{w} = Re \overline{w} = 0 \) for all \( z, t \). Thus, \( \ln w, \ln \overline{w} \) are smooth for all \( w \neq 0 \). So, \( K \) is a product and composition of functions at least \( C^1 \).

Going the other way, suppose \( K(t, z) \in C^1(\mathbb{H}^n - \{0\}) \). Now,

\[
f(\theta) = \frac{K(t, z)\overline{w}^{-k}}{P(z)}.
\]

Fix \( z_0 \) and consider

\[
\theta_0 = -i(\ln(|z_0|^2 + it) - \ln(|z_0|^2 - it)).
\]

We wish to show that \( f \) is continuous at \( \theta_0 \).

Suppose \( |z_0| = r \). Then

\[
\theta_0 = -i(\ln(|z|^2 + it) - \ln(|z|^2 - it)),
\]

for all \( |z| = r \). Recall that \( P(z) \) is a basis element for the space of spherical harmonics, so \( P(z) \) is not identically zero on the sphere \( |z| = r \). Choose \( z_1 \) such that \( |z_1| = r \) and \( P(z_1) \neq 0 \). Then

\[
f(\theta_0) = \frac{K(t, z_1)\overline{w}^{-k}}{P(z_1)}
\]

is defined and \( f \) is continuous at \( \theta_0 \) since \( K, \overline{w}, \) and \( P \) are continuous at \( z_1 \) and \( P(z_1) \neq 0 \). Similarly it can be shown that the derivatives of \( f \) through \( l \) are continuous.

COROLLARY 6.4. For a given distribution \( K(t, z) = f(\theta)\overline{w}^{-k}P(z) \), there exist distributions \( K_n(t, z) = f_n(\theta)\overline{w}^{-k}P(z) \) such that \( f_n(\theta) \in A \) and \( \|K - K_n\|_{C^1} \to 0 \).

P. R. O. O. F. This follows directly from Propositions 6.3 and 6.2.

Using Corollaries 6.4 and 5.7, we will show, for \( K(t, z) = f(\theta)\overline{w}^{-k}P(z) \),

\[
\hat{K}E_\alpha = C \left( \int_{-\pi}^{\pi} f(\theta)K_M(\theta) d\theta \right) \mathcal{W}(P)E_\alpha.
\]

First we need an estimate proven in [5].

Define Hilbert spaces \( H_k^x, k \in \mathbb{R} \), as follows. Consider vectors \( v = \sum_{\alpha} v_{\alpha} E_{\alpha} \) such that \( \sum_\alpha (|\alpha| + 1)^k |v_{\alpha}|^2 < \infty \). These vectors with this norm form a Hilbert space for \( k \in \mathbb{R} \). If \( k \in \mathbb{R} \), set \( H_k^x = H_{|k|}^x \).
Proposition 6.5. Suppose $K$ is a regular homogeneous distribution of order $k$ and $\tilde{K} = J$. Then each $J(\lambda)$ has an extension as a bounded operator from $H_{-k}^\alpha$ to $H_0^\alpha$. Further, there exist constants $c$ and $l$ such that $\|J\|_{-k,0} < c\|K\|_{cl}$. (Here, $\|\cdot\|_{-k,0}$ denotes the operator norm from $H_{-k}^\alpha$ to $H_0^\alpha$ and $\|\cdot\|_{cl}$ denotes the $C^l$ norm over $\{1 \leq |\alpha| \leq 2\}$.

Theorem 6.6. For the regular homogeneous distribution

$$K(t,z) = f(\theta) w^{-k} P(z),$$

$P \in H_{p,q}$, $p + q = \kappa$ and $-2n - 2 < \kappa - 2 \Re k < 0$, the group Fourier transform of $K$ is $\widehat{K}(\lambda) = \hat{J}(\lambda)$, where $\hat{J}$ is defined by

$$\hat{J}(\lambda)E_{\alpha} = C \left( \int_{-\pi}^{\pi} f(\theta) K_M(\theta) d\theta \right) W_\lambda(P) E_{\alpha}.$$

Here, $M = |\alpha| - p$ if $\lambda > 0$, $M = |\alpha| - q$ if $\lambda < 0$, $C = (-1)^{n+1} \frac{n!}{2^{\kappa} \pi^{n+1}} \kappa!$, and

$$where j = n + \kappa + 1 - k.$$

The function $K_M$ is a smooth function of $\theta$ defined in equation (17) above.

Proof. By Proposition 6.2 and Corollary 6.4 there exist distributions $K_n = f_n(\theta) w^{-k} P(z)$ such that $f_n(\theta) \to f(\theta)$ in $C^l([-\pi, \pi])$ and $K_n \to K$ in $C^l(H^n - \{0\})$ for $l$ arbitrarily large (since $K$ is a regular distribution).

By Corollary 5.7, we have

$$\hat{K}_n E_{\alpha} = C \left( \int_{-\pi}^{\pi} f_n(\theta) K_M(\theta) d\theta \right) W_\lambda(P) E_{\alpha}.$$

Hence, for all $\alpha, \beta$,

$$\langle \hat{K}_n E_{\alpha}, E_{\beta} \rangle \to \langle C \left( \int_{-\pi}^{\pi} f(\theta) K_M(\theta) d\theta \right) W_\lambda(P) E_{\alpha}, E_{\beta} \rangle.$$

However, by Proposition 6.5, $\hat{K}_n \to \hat{K}$ in $\|\cdot\|_{-(d_\infty K),0}$, but this implies

$$\langle \hat{K}_n E_{\alpha}, E_{\beta} \rangle \to \langle \hat{K} E_{\alpha}, E_{\beta} \rangle$$

for all $\alpha, \beta$. So $\hat{K} = J$.

7. Functions which are not locally integrable about the origin.

In the previous section, we calculated the group Fourier transform of homogeneous distributions $K(t,z) = f(\theta) w^{-k} P(z)$ where $-2k > -2n - 2$. This hypothesis ensured that $K$ was locally integrable and hence defined a distribution. We now consider $K$ such that $-2k \leq -2n - 2$. In this case $K$ no longer defines a distribution, but it is possible to define a distribution which agrees with $K$ away from 0. We will investigate the group Fourier transform of this new class of distributions. First we need some results and definitions given in [6].

Recall that $B$ is the set of families of operators $R(\lambda)$ where each $R(\lambda), \lambda \in \mathbb{R}^+$, is a bounded operator on $H_0$. Let $\{R_{\alpha,\beta}(\lambda)\}$ be the matrix of $R(\lambda)$ with respect to the orthonormal basis $E_{\alpha}$; that is, $R(\lambda)E_{\alpha} = \sum R_{\alpha,\beta}(\lambda)E_{\beta}$. We denote by $Q$ the subset of $B$ defined by

$$Q = \{ R(\lambda) : R_{\alpha,\beta}(\lambda) \in C^\infty_0(\mathbb{R}^+) \text{ for all } \alpha, \beta, \text{ and for some } N \in \mathbb{N}, R_{\alpha,\beta}(\lambda) = 0 \text{ if } |\alpha| + |\beta| > N \}.$$

Proposition 7.1. For any $R \in Q$, there exists $f \in S(H^n)$ such that $\hat{f} = R$.

Proposition 7.2. Suppose $G$ is a homogeneous function of degree $j$ on $H^n$ which is locally integrable away from 0. Then there exists a number $M(G)$ such that for any $0 < A < B$ we have

$$\int_{A < |\zeta| < B} M(G)(2n + 2 + j)^{-1}(B^{2n+2+j} - A^{2n+2+j})$$

$$= \int_{M(G) \log(B/A)} \frac{1}{M(G)(2n + 2 + j)^{-1}(B^{2n+2+j} - A^{2n+2+j})} \text{ if } j \neq -2n - 2,$$

$$= \int_{M(G) \log(B/A)} \frac{1}{M(G)(2n + 2 + j)^{-1}(B^{2n+2+j} - A^{2n+2+j})} \text{ if } j = -2n - 2.$$

Suppose $G$ is a homogeneous function of degree $j \in \mathbb{C}$ which is smooth away from 0. The preceding proposition allows us to define a distribution $A_G \in S'$ by

$$A_G(\phi) = \int_{|\zeta| \leq N} \left( \phi(\zeta) - \sum_{|\alpha| < \infty} \phi^{(\alpha)}(0) \zeta^\alpha / \alpha! \right) G(\zeta) d\zeta$$

$$+ \sum_{|\alpha| \leq N, |\alpha| > 0} (2n + 2 + |\alpha| + j)^{-1} M(G) \phi^{(\alpha)}(0) / \alpha!$$

$$+ \int_{|\zeta| > 1} \phi(\zeta) G(\zeta) d\zeta$$

where $N$ is chosen arbitrarily with $N \geq 2n + 2 - Re j - 1$. Notice that, if $Re j > -2n - 2$, we can choose $N = -1$ so $A_G = G$. If $Re j \leq -2n - 2$ and $-j - 2n - 2 \notin \mathbb{Z}^+$, then $A_G$ is a distribution which agrees with $G$ away from 0.

Theorem 7.3. Suppose $K(t,z) = f(\theta) w^{-k} P(z)$ where $-2k \Re k \leq -2n - 2$. Further, suppose that $\Im k \neq 0$ then

$$(\hat{A}_K(\phi)) = \int_{-\infty}^{\infty} \sum_{|\alpha| \leq \kappa} (J(\lambda) E_{\alpha}, \phi E_{\alpha})(2|\lambda|)^n d\lambda,$$

where $J$ and $\kappa$ are defined in Theorem 6.6, and $\phi \in Q$.

Proof. The result holds for $-2k \Re k > -2n - 2$ by Theorem 6.6, since in this case $A_K = K$. 


Now, fix $k_0$ such that $\text{Im } k_0 \neq 0$ and $\kappa - 2 \text{Re } k_0 \leq -2n - 2$. Next, fix $N_0$ such that $N_0 > -2n - 2 - (\kappa - 2 \text{Re } k_0) - 1$. Let $A_{\kappa}^{N_0}$ denote $A_{\kappa}$ with the particular choice of $N = N_0$. Then $A_{\kappa}^{N_0}(\phi)$ depends analytically on $k$ where $\text{Im } k \neq 0$ and $\kappa - 2 \text{Re } k > \kappa - 2 \text{Re } k_0 - 1$.

For all $\phi \in \mathcal{Q}$, we have $A_{\kappa}(\phi) = c_n(A_{\kappa} \mid \phi)$. But for $k$ satisfying $\kappa - 2 \text{Re } k > -2n - 2$, $\phi c n$, $\phi k_0 \leq -1$. Hence, by analytic continuation, the statement of the theorem holds for all $k$ satisfying $\text{Im } k \neq 0$, $\kappa - 2 \text{Re } k > \kappa - 2 \text{Re } k_0 - 1$. So in particular the result holds for $k_0$ but $k_0$ was an arbitrary complex number satisfying $\text{Im } k_0 \neq 0$, $\kappa - \text{Re } k_0 \leq -2n - 2$. So the theorem is proved.

References


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Sobolev embeddings with variable exponent

by

DAVID E. EDMUNDS (Brighton) and JIŘÍ RÁKOSNÍK (Práha)

Abstract. Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary and let $p : \Omega \to (1, \infty)$ be Lipschitz-continuous. We consider the generalised Lobachevsky space $L^{p(x)}(\Omega)$ and the corresponding Sobolev space $W^{1,p(x)}(\Omega)$, consisting of all $f \in L^{p(x)}(\Omega)$ with first-order distributional derivatives in $L^{p(x)}(\Omega)$. It is shown that if $1 \leq p(x) < n$ for all $x \in \Omega$, then there is a constant $c > 0$ such that for all $f \in W^{1,p(x)}(\Omega)$,

$$\|f\|_{M,\Omega} \leq c\|f\|_{1,p,\Omega}.$$  

Here $\| \cdot \|_{M,\Omega}$ is the norm on an appropriate space of Orlicz–Musielak type and $\| \cdot \|_{1,p,\Omega}$ is the norm on $W^{1,p(x)}(\Omega)$. The inequality reduces to the usual Sobolev inequality if $\sup \Omega p(x) < n$. Corresponding results are proved for the case in which $p(x) > n$ for all $x \in \Omega$.

1. Introduction. The most common assumptions in existence theorems for the Dirichlet boundary-value problem for the quasi-linear equation

$$-\sum_{i=1}^{n} D_i a_i(x,u(x),\nabla u(x)) + a_0(x,u(x),\nabla u(x)) = f(x), \quad x \in \Omega,$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$, involve the polynomial growth of coefficients:

$$a_i(x,\xi) \leq g(\xi) + c_1|\xi|^{q-1}, \quad g \in L^q(\Omega),$$

$$\sum_{i=0}^{n} a_i(x,\xi)\xi_i \geq c_1|\xi|^p - c_2,$$

for a.a. $x \in \Omega$ and all $\xi \in \mathbb{R}^{n+1}$.

Similarly, regularity problems for variational integrals $\int_{\Omega} F(\nabla u(x)) \, dx$ are solved under the assumption

$$c_1|\xi|^p \leq F(\xi) \leq c_2(1 + |\xi|)^q, \quad \xi \in \mathbb{R}^n.$$

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