Stochastic convolution in separable Banach spaces
and the stochastic linear Cauchy problem

by

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Abstract. Let $H$ be a separable real Hilbert space and let $E$ be a separable real Banach space. We develop a general theory of stochastic convolution of $L(H,E)$-valued functions with respect to a cylindrical Wiener process $\{W_t^H\}_{t \in [0,T]}$ with Cameron–Martin space $H$. This theory is applied to obtain necessary and sufficient conditions for the existence of a weak solution of the stochastic abstract Cauchy problem

\[
\begin{align*}
    dX_t &= AX_t \, dt + BdW_t^H \
    X_0 &= 0 \quad \text{almost surely,}
\end{align*}
\]

where $A$ is the generator of a $C_0$-semigroup $\{S(t)\}_{t \geq 0}$ of bounded linear operators on $E$ and $B \in L(H,E)$ is a bounded linear operator. We further show that whenever a weak solution exists, it is unique, and given by a stochastic convolution

\[
    X_t = \int_0^t S(t-s)B \, dW_s^H.
\]

0. Introduction. Let $H$ be a separable real Hilbert space and let $E$ be a separable real Banach space. In this paper we set up a theory of stochastic convolution for $L(H,E)$-valued functions which enables us to study existence and uniqueness of solutions to the stochastic abstract Cauchy problem

\[
\begin{align*}
    dX_t &= AX_t \, dt + BdW_t^H \
    X_0 &= 0 \quad \text{almost surely.}
\end{align*}
\]

Here $A$ is the generator of a $C_0$-semigroup $\{S(t)\}_{t \geq 0}$ of bounded linear operators on $E$, $B \in L(H,E)$ is a bounded linear operator, and $\{W_t^H\}_{t \in [0,T]}$ is a cylindrical Wiener process with Cameron–Martin space $H$.

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If $E$ is a separable Hilbert space, it is well known that a weak solution of (ACP) exists if and only if the positive self-adjoint operator $Q_T \in \mathcal{L}(E^*, E)$ defined by

$$Q_T x^* = \int_0^T S(t)BB^*S^*(t)x^* \, dt \quad (x^* \in E^*)$$

is of trace class (we do not identify $E$ and its dual $E^*$ here). In this case the weak solution is unique, and given by the Itô type convolution integral

$$X_t = \int_0^t S(t-s)B \, dW^H_s \quad (t \in [0, T]).$$

A detailed account of the theory of the problem (ACP) in Hilbert spaces $E$ is presented in the recent book by Da Prato and Zabczyk [DZ].

Due to the lack of a satisfactory theory of stochastic integration in Banach spaces, it seems impossible to give a straightforward extension of this theory to the case where $E$ is a Banach space. For this, one needs additional assumptions on $E$, such as 2-uniform smoothness (equivalently, martingale type 2). This approach is worked out in [Nh], [Br1], [Br3] and the references therein.

From these works it is well known that the solution of (ACP), if it exists, is an $E$-valued Ornstein–Uhlenbeck process associated with $S$ and $B$, i.e. a centred Gaussian $E$-valued process $\{X_t\}_{t \in [0, T]}$ with covariance given by

$$\mathbb{E}(\langle X_t, x^* \rangle \langle X_s, y^* \rangle) = \int_0^t [BB^*S^*(t-u)x^*, B^*S^*(s-u)y^*]_H \, du.$$  

In certain special situations, vector-valued Ornstein–Uhlenbeck processes have been studied by various methods and various authors; we mention Antoniadis and Carmona [AC], Millet and Smańdziński [MS] and Röckle [Rö]. However, the problem of giving necessary and sufficient conditions in terms of $S$ and $B$ for the existence of such a process in the general case has not been addressed yet.

In this paper we show that it is possible to set up a theory of stochastic convolution in arbitrary separable real Banach spaces $E$. Let us briefly outline its main features. Suppose $H$ is a separable real Hilbert space and $\Phi : (0, T) \to \mathcal{L}(H, E)$ is an operator-valued function satisfying

$$\int_0^T \|\Phi^*(t)x^*\|_H^2 \, dt < \infty, \quad \forall x^* \in E^*.$$  

We show that the formula

$$\langle Q_T x^*, y^* \rangle = \int_0^T [\Phi^*(t)x^*, \Phi^*(t)y^*]_H \, dt \quad (x^*, y^* \in E^*)$$

defines a positive symmetric operator $Q_T \in \mathcal{L}(E^*, E)$. Knowing this, we can consider the reproducing kernel Hilbert space (RKHS) $H_T$ associated with $Q_T$; this is a Hilbert subspace of $E$. Denoting the inclusion operator $H_T \hookrightarrow E$ by $i_T$, we have $Q_T = i_T^* \circ i_T$. We prove the following result (Theorem 2.6 and Proposition 2.8):

**Theorem 0.1.** The following assertions are equivalent:

(i) There exists an $E$-valued centred Gaussian process $\{\xi_t\}_{t \in [0, T]}$ with covariance given by

$$E(\xi_t, x^*) = \int_0^t [\Phi^*(t-u)x^*, \Phi^*(t-u)y^*]_H \, du.$$  

(ii) The inclusion $i_T : H_T \hookrightarrow E$ is $\gamma$-radonifying.

An $E$-valued centred Gaussian process with covariance given by (0.1) will be called an Ornstein–Uhlenbeck process associated with $\Phi$. Note that the second condition is equivalent to $Q_T$ being the covariance operator of a centred Gaussian Borel measure on $E$.

Our second main result (Theorem 3.3) shows that it is possible to obtain Ornstein–Uhlenbeck processes by convolution with a cylindrical Wiener process $\{W^H_t\}_{t \in [0, T]}$.

**Theorem 0.2.** Let $\{W^H_t\}_{t \in [0, T]}$ be a cylindrical Wiener process with Cameron–Martin space $H$. If the inclusion $i_T : H_T \hookrightarrow E$ is $\gamma$-radonifying, then there exists a predictable $E$-valued Ornstein–Uhlenbeck process $\{X_t\}_{t \in [0, T]}$ which satisfies

$$\langle X_t, x^* \rangle = \int_0^t [\Phi(t-s)W^H_s, x^*] \quad a.s. \quad (t \in [0, T], \ x^* \in E^*).$$

Up to a modification, this process is unique.

The weak stochastic convolution on the right hand side is defined in an obvious way (cf. Section 3). This justifies the notation

$$X_t = \int_0^t \Phi(t-s)W^H_s.$$  

If $A$ is the generator of a $C_0$-semigroup $\{S(t)\}_{t \geq 0}$ on $E$ and $B$ is a bounded linear operator from $H$ into $E$, we can apply these results to the operator-valued function $\Phi(t) = S(t) \circ B \in \mathcal{L}(H, E)$. This enables us to derive necessary and sufficient conditions for the existence of weak solutions for the problem (ACP) and study some of their properties. The results can be summarized as follows.

**Theorem 0.3.** The following assertions are equivalent:
(i) The problem \((ACP)\) has a weak solution \(\{X_t\}_{t \in [0,T]}\) on \([0,T].\)

(ii) The inclusion \(i_T : H_T \hookrightarrow E\) is \(\gamma\)-radonifying.

In this situation, the solution is unique, and given by the stochastic convolution

\[
X_t = \int_0^t S(t-s)B \, dW^H_s \quad (t \in [0,T]).
\]

The process \(\{X_t\}_{t \in [0,T]}\) has a version with almost surely square integrable trajectories. If the semigroup generated by \(A\) is analytic, then \(\{X_t\}_{t \in [0,T]}\) has a version with continuous trajectories.

Recalling that a positive symmetric operator on a Hilbert space \(E\) is of trace class if and only if it is the covariance of a centered Gaussian measure on \(E\), we see that our results extend the known existence and uniqueness results for Hilbert spaces mentioned above.

In the final section we apply our theory to the following stochastic heat equation driven by a homogenous space-time Wiener process:

\[
\frac{\partial X}{\partial t}(t,x) = \Delta X(t,x) + \frac{\partial w}{\partial t}(t,x) \quad (t \in [0,T]),
\]

\[
X(0,x) = 0, \quad X(t,0) = X(t,1) = 0.
\]

Some of the questions that led to our research were motivated by the theory of Feynman path integrals and their close relationship to the theory of integrals with respect to Ornstein–Uhlenbeck measures”, i.e. Gaussian measures on spaces of vector-valued functions arising as image measures corresponding to the Cameron–Martin spaces of vector-valued Gaussian processes. It is known that certain equivalent norms on the Cameron–Martin space lead to equivalent image measures (cf. [ABB]). In [BN] we apply the results obtained in the present paper to study equivalence of this type of Gaussian measures in the abstract framework considered here.

1. Preliminaries. In this section we briefly recall some well known facts concerning (cylindrical) Gaussian measures. For more details we refer to [VTC], [Schw1], [Schw2], [Kuo].

Let \(E\) be a real locally convex topological vector space, with topological dual \(E'\). A subset \(C\) of \(E\) is said to be a cylindrical set if it is of the form \(C = \{x \in E : (x,x_1'), \ldots, (x,x_n') \in B\}\) for some \(n \geq 1\), \(x_1', \ldots, x_n' \in E'\), and a Borel set \(B \subset \mathbb{R}^n\). The set of all cylindrical subsets of \(E\) is an algebra of sets and is denoted by \(\mathcal{C}(E)\). A centered cylindrical Gaussian measure on \(E\) is a finitely additive set function \(\mu\) on \(\mathcal{C}(E)\) whose images under the maps \(x \mapsto (x,x_1'), \ldots, (x,x_n')\) are \(\sigma\)-additive Gaussian measures on \(\mathbb{R}^n\), or equi-

alent, whose images under the maps \(x \mapsto (x,x')\) are \(\sigma\)-additive Gaussian measures on \(\mathbb{R}\).

If \(F\) is another locally convex space, and if \(T : E \to F\) is a continuous linear transformation, then the image \(T(\mu) := \mu T^{-1}\) of a centered cylindrical Gaussian measure on \(E\) is a centered cylindrical Gaussian measure on \(F\).

Let \(H\) be a real Hilbert space. By \(\gamma_H\) we denote the standard centered cylindrical Gaussian measure on \(H\), i.e. the centered cylindrical Gaussian measure on \(H\) whose image under every map \(g \mapsto (g,h_1)_H, \ldots, (g,h_n)_H\), with \(\{h_1, \ldots, h_n\}\) orthonormal in \(H\), is the standard Gaussian measure on \(\mathbb{R}^n\).

A continuous linear operator \(Q \in L(E', E)\) is called positive if \(\langle Qx', y' \rangle \geq 0\) for all \((x', y') \in E'\), and symmetric if \(\langle Qx', y' \rangle = \langle Qx, y' \rangle\) for all \((x, y') \in E'\). With every positive symmetric operator \(Q \in L(E', E)\) one can associate a real Hilbert space \(H_Q\) in the following way. On the range of \(Q\) one has a well defined inner product \([\cdot, \cdot]_H\) given by

\[
[Qx', Qy] := \langle Qx', y' \rangle \quad (x', y' \in E').
\]

Denote by \(H_Q\) the Hilbert space completion of range \(Q\) with respect to this inner product; this Hilbert space is called the reproducing kernel Hilbert space (RKHS) associated with \(Q\). If \(E\) is quasi-complete, then the inclusion mapping from range \(Q\) into \(E\) has a continuous extension to an injective linear map \(i : H_Q \to E\). In this way, the pair \((i, H_Q)\) becomes a Hilbert subspace of \(E\). Moreover, upon identifying \(H_Q\) with its dual in the natural way, we then have the operator identity \(Q = i \circ i^*\). In Section 2 these results will be applied to the (quasi-complete) product space \(E = E^{[0,T]}\), with \(E\) a separable real Banach space.

Conversely, if \((i, H)\) is a real Hilbert subspace of \(E\) (i.e. \(i\) is a continuous injective linear map from some real Hilbert space \(H\) into \(E\)), then \(Q := i \circ i^* \in L(E', E)\) is positive and symmetric, and its RKHS equals \(H\).

The relationship between centered cylindrical Gaussian measures and positive symmetric operators in described in the following well known result [VTC, Chapter III].

**Proposition 1.1.** Let \(E\) be a real locally convex topological vector space.

(i) Let \(H\) be a real Hilbert space and let \(T \in L(H, E)\). The image cylindrical measure \(\mu := T(\gamma_H)\) is a centered cylindrical Gaussian measure on \(E\) whose Fourier transform is given by

\[
\mu(x) = \int_{E'} \exp\left(-\frac{1}{2} \langle (T \circ T')x', x' \rangle\right) (x' \in E').
\]

The RKHS \(H_Q\) associated with the positive symmetric operator \(Q = T \circ T' \in L(E', E)\) equals the range of \(T\), which is a Hilbert space under the inner
product

\[ [Tg, Th]_{H_0} = [Pg, Ph]_H, \]

with \( P \) the orthogonal projection in \( H \) onto \((\ker T)^{\perp}\), the orthogonal complement in \( H \) of the kernel of \( T \). Moreover, as a map from \((\ker T)^{\perp}\) onto \( H_0 \), the operator \( T \) is an isometry.

(ii) If \( E \) is quasi-complete and \( Q \in \mathcal{L}(E',E) \) is positive and symmetric, and if \( \mu \) is a centred cylindrical Gaussian measure on \( E \) with Fourier transform

\[ \int \exp(i\langle x, x' \rangle) \, d\mu(x) = \exp \left( -\frac{1}{2} \langle Qx', x' \rangle \right), \quad (x' \in E'), \]

then \( \mu = i(\gamma_H) \), where \( H \) is the RKHS of \( Q \) and \( \gamma : H \to E \) is the natural embedding.

Let \( E \) be a real locally convex topological vector space. A measure \( \mu \) on the \( \sigma \)-algebra \( \sigma(C(E)) \) generated by the algebra \( C(E) \) is called a (centred) Gaussian measure on \( E \) if for all \( x' \in E' \) the image measure \( \langle \mu, x' \rangle := \mu \circ (x')^{-1} \) is a (centred) Gaussian Borel measure on \( \mathbb{R} \). If \( H \) is a real Hilbert space, a continuous linear operator \( T \in \mathcal{L}(H,E) \) is said to be \( \gamma \)-radonifying if the image cylindrical measure \( T(\gamma_H) \) has a (necessarily unique) countably additive extension to a Gaussian measure on \( E \). Note that in general the \( \sigma \)-algebra \( \sigma(C(E)) \) is much smaller than the Borel \( \sigma \)-algebra of \( E \).

The following three examples of \( \gamma \)-radonifying operators will be of importance:

- If \( \mu \) is a centred Gaussian measure on \( E \) with RKHS \( H \), then the inclusion map \( i : H \to E \) is \( \gamma \)-radonifying, and we have \( i(\gamma_H) = \mu \).
- If \( H \) and \( E \) are Hilbert spaces, then \( T \in \mathcal{L}(H,E) \) is \( \gamma \)-radonifying if and only if \( T \) is a Hilbert–Schmidt operator.
- If \( G \) and \( H \) are Hilbert spaces and \( S \in \mathcal{L}(G,H) \) and \( T \in \mathcal{L}(H,E) \) are continuous linear operators, then \( T \circ S \) is \( \gamma \)-radonifying whenever \( T \) is \( \gamma \)-radonifying [Bax], [Ram].

As is common, the dual of a Banach space \( E \) will be denoted by \( E^* \) rather than \( E' \). We will frequently use sequential weak*-approximation arguments in dual Banach spaces. One has to be careful with this, because a weak*-dense linear subspace in the dual \( E^* \) of a Banach space \( E \) need not be weak*-sequentially dense, even if \( E \) is separable. A counterexample is given in [Di]. We get around this in the following way.

**Proposition 1.2.** Let \( E \) be a separable real Banach space and let \( Y \) be a linear subspace of \( E^* \) which is both weak*-dense and weak*-sequentially closed. Then \( Y = E^* \).

**Proof.** The closed unit ball \( B_{E^*} \) is weak*-compact, hence certainly weak*-sequentially closed. It follows that \( B_{E^*} \cap Y \) is weak*-sequentially closed. Because the weak*-topology of \( B_{E^*} \) is metrizable, \( B_{E^*} \cap Y \) is actually weak*-closed. Hence by the Krein–Smulian theorem [DS, Theorem V.5.7], \( Y \) is weak*-closed. Since by assumption \( Y \) is also weak*-dense, we infer that \( Y = E^* \). ■

As a corollary we record:

**Corollary 1.3.** Let \( \mu \) be Borel probability measure on a separable real Banach space \( E \), and suppose there is a weak*-dense linear subspace \( Y \) of \( E^* \) such that the image measures \( \langle \mu, x^* \rangle \) are Gaussian for all \( x^* \in Y \). Then \( \mu \) is a Gaussian measure.

**Proof.** By Zorn’s Lemma there exists a maximal linear subspace \( Y' \) of \( E^* \) with the property that \( \langle \mu, x^* \rangle \) is Gaussian for all \( x^* \in Y' \). Since obviously \( Y' \subset Y \) we see that \( Y' \) is weak*-dense.

Let \( Y'' \) denote the weak*-sequential closure of \( Y' \). Let \( x^* \in Y'' \) be arbitrary and suppose that \( \lim_{n \to \infty} x_n^* = x^* \) in \( E^* \) for some sequence \( (x_n^*) \) in \( Y' \). By the dominated convergence theorem, for the Fourier transforms we have

\[
\lim_{n \to \infty} \langle \mu, x_n^* \rangle(\xi) = \lim_{n \to \infty} \int \exp(i\xi(y, x_n^*)) \, d\mu(y) = \int \exp(i\xi(y, x^*)) \, d\mu(y) = \langle \mu, x^* \rangle(\xi), \quad \forall \xi \in \mathbb{R}.
\]

As is well known [Tv, Lemme 1.5], this implies that \( \langle \mu, x^* \rangle \) is Gaussian.

We have shown that \( \langle \mu, x^* \rangle \) is Gaussian for all \( x^* \in Y'' \). By the maximality of \( Y \) we must have \( Y'' = Y' \), and therefore \( Y \) is weak*-sequentially closed. Proposition 1.2 now finishes the proof. ■

2. The canonical Ornstein–Uhlenbeck process. Throughout the rest of this paper, \( H \) is a separable real Hilbert space and \( E \) is a separable real Banach space. Suppose \( \Phi : (0, T] \to \mathcal{L}(H,E) \) is an operator-valued function on \((0, T]\) with the property that for all \( x^* \in E^* \), \( t \mapsto \Phi^*(t)x^* \) is a strongly measurable \( H \)-valued function satisfying

\[
\int_0^T \|\Phi^*(t)x^*\|_H^2 \, dt < \infty.
\]

By a standard argument, the mapping \( E^* \to L^2((0, T]; H) \) given by \( x^* \mapsto \Phi^*(\cdot)x^* \) is closed, hence bounded by the closed graph theorem. The space of all such \( \Phi \) can be made into a normed linear space, which we denote by \( L^2((0, T]; H, E) \), by defining the norm of \( \Phi \) to be the operator norm of \( \Phi^* \).
regarded as an element of \( L(E^*, L^2((0, T); H)) \):
\[
\|\Phi\|_{L^2((0, T); H, E)} := \sup_{\|x^*\| \leq 1} \left( \int_0^T \|\Phi^*(t)x^*\|^2_H \, dt \right).
\]
For the rest of this section we fix \( \Phi \in L^2((0, T); H, E) \).

**Lemma 2.1.** For all \( x^* \in E^* \) the function \( \Phi(\cdot)x^* \) is a strongly measurable \( E \)-valued function on \((0, T] \).

**Proof.** Fix \( x^* \in E^* \). Choose an orthonormal basis \((h_n)\) in \( H \). Then, for all \( t \in (0, T] \) and \( y^* \in E^* \),
\[
\langle \Phi(t)\Phi^*(t)x^*, y^* \rangle = \langle \Phi^*(t)x^*, \Phi^*(t)y^* \rangle_H = \sum_n [\Phi^*(t)x^*, h_n]_H [\Phi^*(t)y^*, h_n]_H,
\]
which is measurable as a function of \( t \). This shows that \( \Phi(\cdot)x^* \) is weakly measurable. Since by assumption \( E \) is separable, Pettis’s measurability theorem [DU, Chapter 2] implies that this function is actually strongly measurable. \( \square \)

**Proposition 2.2.** For all \( x^* \in E^* \) and \( t \in (0, T] \) there exists a unique element \( Q_t x^* \in E \) satisfying
\[
\langle Q_t x^*, y^* \rangle = \int_0^t \langle \Phi(s)x^*, y^* \rangle \, ds, \quad \forall y^* \in E^*.
\]
The linear operators \( Q_t \) from \( E^* \) to \( E \) obtained in this way are bounded, positive and symmetric.

**Proof.** Fix \( t \in (0, T] \). Define \( Q_t x^* \in E^* \) by:
\[
\langle y^*, Q_t x^* \rangle := \int_0^t \langle \Phi(s)x^*, y^* \rangle \, ds = \int_0^t [\Phi^*(s)x^*, \Phi^*(s)y^*]_H \, ds \quad (y^* \in E^*).
\]
Note that this integral is finite by Hölder’s inequality and the integrability assumption on \( \Phi \). By the boundedness of the map \( x^* \mapsto \Phi^*(\cdot)x^* \) from \( E^* \) into \( L^2((0, T]; H) \), the resulting linear operator \( Q_t : E^* \to E^* \) is bounded. We must prove that \( Q_t \) is actually \( E \)-valued.

Fix \( x^* \in E^* \) arbitrary. We claim that \( Q_t x^* \) acts \( \mu \)-continuously on the closed unit ball \( B_E^* \) of \( E^* \). By the Krein–Shmulian theorem, this implies that \( Q_t x^* \) belongs to \( E \), and the proof will be complete.

Assume, for a contradiction, that the claim is not true. Since \( E \) is separable, the closed unit ball of \( E^* \) is \( \mu \)-sequentially compact, and we can find an \( \varepsilon > 0 \) and a sequence \( (y^*_n) \) in \( B_{E^*} \) that \( \mu \)-converges to some \( y^* \in B_{E^*} \) such that
\[
|\langle y^*_n, Q_t x^* \rangle - \langle y^*, Q_t x^* \rangle| \geq \varepsilon \quad (n \geq 0).
\]
For each \( s \), the adjoint operator \( \Phi^*(s) \) is \( \mu \)-continuous from \( E^* \) into \( H \), and hence \( \mu \)-to-weakly continuous. Therefore,
\[
\lim_{n \to \infty} \langle \Phi(s)y^*_n, x^* \rangle = \langle \Phi(s)y^*, x^* \rangle_H
\]
weakly in \( H \) for all \( s \in (0, T] \), and
\[
\lim_{n \to \infty} \langle \Phi(s)\Phi^*(s)x^*, y^*_n \rangle = \lim_{n \to \infty} \langle \Phi(s)x^*, \Phi^*(s)y^*_n \rangle_H = \langle \Phi(s)x^*, \Phi^*(s)y^* \rangle_H = \langle \Phi(s)\Phi^*(s)x^*, y^* \rangle.
\]
The boundedness of \( (y^*_n) \) in \( E^* \) implies that the function sequence \( \Phi^*(\cdot)y^*_n \) is bounded in \( L^2((0, T]; H) \). Since \( L^2((0, T]; H) \) is reflexive, upon passing to a subsequence we may assume that \( \Phi^*(\cdot)y^*_n \) is weakly convergent in \( L^2((0, T]; H) \) to some limit function \( f \). As \( \Phi^*(\cdot)x^* \in L^2((0, T]; H) \), we then have
\[
\lim_{n \to \infty} \int_0^t \langle \Phi(s)\Phi^*(s)x^*, y^*_n \rangle \, ds = \int_0^t [\Phi^*(s)x^*, f(s)]_H \, ds.
\]
The weak convergence \( \Phi^*(\cdot)y^*_n \to f \) implies further that there exist convex combinations
\[
z^*_n = \sum_{k=n}^{K_n} \lambda_{k,n} y^*_k
\]
such that \( \Phi^*(\cdot)z^*_n \to f \) strongly in \( L^2((0, T]; H) \). Passing, if necessary, to a further subsequence of \( (z^*_n) \), we even have \( \Phi^*(\sigma)z^*_n \to f(\sigma) \) strongly in \( H \) for almost all \( \sigma \in (0, t] \). For any \( \sigma \) with this property,
\[
\lim_{n \to \infty} \langle \Phi(\sigma)\Phi^*(\sigma)x^*, z^*_n \rangle = \langle \Phi(\sigma)x^*, f(\sigma) \rangle_H.
\]
On the other hand, because we take \( z^*_n \) in the convex hull of \( \{y^*_k : k \geq n\} \), by (2.2) we have
\[
\lim_{n \to \infty} \langle \Phi(\sigma)\Phi^*(\sigma)x^*, z^*_n \rangle = \langle \Phi(\sigma)x^*, y^* \rangle
\]
for all \( \sigma \in (0, t] \). From this and (2.4) it follows that
\[
[\Phi^*(\sigma)x^*, f(\sigma)]_H = [\Phi(\sigma)\Phi^*(\sigma)x^*, y^*]
\]
for almost all \( \sigma \in (0, t] \). Combining the above with (2.3) we obtain
\[
\lim_{n \to \infty} \langle y^*_n, Q_t x^* \rangle = \lim_{n \to \infty} \int_0^t \langle \Phi(\sigma)\Phi^*(\sigma)x^*, y^*_n \rangle \, d\sigma
\]
\[
= \int_0^t [\Phi^*(\sigma)x^*, f(\sigma)]_H \, d\sigma
\]
\[
= \int_0^t \langle \Phi(\sigma)\Phi^*(\sigma)x^*, y^* \rangle \, d\sigma = \langle y^*, Q_t x^* \rangle.
\]
But this contradicts (2.1). \( \square \)
Proposition 2.2 shows that for all \( t \in (0,T) \) we have a well defined bounded linear operator \( Q_t \in \mathcal{L}(E^*, E) \), which can be represented as a Pettis integral by
\[
Q_t x^* = \int_0^t \Phi(s) \Phi^*(s) x^* \, ds \quad (x^* \in E^*).
\]
Clearly, \( Q_t \) is positive and symmetric; we denote by \( (i_t, H_t) \) its RKHS (cf. Section 1 for the definition). If \( 0 < s \leq t \leq T \), then for all \( x^* \in E^* \) we have \( \|Q_s x^*\|_{H_s} \leq \|Q_t x^*\|_{H_t} \), which implies that there is a natural inclusion \( H_s \hookrightarrow H_t \) (cf. [Ne1], where it is shown that this inclusion is in fact a contraction).

Just as in Lemma 2.1 one proves:

**Lemma 2.3.** For all \( t \in (0, T) \) and \( x^* \in E^* \) the function \( s \mapsto \Phi(t-s) \Phi^*(s) x^* \) is a strongly measurable \( E \)-valued function on \( [0, t] \).

For each \( t \in (0, T) \) we let \( \mathcal{H}_t = \mathcal{H}_t^\Phi \) denote the closure in \( L^2((0, T); H) \) of its linear subspace \( \left\{ \chi_{(0,t]} \Phi^*(\cdot) x^* : x^* \in E^* \right\} \).

**Lemma 2.4.** For each \( t \in (0, T) \) there exists a unique bounded linear operator \( I_{x,t} : \mathcal{H}_t \to H_t \), which satisfies
\[
[I_{x,t} \chi_{(0,t]} \Phi^*(\cdot) x^*, y^*]_{H_t} = \int_0^t \langle \Phi(t-s) \Phi^*(s) x^*, y^* \rangle \, ds, \quad \forall x^*, y^* \in E^*.
\]

**Proof.** By the Cauchy–Schwarz inequality and the identity
\[
\|i_t^* y^*\|_{H_t} = \|\chi_{(0,t]} \Phi^*(\cdot) y^*\|_{L^2((0,t]; H)}
\]
we have
\[
\left| \int_0^t \langle \Phi(t-s) \Phi^*(s) x^*, y^* \rangle \, ds \right|
\]
\[
= \int_0^t \left| \langle \Phi^*(s) x^*, \Phi(t-s) y^* \rangle \right|_{H_t} \, ds
\]
\[
\leq \|\chi_{(0,t]} \Phi^*(\cdot) x^*\|_{L^2((0,t]; H)} \|\chi_{(0,t]} \Phi^*(\cdot) y^*\|_{L^2((0,t]; H)}
\]
\[
= \|\chi_{(0,t]} \Phi^*(\cdot) x^*\|_{\mathcal{H}_t} \|i_t^* y^*\|_{H_t}.
\]
It follows that the map
\[
i_t^* y^* \mapsto \int_0^t \langle \Phi(t-s) \Phi^*(s) x^*, y^* \rangle \, ds
\]
defines a bounded linear functional on \( H_t \) of norm \( \leq \|\chi_{(0,t]} \Phi^*(\cdot) x^*\|_{\mathcal{H}_t} \).

By the Riesz representation theorem, this functional can be identified with an element of \( H_t \); we will denote it by \( i_{x,t} \chi_{(0,t]} \Phi^*(\cdot) x^* \). In this way we obtain a bounded linear operator \( I_{x,t} \) of norm \( \leq 1 \) from the linear span of \( \left\{ \chi_{(0,t]} \Phi^*(\cdot) x^* : x^* \in E^* \right\} \) into \( H_t \). Since this span is dense in \( \mathcal{H}_t \), the result is proved.

From the identity
\[
\langle (i_t \circ I_{x,t}) \chi_{(0,t]} \Phi^*(\cdot) x^*, y^* \rangle = [I_{x,t} \chi_{(0,t]} \Phi^*(\cdot) x^*, i_t^* y^*]_{H_t}
\]
\[
= \int_0^t \langle \Phi(t-s) \chi_{(0,t]} \Phi^*(\cdot) x^*, y^* \rangle \, ds
\]
and a continuity argument we see that \( i_t \circ I_{x,t} \) can be represented as a Pettis integral by
\[
(i_t \circ I_{x,t}) g = \int_0^t \Phi(t-s) \Phi^*(s) g(s) \, ds \quad (g \in \mathcal{H}_t).
\]
Noting that \( f \mapsto \chi_{(0,t]} f \) defines a contraction from \( \mathcal{H}_T \) onto \( \mathcal{H}_t \), we can define a continuous linear operator \( I_{x,T} : \mathcal{H}_T \to E^{[0,T]} \) by
\[
(I_{x,T} f)(t) := \begin{cases} 0, & t = 0, \\ (i_t \circ I_{x,t}) \chi_{(0,t]} f, & t \in (0, T) \end{cases} \quad (f \in \mathcal{H}_T).
\]

**Theorem 2.5.** If the embedding \( i_T : H_T \hookrightarrow E \) is \( \gamma \)-radonifying, then so is the operator \( I_{x,T} : \mathcal{H}_T \to E^{[0,T]} \).

**Proof.** We noted earlier that for each \( 0 < t \leq T \) there is a natural inclusion \( i_t : H_t \hookrightarrow H_T \). Composing this with the inclusion \( i_T : H_T \to E \) we obtain a factorization \( i_t = i_T \circ i_t \). Since \( i_T \) is \( \gamma \)-radonifying by assumption, it follows that each of the inclusions \( i_t \) is \( \gamma \)-radonifying.

Let \( \nu := \nu_{x,T} := I_{x,T} \gamma_{\mathcal{H}_T} \) denote the image cylindrical measure on \( E^{[0,T]} \) under \( I_{x,T} \) of the standard cylindrical Gaussian measure \( \gamma_{\mathcal{H}_T} \) of \( \mathcal{H}_T \). Let \( \delta_t : E^{[0,T]} \to E \) denote the point evaluation at \( t \), and let \( \nu_t := \delta_t(\nu) \) be the corresponding image cylindrical measure on \( E \). By Proposition 1.1 the covariance operator \( R_t \in \mathcal{L}(E^*, E) \) of \( \nu_t \) is given by \( R_t = \delta_t \circ I_{x,T} \circ I_{x,T} \circ \delta_t \).

For \( y^* \in E^* \) and \( f = \Phi^*(\cdot) x^* \in \mathcal{H}_T \) we have
\[
[I_{x,T} \circ \delta_t^* y^*, f]_{\mathcal{H}_T} = \int_0^t \langle \Phi(t-s) \Phi^*(s) x^*, y^* \rangle \, ds
\]
\[
= \int_0^t \langle \Phi(t-s) \Phi^*(s) x^*, y^* \rangle \, ds
\]
\[
= \int_0^t \langle \chi_{(0,t]} \Phi^*(\cdot) x^*, y^* \rangle_{\mathcal{H}_T}.
\]
Therefore,
\[
(I_{x,T} \circ \delta_t^* y^*) = \chi_{(0,t]} \Phi^*(\cdot) x^* \]

(2.5)
and for all \( x^*, y^* \in E \) we obtain
\[
\langle R_t x^*, y^* \rangle = \left[ (I_{\sigma_t^*} \circ \delta_t^*) x^*, (I_{\sigma_t^*} \circ \delta_t^*) y^* \right]_{\gamma H_t}
\]
\[
= \left[ \chi_{\{0,\theta\}} \Phi^*(t - \cdot) x^*, \chi_{\{0,\theta\}} \Phi^*(t - \cdot) y^* \right]_{\gamma H_t}
\]
\[
= \int_0^t \langle \Phi(s) \Phi^*(s) x^*, y^* \rangle \, ds = \langle Q_t x^*, y^* \rangle.
\]

But \( Q_t \) is the covariance operator of the Gaussian Borel measure \( \mu_t := \gamma_{t}(\gamma_{H_t}) \), and it thus follows that \( \nu_t = \mu_t \) as cylindrical measures on \( E \). We conclude that \( \nu_t \) extends to a centered Gaussian Borel measure on \( E \).

Now suppose \( 0 \leq t_1 < \ldots < t_n \leq T \) are fixed and consider the canonical projection \( \delta_{t_1, \ldots, t_n} : E^{[0,T]} \to E^n, \, f \mapsto (f(t_1), \ldots, f(t_n)) \). Let \( \nu_{\{t_1, \ldots, t_n\}} := \delta_{t_1, \ldots, t_n}(\nu) \). By a result of Dudley, Feldman and Le Cam [DFL, Lemma 5], the fact that each \( \nu_{t_k} \) extends to a centered Gaussian Borel measure on \( E \) implies that \( \nu_{\{t_1, \ldots, t_n\}} \) extends to a centered Gaussian Borel measure on \( E^n \). By the Kolmogorov consistency theorem the projective limit of these measures exists and defines a probability measure \( \tilde{\nu} \) on the product \( \sigma \)-algebra \( B(E^{[0,T]}) \) of \( E^{[0,T]} \). But since this measure is completely determined by its finite marginals \( \nu_{\{t_1, \ldots, t_n\}} \) it follows that \( \tilde{\nu} = \nu \). This proves that \( \nu \) extends to a Gaussian measure on \( (E^{[0,T]}, B(E^{[0,T]})) \).

Suppose the embedding \( \iota_T : H_T \to E \) is \( \gamma \)-radonifying and let \( \nu_{\Phi} := I_{\Phi}(\gamma_{H_T}) \). By Theorem 2.5, this is a Gaussian measure on \( (E^{[0,T]}, B(E^{[0,T]})) \). On the resulting probability space \( (\Omega, \mathcal{F}, \mathbb{P}) = (E^{[0,T]}, B(E^{[0,T]}), \nu_{\Phi}) \) we consider the canonical process \( \xi_t = \{\xi_t \}_{t \in [0,T]} \) defined by point evaluation:
\[
\xi_t(\omega) := \omega(t) \quad (t \in [0,T]).
\]

**Theorem 2.6.** Suppose the embedding \( \iota_T : H_T \to E \) is \( \gamma \)-radonifying. The canonical process \( \{\xi_t \}_{t \in [0,T]} \) is an \( E \)-valued Gaussian process with covariance
\[
\mathbb{E}(\langle \xi_s, x^* \rangle \langle \xi_t, y^* \rangle) = \int_0^T \langle \Phi^*(t - u) x^*, \Phi^*(s - u) y^* \rangle_H \, du.
\]

**Proof.** We compute, using (2.5) and Proposition 1.1,
\[
\mathbb{E}(\langle \xi_s, x^* \rangle \langle \xi_t, y^* \rangle) = \langle (I_{\sigma_s^*} \circ \delta_t^*) x^*, (I_{\sigma_s^*} \circ \delta_t^*) y^* \rangle_{\gamma H_t}
\]
\[
= \langle (I_{\sigma_t^*} \circ \delta_s^*) x^*, (I_{\sigma_t^*} \circ \delta_s^*) y^* \rangle_{\gamma H_t}
\]
\[
= \int_0^T \langle \Phi^*(t - u) x^*, \Phi^*(s - u) y^* \rangle_H \, du.
\]

**Definition 2.7.** An \( E \)-valued stochastic process \( \{X_t \}_{t \in [0,T]} \) will be called an **Ornstein-Uhlenbeck process associated with the operator-valued function**
\[
\Phi \in L^2([0,T]; H, E) \quad \text{if it is centered Gaussian with covariance given by}
\]
\[
\mathbb{E}(\langle X_t, x^* \rangle \langle X_s, y^* \rangle) = \int_0^T \langle \Phi^*(t - u) x^*, \Phi^*(s - u) y^* \rangle_H \, du.
\]

The canonical process \( \{\xi_t \}_{t \in [0,T]} \) of Theorem 2.6 will be called the **canonical Ornstein-Uhlenbeck process associated with \( \Phi \).**

We close this section with the following converse of Theorem 2.6:

**Proposition 2.8.** Suppose \( \{X_t \}_{t \in [0,T]} \) is an Ornstein-Uhlenbeck process with respect to a function \( \Phi \in L^2([0,T]; H, E) \). Then the inclusion mapping \( \iota_T : H_T \to E \) is \( \gamma \)-radonifying.

**Proof.** Let \( \mu_T \) denote the distribution of the \( E \)-valued random variable \( X_T \). Then \( \mu_T \) is a centered Gaussian Borel measure on \( E \) whose covariance operator \( R_T \in L(E^*, E) \) satisfies
\[
\langle R_T x^*, x^* \rangle = \mathbb{E}(\langle X_T, x^* \rangle^2) = \int_0^T \langle \Phi^*(T - u) x^*, \Phi^*(T - u) x^* \rangle_H \, du = \langle Q_T x^*, x^* \rangle.
\]

This implies that \( Q_T = R_T \), from which we infer that \( Q_T = \mu_T \) is the covariance operator of \( X_T \). On the other hand, \( Q_T = \iota_T \circ \gamma_{H_T} \) is the covariance operator of the image cylindrical measure \( \iota_T(\gamma_{H_T}) \). Since a cylindrical measure is uniquely determined by its covariance operator, it follows that \( \iota_T(\gamma_{H_T}) = \mu_T \) as cylindrical measures. This implies that \( \iota_T(\gamma_{H_T}) \) has a \( \sigma \)-additive extension to a Borel measure on \( E \), and thus \( \iota_T \) is \( \gamma \)-radonifying.

**3. Stochastic convolution.** As before, we let \( E \) be a separable real Banach space and \( H \) a separable real Hilbert space.

In this section we shall investigate under what conditions it is possible to define a stochastic convolution of an operator-valued function \( \Phi : [0,T] \to \mathcal{L}(H, E) \) with respect to a cylindrical Wiener process \( \{W_t^H \}_{t \in [0,T]} \) with Cameron-Martin space \( H \). We start with a definition.

**Definition 3.1.** Let \( \Omega, \mathcal{F}, \{\mathcal{F}_t \}_{t \in [0,T]} \mathbb{P} \) be a filtered probability space. A **cylindrical Wiener process with Cameron-Martin space \( H \)** is a family \( \{W_t^H \}_{t \in [0,T]} \) of bounded linear operators from \( H \) into \( L^2(E) \) with the following properties:

(i) For all \( h \in H \), \( \{W_t^H h \}_{t \in [0,T]} \) is a real-valued Brownian motion adapted to the filtration \( \{\mathcal{F}_t \}_{t \in [0,T]} \).

(ii) For all \( t, s \in [0,T] \) and \( h, g \in H \) we have
\[
\mathbb{E}(W_t^H h \cdot W_s^H g) = (t \wedge s)[h, g]_H.
\]

Instead of \( W_t^H h \) we will usually write \( [W_t^H, h] \).
Consider an operator-valued function $\Phi \in L^2((0, T]; H, E)$ (we recall that this space has been defined at the beginning of Section 2) and let $\{W^H_t\}_{t \in [0, T]}$ be a cylindrical Wiener process with Cameron–Martin space $H$. We briefly outline how to define, for all $x^* \in E^*$, a stochastic Itô type integral
\[
\int_0^T \langle \Phi(s) dW^H_s, x^* \rangle := [W^H_t, U^* x^*] - [W^H_0, U^* x^*].
\]
If $\Phi(s) = \chi_{(t_0, t_1)}(s) U$ for some fixed $U \in L(H, E)$, we put
\[
\int_0^T \langle \Phi(s) dW^H_s, x^* \rangle := [W^H_t, U^* x^*] - [W^H_0, U^* x^*].
\]
Extending this definition by linearity, we obtain a stochastic integral for $L(H, E)$-valued step functions. For such a step function $\Phi$ it is straightforward to verify that
\[
\mathbb{E}\left\{ \left( \int_0^T \langle \Phi(s) dW^H_s, x^* \rangle \right)^2 \right\} = \|\Phi(\cdot) x^*\|^2_{L^2(\Omega, \mathcal{F}, \mathbb{P}, (0, T], H)}.
\]

The construction is completed by the following observation:

**Lemma 3.2.** For each $\Phi \in L^2((0, T]; H, E)$ and $x^* \in E^*$ there exists a sequence of step functions $(\Phi_n)$ in $L^2((0, T]; H, E)$ such that
\[
\lim_{n \to \infty} \|\Phi_n(\cdot) x^* - \Phi(\cdot) x^*\|_{L^2(\Omega, \mathcal{F}, \mathbb{P}, (0, T], H)} = 0.
\]

**Proof.** Let $H'$ be the closed linear subspace in $H$ generated by the set $\{\Phi(t) x^* : t \in [0, T]\}$. Choose a sequence $(\phi_n)$ in $L^2((0, T]; H')$ consisting of step functions of the form
\[
\phi_n(\cdot) = \sum_{k=1}^{N_n} \chi_{(t_{n,k}, t_{n,k+1})}(\cdot) \odot h_{n,k}^t,
\]
such that $\lim_{n \to \infty} \phi_n(\cdot) = \Phi(\cdot) x^*$ almost surely and in $L^2((0, T]; H')$. There is no loss in generality to assume that each $h_{n,k}^t$ is in the linear span of $\{\Phi(t) x^* : t \in [0, T]\}$, say $h_{n,k}^t = \sum_{k=1}^{N_n} \Phi(t_{n,k}) x^*$. Defining $U_{n,k} = \sum_{k=1}^{N_n} \Phi(t_{n,k}) x^*$, and
\[
\Phi_n(\cdot) = \sum_{k=1}^{N_n} \chi_{(t_{n,k}, t_{n,k+1})}(\cdot) \odot U_{n,k},
\]
we have $\Phi_n(\cdot) x^* = \phi_n(\cdot)$ and the lemma follows.

For $t \in (0, T]$ and $\Phi \in L^2((0, T]; H, E)$ we have $\chi_{(0, t]} \Phi \in L^2((0, T]; H, E)$. This allows us to define
\[
\int_0^t \langle \Phi(s) dW^H_s, x^* \rangle := \int_0^t \langle \chi_{(0, t]}(s) \Phi(s) dW^H_s, x^* \rangle.
\]

For the rest of this section we fix $\Phi \in L^2((0, T]; H, E)$ and a cylindrical Wiener process $\{W^H_t\}_{t \in [0, T]}$ with Cameron–Martin space $H$. As before, we let
\[
Q_T x^* = \int_0^T \Phi(s) \Phi^*(s) x^* ds
\]
and denote by $H_T$ the RKHS associated with $Q_T$; for the natural embedding map $i_T : H_T \hookrightarrow E$ we then have $Q_T = i_T \circ i_T^*.$

**Theorem 3.3.** If the inclusion $i_T : H_T \hookrightarrow E$ is $\gamma$-rilonifying, then there exists a predictable $E$-valued process $\{X_t\}_{t \in [0, T]}$, adapted to the filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$, such that for all $x^* \in E^*$ and $t \in [0, T]$ we have
\[
\mathbb{E}\{X_t, x^*\} = \int_0^t \langle \Phi(t-s) dW^H_s, x^* \rangle \quad a.s.
\]

Up to a modification this process is unique. For all $x^*, y^* \in E^*$ and $0 \leq s, t \leq T$ we have
\[
\mathbb{E}\{X_t, x^*\} = \int_0^t \langle \Phi(t-s) x^*, \Phi^*(s-u) y^* \rangle_H du,
\]
i.e., the process $\{X_t\}_{t \in [0, T]}$ is an Ornstein–Uhlenbeck process associated with $\Phi$.

**Proof.** Uniqueness up to a modification is obvious from the Hahn–Banach theorem and the separability of $E$.

Let $j : E \hookrightarrow \tilde{E}$ be a continuous dense embedding of $E$ into a separable real Hilbert space $\tilde{E}$. As is well known, such a pair $(j, \tilde{E})$ always exists (for instance, let $(x_n^* \in E^*)$ be a weak*–dense sequence in the dual unit ball $B_{E^*}$, let $(\lambda_n)$ be a summable sequence of strictly positive real numbers and define $\tilde{E}$ to be the completion of $E$ with respect to the inner product $(x, y)_\tilde{E} := \sum_{n=1}^\infty \lambda_n (x, x_n^*) (y, x_n^*)$, cf. [Kuo, p. 154]).

For $t \in (0, T]$ define $\tilde{\Phi}(t) \in L(\tilde{H}, \tilde{E})$ by
\[
\tilde{\Phi}(t) := j \circ \Phi(t).
\]
It is immediate that $\tilde{\Phi} \in L^2((0, T]; H, \tilde{E})$. For $t \in (0, T]$ let $\tilde{Q}_t \in L(\tilde{E}^*, \tilde{E})$ be defined by
\[
\tilde{Q}_t \tilde{x}^* := \int_0^t \langle \tilde{\Phi}(s) \tilde{\Phi}^*(s) \tilde{x}^* \rangle ds.
\]
We have $\tilde{Q}_t = j \circ Q_t \circ j^*$. Let $(\tilde{v}_T, \tilde{H}_T)$ denote the RKHS associated with $\tilde{Q}_T$. The map $k_T : \tilde{Q}_T \tilde{x}^* \mapsto Q_T x^*$ extends to an isometry from $\tilde{H}_T$ onto $H_T$, and we have $i_T = j \circ i_T \circ k_T$. It follows that $i_T$ is $\gamma$-rilonifying (cf. Section 1).
The space $\tilde{E}$ being a Hilbert space, we may define an $\tilde{E}$-valued process $\{\tilde{X}_t\}_{t \in [0,T]}$ by the Hilbert space-valued stochastic Itô convolution integral

$$\tilde{X}_t = \int_0^t \tilde{\varphi}(t-s) \, dW_s^H$$

(cf. [DZ, Chapter 4]); this process is predictable and adapted to the filtration $\{\mathcal{F}_t\}_{t \in [0,T]}$.

We denote by $\mu_\tilde{E}$ the distribution of the $\tilde{E}$-valued random variable $\tilde{X}_t$. This is a centred Gaussian Borel measure on $\tilde{E}$. By the theory of stochastic convolutions in Hilbert spaces, $\{\tilde{X}_t\}_{t \in [0,T]}$ is an Ornstein–Uhlenbeck process associated with the function $\tilde{\varphi}$; in particular, the covariance operator of $\mu_\tilde{E}$ equals $\tilde{Q}_t$.

By a theorem of Kuratowski [VTC, Chapter 1], $j\tilde{E}$ is a Borel subset of $\tilde{E}$. We are going to show that $\mu_{j\tilde{E}}(j\tilde{E}) = 1$.

Since by assumption the inclusion map $i_T : H_T \hookrightarrow \tilde{E}$ is $\gamma$-radonifying, the remark preceding Lemma 2.3 and the results mentioned in Section 1 show that for each $t \in [0,T]$ the inclusion map $i_t : H_t \hookrightarrow \tilde{E}$ is $\gamma$-radonifying as well. Let $\nu_t := i_t(\gamma_{H_t})$, and let $\tilde{\nu}_t := j(\nu_t)$; these are centred Gaussian Borel measures on $E$ and $\tilde{E}$, respectively. The covariance operator $\tilde{R}_t$ of $\tilde{\nu}_t$ is given by

$$\langle \tilde{R}_t \tilde{x}, \tilde{y} \rangle = \int_0^t \langle \tilde{\varphi}^*(s) \tilde{x}, \tilde{\varphi}^*(s) \tilde{y} \rangle_H \, ds = \langle \tilde{Q}_t \tilde{x}, \tilde{y} \rangle.$$

It follows that $\tilde{R}_t = \tilde{Q}_t$. Since a centred Gaussian Borel measure is completely determined by its covariance, we conclude that $\tilde{\nu}_t = \tilde{\mu}_t$. But from $\tilde{\nu}_t = j(\nu_t)$ it follows that $\tilde{\nu}_t(j\tilde{E}) = \nu_t(E) = 1$. This proves that $\mu_{j\tilde{E}}(j\tilde{E}) = 1$.

As a consequence we have $\tilde{X}_t \in j\tilde{E}$ almost surely. This allows us to define an $\mathcal{F}_t$-measurable $E$-valued random variable $X_t$ by insisting that $jX_t = \tilde{X}_t$. The resulting adapted process $\{X_t\}_{t \in [0,T]}$ is predictable.

The distribution $\mu_\tilde{E}$ of $X_t$ is a probability Borel measure on $E$ which satisfies $j(\mu_\tilde{E}) = \mu_t$. For all $x^* \in E^*$ of the form $x^* = j^*\tilde{x}$ for some $\tilde{x} \in \tilde{E}$ we have $\langle \mu_t, x^* \rangle = \langle \mu_\tilde{E}, \tilde{x} \rangle$, the right hand side being a centred Gaussian Borel measure on $\tilde{E}$. Because the subspace $j^*\tilde{E}^*$ is weak-*dense in $E^*$; the measure $\mu_t$ is centred Gaussian by Corollary 1.3.

Next we prove (3.2). First note that for all $x^* = j^*\tilde{x}$ with $\tilde{x} \in \tilde{E}$ we have

$$\langle X_t, x^* \rangle = \langle \tilde{X}_t, \tilde{x} \rangle = \int_0^t \langle \tilde{\varphi}(t-s) \, dW_s^H, \tilde{x} \rangle = \int_0^t \langle \varphi(t-s) \, dW_s^H, x^* \rangle.$$

Therefore the subspace $Y$ consisting of all $x^* \in E^*$ for which (3.2) holds is weak-*dense. In order to prove that $Y = E^*$, by Proposition 1.2 it suffices to check that $Y$ is weak-*sequentially closed.

Let $(x^*_n)$ be a sequence in $Y$ converging to some $x^* \in E^*$ in the weak-*topology. We will show that $x^* \in Y$.

First we note that for all $t \in [0,T]$ we have $\lim_{n \to \infty} \varphi^*(t)x^*_n = \varphi^*(t)x^*$ weakly in $H$. The sequence $(x^*_n)$ being bounded, the sequence $(\varphi^*(\cdot)x^*_n)$ is bounded in $L^2((0,T);H)$. Upon passing to a weakly convergent subsequence we may assume that $\lim_{n \to \infty} \varphi^*(\cdot)x^*_n = f$ weakly for some $f \in L^2((0,T);H)$. By a convex combination argument as in the proof of Proposition 2.2, we find a sequence $(y^*_n)$ in $Y$ such that $\lim_{n \to \infty} y^*_n = x^*$ weak-* and $\lim_{n \to \infty} \varphi^*(\cdot)y^*_n = f$ strongly in $L^2([0,T];H)$. Upon passing to a pointwise a.e. convergent subsequence we conclude that

$$f = \lim_{n \to \infty} \varphi^*(\cdot)y^*_n = \varphi^*(\cdot)x^* \; a.e.$$

Next we note that

$$\langle X_t, x^* \rangle = \lim_{n \to \infty} \langle X_t, y^*_n \rangle = \lim_{n \to \infty} \int_0^t \varphi(t-s) \, dW_s^H, y^*_n \rangle \; a.e.$$

But by (3.1), which in view of Lemma 3.2 extends to arbitrary $\varphi \in L^2((0,T);H,E)$,

$$\lim_{n \to \infty} \mathbb{E} \left( \int_0^t \langle \varphi(t-s) \, dW_s^H, y^*_n - x^* \rangle^2 \right) = \lim_{n \to \infty} \| \varphi^*(\cdot)(y^*_n - x^*) \|^2_{L^2([0,T];H)} = 0.$$

Therefore,

$$\lim_{n \to \infty} \int_0^t \langle \varphi(t-s) \, dW_s^H, y^*_n \rangle = \int_0^t \langle \varphi(t-s) \, dW_s^H, x^* \rangle \; in \; L^2([0,T]).$$

Upon passing once more to a pointwise a.e. convergent subsequence if necessary, we conclude that (3.2) follows from (3.4) and (3.5).

It remains to show that (3.3) holds, i.e. that $\{X_t\}_{t \in [0,T]}$ is an Ornstein–Uhlenbeck process associated with $\varphi$. First let $j^*\tilde{y}^* \in j^*\tilde{E}^*$ be fixed and let $Y$ denote the set of all $x^* \in E^*$ such that

$$\mathbb{E} \langle X_t, x^* \rangle \langle Y, x^* \rangle = \int_0^t \left[ \langle \varphi^*(t-u) x^*, \varphi^*(s-u) j^*\tilde{y}^* \rangle \right] \, du$$

holds for all $t, s \in [0,T]$. Since $\{X_t\}_{t \in [0,T]}$ is an Ornstein–Uhlenbeck process with values in $\tilde{E}$ we have $j^*\tilde{E}^* \subset Y$ and therefore $Y$ is a weak-*dense linear subspace of $E^*$. By the dominated convergence theorem it is also weak-*sequentially closed. Hence by Proposition 1.2, $Y = E^*$.

Let $Z$ denote the set of all $y^* \in E^*$ such that (3.3) holds for all $x^* \in E^*$ and all $t, s \in [0,T]$. By what we already know, $j^*\tilde{E}^* \subset Z$ and therefore $Z$ is a weak-*dense linear subspace of $E^*$. Once more the dominated convergence theorem shows that $Z$ is also weak-*sequentially closed, and therefore
$Z = E^*$. This proves that $\{X_t\}_{t \geq 0}$ is an Ornstein–Uhlenbeck process with covariance given by (3.3).

Remark. By the Kolmogorov scheme, with the process $\{X_t\}_{t \in [0,T]}$ one can associate a canonical process on the probability space $(E^{0,T}, \nu)$, where $\nu$ is the measure obtained as the projective limit of the finite-dimensional distributions of $\{X_t\}_{t \in [0,T]}$. In this way we just obtain the canonical process $\{\xi_t\}_{t \in [0,T]}$ of Section 2.

Definition 3.4. The predictable $E$-valued process $\{X_t\}_{t \in [0,T]}$ constructed in Theorem 3.3 will be called the stochastic convolution of $\Phi$ with respect to $\{W^H_t\}_{t \in [0,T]}$; notation:

$$X_t = \int_0^t \Phi(t-s) \, dW^H_s.$$

4. Path regularity. In this section we discuss path regularity of the stochastic convolution process

$$X_t = \int_0^t \Phi(t-s) \, dW^H_s$$

under the assumptions that $\{W^H_t\}_{t \in [0,T]}$ is a cylindrical Wiener process with Cameron–Martin space $H$ and $\Phi \in L^2 \left( (0,T); H, E \right)$ is such that the embedding $\gamma_T: H_T \hookrightarrow E$ is $\gamma$-radonifying.

We begin with some preparations. As before, $\mu_t$ denotes the distribution of $X_t$; this is the centred Gaussian Borel measure on $E$ whose covariance operator is $Q_t$. The following inequality is a direct consequence of [Nh, Lemma 28] and the observation that $\|Q_t x^*\|_{H_t} \leq \|Q_T x^*\|_{H_T}$ whenever $0 < t \leq T$.

**Proposition 4.1.** If $0 < t < T$, then

$$\int_E \|x\|^2 \, d\mu_t(x) \leq t \int_E \|x\|^2 \, d\mu_T(x).$$

**Proposition 4.2.** The process $\{X_t\}_{t \in [0,T]}$ has a strongly measurable modification such that for almost all $\omega \in \Omega$,

$$\int_0^T \|X_t(\omega)\|^2 \, dt < \infty.$$

**Proof.** The process $\{X_t\}_{t \in [0,T]}$ has a predictable, and therefore a progressively measurable, modification. Hence by Fubini’s theorem, the paths of this modification are strongly measurable almost surely, and by Proposition 4.1 we have, for almost all $\omega \in \Omega$,

$$\int_0^T \|X_t(\omega)\|^2 \, dt < \infty.$$
for some \( M \geq 0 \) and all \( t, s \in [0, T] \). In particular,
\[
\mathbb{E}(X_t - X_s, x^*)^2 \leq M|t - s|^\theta
\]
for all \( t, s \in [0, T] \) and \( x^* \in E^* \) with \( \|x^*\| \leq 1 \).

To finish the proof we proceed as in [MS, Proposition 3.1] and check that the assumptions of [Ca, Proposition 5] are satisfied. The existence of a continuous modification then follows. For the reader’s convenience we give the details.

First we consider the Gaussian process \( X_T = \{(X_T, x^*)\}_{x^* \in U} \) indexed by the closed unit ball \( U \) of \( E^* \). This process has weak*-continuous paths. Putting
\[
\Gamma(t, s; x^*, y^*) := \int_0^\infty \left[ \Phi(t-u)x^*, \Phi(s-u)y^* \right] d\mu_u,
\]
for \( 0 \leq t \leq T \) we have
\[
\Gamma(t, t; x^* - y^*, x^* - y^*) = \left\| \Phi^*(v)(x^* - y^*) \right\|_H^2 dv
\]
\[
\leq \int_0^T \left\| \Phi^*(v)(x^* - y^*) \right\|_H^2 dv
\]
\[
= \langle Q_T(x^* - y^*), (x^* - y^*) \rangle
\]
\[
= \mathbb{E}(X_T, x^* - y^*)^2.
\]
This yields the first condition of [Ca, Proposition 5].

Since the function \( (t, s) \to (M/2)(t^\theta + s^\theta - |t - s|^\theta) \) is symmetric and positive definite, there exists a centered real-valued Gaussian process \( \{Z_t\}_{t \in [0, T]} \) with
\[
\mathbb{E}(Z_t, Z_s) = \frac{M}{2}(t^\theta + s^\theta - |t - s|^\theta).
\]
Then
\[
\mathbb{E}|Y_t - Y_s|^p = C_p|t - s|^\theta p.
\]
This process being Gaussian, we have
\[
\mathbb{E}|Y_t - Y_s|^2p = C_p|t - s|^\theta p
\]
and by taking \( p \) large enough we see that it has a continuous modification. By the computations above, for \( 0 \leq s \leq t \leq T \) and \( x^* \in U \) we have
\[
\Gamma(t, t; x^*, x^*) - 2\Gamma(t, s; x^*, x^*) + \Gamma(s, s; x^*, x^*)
\]
\[
= \mathbb{E}(X_t - X_s, x^*)^2 \leq M|t - s|^\theta = \mathbb{E}|Y_t - Y_s|^2.
\]
This proves the second condition of [Ca, Proposition 5].

In particular, it follows from this proposition that the process \( \{X_t\}_{t \in [0, T]} \) has a continuous modification if there exists a constant \( M \) such that
\[
\|\Phi(t) - \Phi(s)\| \leq M|t - s|, \quad t, s \in (0, T].
\]

Remark 4.4. If the conditions (i) and (ii) in Proposition 4.3 hold for a single \( x^* \in E^* \), then \( \{X_t, x^*\}_{t \in [0, T]} \) admits a continuous version. This follows upon taking \( r \) large in (4.1) and applying the Kolmogorov–Chentsov theorem. In particular, if there exists a constant \( M \) such that
\[
\|\Phi(t)x^* - \Phi(s)x^*\| \leq M|t - s|, \quad t, s \in (0, T],
\]
then the process \( \{X_t, x^*\}_{t \in [0, T]} \) has a continuous modification.

5. Weak solutions of the stochastic Cauchy problem. In this section we will apply our theory to the study of the following stochastic abstract Cauchy problem:

\[
(ACP) \quad dX_t = AX_t dt + B dW^H_t \quad (t \in [0, T]),
\]
\[
X_0 = 0 \quad \text{a.s.}
\]
Here \( A \) is the generator of a \( C_0 \)-semigroup \( S = \{S(t)\}_{t \geq 0} \) on a separable real Banach space \( E \), \( B \) is a bounded linear operator from a separable real Hilbert space \( H \) into \( E \), and \( \{W^H_t\}_{t \in [0, T]} \) is a cylindrical Wiener process with Cameron–Martin space \( H \).

In this setting we may define an operator-valued function \( \Phi : (0, T] \to \mathcal{L}(H, E) \) by
\[
\Phi(t) = S(t) \circ B \quad (t \in [0, T]).
\]
Clearly we have \( \Phi \in L^2((0, T]; H, E) \). The operators \( Q_t \in \mathcal{L}(E^*, E) \) are given by
\[
Q_t x^* = \int_0^t S(s)QS^*(s)x^* ds \quad (x^* \in E^*, t \in (0, T]),
\]
where \( Q = B \circ B^* \). This integral can be shown to exist in the sense of Bochner [Nel], but this will not play a role in what follows. As before, we let \( (\mathcal{I}_T, \mathcal{H}_T) \) denote the RKHS associated with \( Q_T \).

Definition 5.1. A weak solution of (ACP) is a predictable \( E \)-valued stochastic process \( \{X_t\}_{t \in [0, T]} \) such that for all \( x^* \in D(A^*) \) the function \( s \mapsto \langle X_s, A^*x^* \rangle \) is almost surely integrable on \([0, T] \) and
\[
\langle X_t, x^* \rangle = \int_0^t \langle X_s, A^*x^* \rangle ds + [W^H_t, B^*x^*] \quad (t \in [0, T]).
\]
Remark. Although we do not assume that a weak solution \( \{X_t\}_{t \in [0,T]} \) has a (weakly) continuous version, it is an immediate consequence of our definition and Definition 3.1 that the process \( \{ (X_t, x^*) \}_{t \in [0,T]} \) does have a continuous version for every \( x^* \in D(A^*) \).

The proofs of our main results depend on the following extension result for \( C_0 \)-semigroups \([\text{Ne}2]\):

Proposition 5.2. There exists a separable real Hilbert space \( \widetilde{E} \), a continuous and dense embedding \( j : E \hookrightarrow \widetilde{E} \), and a \( C_0 \)-semigroup \( \tilde{S} \) on \( \widetilde{E} \) such that \( j \circ S(t) = \tilde{S}(t) \circ j \) for all \( t \geq 0 \).

Theorem 5.3. If the embedding \( \iota_T : H_T \hookrightarrow E \) is \( \gamma \)-radonifying, then the process \( \{ X_t \}_{t \in [0,T]} \) defined by stochastic convolution,

\[
X_t = \int_0^t S(t-s)B \ dW_s^H \quad (t \in [0,T]),
\]

is a weak solution of (ACP). This process has a strongly measurable modification that satisfies

\[
\int_0^T \|X_t\|^2 \ dt < \infty
\]

almost surely.

Proof. By Proposition 4.2, with \( \Phi(t) = S(t) \circ B \), the process \( \{ X_t \}_{t \in [0,T]} \) has a strongly measurable modification which satisfies \( \int_0^T \|X_t\|^2 \ dt < \infty \) almost surely.

Let \( j : E \hookrightarrow \widetilde{E} \) denote the embedding of Proposition 5.2 and let \( \tilde{S} \) denote the \( C_0 \)-extension of \( S \) to \( \widetilde{E} \). By the theory of (ACP) in Hilbert spaces \([\text{DZ, Chapter 5}]\), the \( \widetilde{E} \)-valued process \( \{ \tilde{X}_t \}_{t \in [0,T]} \) defined by the Hilbert space stochastic Itô convolution integral

\[
\tilde{X}_t = \int_0^t \tilde{S}(t-s)\tilde{B} \ dW_s^H,
\]

where \( \tilde{B} = j \circ B \), is a weak solution in \( \widetilde{E} \) of the problem

\[
d\tilde{X}_t = \tilde{A}\tilde{X}_t \ dt + \tilde{B}dW_s^H \quad (t \in [0,T]),
\]

\[
\tilde{X}_0 = 0 \quad \text{a.s.}
\]

Here \( \tilde{A} \) is the generator of \( \tilde{S} \). As we have seen in the proof of Theorem 3.3, for all \( t \in [0,T] \) we have \( \tilde{X}_t = jX_t \).

For all \( v^* \in D(A^*) \) we have \( j^*v^* \in D(A^*) \) and \( A^*(j^*v^*) = j^*(\tilde{A}^*v^*) \). This implies that for all elements in \( v^* \in D(A^*) \) of the form \( v^* = j^*\tilde{v}^* \) for some \( \tilde{v}^* \in D(\tilde{A}^*) \) we have

\[
\langle X_t, v^* \rangle = \langle \tilde{X}_t, \tilde{v}^* \rangle = \int_0^t \left( \tilde{X}_s, \tilde{A}^*\tilde{v}^* \right) \ ds + [W_s^H, \tilde{B}^*\tilde{v}^*]
\]

\[
= \int_0^t \left( X_s, A^*v^* \right) \ ds + [W_s^H, B^*v^*] \quad (t \in [0,T]).
\]

Fix \( \lambda \in \rho(A) \). Let \( Y \) denote the set of all \( v^* \in E^* \) such that (5.1) holds for the element \( x^* := (\lambda - A^*)^{-1}v^* \in D(A^*) \). By the above, \( Y \) is a linear subspace of \( E^* \) containing the weak*-dense subspace \( j^*E^* \).

We will show next that \( Y \) is weak*-sequentially closed. Let \( \{ \tilde{x}_n^* \} \) be a sequence in \( Y \) converging weak* to some \( x^* \in E^* \). Then the functional \( y_n^* := (\lambda - A^*)^{-1}x_n^* \) belongs to \( D(A^*) \) and the sequence \( \{ y_n^* \} \) converges weak* to \( y^* := (\lambda - A^*)^{-1}x^* \). Hence for all \( \omega \) we have

\[
\lim_{n \to \infty} \langle X_t(\omega), y_n^* \rangle = \langle X_t(\omega), y^* \rangle.
\]

Moreover, from \( A^*y_n^* = A^*(\lambda - A^*)^{-1}x_n^* = \lambda(\lambda - A^*)^{-1}x_n^* - x_n^* \) we see that \( A^*y_n^* \) converges weak* to \( \lambda(\lambda - A^*)^{-1}x^* - x^* = A^*y^* \). By dominated convergence, for all \( \omega \in \Omega \) we have

\[
\lim_{n \to \infty} \int_0^t \langle X_s(\omega), A^*y_n^* \rangle \ ds = \int_0^t \langle X_s(\omega), A^*y^* \rangle \ ds.
\]

The weak*-to-weak continuity of \( B^* \) implies that \( B^*y_n^* \to B^*y^* \) weakly in \( H \). Since bounded linear operators are weakly continuous, it follows that

\[
\lim_{n \to \infty} [W_t^H, B^*y_n^*] = [W_t^H, B^*y^*] \quad \text{weakly in } L^2(P).
\]

On the other hand, combining (5.2) with (5.3) and (5.4) we deduce that for all \( \omega \in \Omega \) the limit

\[
\lim_{n \to \infty} [W_t^H, B^*y_n^*] (\omega) = Y(\omega)
\]

exists. With a convex combination argument as in the proof of Proposition 2.2, together with (5.5) this shows that \( Y = [W_t^H, B^*y^*] \) a.e. Hence for almost all \( \omega \in \Omega \) we have

\[
\lim_{n \to \infty} [W_t^H, B^*y_n^*] (\omega) = [W_t^H, B^*y^*] (\omega).
\]

By (5.2), (5.3), (5.4), (5.6) and dominated convergence we finally obtain

\[
\lim_{n \to \infty} \langle X_t, v^* \rangle = \lim_{n \to \infty} \left( \int_0^t \langle X_s, A^*y_n^* \rangle \ ds + [W_s^H, B^*y_n^*] \right)
\]

\[
= \int_0^t \langle X_s, A^*y^* \rangle \ ds + [W_s^H, B^*y^*] \]
almost everywhere. This shows that $x^* \in Y$, and $Y$ is weak*-sequentially closed as claimed.

By Proposition 1.2, $Y = E^*$ and the proof is complete. ■

**Remark.** As we noted above, the fact that $\{X_t\}_{t \in [0,T]}$ is a weak solution implies that for each $x^* \in D(A^*)$, the scalar process $\{(X_t, x^*)\}_{t \in [0,T]}$ has a continuous modification. This can also be seen from the observation in Remark 4.4. Indeed, if $x^* \in D(A^*)$, the identity

$$S^*(t)x^* - x^* = \int_0^t S^*(s)A^*x^* \, ds$$

shows that the orbit $t \mapsto S^*(t)x^*$ is Lipschitz continuous on the bounded interval $[0,T]$.

Theorem 5.3 admits the following converse:

**Theorem 5.4.** Suppose (ACP) admits a weak solution $\{X_t\}_{t \in [0,T]}$. Then the embedding $x^* : H_T \to E$ is $\gamma$-radonifying and $\{X_t\}_{t \in [0,T]}$ is an Ornstein–Uhlenbeck process.

**Proof.** Let $j : E \to \tilde{E}$ and $\tilde{S}$ be as in Proposition 5.2. By the results of [BRS], $\tilde{E}$ may be densely embedded into another separable real Hilbert space $\tilde{E}$ in such a way that $\tilde{S}$ extends to a $C_0$-semigroup $\tilde{S}$ on $\tilde{E}$ and the embedding $j : E \to \tilde{E}$ is Hilbert–Schmidt. Let $\tilde{j} := j \circ j$.

The operator $\tilde{B} := j \circ B = j \circ \tilde{B} : H \to \tilde{E}$ is Hilbert–Schmidt, being the composition of the bounded operator $B = j \circ B$ and the Hilbert–Schmidt operator $\tilde{j}$. It follows that $\tilde{Q} := \tilde{B} \circ \tilde{B}^*$ is of trace class. Define the positive selfadjoint operator $\tilde{Q}_T$ on $\tilde{E}$ by

$$\tilde{Q}_T \tilde{h} = \int_0^T \tilde{S}(s)\tilde{Q} \tilde{S}^*(s) \tilde{h} \, ds \quad (\tilde{h} \in \tilde{E}).$$

Then it is easy to check (cf. [Ne]) that $\tilde{Q}_T$ is of trace class as well.

It now follows from the general theory of stochastic equations in Hilbert spaces [DZ, Chapter 5] that the stochastic convolution process $\tilde{X}_t = \int_0^t \tilde{S}(t-s) \tilde{B} \, dw_t^H$ is the unique weak solution to the problem

$$d\tilde{X}_t = \tilde{A}\tilde{X}_t \, dt + \tilde{B} \, dw_t^H \quad (t \in [0,T]),$$

$$\tilde{X}_0 = 0 \quad a.s.$$  

But the process $\{\tilde{X}_t\}_{t \in [0,T]}$ is a weak solution of this problem as well, and hence by uniqueness it follows that $\tilde{X}_t = j \tilde{X}_t$ for all $t \in [0,T]$. We conclude that $\{j \tilde{X}_t\}_{t \in [0,T]}$ is an $E$-valued Ornstein–Uhlenbeck process, this being true for $\{\tilde{X}_t\}_{t \in [0,T]}$. This implies that for all $\tilde{x}^*, \tilde{y}^* \in \tilde{E}^*$ and $t, s \in [0, T]$ we have

$$\mathbb{E}(\langle X_t, \tilde{x}^* \rangle, \langle X_s, \tilde{y}^* \rangle)$$

$$= \mathbb{E}(\langle \tilde{X}_t, \tilde{x}^* \rangle, \langle \tilde{X}_s, \tilde{y}^* \rangle)$$

$$= \int_{\tilde{H}_T} \langle \tilde{B}^* \tilde{S}^*(t-u) \tilde{x}^*, \tilde{B}^* \tilde{S}^*(s-u) \tilde{y}^* \rangle \, du$$

$$= \int_{H_T} \langle B^* S^*(t-u) \tilde{x}^*, B^* S^*(s-u) \tilde{y}^* \rangle \, du.$$

The linear subspace $Y = \{\tilde{j} \tilde{x}^* : \tilde{x}^* \in \tilde{E}^*\}$ is weak*-dense in $E^*$, $\tilde{j}$ being a dense embedding.

We claim that $\{X_t\}_{t \in [0,T]}$ is a Gaussian process. To see this, fix $t \in [0,T]$ and let $\mu_t$ and $\tilde{\nu}_t$ be the distributions of $X_t$ and $\tilde{X}_t$, respectively. These are Borel probability measures on $E$ and $\tilde{E}$, respectively, and we have $\tilde{\nu}_t = \tilde{j}(\mu_t)$. Moreover, because $\{X_t\}_{t \in [0,T]}$ is an Ornstein–Uhlenbeck process, hence a Gaussian process, $\mu_t$ is a Gaussian measure. Hence for all $y^* = j^* \tilde{x}^*$ in the weak*-dense subspace $Y$ of $E^*$, the image measures $\mu_t(y^*) = (\tilde{\mu}_t, \tilde{x}^*)$ are Gaussian on $\tilde{E}$. By Corollary 1.3, this implies that $\mu_t$ is Gaussian, and the claim is proved.

The process $\{X_t\}_{t \in [0,T]}$ being Gaussian, the weak second moments $\mathbb{E}(\langle X_t, x^* \rangle^2)$ are finite for all $t \in [0,T]$ and $x^* \in E^*$. Departing from (5.7), the proof that $\{X_t\}_{t \in [0,T]}$ is an Ornstein–Uhlenbeck process now proceeds along the lines of the proof of Theorem 3.3. ■

**Concerning uniqueness of weak solutions, we have the following result: **

**Theorem 5.5.** Let $X^{(0)} = \{X^{(0)}_t\}_{t \in [0,T]}$ and $X^{(1)} = \{X^{(1)}_t\}_{t \in [0,T]}$ be two weak solutions of (ACP). Then $X^{(0)}$ and $X^{(1)}$ are versions of each other.

**Proof.** This follows immediately by embedding $E$ into a Hilbert space $\tilde{E}$ in the way described in the proof of Theorem 5.4 and the fact that the corresponding uniqueness result for weak solutions holds in the Hilbert space setting. ■

So far, we were concerned only with solutions on a finite time interval $[0,T]$. By obvious modifications, the theory extends to the interval $[0, \infty)$. In particular, a weak global solution of (ACP) exists if and only if for all $T > 0$ the associated inclusion mapping $i_T : H_T \looparrowright E$ is $\gamma$-radonifying; in this case the solution is unique, and given by stochastic convolution.

Under this assumption, for each $t > 0$ we let $\mu_t = i_t(\gamma_{i_T})$ denote the corresponding centered Gaussian measure on $E$; we further set $\mu_0 = \delta_0$, the Dirac measure concentrated at 0. For each $t \geq 0$ we define a linear contraction $P(t)$ on the space $B_0(E)$ of bounded real-valued Borel functions
on $E$ by the formula
\[ P(t)f(x) = \int_E f(S(t)x + y) d\mu_t(y) \quad (x \in E, \; f \in B_b(E)). \]
From the identity
\[ Q_{t+s} = Q_t + S(t)Q_sS^*(t) \]
we see that
\[ \mu_{t+s} = \mu_t \ast S(t)\mu_s, \]
from which it easily follows that $P(t+s) = P(t) \circ P(s)$ for all $t, s \geq 0$. Thus the family $\{P(t)\}_{t \geq 0}$ is a semigroup of contractions on $B_b(E)$. This semigroup has been studied in some detail in [Ne1] from a functional-analytic point of view. We conclude this section by showing that it arises as the transition semigroup of the weak solution of the stochastic Cauchy problem (ACP):

**Proposition 5.6.** Let $\{X_t\}_{t \geq 0}$ be a weak solution of the problem (ACP). For all $t \in [0, T]$ we have, for all $x \in E$,
\[ P(t)f(x) = E(f(X_{t,x})), \]
where $X_{t,x} := S(t)x + X_t$.

**Proof.** Fix $t \in [0, T]$. Recalling that $\mu_t$ is the distribution of $X_t$, for all $x \in E$ we have
\[ \mathbb{E}(f(X_{t,x})) = \int f(S(t)x + X_t) d\mu_t(x) = \int f(S(t)x + y) d\mu_t(y) = P(t)f(x). \]
We point out that the weak solution is always a Markov process. This can be seen directly as in the proof of Proposition 5.6 or by using the fact that this is true for the Hilbert space case and using the argument in the proof of Theorem 5.5.

6. The analytic case. The results of the previous section do not take into account possible regularization effects of the semigroup $S$. We will now present a result in this direction for the case where $S$ is an analytic semigroup. Roughly speaking it turns out that if $S$ maps $E$ into some smaller space $F$, then under some natural assumptions the weak solution of (ACP) is also $F$-valued.

**Theorem 6.1.** Suppose that $F$ and $E$ are separable real Banach spaces, with $F$ continuously embedded in $E$. Let $S_E = \{S_E(t)\}_{t \geq 0}$ be a $C_0$-semigroup on $E$, with generator $A_E$ such that $S_E(t)E \subset F$ for all $t > 0$. Denote by $S_{EF}(t)$ the operator $S_E(t)$ regarded as a bounded linear operator from $E$ into $F$, and let $S_{EF}(t)$ denote the restriction of $S_{EF}(t)$ to $F$. Let $B$ be a bounded linear operator from a separable real Hilbert space $H$ into $E$, and let $Q = B \circ B^*$. Let $\mathcal{Q}_T \in \mathcal{L}(F^*, F)$ be the positive symmetric operator defined by the Pettis integral
\[ \mathcal{Q}_T := \frac{1}{T} \int_0^T S_{EF}(t)Q S_{EF}^*(t) dt \quad (x^* \in F^*). \]
Let $(i_T, H_T)$ be the RKHS associated with $Q_T$. Assume that:

(i) For each $x^* \in F^*$, the function $t \mapsto B^*S_{EF}^*(t)x^*$ is strongly measurable and
\[ \int_0^T \|B^*S_{EF}^*(t)x^*\|^2_H dt < \infty. \]

(ii) The semigroup $S_F = \{S_F(t)\}_{t \geq 0}$ is an analytic $C_0$-semigroup on $F$, with generator $A_F$, and there exist $\lambda \in \rho(A_F)$ and $\theta \in (0, 1]$ such that
\[ \int_0^T \| (\lambda - A_F)^\theta S_{EF}(t) \|^2_{L(E,F)} dt < \infty. \]

(iii) The embedding $i_T : H_T \hookrightarrow F$ is a $\gamma$-radonifying.

Under these assumptions there is an $F$-valued stochastic process $\{X_t\}_{t \in [0, T]}$ with covariance
\[ E(\langle X_{t,x}^*, y^* \rangle) = \int_0^T \langle [B^*S_{EF}^*(t-u)x^*, B^*S_{EF}^*(t-u) y^*]_H du \quad (x^*, y^* \in F^*). \]
This process has a continuous modification. As an $E$-valued process, it is a weak solution to the stochastic abstract Cauchy problem
\[ dX_t = A_EX_t dt + B dW^H_t, \quad t \in [0, T], \quad X_0 = 0. \]

**Proof.** By (i), (iii), and Theorem 3.3 applied to the $\mathcal{L}(H,F)$-valued function $t \mapsto S_{EF}(t) \circ B$, there exists an $F$-valued process Ornstein–Uhlenbeck process $\{X_t\}_{t \in [0, T]}$ with covariance given by (6.3) and we have
\[ E(\langle X_{t,x}^*, y^* \rangle) = \int_0^t \langle [S_{EF}(t-s)B dW^H_s, x^*] \quad (t \in [0, T], x^* \in F^*). \]

We shall prove that the process $\{X_t\}_{t \in [0, T]}$ has a continuous version. We argue as in [MS, Remark 3.2]. Fix $\lambda \in \rho(A_F)$ and $\theta \in (0, 1]$ as in assumption (ii). For all $x^* \in F^*$ we have
By a weak solution of (6.5) we understand a weak solution to the problem

\[\begin{align*}
  dX_t &= AX_t + f(t, X_t) \, dt, \\
  X_0 &= \xi,
\end{align*}\]

where \( \Delta \) is the Dirichlet Laplacian in \( E = L^2[0,1] \) and \( \{W_t\}_{t \in [0,T]} \) is a cylindrical Wiener process with Cameron–Martin space \( H = E = L^2[0,1] \).

For \( \beta \in [0,1] \) let

\[ c^\beta_0[0,1] = \{u \in C^\beta[0,1] : u(0) = u(1) = 0\}, \]

where \( C^\beta[0,1] \) is the little Hölder space of all continuous functions \( f \) on \( [0,1] \) for which

\[ \|f\|_{C^\beta[0,1]} := \sup_{t \in [0,1]} |f(t)| + \sup_{0 \leq s < t \leq 1} \frac{|f(t) - f(s)|}{(t - s)^\beta} < \infty \]

and

\[ \lim_{\delta \to 0} \sup_{|t - s| \leq \delta} \frac{|f(t) - f(s)|}{(t - s)^\beta} = 0. \]

Theorem 6.3. The problem (6.5) has a unique global weak solution \( \{X_t\}_{t \geq 0} \). For each \( t \geq 0 \) the random variable \( X_t \) takes values in \( c^\beta_0[0,1] \) almost surely. As a \( c^\beta_0[0,1] \)-valued process, \( \{X_t\}_{t \geq 0} \) has a continuous modification.

Proof. For \( p \in [1,\infty) \), let \( A_p = \Delta \) be the Laplacian on \( L^p[0,1] \) with Dirichlet boundary conditions, i.e., \( D(A_p) = H_0^{1,p}[0,1] \cap H^{2,p}[0,1] \), and let \( S_p \) denote the heat semigroup on \( L^p[0,1] \), i.e. the analytic \( C_0 \)-semigroup on \( L^p[0,1] \) generated by \( A_p \).

Let \( T > 0 \) be arbitrary. As is well known (see e.g. [DZ]), the RKHS corresponding to the selfadjoint operator \( R_T \in L(L^2[0,1]) \) defined by

\[ R_T f = \int_0^T S_T(t)S_T(t)f \, dt \quad (f \in L^2[0,1]) \]

equals \( H_{1/2}^{1/2}[0,1] \). The inclusion \( H_{1/2}^{1/2}[0,1] \hookrightarrow L^2[0,1] \) is Hilbert–Schmidt and hence \( \gamma \)-radonifying. Hence (6.5), and therefore (6.6), has a unique global weak solution \( \{X_t\}_{t \geq 0} \).

Fix \( \alpha \in (0, 1/4) \) and \( 2 < p < \infty \) be such that \( 2\alpha > 1/p \). We are going to check first that Theorem 6.1 applies, with \( H = E = L^2[0,1] \), \( B : H \to E \) the identity operator, \( F = H_{1/2}^{1/2}[0,1] \), and \( S_2 = S_2 \).

The restriction \( S_{2\alpha,p} \) of \( S_2 \) to \( H_{2\alpha,p} \) is strongly continuous and analytic on \( H_{2\alpha,p}^\alpha[0,1] \). Notice that, with the notation of Theorem 6.1, \( S_{2\alpha,p} \) equals the semigroup \( S_F \). Let \( A_{2\alpha,p} \) be its generator. Put \( \delta := 1/2 - 1/p \) and...
note that $2(\alpha + \delta) < 1$ since we assume that $\alpha \in (0, 1/4)$. Choose $\theta \in (0, 1)$ so small that $2(\alpha + \delta + 2\theta) < 1$. Suppressing subscripts, we then have, with a suitable choice of $0 < \eta < \delta + \theta$, \[
abla \|S(t)\|_{L^{L^p[0,1],L^p[0,1]}} \leq Ct^{-\eta} \quad (t \in (0, 1]), \]
\[
abla \|S(t)\|_{L^{L^p[0,1],L^{2\alpha,p}[0,1]}} \leq Ct^{-\alpha} \quad (t \in (0, 1]), \]
\[
abla \|(-A_{2\alpha,p})^\theta S(t)\|_{L^{L^p[0,1],L^{2\alpha,p}[0,1]}} \leq Ct^{-\alpha - \theta} \quad (t \in (0, 1]). \]

The first of these estimates follows from \[
\|S(t)f\|_{L^{L^p[0,1]}} \leq C\|S(t)\|_{L^{L^p[0,1],L^{2\alpha,p}[0,1]}} \leq Ct^{-\eta} \|f\|_{L^{L^p[0,1]}} \quad (t > 0), \]

and interpolation; here we use the fact that by assumption $\delta + \theta > 1/4$, so that $H_0^{2(\delta + \theta)}[0, 1] \hookrightarrow L^\infty[0, 1]$ by the Sobolev embedding theorem. The second and third estimates follow from general results about analytic semigroups.

The first two estimates show that assumption (i) of Theorem 6.1 holds. From the first and third estimates we infer that \[
\int_0^1 \|(-A_{2\alpha,p})^\theta S(t)^2\|_{L^{L^p[0,1],L^{2\alpha,p}[0,1]}} dt \leq C \int_0^1 t^{-2(\alpha + \delta + \theta)} dt < \infty, \]

which shows that assumption (ii) of Theorem 6.1 is satisfied (cf. the remark following the formulation of the theorem). By [Br1] and a subspace argument, the inclusion $\iota_{2\alpha,p}: H_0^{2\alpha}[0, 1] \hookrightarrow H_0^{2\alpha}[0, 1]$ is $\gamma$-radonifying; this proves assumption (iii) of Theorem 6.1. Hence by Theorem 6.1 the weak solution $(X_t)_{t \geq 0}$ of (6.5) has a modification that is a continuous $H_0^{2\alpha,p}[0, 1]$-valued process with covariance \[
\mathbb{E}(X_t, \varphi)(X_s, \psi) = \int_0^\alpha \int_0^\alpha \left[ \langle x_{2\alpha,p} S_{2\alpha,p}(t-u) \varphi, x_{2\alpha,p} S_{2\alpha,p}(s-u) \psi \rangle \right] H_0^{2\alpha}[0, 1] du \] for all $t, s \geq 0$ and $\varphi, \psi \in (H_0^{2\alpha,p}[0, 1])^*$. Now fix $\beta \in (0, 1/2)$. Choose $\alpha \in (0, 1/4)$ and $p > 2$ in such a way that $2\alpha > \beta + 1/p$. By the Sobolev embedding theorem we then have a continuous inclusion $H_0^{2\alpha,p}[0, 1] \hookrightarrow C_0^\beta[0, 1]$. Combining this with the above, we deduce that $(X_t)_{t \geq 0}$ takes values in $C_0^\beta[0, 1]$, and that it is continuous as a $C_0^\beta[0, 1]$-valued process.

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References


On the existence for the Cauchy–Neumann problem for the Stokes system in the $L_p$-framework

by

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Abstract. The existence for the Cauchy–Neumann problem for the Stokes system in a bounded domain $\Omega \subset \mathbb{R}^3$ is proved in a class such that the velocity belongs to $W^{2,1}_r(\Omega \times (0,T))$, where $r > 3$. The proof is divided into three steps. First, the existence of solutions is proved in a half-space for vanishing initial data by applying the Marcinkiewicz multiplier theorem. Next, we prove the existence of weak solutions in a bounded domain and then we regularize them. Finally, the problem with nonvanishing initial data is considered.

1. Introduction. In a bounded domain $\Omega$ in $\mathbb{R}^3$ with boundary $S$ we consider the initial-boundary value problem for the Stokes system:

$$
\begin{align*}
\n u_t - \nu \Delta u + \nabla p &= F, \\
\text{div } u &= 0, \\
\bar{n} \cdot T(u,p)|_{S_T} &= H, \\
\n u|_{t=0} &= u_0,
\end{align*}
$$

(1.1)

where $T(u,p) = (T_{ij}(u,p))_{i,j=1,2,3} = \{\nu(\partial_i u_j + \partial_j u_i) - \rho \delta_{ij}\}$ is the stress tensor, $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ the velocity vector, $p(x,t)$ the pressure, $\nu > 0$ the constant viscosity coefficient and $\bar{n}$ the exterior normal vector to $S$.

To solve (1.1) we have to impose the following compatibility conditions on the initial and boundary data:

$$
\begin{align*}
\text{div } u_0(x) &= G(x,0), \\
\bar{n} \cdot T(u_0,p_0)|_{S} &= H(x,0),
\end{align*}
$$

(1.2)

where $p_0$ is defined by $\bar{n} \cdot T(u_0,p_0) \cdot \bar{n} = H(0) \cdot \bar{n}$ on $S$. From (1.2) we get the initial boundary condition $p|_{t=0} = p_0$.