On the existence for the Cauchy–Neumann problem for the Stokes system in the $L_p$-framework

by

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Abstract. The existence for the Cauchy–Neumann problem for the Stokes system in a bounded domain $\Omega \subset \mathbb{R}^3$ is proved in a class such that the velocity belongs to $W^{2,1}_r(\Omega \times (0, T))$, where $r > 3$. The proof is divided into three steps. First, the existence of solutions is proved in a half-space for vanishing initial data by applying the Marcinkiewicz multiplier theorem. Next, we prove the existence of weak solutions in a bounded domain and then we regularize them. Finally, the problem with nonvanishing initial data is considered.

1. Introduction. In a bounded domain $\Omega$ in $\mathbb{R}^3$ with boundary $S$ we consider the initial-boundary value problem for the Stokes system:

\[ u_t - \nu \Delta u + \nabla p = F, \]
\[ \text{div } u = G, \]
\[ \bar{n} \cdot T(u, p)|_{S_T} = H, \]
\[ u|_{t=0} = u_0, \]

where $T(u, p) = \{T(u, p)\}_{i,j=1,2,3} = \{\nu(\partial_i u_j + \partial_j u_i) - p \delta_{ij}\}$ is the stress tensor, $u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$ the velocity vector, $p(x, t)$ the pressure, $\nu > 0$ the constant viscosity coefficient and $\bar{n}$ the exterior normal vector to $S$.

To solve (1.1) we have to impose the following compatibility conditions on the initial and boundary data:

\[ \text{div } u_0(x) = G(x, 0), \]
\[ \bar{n} \cdot T(u_0, p_0)(x)|_{S} = H(x, 0), \]

where $p_0$ is defined by $\bar{n} \cdot T(u_0, p_0) \cdot \bar{n} = H(0) \cdot \bar{n}$ on $S$. From (1.2) we get the initial boundary condition $p|_{t=0} = p_0$.

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[78]
We prove the existence of solutions for system (1.1). The main result of this paper is the following

**Theorem 1.** Let $r > 3$, $S \in W_r^{2-1/r}$, $F \in L_r(\Omega_T)$, $G \in W_r^{1,0}(\Omega_T)$,

$$\frac{\partial G}{\partial t} - \text{div} F = \text{div} B + A, \quad A, B \in L_r(\Omega_T),$$

diam supp $A < \lambda$, $u_0 \in W_r^{2-2/r}(\Omega)$, $H \in W_r^{1-1/r,1/2-1/(2r)}(S_T)$.

Then there exists a unique solution of problem (1.1)-(1.2) such that:

$$u \in W_r^{2,1}(\Omega_T), \quad p \in W_r^{1,0}(\Omega_T), \quad p \in W_r^{2-1/r,1/2-1/(2r)}(S_T),$$

and the following estimate holds:

$$\|u\|_{W_r^{2,1}(\Omega_T)} + \|p\|_{W_r^{1,0}(\Omega_T)} + \|p\|_{W_r^{2-1/r,1/2-1/(2r)}(S_T)} \leq C(T)\|F\|_{L_r(\Omega_T)} + \|G\|_{W_r^{1,0}(\Omega_T)} + \|B\|_{L_r(\Omega_T)}$$

$$+ \lambda\|u_0\|_{W_r^{2-1/r}(\Omega)} + \|u_0\|_{W_r^{2-2/r}(\Omega)} + \|H\|_{W_r^{1-1/r,1/2-1/(2r)}(S_T)},$$

where $\Omega_T = \Omega \times (0,T)$, $S_T = S \times (0,T)$ and $C(T)$ is an increasing positive function of $T$.

Theorem 1 can be found in [7] without proof.

The aim of the paper is to present a new approach to obtaining $L_r$-estimates for solutions of evolution equations. We apply our technique to the Stokes system. To prove existence of solutions to (1.1) we use the technique of regularizers. Therefore we consider problem (1.1) locally in a neighbourhood of either an interior point or a boundary point. The boundary neighbourhood problem (1.1) is transformed to a problem in the half-space $x_3 > 0$. By applying the Fourier transform with respect to time and tangent directions, problem (1.1) becomes a system of ordinary differential equations (see (3.2)) whose solutions have the form (3.5). Solonnikov [9] calculates explicitly the inverse Fourier transform of solutions (3.5) and expresses them in the form of potentials in the half-space $x_3 > 0$. Then he estimates them in suitable norms. In our case we directly estimate the solutions of the ordinary differential equations (3.5) using the Marchuk-Sobolev multiplier theorem [2, 3].

Moreover, in [10] the existence of solutions to (1.1) is proved in Hölder spaces in a domain $\Omega$ which can be either bounded or unbounded.

The result of this paper has an auxiliary character. Our ultimate goal is to prove stability of an equilibrium solution to the free boundary problem for a self-gravitating incompressible fluid. First we have to prove existence of local solutions to the corresponding Navier-Stokes problem (see [4]).

Next we are going to prove existence of global-in-time solutions of the following problem:

$$v_t + v \cdot \nabla v - \text{div} \nabla = k\nabla \left( \sum_{\nu} \frac{dy}{\partial_t \|x - y\|} \right) \text{ in } \Omega_t,$$

$$\text{div} v = 0 \quad \text{in } \Omega_t,$$

$$\n \cdot \nabla = -p_0 \n \quad \text{on } \Sigma_t = \partial \Omega_t,$$

$$v|_{t=0} = v_0, \quad \Omega_t|_{t=0} = 0,$$

$$v \cdot n = \text{velocity of the boundary } \Sigma_t,$$

where $v(x,t)$ is the velocity of the fluid, $p(x,t)$ the pressure, $\nabla v(p,v)$ the stress tensor, $k$ the constant of gravity, $p_0$ the external constant pressure.

By an equilibrium solution of problem (1.4) we mean the following solution of (1.4):

$$v = 0, \quad \Omega_t = B_R,$$

$$\nabla p(x) = k \int_{B_R} \frac{dy}{\|x - y\|} \quad \text{in } B_R,$$

$$p = p_0 \quad \text{on } \partial B_R,$$

where $B_R$ is a ball of radius $R$.

In [5] we prove stability of solutions of (1.5). For this purpose the $L^p$-approach is much more appropriate, because all considerations are simpler and shorter.

Summarizing, to show stability in [5] we need the result of this paper.

2. **Notation.** In our considerations we will need the anisotropic Sobolev spaces $W_r^{m,n}(Q_T)$, where $m, n \in \mathbb{R}_+ \cup \{0\}$, $r \geq 1$ and $Q_T = Q \times (0,T)$, with the norm

$$\|u\|_{W_r^{m,n}(Q_T)} = \int_0^T \int_Q |u(x,t)|^r \, dx \, dt$$

$$+ \sum_{0 \leq |m'| \leq |m|} 0 \|Q_T \int_Q \frac{|D_{x'}^m u(x,t) - D_{x'}^m u(x',t)|^r}{|x - x'|^{s + (|m| - |m'|)|}} \, dx' \, dt$$

$$+ \sum_{0 \leq |m'| \leq |m|} 0 \|Q_T \int_Q \frac{|D_t^n u(x,t)|^r}{|t - t'|^{s + (n - |n'|)|}} \, dt' \, dx$$

$$+ \int_Q \int_0^T \frac{|D_{x'}^n u(x,t) - D_{x'}^n u(x',t')|^r}{|t - t'|^{s + (n - |n'|)|}} \, dt' \, dx.'
where $s = \dim Q$, $[\alpha]$ is the integral part of $\alpha$, $D^l_x = \partial_{x_1}^{l_1} \ldots \partial_{x_n}^{l_n}$, where $l = (l_1, \ldots, l_n)$ is a multiindex. If $Q$ is a manifold the above norm is defined using a partition of unity.

In the case when $Q_T = \mathbb{R}^n \times \mathbb{R}$ we can apply the Fourier transform and define the Bessel-potential spaces given by the norm
\[
\|u\|_{H^{m,n}(\mathbb{R}^{n+1})} = \|u\|_{L^p(\mathbb{R}^{n+1})} + \|\mathcal{F}_{t,x}^{-1}[\mathcal{F}_{t,x}^m \mathcal{G}(\xi, \zeta_0)]\|_{L^p(\mathbb{R}^{n+1})} + \|\mathcal{F}_{t,x}^{-1}[\mathcal{F}_{t,x}^m \mathcal{G}(\xi, \zeta_0)]\|_{L^p(\mathbb{R}^{n+1})},
\]
where $\mathcal{G}(\xi, \zeta_0)$ is the Fourier transform of $u(x, t)$,
\[
\mathcal{G}(\xi, \zeta_0) = \int e^{-i\xi \cdot x} u(x, t) \, dx \, dt \equiv \mathcal{F}_{t,x}[u](\xi, \zeta_0),
\]
and $\mathcal{F}^{-1}$ the inverse transformation
\[
\mathcal{F}^{-1}[\mathcal{G}](x, t) = (2\pi)^{-\frac{n+1}{2}} \int e^{i\xi \cdot x} \mathcal{G}(\xi, \zeta_0) \, d\xi \, d\zeta_0
\]
where $\xi = (\xi_1, \ldots, \xi_n)$ and $\xi : x = x_1 + \ldots + x_n$.

If $m, n \in \mathbb{N}$ then $H^{m,n}(Q_T) = H^m(\mathbb{R}) \times H^n(\mathbb{R})$ (see [11]).

In the proof we will use the following results.

**Theorem 2.1** (Marcinkiewicz theorem, see [3]). Suppose that a function $f : \mathbb{R}^n \to \mathbb{C}$ is smooth enough and there exists a constant $M > 0$ such that for every $x \in \mathbb{R}^n$ we have
\[
|\partial^k f| \leq M, \quad 0 \leq k \leq m, \quad 1 \leq j_1 < \ldots < j_k \leq m.
\]

Then the operator
\[
Pf(z) = (2\pi)^{-m} \int_{\mathbb{R}^m} dy \int_{\mathbb{R}^m} e^{i\xi \cdot y} f(y) \int_{\mathbb{R}^m} e^{-i\xi \cdot z} g(x) \, dx
\]
is bounded in $L^p(\mathbb{R}^m)$ and
\[
\|Pf\|_{L^p(\mathbb{R}^m)} \leq A_{p,m} M \|g\|_{L^p(\mathbb{R}^m)}.
\]

**Proposition 2.2** (see [1]). Let $u \in W^{m,n}_{r,s}(\Omega_T)$. If
\[
\kappa = \sum_{i=1}^n \left( \alpha_i + \frac{1}{r} - \frac{1}{q} \right) \frac{1}{m} + \left( \beta + \frac{1}{r} - \frac{1}{q} \right) \frac{1}{n} < 1
\]
then the following estimate holds:
\[
\|D^\alpha_x D^\beta_t u\|_{L^q(\Omega_T)} \leq \varepsilon \|u\|_{W^{m,n}(\Omega_T)} + c(\varepsilon) \|u\|_{L^q(\Omega_T)}
\]
where $q \geq r \geq 2$, $\varepsilon \in (0, 1)$ and $c(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

**Proposition 2.3.** Let $r \geq 2$, $u \in W^{2,1}_{r,s}(\Omega_T)$ and $u|_{t=0} = 0$. Then
\[
\|u\|_{W^{2,1}(\Omega_T)} \leq c(T^{1/r} + T^{3/2}) \|u\|_{W^{2,1}(\Omega_T)}.
\]

**Proof.** First we note that
\[
\int_0^T \int_0^T \int |u(x, t) - u(x, t')|^r dt' \, dt \, dx \\ \leq T^{\gamma/2} \int_0^T \int_0^T \int |u(x, t) - u(x, t')|^r dt' \, dt \, dx \leq T^{\gamma/2} \|u\|^{r+1}_{W^{2,1} \left( T^2 \right)},
\]

Next we have
\[
\|u\|_{W^{2,1}(\Omega_T)} \leq C \|u\|_{L^m(\Omega_T)} \leq C \|u\|_{W^{2,1}(\Omega_T)},
\]
where in the last inequality we use $u|_{t=0} = 0$. From (2.3) and (2.4) we get the assertion.

In our considerations we will use well known results such as imbedding theorems for Sobolev spaces. All constants are denoted by $c$.

3. Problem in the half-space. The first step to solve problem (1.1) is to consider this system in the half-space $x_3 \geq 0$ with $u_0 = 0$ in (1.1). To solve it we have to consider two cases. The first is when $F = 0$ in the half-space and the second when the whole space without boundary conditions is examined.

The problem (1.1) in the half-space $x_3 > 0$ with vanishing forces reads
\[
u_t - \nu \Delta u + \nabla p = 0,
\]
\[
\text{div } u = 0,
\]
\[
\nu(u_{3,1} + u_{3,3})|_{x_3 = 0} = H_1,
\]
\[
\nu(u_{3,2} + u_{3,3})|_{x_3 = 0} = H_2,
\]
\[
(2u_{3,3} - p)|_{x_3 = 0} = H_3,
\]
\[
u|_{x_3 = 0} = 0.
\]

To solve (3.1) we apply the Fourier transform
\[
\begin{align*}
\nu(s, \xi', x_3) &= \int_0^{\infty} e^{-st} \int_{\mathbb{R}^3} e^{i\xi' \cdot \xi} u(t, x) \, dx' \, dt' \, ds, \\
\varphi(s, \xi', x_3) &= \int_0^{\infty} e^{-st} \int_{\mathbb{R}^3} e^{i\xi' \cdot \xi} p(t, x) \, dx' \, dt' \, ds,
\end{align*}
\]
where $s = i\xi_0$ and $\xi' = (\xi_1, \xi_2, \xi_3') = (x_1, x_2)$.

Assuming that $H \in W^{-1/r, 1/2-1/(2r)}(\mathbb{R}^3) (x_3 = 0)$, $r > 3$ and $H|_{x_3 = 0} = 0$ we can extend $H$ by zero for $t < 0$. Therefore we can look for solutions of (3.1) vanishing for $t < 0$. 

After the transformation system (3.1) takes the form
\[
\begin{align*}
\nu \left( -\frac{d^2}{dx_3^2} + r^2 \right) v_1 + i\xi_1 q &= 0, \\
\nu \left( -\frac{d^2}{dx_3^2} + r^2 \right) v_2 + i\xi_2 q &= 0, \\
\nu \left( -\frac{d^2}{dx_3^2} + r^2 \right) v_3 + \frac{dq}{dx_3} &= 0, \\
i\xi_1 v_1 + i\xi_2 v_2 + \frac{dv_3}{dx_3} &= 0,
\end{align*}
\]
(3.2)
\[
\begin{align*}
\nu \left( \frac{dv_1}{dx_3} + i\xi_1 v_3 \right) \bigg|_{x_3=0} &= h_1, \\
\nu \left( \frac{dv_2}{dx_3} + i\xi_2 v_3 \right) \bigg|_{x_3=0} &= h_2, \\
2\nu \frac{dv_3}{dx_3} - \phi \bigg|_{x_3=0} &= h_3, \\
v \to 0, q \to 0 & \text{ as } x_3 \to \infty,
\end{align*}
\]
where \( r^2 = s/\nu + |\xi'|^2, \arg r \in (-\pi/4, \pi/4) \). Solving (3.2) with (3.2) we get (see also [6])
\[
\begin{align*}
v &= \Phi(s, \xi')e^{-r_{\xi}z_3} + \phi(s, \xi')(i\xi_1, i\xi_2, -|\xi'|)e^{i|\xi'|z_3}, \\
q &= -s\phi(s, \xi')e^{-|\xi'|z_3},
\end{align*}
(3.3)
where \( \Phi = (\Phi_1, \Phi_2, (i\xi_1 \Phi_1 + i\xi_2 \Phi_2)/r) \). From boundary conditions (3.2) we obtain
\[
\begin{align*}
\nu[-r\Phi_1 - 2i\xi_1 |\xi'| \phi + (i\xi_1/r)(i\xi_1 \Phi_1 + i\xi_2 \Phi_2)] &= h_1, \\
\nu[-r\Phi_2 - 2i\xi_2 |\xi'| \phi + (i\xi_2/r)(i\xi_2 \Phi_1 + i\xi_2 \Phi_2)] &= h_2, \\
s\phi + 2\nu(|\xi'|^2 \phi - i\xi_1 \Phi_1 - i\xi_2 \Phi_2) &= h_3.
\end{align*}
(3.4)
Solving system (3.4) and using (3.3) we get a solution of (3.2):
\[
\begin{align*}
v_1(s, \xi', x_3) &= -\frac{h_1}{\nu r}e^{-r_{\xi}z_3} + \frac{i\xi_1 s}{\nu^2 r P(r + |\xi'|)} \\
&\times \left[ |\xi'|^2 r - 3r \right)(i\xi_1 h_1 + i\xi_2 h_2) + r(r - |\xi'|)h_3 e^{-r_{\xi}z_3} \\
&+ \frac{i\xi_1 s}{\nu^2 r P(r + |\xi'|)} \\
&\times \left[ 2r(i\xi_1 h_1 + i\xi_2 h_2) - (r^2 + |\xi'|^2)h_3 \right] \\
&\times e^{-r_{\xi}z_3} - e^{-|\xi'|z_3} \\
&\times \frac{r}{r - |\xi'|},
\end{align*}
(3.5)
\]
\[
\begin{align*}
v_2(s, \xi', x_3) &= -\frac{h_2}{\nu r}e^{-r_{\xi}z_3} + \frac{i\xi_2 s}{\nu^2 r P(r + |\xi'|)} \\
&\times \left[ (|\xi'|^2 - 3r)(i\xi_1 h_1 + i\xi_2 h_2) + r r - |\xi'|)h_3 e^{-r_{\xi}z_3} \\
&+ \frac{i\xi_2 s}{\nu^2 r P(r + |\xi'|)} \\
&\times \left[ 2r(i\xi_1 h_1 + i\xi_2 h_2) - (r^2 + |\xi'|^2)h_3 \right] \\
&\times e^{-r_{\xi}z_3} - e^{-|\xi'|z_3} \\
&\times \frac{r}{r - |\xi'|},
\end{align*}
(3.6)
\]
\[
\begin{align*}
v_3(s, \xi', x_3) &= -\frac{h_3}{s} \left[ -\frac{1}{\nu P} \left[ (r - |\xi'|^2)(i\xi_1 h_1 + i\xi_2 h_2) + |\xi'| (s/\nu) h_3 \right] e^{-r_{\xi}z_3} \\
&- \frac{s}{\nu^2 P(r + |\xi'|)} \\
&\times \left[ 2r(i\xi_1 h_1 + i\xi_2 h_2) - (r^2 + |\xi'|^2)h_3 \right] \\
&\times e^{-r_{\xi}z_3} - e^{-|\xi'|z_3} \\
&\times \frac{r}{r - |\xi'|},
\end{align*}
(3.7)
\]
\[
q(s, \xi', x_3) = \frac{rs}{\nu^2 P} \left[ 2r(i\xi_1 h_1 + i\xi_2 h_2) - (r^2 + |\xi'|^2)h_3 \right] e^{-|\xi'|z_3},
\]
where
\[
P = (r^2 + |\xi'|^2) - 4r|\xi'|^3.
\]
\textbf{Lemma 3.1.} For \( P \) defined by (3.6) the following estimates hold:
\[
|P| \geq \frac{2}{\nu} |s| \cdot |\xi'|^2, \quad |s|^2 \leq 3\nu^2 |P|.
\]
\textbf{The proof can be found in [8].}

\textbf{Lemma 3.2.} Let \( h \in W_{r,1}^{-1/2,1/2-1/2}(\mathbb{R}) \). Then there exists a solution of problem (3.1) such that \( u(t, x) \in W_{r,1}^{0,0}(\mathbb{R}^4) \) and \( p(t, x) \in W_{r,1}^{0,0}(\mathbb{R}^4) \), and the following estimates are valid:
\[
\begin{align*}
\|D^2 u\|_{L_r(\mathbb{R}^4)} + \|D u\|_{L_r(\mathbb{R}^4)} &\leq c\|h\|_{W_{r,1}^{-1/2,1/2-1/2}(\mathbb{R})}, \\
\|D p\|_{L_r(\mathbb{R}^4)} &\leq c\|h\|_{W_{r,1}^{-1/2,1/2-1/2}(\mathbb{R})}, \\
\|u\|_{W_{r,1}^{-1/2,1/2-1/2}(\mathbb{R})} &\leq c\|h\|_{W_{r,1}^{-1/2,1/2-1/2}(\mathbb{R})}, \\
\|p\|_{W_{r,1}^{-1/2,1/2-1/2}(\mathbb{R})} &\leq c\|h\|_{W_{r,1}^{-1/2,1/2-1/2}(\mathbb{R})},
\end{align*}
\]
where \( D^4 = \mathbb{R}^2 \times [0, \infty) \times \mathbb{R}_3 \).
Proof. First we take the pressure (3.5) with \( h_2 = h_3 = 0 \). Then

\[
q(s, \xi', x_3) = 2 \frac{s_1 \xi_1 s_2}{\nu P} e^{-|\xi'|^2 s_1 h_1(s, \xi')}. \tag{3.8}
\]

From Lemma 3.1 one can check that for some \( M \),

\[
|s_1 \xi_1 s_2| \left| \frac{\partial^k}{\partial s^l \partial \xi_1^m \partial \xi_2^n} \frac{\xi_1 s_2}{\nu P} \right| < M,
\]

where \( l = 0 \) or 1, so that the condition of Theorem 2.1 is satisfied. To see this we note that

\[
\left| P \xi' \frac{1}{P} \right| = \left| \frac{P \xi'}{P} \right| = \left| \frac{2(r^2 + |\xi'|^2) - 4 |\xi'|^1}{P} \right| < M,
\]

\[
\left| P \xi' \frac{1}{P} \right| = \left| \frac{8 |\xi'|^2 r^3 + 4 s |\xi'|^3 s - 4 |\xi'|^4 s^2}{r P} \right| < M,
\]

which gives, by Lemma 3.1, the desired estimate.

We consider the norm of derivatives of the pressure

\[
D_{s_1} P(t, x) = \int \frac{e^{s_1 \xi'} 2 s_1 \xi_1 s_2}{\nu P} e^{-|\xi'|^2 s_1 h_1(s, \xi')} \frac{d \xi_0}{\xi_0}.
\]

We know that

\[
K^1 = F_{\xi_1 s_1}^{-1}[e^{-|\xi'|^2 s_1}] = \frac{x_3}{(x_1^2 + x_2^2 + x_3^2)^{3/2}},
\]

and

\[
F_{\xi_1 s_1}^{-1}[\xi_1 e^{-|\xi'|^2 s_1} \xi_1] = \frac{x_3 x_2}{(x_1^2 + x_2^2 + x_3^2)^{5/2}}.
\]

We also have

\[
\int \frac{K^1 x_3}{\nu P} d\xi' = 0.
\]

Hence to estimate \( \|D_{s_1} P\|_{L_\infty(D^4)} \) we have to show that

\[
I_1(t, x) = \int \frac{d\xi'}{\nu P} K^1 \xi_1 s_1 (y', x_3)[h(t, x'-y') - h(t, x')] \tag{3.13}
\]

belongs to \( L_r(D^4) \). By (3.12) we have

\[
F_{\xi_1 s_1}^{-1}[\xi_1 e^{-|\xi'|^2 s_1} h(s, \xi')] = I_1.
\]

Hence we have

\[
\|I_1\|_{L_r(D^4)} \leq 3 \int \frac{|y'| x_3}{(y'^2 + x_3^2)^{3/2}} N(y') \frac{d\xi'}{y'^\alpha}, \tag{3.14}
\]

where

\[
N(y') = \|h(x' - y', t) - h(x', t)\|_{L_r(\mathbb{R}^4 \times \mathbb{R}^4)}.
\]

Applying the Hölder inequality to (3.14) we get

\[
\|I_1\|_{L_r(D^4)} \leq 3 \int_0^\infty \int_{\mathbb{R}^4} \frac{|y'| x_3}{(y'^2 + x_3^2)^{3/2}} N(y') \left( \int_{\mathbb{R}^4} \frac{x_3 d\xi'}{(y'^2 + x_3^2)^{3/2}} \right)^{1/r} \frac{d\xi'}{y'^\alpha}, \tag{3.15}
\]

where we have used the fact that \( \int_{\mathbb{R}^4} (y'^2 + x_3^2)^{-3/2} d\xi' = 2\pi \).

We examine the integral

\[
J_1 = \int_0^\infty \frac{|y'| x_3}{(y'^2 + x_3^2)^{3/2}} d\xi', \tag{3.16}
\]

Taking \( w = x_3/|y'| \) we get

\[
J_1 = \int_0^\infty \frac{w |y'|^{r + 2 + \alpha}}{(y'^2 + x_3^2)^{r + 3/2}} \frac{dw}{w^{2 + \alpha}},
\]

so \( J_1 \) does not depend on \( |y'| \) if

\[
r + 2 + \alpha = 2r + 3
\]

or

\[
\alpha = r + 1
\]

and is finite when

\[
r + 2 + \alpha = 2r + 3 - 1 > 1,
\]

which holds for our \( r \). Therefore from (3.15) we get

\[
\|I_1\|_{L_r(D^4)} \leq 3 \int_0^\infty \int_{\mathbb{R}^4} \frac{|h(t, x' - y') - h(t, x')|^r}{|y'|^\alpha} \frac{d\xi'}{y'^\alpha},
\]

so by definition (2.1), the fact that \( h \in W^{1-1/r, 1, \infty}_r(\mathbb{R}^2 \times \mathbb{R}) \) (note also that \( 2 + r(1 - 1/r) = 1 + r \)) and (3.17) we have

\[
\|I_1\|_{L_r(D^4)} \leq c \|h\|_{W^{1-1/r, 0}_r(\mathbb{R}^4)}.
\]

Similar considerations can be applied in the cases \( h_2 \neq 0 \) and \( h_3 \neq 0 \). When we consider the derivation with respect to \( x_3 \) we use \( \partial_{x_3} (e^{-|\xi'|^2 s_1}) = -|\xi'|^2 e^{-|\xi'|^2 s_1} \), which reduces our considerations to cases with \( x' \). Finally, we get

\[
\|D_{s_1} P\|_{L_r(D^4)} \leq c \|h\|_{W^{1-1/r, 0}_r(\mathbb{R}^4)},
\]

which gives (3.7).
Now we take the velocity. First we assume that $h_2 = h_3 = 0$ and we consider the terms from (3.5)_{1,2,3} with $e^{-r\xi}$. We take the first term; the others can be estimated in the same way. We consider

$$u_1(t, x) = \int e^{it} e^{i\omega \xi} \left[ -\frac{1}{v_i} e^{-r\xi} h_1(s, \xi') \right] d\xi' d\xi$$

and

$$D_{x_2}^2 u_1(t, x) = \int e^{it} e^{i\omega \xi} \left[ -\frac{\xi_1}{v_i} e^{-r\xi} h_1(s, \xi') \right] d\xi' d\xi'.$$

We have

$$K^2 = \mathcal{F}_{t, v}^{-1} e^{-r\xi} = -\frac{x_3}{v_5/2 + 5/2} \exp \left( -\frac{|x|^2}{4v_t} \right),$$

$$\mathcal{F}_{t, v}^{-1} \xi_1 e^{-r\xi} = K_{x_1} \frac{x_3}{2v_7/2 + 7/2} \exp \left( -\frac{|x|^2}{4v_t} \right),$$

and

$$\left\{ K_{x_1, x_2}^{K_2} dx' = 0. $$

Since $|s_i^i \xi_{i}^j e^{\xi'} | \cdot |\partial_{j} \partial_{k} \partial_{l} \partial_{m} g(\xi_t / \xi) | \leq c$, by Theorem 2.1, acting as in the pressure case we have to consider only

$$I_2(t, x) = \int \text{d}y' \int \text{d}t' K_{y_1}^{y_1}(t', y', x_1) [h(x' - y', t' - t) - h(x', t' - t)].$$

In view of the Minkowski inequality, we get

$$\|I_2\|_{L_r(\mathbb{R}^3)} \leq \int \text{d} t \int \text{d}y' \| K_{y_1} \|_{N(y')} \left( \int \text{d} t \int \text{d}x' \frac{1}{v_5/2 + 5/2} \exp \left( -\frac{|y|^2}{4v_t} \right) \right)^{1/r}$$

$$\times \left( \int \text{d} t \int \text{d}x' \frac{1}{v_5/2 + 5/2} \exp \left( -\frac{|x|^2}{4v_t} \right) \right)^{1/r}$$

where $m$ will be specified later, $r^* = r/(r - 1), N(y') = \|h(x' - y', t' - t) - h(x', t' - t)\|_{L_r(\mathbb{R}^3 \times \mathbb{R}^3)}$ and for the last integral in (3.25) we have

$$\int \text{d} t \int \text{d}x' \frac{1}{v_5/2 + 5/2} \exp \left( -\frac{|y|^2}{4v_t} \right)$$

$$\times \left( \int \text{d} t \int \text{d}x' \frac{1}{v_5/2 + 5/2} \exp \left( -\frac{|x|^2}{4v_t} \right) \right)^{1/r}$$

$$\leq c \int \text{d} t \int \text{d}x' t^{1/2} x_1^r \exp \left( -\frac{x_3^2}{4v_t} \right) \exp \left( -\frac{x_3^2}{4v_t} \right).$$

Taking $w = x_1/t^{1/2}$ we have

$$\int_0^\infty \text{d} t \frac{1}{v_5/2 + 5/2} \exp \left( -\frac{x_3^2}{4v_t} \right) \exp \left( -\frac{x_3^2}{4v_t} \right)$$

$$= c \int_0^\infty \text{d} t \frac{t^{r^*+1}}{v_5/2 + 5/2} \exp \left( -\frac{x_3^2}{4v_t} \right).$$

Taking $t = |x_3|^2 / w$ we get

$$\int_0^\infty \text{d} t \frac{t^{r^*+1}}{v_5/2 + 5/2} \exp \left( -\frac{x_3^2}{4v_t} \right)$$

$$= c \int_0^\infty \text{d} t \frac{t^{r^*+2}}{v_5/2 + 5/2} \exp \left( -\frac{x_3^2}{4v_t} \right).$$

Thus we have to assume

$$2 + r^* + 2 + r^* m < 1,$$

so

$$m > \frac{2}{r^*} + 1 - \frac{5}{2} - \frac{2}{r}.$$

Hence if $m$ satisfies (3.26), then

$$\int_0^\infty \text{d} x_3 x_3^{r^*+2} \exp \left( -\frac{x_3^2}{4v_t} \right)^{1/r^*} \leq c_{x_3} \left( \frac{t^{r^*+2} m}{r^*} \right)^{1/r^*}.$$

We have

$$[(4 + r^* - 2 r^* m) / r^*] r = [4 + r^* + 1 - 2m] r = [5 - 4 r/2 - 2 m] r = 5 r - 2 m r - 4.$$

Applying (3.25) and (3.27) we get

$$\|I_2\|_{L_r(\mathbb{R}^3)} \leq c \int_0^\infty \text{d} x_3 x_3^{r^*+r^*+2 - 2 m - 4} \int_0^\infty \text{d} y' \frac{1}{t^{(7/2 - 2 m)}} \exp \left( -\frac{x_3^2}{4v_t} \right)$$

$$\times \exp \left( -\frac{|y'|^2}{4v_t} \right) \frac{|y'|^{r^*+2} N_r(y')}{|y'|^2}.$$

Hence we have to consider

$$\int_0^\infty \text{d} x_3 x_3^{r^*+r^*+2 - 2 m - 4}$$

$$\times \int_0^\infty \text{d} t \frac{1}{t^{(7/2 - 2 m)}} \exp \left( -\frac{x_3^2}{4v_t} \right) \exp \left( -\frac{|y'|^2}{4v_t} \right)$$

$$= c \int_0^\infty \text{d} t \frac{t^{r^*+r^*+2 + 1/2}}{t^{(7/2 - 2 m)}} \exp \left( -\frac{|y'|^2}{t} \right).$$
Taking $t = |y'|^2/w$ we see that the above integral is equal to
\[
\int_0^\infty \frac{dy'}{dy'} \left( \left| \frac{y'}{2^{3-r-mr-3/2-(7/2-m)r}} \right| \right) e^{-w/(4v)} \frac{e^{-r|y'|^2}}{r - |\xi'|^2}
\]
\[
= \int_0^\infty \frac{1}{dy'} 2^{3-r-mr-3/2-(7/2-m)r} e^{-w/(4v)} dy' = \frac{1}{2^{2+6r-2mr-3-(7-2)m}r + \alpha} e^{-w/(4v)}.
\]
The conditions for independence from $|y'|$ and for finiteness of (3.29) are
\[
2 + 6r - 2mr - 3 - (7 - 2m)r + \alpha = 0
\]
or
\[
\alpha = 1 + r
\]
and
\[
2 + 3r - mr - 3/2 - (7/2 - m)r < 1,
\]
which is true for $m$ satisfying (3.26), and this implies boundedness of (3.29).

Thus from (3.28), (3.30) and definition (2.1) we get
\[
\|I_2\|_{L^1(R^3)} \leq C \|h\|_{W^{1,1/2}(R^3)}
\]
and in the same way as in the case of the pressure we conclude that
\[
\|D^2_{t,u_2}(x,t)\|_{L^1(R^3)} \leq C \|h\|_{W^{1,1/2}(R^3)}
\]

To show regularity with respect to $t$ we consider
\[
D^2_{t,u_2}(t,x) = \mathcal{F}_{t,y'}^{-1} \left[ \frac{e^{-r|y'|^2}}{r - |\xi'|^2} \right] h_1(s,\xi') d\xi' dt.
\]

Using
\[
\mathcal{F}_{t,y'}^{-1} \left[ \frac{e^{-r|y'|^2}}{r - |\xi'|^2} \right] = C \frac{1}{\xi'^2} \exp \left( - \frac{|\xi'|^2}{4v} \right)
\]
and repeating the considerations for $D^2_{y_2,u_1}$ we get
\[
\|D^2_{t,u_2}(x,t)\|_{L^1(R^3)} \leq C \|h\|_{W^{1,1/2}(R^3)}.
\]

Now we take only one term with $e^{-r|y'|^2} - e^{-|\xi'|^2}$ because others can be estimated similarly using Lemma 3.1. Choosing $h_2 = h_3 = 0$ we consider
\[
u_2(t,x) = \mathcal{F}_{t,y'}^{-1} \left[ \frac{i\xi_1^s}{\nu^2 P(r + |\xi'|^2)} \right] \left[ 2ri\xi_1h_1 \right] e^{-r|y'|^2} - e^{-|\xi'|^2}.
\]

By Lemma 3.1,
\[
\frac{i\xi_1^s}{\nu^2 P(r + |\xi'|^2)} 2ri\xi_1
\]

satisfies the conditions of Theorem 2.1.

We need to know that (see [6])
\[
K^2 = \mathcal{F}_{t,y_3}^{-1} \left[ \frac{e^{-r|y'|^2} - e^{-|\xi'|^2}}{r - |\xi'|^2} \right] = \mathcal{F}_{t,y_3}^{-1} \left[ \frac{e^{-|\xi'|^2}}{r - |\xi'|^2} \right] dt
\]
\[
= \int_0^\infty \frac{y}{\xi'^2} e^{-y/(4v)} e^{-|\xi'|^2} d\xi' dt
\]
\[
= \int_0^\infty \frac{y}{\xi'^2} e^{-y/(4v)} \frac{x_3 - y}{(x_3 - x_1)^2 + (x_3 - x_2^2 + (x_3 + z)^2)^2} e^{-|\xi'|^2} d\xi' dt
\]

We have
\[
(3.33) \quad D^2_{t,u_2}(x,t) = \mathcal{F}_{t,y'}^{-1} \left[ \frac{i\xi_1^s}{\nu^2 P(r + |\xi'|^2)} \right] \left[ 2ri\xi_1h_1 \right] \frac{e^{-r|y'|^2} - e^{-|\xi'|^2}}{r - |\xi'|^2}
\]

and (see [6])
\[
(3.34) \quad |D^2_{x,y} K^2| \leq \frac{C}{\xi'^2/(x_3 + t)^2}
\]

and
\[
(3.35) \quad \int_{R^3} D^2_{x,y} K^3 dx' = 0.
\]

Therefore, similarly to the case of $u_1$, we introduce
\[
(3.36) \quad I_3(t,x) = \int_{R^3} dt' \int_{R^2} dy' D^2_{x,y} K^3(x',y',x_3)[h(x' - y', t - t') - h(x', t - t')]
\]

and get
\[
(3.37) \quad \|I_3\|_{L^1(R^3)} \leq \int_0^\infty dt \int_{R^2} dy' \frac{dy'}{\xi'^2/(y_3^2 + x_3^2 + t)^2} N(y')
\]
\[
\leq \left( \int_{R^3} dt \int_{R^2} dy' \frac{1}{\xi'^2/(y_3^2 + x_3^2 + t)^2} N(y') \right)^{1/r}
\]
\[
\times \left( \int_{R^3} dt \int_{R^2} dy' \frac{1}{\xi'^2/(y_3^2 + x_3^2 + t)^{2m}} \right)^{1/r'}
\]

For the second integral in (3.37) we have
\[
\int_{R^3} dt \int_{R^2} dy' \frac{1}{\xi'^2/(y_3^2 + x_3^2 + t)^{mr}} \leq \int_0^\infty dt \frac{1}{\xi'^2/(x_3^2 + t)^{mr}} \leq C \int_0^\infty dt \frac{1}{\xi'^2/(x_3^2 + t)^{mr-1}}
\]
If we take \( w = t/x_3 \), the last integral becomes
\[
\int_0^\infty \frac{dw}{x_3 w^{1/2}(1+w)^{m-1} x_3^{2r-2}}.
\]
Hence when choosing
\[
m r^* - 1 > \frac{1}{2},
\]
\[
m > \frac{3}{2} \frac{r-1}{r},
\]
we get
\[
\left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{t^{1/2}(y^2 + x_3^2 + t)^{m-3}} \right)^{1/r^*} \leq c x_3^{(3-2mr^*)/r^*}.
\]
Thus by (3.37) and (3.39) we have
\[
\|I_3\|_{L^r(\mathbb{R}^4)} \leq c \int_0^\infty dx_3 x_3^{(3-2mr^*)(r-1)}
\]
\[
\times \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{t^{1/2}(y^2 + x_3^2 + t)^{(2-m) r}} dy't^{1/2}(y^2 + x_3^2 + t)^{(2-m) r} |y'|^\alpha /|y'|^\alpha.
\]
We consider
\[
\|D_{x_3} u\|_{L^r(\mathbb{R}^4)} \leq c \|H\|_{W^{1/2,1/2}_r(\mathbb{R}^4)}
\]
and from (3.47) together with (3.20), (3.31), (3.32), (3.46) and from equation (3.1), we get estimates for \( \|D_{x_3} u\|_{L^r} \). Hence we have
\[
u \in W^{2,1}_{r, \text{loc}}(\mathbb{R}^4),
\]
which gives (3.7) and (3.7). From the boundary conditions we get (3.7.4). Lemma 3.2 is proved.

The next step towards solving (1.1) in the half-space is to solve the following problem in the whole space:
\[
u_{x_3} - \nu \Delta u + \nabla p = f,
\]
\[
\text{div } u = 0,
\]
\[
u_{|t=0} = 0.
\]

**Lemma 3.3.** Let \( f \in L_r(\mathbb{R}_+) \) then there exists a unique solution of (3.49) such that \( u \in W^{2,1}_{r, \text{loc}}(\mathbb{R}^4) \) and \( p \in W^{1,0}_{r, \text{loc}}(\mathbb{R}^4) \) and the following estimate holds:
\[
\|D_{x_3} u\|_{L^r(\mathbb{R}^4)} + \|D_{x_3} u\|_{L^r(\mathbb{R}^4)} + \|D_{x_3} p\|_{L^r(\mathbb{R}^4)} \leq c \|f\|_{L^r(\mathbb{R}^4)}.
\]
Moreover if \( f \in L^1_{r, \text{div}}(\mathbb{R}^3 \times \mathbb{R}) \) then \( p \equiv 0 \), where
\[
L^1_{r, \text{div}}(\mathbb{R}^3 \times \mathbb{R}) = \{ f \in C^0(\mathbb{R}^3 \times \mathbb{R}) : \text{div } f = 0 \}
\]
Proof. After taking the Fourier transform system (3.49) reads

\[(s + iv)\xi^2 v_a + i\xi_a q = f_a, \quad \sum_a \xi_a v_a = 0\]

or

\[AX = \begin{bmatrix}
\nu s^2 & 0 & 0 & i\xi_1 \\
0 & \nu s^2 & 0 & i\xi_2 \\
0 & 0 & \nu s^2 & i\xi_3 \\
\xi_1 & \xi_2 & \xi_3 & 0
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
q
\end{bmatrix} = \begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
f
\end{bmatrix} = F.
\]

Thus the solution is \(X = A^{-1}F\), where

\[A^{-1} = \begin{bmatrix}
\frac{\xi_1^2 + s^2}{\nu s^2} & -\frac{\xi_1 \xi_2}{\nu s^2} & -\frac{\xi_1 \xi_3}{\nu s^2} & \frac{\xi_1}{\nu s^2} \\
-\frac{\xi_2 \xi_1}{\nu s^2} & \frac{\xi_2^2 + s^2}{\nu s^2} & -\frac{\xi_2 \xi_3}{\nu s^2} & \frac{\xi_2}{\nu s^2} \\
-\frac{\xi_3 \xi_1}{\nu s^2} & -\frac{\xi_3 \xi_2}{\nu s^2} & \frac{\xi_3^2 + s^2}{\nu s^2} & \frac{\xi_3}{\nu s^2} \\
-\frac{1}{\nu s^2} & -\frac{1}{\nu s^2} & -\frac{1}{\nu s^2} & \frac{1}{\nu s^2}
\end{bmatrix}.
\]

It is seen that estimate (3.50) is true and if \(f \in L_r^{\text{div}}(\text{div} f = 0)\) then \(p \equiv 0\). From (3.49), we get the assertion.

Using Lemmas 3.2 and 3.3 we deduce the main result of this section:

**Lemma 3.4.** Let \(r \geq 2, f \in L_r(\mathbb{D}^d), g \in W_r^2(\mathbb{D}^d), g - \text{div} f = \text{div} B + A, A, B \in L_r(\mathbb{D}^d), \text{diam}supp\ A < \lambda\) and \(h \in W_r^{1-1/r,1/2-1/(2r)}(\mathbb{R}_+^3 = 0)\). Then there exists a unique solution of the system

\[u_t - \nu \Delta u + \nabla p = f,\]
\[
\text{div} u = g,\]
\[
e_3 \cdot T(u, p) = h,\]
\[
u|_{t=0} = 0,
\]

such that \(u \in W_r^{2,1}(\mathbb{D}^d), p \in W_r^{1,0}(\mathbb{D}^d)\) and \(p \in W_r^{1-1/r,1/2-1/(2r)}(\mathbb{R}_+^3 = 0)\) and the following estimate holds:

\[(3.52) \quad \|D_t u\|_{L_r(\mathbb{D}^d)} + \|D_t^2 u\|_{L_r(\mathbb{D}^d)} + \|D_x p\|_{L_r(\mathbb{D}^d)} + \|p\|_{W_r^{1-1/r,1/2-1/(2r)}(\mathbb{R}_+^3, \times [0, T])} \leq c(T)\|f\|_{L_r(\mathbb{D}^d)} + \|B\|_{L_r(\mathbb{D}^d)} + \lambda\|A\|_{L_r(\mathbb{D}^d)} + \|h\|_{W_r^{1-1/r,1/2-1/(2r)}(\mathbb{R}_+^3 = 0)} + c\|\nabla u\|_{W_r^{2,1}}.
\]

where \(\mathbb{D}^d = \mathbb{R}_+^3 \times [0, \infty)_{x_3} \times [0, T]_t\).

**Proof.** First we consider the following problem:

\[(3.53) \quad \Delta w = g,\]
\[
w_{|x_3=0} = 0,\]
\[
w \to 0 \quad \text{as} \quad x_3 \to \infty.
\]

We easily see that a solution of (3.53) exists and

\[(3.54) \quad \|\nabla w\|_{W_r^{2,1}(\mathbb{D}^d)} \leq c\|g\|_{W_r^{2,1}(\mathbb{D}^d)}.
\]

Next, we differentiate (3.53) with respect to \(t\) to get

\[(3.55) \quad \Delta w_t = \text{div}(f + B) + A
\]

with the same boundary conditions. To estimate \(\|\nabla w_t\|_{L_r}\) we look for \(\nabla w_e\) in the form

\[(3.56) \quad \nabla w_t = \nabla w_{t1} + \nabla w_{t2},\]

where

\[(3.57) \quad \Delta w_{t1} = \text{div}(f + B),\]
\[(3.58) \quad \Delta w_{t2} = A,
\]

and for (3.57) and (3.58) we have the same boundary conditions as in (3.53). From (3.57) we get

\[(3.59) \quad \|\nabla w_{t1}\|_{L_r(\mathbb{D}^d)} \leq c(\|f\|_{L_r(\mathbb{D}^d)} + \|B\|_{L_r(\mathbb{D}^d)} + c\|\|A\|_{L_r(\mathbb{D}^d)})
\]

and from (3.58),

\[(3.60) \quad \|\nabla w_{t2}\|_{L_r(\mathbb{D}^d)} \leq c\|A\|_{L_r(\mathbb{D}^d)}.
\]

Hence by (3.54), (3.56), (3.59) and (3.60) we obtain

\[(3.61) \quad \|\nabla w\|_{W_r^{2,1}(\mathbb{D}^d)} \leq c(\|g\|_{W_r^{2,1}(\mathbb{D}^d)} + \|f\|_{L_r(\mathbb{D}^d)} + \|B\|_{L_r(\mathbb{D}^d)} + \lambda\|A\|_{L_r(\mathbb{D}^d)})
\]

If we take \(u = v + \nabla w\), system (3.53) reduces to

\[(3.62) \quad \begin{align*}
u_t - \nu \Delta u + \nabla p &= f', \\
\text{div} v &= 0, \\
e_3 \cdot T(v, p) &= h', \\
v_{|t=0} &= 0,
\end{align*}
\]

where \(f' = f - \nu \Delta u + \nabla \nu w + h' - \lambda e_3 \cdot T(u, 0)\) and

\[(3.63) \quad \|f'\|_{L_r} \leq \|f\|_{L_r} + c\|\|\nabla u\|_{W_r^{2,1}}.
\]

Next we consider the following problem:

\[(3.64) \quad \Delta p'' = \text{div} f',
\]
\[
p''|_{x_3=0} = 0, \\
p'' \to 0 \quad \text{as} \quad x_3 \to \infty.
\]

We see that \(p'' \in W_r^{1,0}(\mathbb{D}^d)\) and

\[(3.65) \quad \|\nabla p''\|_{L_r(\mathbb{D}^d)} \leq c\|f'\|_{L_r(\mathbb{D}^d)}.
\]
Now we take $p$ in the form
\begin{equation}
(3.66)
  p = p' + p''
\end{equation}
where $p''$ is the solution of (3.64). Hence problem (3.62) can be reduced to
\begin{align}
  u_t - \nu \Delta u + \nabla p' &= f_0, \\
  \text{div } v &= 0, \\
  \bar{\Omega} \cdot \nabla (u, p') &= h', \\
  u|_{t=0} &= 0,
\end{align}
(3.67)
where $f_0 = f' - \nabla p''$ and it is easy to see that $\text{div } f_0 = 0$.
From Lemmas 3.2 and 3.3 (the case with $\text{div } f = 0$) we get a solution of (3.56) which satisfies
\begin{align}
(3.68) \quad &\|D_1 v\|_{L^2(\Omega^k_2)} + \|D_2 v\|_{L^2(\Omega^k_2)} + \|D_3 p'\|_{L^2(\Omega^k_2)} \\
&+ \|p'\|_{W^{2-1/r, 1/2-1/(2r)}(\Omega^k_2 \times [0, T])} \\
&\leq c(T) [\|f_0\|_{L^2(\Omega^k_2)} + \|h'\|_{W^{2-1/r, 1/2-1/(2r)}(\Omega^k_2)}].
\end{align}
The constant $c(T)$ depends on $T$ because we take the trace of the solution from Lemma 3.3 and the whole norm depends on $T$ by (3.50).
By (3.61), (3.63), (3.65), (3.66) and (3.68) we get (3.52).

**4. Problem in a bounded domain.** In this section we prove the existence of solutions of the following problem in the bounded domain $\Omega_T$:
\begin{align}
  u_t - \nu \Delta u + \nabla p &= f, \\
  \text{div } u &= 0, \\
  \bar{\Omega} \cdot \nabla (u, p)|_{\partial \Omega_T} &= 0, \\
  u|_{t=0} &= 0.
\end{align}
(4.1)

Restrict our considerations to the case when
\begin{align}
  f \in L^2_{\text{div}}(\Omega_T) = \{ f \in C^\infty(\Omega_T) : \text{div } f = 0 \}.
\end{align}

**Lemma 4.1.** If $f \in L^2_{\text{div}}(\Omega_T)$ then there exists a unique solution of (4.1) such that $u \in W^{2,1}(\Omega_T)$ and $p \in W^{1,4}(\Omega_T) \cap W^{2,1/4}(\Omega_T)$, and the following estimate holds:
\begin{align}
  |u|_{W^{2,1}(\Omega_T)} + |p|_{W^{1,4}(\Omega_T)} + |p|_{W^{2,1/4}(\Omega_T)} \leq c(T) \|f\|_{L^2(\Omega_T)}.
\end{align}

**Proof.** In [10] we have existence of solutions for (4.1) and the estimate
\begin{align}
  |u|_{W^{2,1}(\Omega_T)} + |p|_{W^{1,4}(\Omega_T)} \leq c(T) \|f\|_{L^2(\Omega_T)}.
\end{align}

By the trace theorem and boundary condition (4.1) we get $p \in W^{2,1/4}(\Omega_T)$.

From (4.1) and $\text{div } f = 0$ we get the following problem for $p$:
\begin{align}
  \Delta p &= 0, \\
  p|_{\partial \Omega_T} &\in W^{1/2,1/4}_2(\partial \Omega_T).
\end{align}
(4.4)

We see that the solution $p$ of (4.4) belongs to $W^{1,1/4}_2(\Omega_T)$. Thus by (4.3) and (4.4) we get (4.2).

Now we want to show regularity of the weak solution to problem (4.1).
We introduce two collections of open sets: $\{\Omega^{(k)}\}$ and $\{\Omega^{(k)}\}$ such that
$\omega^{(k)} \subset \Omega^{(k)} \subset \Omega$, $\bigcup_k \omega^{(k)} = \Omega$ and $\bigcup_k \Omega^{(k)} = \Omega$ with $k \in \mathcal{M} \cup \mathcal{N}$ where
$\Omega^{(k)} \cap S \neq \emptyset$ if $k \in \mathcal{M}$ and $\omega^{(k)} \cap S \neq \emptyset$ if $k \in \mathcal{N}$.
We assume that
\begin{align}
  \sup_{k} \text{diam } \Omega^{(k)} \leq 2\lambda
\end{align}
for some $\lambda$ small enough. Let $\zeta^{(k)}$ be a smooth function such that $0 \leq \zeta^{(k)} \leq 1$ and $\zeta^{(k)}(x) = 1$ for $x \in \omega^{(k)}$, $\zeta^{(k)}(x) = 0$ for $x \in \Omega \setminus \Omega^{(k)}$ and $|D_1 \zeta^{(k)}(x)| \leq c/|x|$ and $1 \leq \sum_k (|\zeta^{(k)}|)^2 \leq N_0$. We will omit $(k)$ if it causes no confusion.

By $\xi^{(k)}$ we denote a center (a point inside) of $\omega^{(k)}$ for $k \in \mathcal{M}$ and a center of $\omega^{(k)} \cap S$ for $k \in \mathcal{N}$.

Let us consider a local coordinate system $y = (y_1, y_2, y_3)$ with center at $\xi^{(k)}$. If $k \in \mathcal{N}$ then the part $S^{(k)} = S \cap \Omega^{(k)}$ of the boundary is described by $y_3 = F(y_1, y_2)$. We choose the coordinates such that $F(0) = 0$ and $\nabla F(0) = 0$. From $S \in W^{2-1/r}_{\text{rad}}(\Omega^k_2)$ we see that $F \in W^{2-1/r}_{\text{rad}}$. Extending $F$ to $\overline{F}$ in such a way that
\begin{align}
  \overline{F}(y_1, y_2, 0) = F(y_1, y_2) \quad \text{and} \quad \overline{F} \in W^{2}_{\text{rad}}
\end{align}
we have
\begin{align}
  \overline{F} \in C^{1+\alpha} \quad \text{with } \alpha < 1 - 3/r, \quad |\nabla \overline{F}| \leq c\lambda^\alpha.
\end{align}

Now we can transform $\Omega^{(k)}$ into the half-space by the transformation
\begin{align}
  z = \Phi_{(k)}(y) = (\text{Id} - \overline{F})(y).
\end{align}

Let $y = Y_k(x)$ be a transformation to the local coordinates $y$ which consists of translations and rotations.

In our considerations we need some smallness argument, hence we define
\begin{align}
  \beta = \beta + c(\delta)T^\alpha + \frac{T^{1/k} + T^{1/2}}{\lambda} + \lambda\varepsilon/k,
\end{align}
where $0 < \alpha < 1 - 3/r$, $\varepsilon, k, \alpha, \delta$ and $c(\delta)$ will be defined by (4.12), (4.19) and (4.23).
Let us introduce the variables $U = u\zeta$ and $P = pq$. Assume that $\zeta = \zeta^{(1)}$ and $l \in \mathcal{M}$. Then the equations (4.1) take the form
\begin{equation}
U_2 - \nu \Delta U + \nabla P = -2\nu \nabla \zeta \cdot \nabla u - \nu u \Delta \zeta + p \nabla \zeta + f \zeta \equiv f',
\end{equation}
(4.6)
\begin{equation}
\text{div } U = u \cdot \nabla \zeta \equiv g',
\end{equation}
(4.7)
\begin{equation}
\nabla \zeta \cdot (u_2 - f) = \nabla \zeta \cdot (\nu \Delta u - \nabla p) = \nabla \cdot \left[ \nu (\nabla \zeta \times \text{rot } u) - p \nabla \zeta \right] + p \Delta \zeta,
\end{equation}
(4.8)
where
\begin{equation}
B' = \zeta B + 2\nu \nabla \zeta \cdot \nabla u + \nu u \Delta \zeta - 2p \nabla \zeta + \nu (\nabla \zeta \times \text{rot } u),
\end{equation}
(4.9)
\begin{equation}
A' = \zeta A - B \cdot \nabla \zeta + p \Delta \zeta.
\end{equation}

Now we obtain a condition on new functions $f', g', A', B'$, where $A'$ and $B'$ are to be defined. For this purpose we consider
\begin{equation}
g' = -\text{div } f' = \zeta (g - \text{div } f) + \nabla \zeta \cdot (u_2 - f) + \nabla [2\nu \nabla \zeta \cdot \nabla u + \nu u \Delta \zeta - p \nabla \zeta]
\end{equation}
and since
\begin{equation}
\nabla \zeta \cdot (u_2 - f) = \nabla \zeta \cdot (\nu \Delta u - \nabla p) = \nabla \cdot \left[ \nu (\nabla \zeta \times \text{rot } u) - p \nabla \zeta \right] + p \Delta \zeta,
\end{equation}

we have
\begin{equation}
g' = -\text{div } f' = \nabla \zeta \cdot (\nu \Delta u - \nabla p) = \nabla \cdot [\nu (\nabla \zeta \times \text{rot } u) - p \nabla \zeta] + p \Delta \zeta,
\end{equation}
(4.10)
where
\begin{equation}
B' = \zeta B + 2\nu \nabla \zeta \cdot \nabla u + \nu u \Delta \zeta - 2p \nabla \zeta + \nu (\nabla \zeta \times \text{rot } u),
\end{equation}
(4.11)
\begin{equation}
A' = \zeta A - B \cdot \nabla \zeta + p \Delta \zeta.
\end{equation}

Since we are looking for $u \in W^{2,1}_{r}$ and $\nabla p \in L_r$, we apply Lemma 3.4, thus we have to examine the norms $\|A'\|_{L_r}$ and $\|B'\|_{L_r}$.

By Lemma 3.4 we see that the solutions of (4.6) satisfy the following estimate:
\begin{equation}
\|f'\|_{L_k(\mathcal{M})} + \|P\|_{W^{1,0}_{k}(\mathcal{M})} \leq C(T)\left(\|f\|_{L_k(\mathcal{M})} + \|g'\|_{W^{1,0}_{k}(\mathcal{M})} + \lambda \|A'\|_{L_k(\mathcal{M})} + \|B'\|_{L_k(\mathcal{M})}\right),
\end{equation}
where $k > 2$.

By Proposition 2.3 we have
\begin{equation}
\|f'\|_{L_k(\mathcal{M})} \leq c\left(\|f\|_{L_k(\mathcal{M})} + \frac{T^{1/2}}{\lambda} \|u\|_{W^{2,1}_{k}(\mathcal{M})} + \frac{1}{\lambda} \|P\|_{L_k(\mathcal{M})}\right),
\end{equation}
(4.12)
From the imbedding theorem [1, Chap. 18] we see that

\begin{equation}
\|P\|_{L_k(\mathcal{M})} \leq c\|P\|_{W^{1,1/4}_{k}(\mathcal{M})},
\end{equation}
(4.13)

since $(1/2 - 1/k)3 + (1/2 - 1/k)2 < 1$. This implies that now we can consider only the case $2 < k < 10/3$. Then by Proposition 2.3 we obtain
\begin{equation}
\|f'\|_{W^{1,0}_{k}(\mathcal{M})} \leq \frac{T^{1/2}}{\lambda} \|u\|_{W^{2,1}_{k}(\mathcal{M})},
\end{equation}
(4.14)
\begin{equation}
\|A'\|_{L_k(\mathcal{M})} \leq c\left(\|A\|_{W^{1,1/4}_{k}(\mathcal{M})} + \frac{1}{\lambda} \|B\|_{L_k(\mathcal{M})} + \frac{T^{1/2}}{\lambda} \|P\|_{W^{1,1/4}_{k}(\mathcal{M})}\right),
\end{equation}
(4.15)
\begin{equation}
\|B'\|_{L_k(\mathcal{M})} \leq c\left(\|B\|_{L_k(\mathcal{M})} + \frac{T^{1/2}}{\lambda} \|u\|_{W^{2,1}_{k}(\mathcal{M})} + \frac{1}{\lambda} \|P\|_{W^{1,1/4}_{k}(\mathcal{M})}\right).
\end{equation}
(4.16)

By (4.11)–(4.13) inequality (4.10) reads
\begin{equation}
\|f'\|_{L_k(\mathcal{M})} + \|P\|_{W^{1,0}_{k}(\mathcal{M})} \leq C(T)\left(\|f\|_{L_k(\mathcal{M})} + \frac{T^{1/2}}{\lambda} \|u\|_{W^{2,1}_{k}(\mathcal{M})} + \lambda \|A\|_{W^{1,1/4}_{k}(\mathcal{M})} + \frac{T^{1/2}}{\lambda} \|P\|_{W^{1,1/4}_{k}(\mathcal{M})}\right).
\end{equation}
(4.17)

We would like to have a similar estimate when $k \in \mathcal{N}$. We write system (4.1) in $z$-coordinates:
\begin{equation}
U_1 - \nu \Delta_1 U + \nabla_z P = f' + L_1(\partial_z - \nabla \zeta \cdot \partial_z)(U, P) - L_1(\partial_z)(U, P) \equiv f'',
\end{equation}
(4.18)
\begin{equation}
\text{div }_z U = g' + \nabla_z \alpha \cdot \partial_z U \equiv g'',
\end{equation}
(4.19)
where
\begin{equation}
L_1(\partial_z)(u, p) = u_2 - \nu \Delta_z u + \nabla_z p.
\end{equation}
(4.20)

To apply Lemma 3.4 we need new $A''$ and $B''$ which satisfy
\begin{equation}
\partial_z g'' - \text{div } f'' = \text{div } B'' + A''.
\end{equation}
(4.21)
First we note that (the second term of the r.h.s. of (4.15))
\begin{equation}
\bar{F}_{i,j}U_{j,i} = \partial_z(\bar{F}_{i,j}U_{j,i}) - U_j \bar{F}_{i,j,i}.
\end{equation}
(4.22)
To obtain the new $B''$ we have to consider
\begin{equation}
\Delta b = U_j \bar{F}_{i,j,i} - b|_{z=0} = 0,
\end{equation}
(4.23)
\begin{equation}
b \to 0 \text{ as } z \to \infty.
\end{equation}
(4.24)

We see that
\begin{equation}
\|U_j \bar{F}_{i,j,i}\|_{L_k(\mathcal{M})} \leq c\|U_j\|_{L_k(\mathcal{M})},
\end{equation}
where $l = rk/(r + k)$, hence solving (4.21) we get
\begin{equation}
\|\nabla b\|_{W^{1,1/4}_{k}(\mathcal{M})} \leq c\|U_j\|_{L_k(\mathcal{M})}.
\end{equation}
(4.25)
Since $3l/(3 - l) > k' + \epsilon = k$, by Proposition 2.2 we have
\begin{equation}
\|\nabla b\|_{L^k(0, \tau; L^2(U))} \leq c \|U\|_{L^k(0, \tau; L^2(U))} \leq c\lambda^{3k/((k' + 1)k)} \|U\|_{L^k(U)}.
\end{equation}

Then we define $A''$ and $B''$ by
\begin{equation}
A'' = B' - L_1(\partial_x - \nabla \bar{F} \cdot \partial_x)(U, P) - L_1(\partial_x)(U, P) + \bar{F} \nabla U - \nabla b,
B'' = B - L_1(\partial_x - \nabla \bar{F} \cdot \partial_x)(U, P) - L_1(\partial_x)(U, P) + \bar{F} \nabla U - \nabla b,
\end{equation}
and we also need estimates for components of $B''$, hence we examine
\begin{equation}
\|L_1(\partial_x - \nabla \bar{F} \cdot \partial_x)(U, P) - L_1(\partial_x)(U, P)\|_{L^2(U)} \leq c\|\nabla \bar{F}\|_{L^2(U)} \|\nabla U\|_{L^2(U)} + c\|\nabla \bar{F}\|_{L^2(U)} \|\nabla U\|_{L^2(U)}.
\end{equation}

To estimate the first term of the r.h.s. of (4.21) we note that
\begin{equation}
\|\nabla U\|_{L^2(U)} \leq \|\nabla \bar{F}\|_{L^2(U)} \|\nabla U\|_{L^2(U)},
\end{equation}
where $\delta, \alpha > 0$ and $c(\delta)$ tends to infinity as $\delta \to 0$.

By (4.22) we get
\begin{equation}
\|\nabla U\|_{L^2(U)} \leq c(\delta) T^n \|U\|_{W^{2,1}},
\end{equation}
and then from (4.25) we get
\begin{equation}
\|U\|_{W^{2,1}} + \|P\|_{W^{1,0}} + \|P\|_{W^{1-1/2,1/2-1/(3k)}_{(S)}} \leq c(\delta) T^n \|U\|_{W^{2,1}} + \|P\|_{W^{1-1/2,1/2-1/(3k)}_{(S)}}.
\end{equation}

Then by Lemma 3.4 and (4.24) we obtain
\begin{equation}
\|U\|_{W^{2,1}} + \|P\|_{W^{1,0}} + \|P\|_{W^{1-1/2,1/2-1/(3k)}_{(S)}} \leq c(\delta) T^n \|U\|_{W^{2,1}} + \|P\|_{W^{1-1/2,1/2-1/(3k)}_{(S)}}.
\end{equation}

If we take $\beta$ so small that
\begin{equation}
\beta \equiv c(\delta) T^n + \lambda^\alpha + \frac{T^{1/2} + T^{1/k}}{\lambda} + \lambda^{\epsilon/k} < \frac{1}{2},
\end{equation}
then from (4.25) we get
\begin{equation}
\|U\|_{W^{2,1}} + \|P\|_{W^{1,0}} + \|P\|_{W^{1-1/2,1/2-1/(3k)}_{(S)}} \leq c(\delta) T^n \|U\|_{W^{2,1}} + \|P\|_{W^{1-1/2,1/2-1/(3k)}_{(S)}}.
\end{equation}

The above considerations lead to the following lemma.

**Lemma 4.2.** If $A, B \in L^k(\Omega_T)$ and $f \in L^k_{\text{div}}(\Omega_T)$ then the weak solution $(u, p)$ of (4.1) satisfies
\begin{equation}
\|\nabla U\|_{W^{1,0}} + \|P\|_{W^{1,0}} + \|p\|_{W^{1-1/2,1/2-1/(3k)}_{(S)}} \leq c(\delta) T^n \|U\|_{W^{2,1}} + \|P\|_{W^{1-1/2,1/2-1/(3k)}_{(S)}} + \lambda^{\epsilon/k} + \lambda A|_{L^2(\Omega_T)}.
\end{equation}

If $\beta$ is small enough.

**Proof.** First we note that $dx = \beta \cdot dx$ and $|J_{\beta|1} - 1| \leq c\lambda^\alpha$. Thus if we denote $U(\tau) = \zeta(\tau) u$ and $P(\tau) = \zeta(\tau) p$ we get
\begin{equation}
\|U\|_{W^{2,1}_{(\tau)}} \leq \sum_{i \in \mathcal{M}, \lambda, k} \|U(\tau)\|_{W^{2,1}_{(\tau)}}.
\end{equation}

Then by (4.11)–(4.14), (4.26) and properties of functions $\zeta(\tau)$ we obtain
\begin{equation}
\|U\|_{W^{2,1}_{(\tau)}} + \|P\|_{W^{1,0}_{(\tau)}} + \|p\|_{W^{1-1/2,1/2-1/(3k)}_{(S)}} \leq cN_0 \left( \|f\|_{L^2} + \|B\|_{L^2} + \lambda A|_{L^2} + \frac{T^{1/2} + T^{1/2}}{\lambda} \right) \|U\|_{W^{2,1}_{(\tau)}} + \|p\|_{W^{1-1/2,1/2-1/(3k)}_{(S)}}.
\end{equation}

and if $(T^{1/2} + T^{1/2})cN_0 \lambda < 1/2$ we get
\begin{equation}
\|U\|_{W^{2,1}_{(\tau)}} + \|P\|_{W^{1,0}_{(\tau)}} + \|p\|_{W^{1-1/2,1/2-1/(3k)}_{(S)}} \leq cN_0 \left( \|f\|_{L^2} + \|B\|_{L^2} + \lambda A|_{L^2} + \|p\|_{W^{1-1/2,1/2-1/(3k)}_{(S)}} \right).
\end{equation}

Recalling (4.12) we have (4.30) only for $2 < k < 10/3$. 


By (4.1) and (4.30) we also obtain

\[ \Delta p = 0, \]

\[ p \in W^{1-1/k, 1/2-1/(2k)}_k(S_T), \]

which gives \( p \in W^{1,1/2-1/(2k)}(\Omega_T) \).

We put \( k_1 = 5/2 < 10/3 \). Then by (4.30) and (4.31) we have

\[ \|u\|_{W^{k,2}_{k_1}}(\Omega_T) + \|p\|_{W^{1-1/(2k), 1/2-1/(2k)}_k(\Omega_T)} + \|p\|_{W^{1-1/k, 1/2-1/(2k)}(S_T)} \leq c(\|f\|_{L^1_{k_1}} + \|B\|_{L_{k_1}} + \|A\|_{L_0} + \|p\|_{W^{1-1/(2k), 1/2-1/(2k)}_k(\Omega_T)}), \]

where \( k_0 = 2 \).

Now we repeat the above considerations having estimate (4.32). Imbedding (4.12) can be replaced by

\[ \|p\|_{L_0} \leq c\|p\|_{W^{1,1/2-1/(2k)}_k(\Omega_T)}. \]

Since \( k_1 = 5/2 \), we have the following condition on the new \( k \):

\[ 3\left(2 - \frac{1}{k}\right) + \frac{10}{3} \left(\frac{2}{5} - \frac{1}{k}\right) < 1, \]

which gives \( 5/2 < k < 30/7 \), and we put \( k_2 = \min\{r, 4\} \). Then we obtain estimate (4.32) with \( k_1 = k_2 \) and \( k_0 = k_1 \).

If \( r > 4 \) we again repeat the above considerations and obtain, having \( p \in W^{1,1/2-1/(2k)}_k(\Omega_T) \), a new \( k \) which has to satisfy \( 4 < k < 16 \) and we put \( k_3 = \min\{r, 15\} \), and if \( r > 15 \) we repeat all considerations and obtain (4.27), which follows from (4.32) with \( k_1 = r \) and \( k_0 = k_3 \), because \( W^{1,1/2-1/30}_{15} \subset L_0 \). In the last step \( \beta'(r) > 0 \) (see (4.5)), hence \( T(r), \lambda(r) > 0 \). The proof of Lemma 4.2 is finished.

5. Proof of Theorem 1. Our aim is to show existence of a solution of problem (1.1):

\[ u_t - \nu \Delta u + \nabla p = F, \]

\[ \text{div} u = G, \]

\[ \vec{n} \cdot \mathbf{T}(u, p)|_{S_T} = H, \]

\[ u|_{t=0} = u_0. \]

By (1.2) and the assumptions of Theorem 1, \( p_0 = p|_{t=0} \in W^{1-3/r}_r(\Omega_T) \). We take \( \vec{u}_0 \in W^{2-1/r}_r(\Omega_T) \) and \( \vec{p}_0 \in W^{1,0}_{k_0}(\Omega_T) \cap W^{1-1/r, 1/2-1/(2r)}_k(S_T) \) such that

\[ \Delta \vec{p}_0 = 0, \]

\[ \vec{p}_0|_{S} = \vec{p}_0, \]

where \( \vec{p}_0 \in W^{3-1/r, 1/2-1/(2r)}_k(S_T) \) is an extension of \( p_0 \) and \( \vec{u}_0(x, 0) = u_0(x) \) and \( \|\vec{u}_0\|_{W^{2,1}_r(\Omega_T)} \leq c\|u_0\|_{W^{3-1/r}_r(\Omega_T)} \), \( \vec{p}_0(x, 0) = p_0(x) \) and \( \|\vec{p}_0\|_{W^{1-1/r, 1/2-1/(2r)}_k(S_T)} \leq c\|p_0\|_{W^{3-1/r}_r(S_T)} \), \( \|\vec{p}_0\|_{W^{r/2, 0}(\Omega_T)} \leq c\|p_0\|_{W^{3-1/r}_r(S_T)} \), \( \|\vec{p}_0\|_{W^{2-2/r}_r(\Omega_T)} + c\|H\|_{W^{1-1/r, 1/2-1/(2r)}_k(S_T)} \)

If we assume that the solution of (1.1) has the form

\[ u = u' + \vec{u}_0, \quad p = p' + \vec{p}_0, \]

then problem (1.1) reduces to

\[ u' - \nu \Delta u' + \nabla p' = F', \]

\[ \text{div} u = G', \]

\[ \vec{n} \cdot \mathbf{T}(u', p')|_{S_T} = H', \]

\[ u'|_{t=0} = 0, \]

where (using (5.1))

\[ F' = F - (\vec{u}_0 - \nu \Delta \vec{u}_0) - \nabla \vec{p}_0, \]

\[ G' = G - \text{div} \vec{u}_0, \]

\[ H' = H - \vec{n} \cdot \mathbf{T}(\vec{u}_0, \vec{p}_0), \]

\[ \|F'\|_{L_r} \leq \|F\|_{L_r} + c\|u_0\|_{W^{2-1/r}_r}, \]

\[ \|G'\|_{W^{r/2, 0}_k} \leq c\|G\|_{W^{r/2, 0}_k} + c\|\vec{u}_0\|_{W^{3-1/r}_r} \]

\[ \|H'\|_{W^{1-1/r, 1/2-1/(2r)}_k} \leq c\|H\|_{W^{3-1/r, 1/2-1/(2r)}_k} + c\|u_0\|_{W^{3-1/r}_r} \]

where \( G' = \text{div} B' + A \) and \( B' = B - \vec{u}_0 \). By (1.2) we see that \( G'|_{t=0} = 0, H'|_{t=0} = 0 \) and \( p'|_{t=0} = 0 \). Taking \( w = \nabla \Phi \), where

\[ \Delta \Phi = G', \]

\[ \Phi|_{S_T} = 0, \]

we have \( w = \nabla \Phi \in W^{2,0}_r(\Omega_T) \), but from \( \Delta \Phi = \text{div}(B + f) + A \), as in the proof of Lemma 3.4 (see (3.53)–(3.61), we get

\[ \|u_t\|_{L_r} \leq c(\|f\|_{L_r} + \|B\|_{L_r} + \lambda\|A\|_{L_r}). \]

So \( w \in W^{2,1}_r \) and

\[ \|w\|_{W^{2,1}_r(\Omega_T)} \leq c(\|g\|_{W^{2,0}(\Omega_T)} + \|f\|_{L_r} + \|B\|_{L_r} + \lambda\|A\|_{L_r}) \]

and from \( G'|_{t=0} = 0 \) we have \( w|_{t=0} = 0 \). Thus we can take

\[ u' = u'' + w, \]

where \( u'' \in W^{3-1/r, 1/2-1/(2r)}_k(S_T) \) is an extension of \( u_0 \) and \( \vec{u}_0(x, 0) = u_0(x) \) and \( \|\vec{u}_0\|_{W^{2,1}_r(\Omega_T)} \leq c\|u_0\|_{W^{3-1/r}_r(\Omega_T)} \), \( \vec{p}_0(x, 0) = p_0(x) \) and \( \|\vec{p}_0\|_{W^{1-1/r, 1/2-1/(2r)}_k(S_T)} \leq c\|p_0\|_{W^{3-1/r}_r(S_T)} \), \( \|\vec{p}_0\|_{W^{r/2, 0}(\Omega_T)} \leq c\|p_0\|_{W^{3-1/r}_r(S_T)} \), \( \|\vec{p}_0\|_{W^{2-2/r}_r(\Omega_T)} + c\|H\|_{W^{1-1/r, 1/2-1/(2r)}_k(S_T)} \)
and reduce problem (5.3) to

\begin{align}
\frac{\partial u''}{\partial t} - \nu \Delta u'' + \nabla p' &= F''', \\
\nabla \cdot u'' &= 0,
\end{align}

(5.7)

\[ u'|_{t=0} = 0, \]

where, by (5.5),

\[
F'' = F' - (w_0 - \nu \Delta u), \quad \|F''\|_{L^2} \leq \|F'\|_{L^2} + \|\nu\|_{W^{2,1}},
\]

\[
H'' = H' - \nabla \cdot \nabla (w, 0), \quad \|H''\|_{W^{2,1}} \leq \|H'\|_{W^{2,1}} + \|\nu\|_{W^{2,1}}.
\]

To reduce (5.7) to (4.1) we have to solve the following problem:

\begin{align}
\Delta p'' &= \text{div } F''', \\
p''|_{S_t} &= H'',
\end{align}

(5.9)

which gives

\[
\|\nabla p''\|_{L^2} \leq c(\|F''\|_{L^2} + \|H''\|_{W^{2,1}}).
\]

(5.10)

Putting \( p' = p'' + p''' \) from (5.7) we get

\[
\frac{\partial u''}{\partial t} - \nu \Delta u'' + \nabla p''' = F''',
\]

\[
\nabla \cdot u'' = 0,
\]

\[
\nabla \cdot \nabla (u'', p''') = 0,
\]

(5.12)

\[ u'|_{t=0} = 0, \]

where \( F''' = F'' - \nabla p'' \) and \( \text{div } F''' = 0 \).

By Lemma 4.2, (5.1), (5.5), (5.9) and (5.10), we obtain a solution of system (5.12) which in view of (5.2), (5.6) and (5.11) gives a solution of system (1.1) given by

\[
u = u_0 + u'' + w \quad \text{and} \quad p = p'' + p''' + p_0,
\]

(5.13)

which satisfies (1.2) for \( T \leq T_0 \). Now it is enough to continue the solution to the intervals \([T_0, 2T_0], [2T_0, 3T_0], \ldots\). This proves (1.3) and concludes our considerations.

References


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