Eigenvalue problems with indefinite weight

by

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Abstract. We consider the linear eigenvalue problem $-\Delta u = \lambda V(x)u$, $u \in D_0^{1,2}(\Omega)$, and its nonlinear generalization $-\Delta_p u = \lambda V(x)|u|^{p-2}u$, $u \in D_0^{1,p}(\Omega)$. The set $\Omega$ need not be bounded, in particular, $\Omega = \mathbb{R}^N$ is admitted. The weight function $V$ may change sign and may have singular points. We show that there exists a sequence of eigenvalues $\lambda_n \to \infty$.

1. Introduction. In this paper we shall be concerned with the linear eigenvalue problem

\begin{equation}
-\Delta u = \lambda V(x)u, \quad u \in D_0^{1,2}(\Omega),
\end{equation}

$\Omega$ open in $\mathbb{R}^N$, $N \geq 3$, and its nonlinear generalization

\begin{equation}
-\Delta_p u = \lambda V(x)|u|^{p-2}u, \quad u \in D_0^{1,p}(\Omega),
\end{equation}

where $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian, $1 < p < N$, and $\Omega$ is open in $\mathbb{R}^N$. Observe that $\Omega$ may be unbounded, and in particular, it may be equal to $\mathbb{R}^N$. We assume that $V \in L^1_{\text{loc}}(\Omega)$, $V = V^+ - V^-$ (as usual, $V^\pm(x) := \max\{\pm V(x), 0\}$) and $V^+ = V_1 + V_2$, where $V_i \in L^{N/p}(\Omega)$, $|x|^p V_2(x) \to 0$ as $|x| \to \infty$ and for each $y \in \bar{\Omega}$, $|x - y|^p V_2(x) \to 0$ as $x \to y$ (in the linear case (1), $p = 2$ in the conditions on $V^+$). Under these hypotheses we show that (1) and (2) have a sequence of eigenvalues $\lambda_n \to \infty$. This generalizes several earlier results. In particular, for $\Omega = \mathbb{R}^N$ it was shown in [3, 4] that (1) has a principal eigenvalue $\lambda_1$ if $V$ is sufficiently smooth and satisfies an appropriate condition at infinity, and in [1] existence of infinitely many eigenvalues $\lambda_n \to \infty$ of (1) was established under...
the assumption that \( V \in L^\infty(\mathbb{R}^N) \) and \( V^+ \in L^{N/2}(\mathbb{R}^N) \). In \([18]\) several results on the existence and nonexistence of a principal eigenvalue of \((1)\) were obtained for nonnegative weight functions \( V \) of Hardy type. In this case even if a principal eigenvalue exists, one cannot expect to have a sequence of eigenvalues \( \lambda_n \to \infty \). Equation \((2)\) for \( \Omega = \mathbb{R}^N \) was studied in \([2]\), where it was demonstrated that if \( V \in L^\infty(\mathbb{R}^N) \) and \( V^+ \in L^{N/2}(\mathbb{R}^N) \), then there is a sequence \( \lambda_n \to \infty \) (see also \([8, 10]\)). Furthermore, it was shown in \([7]\) that \((2)\) has a principal eigenvalue whenever \( V \in L^N(\mathbb{R}^N) \cap L^{(N+2)/p}(\mathbb{R}^N) \), for some \( \delta > 0 \). More references concerning \((1)-(2)\), in particular to earlier work on bounded \( \Omega \), may be found in the papers cited above.

The paper is organized as follows: In Section 2 we prove the existence of infinitely many eigenvalues of \((1)\). Our argument is fairly elementary and is based on a simple minimization procedure. We also show that under an additional assumption on \( V \) the principal eigenvalue of \((1)\) is simple. In Section 3 we give a few examples demonstrating that our hypotheses on \( V \) are in a sense optimal. Finally, in Section 4 we are concerned with the nonlinear problem \((2)\). Again, a simple minimization argument shows the existence of a principal eigenvalue \( \lambda_1 \). However, since the equation is nonlinear now, it is not clear whether higher eigenvalues can be obtained by minimization. Therefore we use a different approach, based on minimax methods in critical point theory.

**Notation.** \( B(x, r) \) and \( B(x, r) \) denote respectively the open and the closed ball centered at \( x \) and having radius \( r \). \( \| \cdot \| \) is the usual norm in \( L^p(\Omega) \), \( D(\Omega) \) and \( D_0^{1,2}(\Omega) \) are the test functions in \( \Omega \) and \( D_0^{1,2}(\Omega) \) is the closure of \( D(\Omega) \) in the norm \( \| u \| := \| \nabla u \| \). A functional \( \chi: X \to \mathbb{R} \) is weakly continuous if \( u_n \to u \) implies that \( \chi(u_n) \to \chi(u) \).

### 2. Eigenvalues of the Laplacian

In this section we consider the linear eigenvalue problem

\[
-\Delta u = \lambda V(x) u, \quad u \in D_0^{1,2}(\Omega),
\]

where \( \Omega \) is an open subset of \( \mathbb{R}^N \), \( N \geq 3 \). Possibly \( \Omega = \mathbb{R}^N \). Our basic assumption is

\[
(\text{H}) \quad V \in L^1_{\text{loc}}(\Omega), \quad V^+ = V_1 + V_2 \neq 0, \quad V_1 \in L^{N/2}(\Omega),
\]

\[
\lim_{|x-y| \to \infty} |x-y|^2 V_2(x) = 0 \quad \text{for every } y \in \bar{\Omega}, \quad \lim_{|x-y| \to \infty} |x|^2 V_2(x) = 0.
\]

In order to find the principal eigenvalue of \((3)\) we solve the following minimization problem:

\[
(P_1) \quad \text{minimize } \int_{\Omega} |\nabla u|^2 dx, \quad u \in D_0^{1,2}(\Omega), \quad \int_{\Omega} V u^2 dx = 1.
\]

We shall use the following notation:

\[
X := D_0^{1,2}(\Omega), \quad \varphi(u) := \int_{\Omega} |\nabla u|^2 dx, \quad \psi(u) := \int_{\Omega} V u^2 dx.
\]

**Lemma 2.1.** Under assumption \((\text{H})\), \( \int_{\Omega} V^+ u^2 dx \) is weakly continuous.

**Proof.** By \([20, \text{Lemma 2.13}]\), \( \int_{\Omega} V u^2 dx \) is weakly continuous.

In order to prove that \( \int_{\Omega} V u^2 dx \) is weakly continuous, let us recall the Hardy inequality in \( D_0^{1,2}(\mathbb{R}^N) \):

\[
\int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx \leq \frac{4}{(N-2)^2} \int_{\mathbb{R}^N} |\nabla u|^2 dx.
\]

Let \( u_n \to u \) and \( \varepsilon > 0 \). By assumption, there exists \( R > 0 \) such that if \( x \in \Omega \) and \( |x| \geq R \), then \( |x|^2 V_2(x) \leq \varepsilon \). Define

\[
\Omega_1 := \Omega \setminus B[0, R], \quad \Omega_2 := \Omega \cap B(0, R), \quad c := \frac{2}{N-2} \sup_n \| u_n \|.
\]

The Hardy inequality implies that

\[
\int_{\Omega_1} V u_n^2 dx \leq \varepsilon \int_{\Omega_1} \frac{u_n^2}{|x|^2} dx \leq \varepsilon c^2,
\]

and similarly,

\[
\int_{\Omega_1} V u_n^2 dx \leq \varepsilon c^2.
\]

By compactness, there is a finite covering of \( \Omega_2 \) by closed balls \( B[x_1, r_1], \ldots, B[x_k, r_k] \) such that, for \( 1 \leq j \leq k \),

\[
|x - x_j| \leq r_j \Rightarrow |x - x_j|^2 V_2(x) \leq \varepsilon.
\]

There exists \( \tau > 0 \) such that, for \( 1 \leq j \leq k \),

\[
|x - x_j| \leq \tau \Rightarrow |x - x_j|^2 V_2(x) \leq \varepsilon/\kappa.
\]

Define \( A := \bigcup_{j=1}^k B[x_j, \tau] \). Then by the Hardy inequality,

\[
\int_{A} V u_n^2 dx \leq \varepsilon c^2, \quad \int_{A} V u^2 dx \leq \varepsilon c^2.
\]

It follows from \((6)\) that \( V_2 \in L^\infty(\Omega_2 \setminus A) \). Since \( \Omega_2 \setminus A \) is bounded, \( V_2 \in L^{N/2}(\Omega_2 \setminus A) \) so that by \([20, \text{Lemma 2.13}]\),

\[
\int_{A \setminus A} V u_n^2 dx \to \int_{A \setminus A} V u^2 dx.
\]

We deduce from \((4)\), \((5)\), \((7)\) and \((8)\) that \( \int_{\Omega} V u_n^2 dx \to \int_{\Omega} V u^2 dx. \)
THEOREM 2.2. Under assumption (H), problem (P1) has a solution $e_1 \geq 0$. Moreover, $e_1$ is an eigenfunction of (3) corresponding to the eigenvalue $\lambda_1^i := \int_\Omega |\nabla e_1|^2 \, dx$.

Proof. Let $(u_n)$ be a minimizing sequence for (P1). Since $(u_n)$ is bounded in $X$, we may assume that $u_n \to u$. Hence we obtain

$$\int_\Omega |\nabla u|^2 \, dx \leq \liminf_{n \to \infty} \int_\Omega |\nabla u_n|^2 \, dx = \inf \{ \text{the set of } P_1 \}. $$

Since $\int_\Omega V^{-u_n^2} \, dx = \int_\Omega V^{+u_n^2} \, dx - 1$, the preceding lemma and Fatou’s lemma imply that $\int_\Omega V^{-u_n^2} \, dx \leq \int_\Omega V^{+u_n^2} \, dx - 1$, i.e., $\int_\Omega V u_n^2 \, dx \geq 1$. It is then clear that $u$ is a solution of (P1). Moreover, since also $|u|$ is a solution, we may assume $u \geq 0$.

Since for every $v \in D(\Omega)$,

$$\frac{d}{dx} \bigg|_{x=0} \frac{\psi(u + \varepsilon v)}{\psi(u + \varepsilon v)} = 0,$$

$u$ is an eigenfunction of (3) corresponding to the eigenvalue $\int_\Omega |\nabla u|^2 \, dx$.

In order to find the other positive eigenvalues of (3) we solve the problems

$$(P_n) \quad \text{minimize } \int_\Omega |\nabla u|^2 \, dx, \quad u \in D^{2,2}_0(\Omega),$$

$$\int_\Omega \nabla u \cdot \nabla e_j \, dx = \ldots = \int_\Omega \nabla u \cdot \nabla e_{n-1} \, dx = 0, \quad \int_\Omega V u^2 \, dx = 1,$$

where $e_j$ is the solution of (P1), $1 \leq j \leq n - 1$.

THEOREM 2.3. Under assumption (H), for every $n \geq 2$, problem (Pn) has a solution $e_n$. Moreover, $e_n$ is an eigenfunction of (3) corresponding to the eigenvalue $\lambda_n := \int_\Omega |\nabla e_n|^2 \, dx$, and $\lambda_n \to \infty$ as $n \to \infty$.

Proof. The existence of $e_n$ is proved as in Theorem 2.2. An elementary argument (see [19, Lemma 4.4]) shows that $e_n$ is an eigenfunction of (3) corresponding to the eigenvalue $\lambda_n := \int_\Omega |\nabla e_n|^2 \, dx$.

The sequence $f_n := e_n/\sqrt{\lambda_n}$ is orthonormal in $X$ so that $f_n \to 0$.

Since $\lambda_{n-1} = \int_\Omega |\nabla f_n|^2 \, dx = \int_\Omega V f_n^2 \, dx$, Lemma 2.1 implies that $0 \leq \lim_{n \to \infty} \lambda_{n-1} = \lim_{n \to \infty} \int_\Omega V f_n^2 \, dx \leq 0$.

REMARKS 2.4. (a) If $-V$ satisfies (H), then problem (3) has infinitely many negative eigenvalues $0 > \lambda_1 \geq \lambda_2 \geq \ldots$. Moreover, $\lambda_{n,m} \to -\infty$ as $m \to \infty$ and the eigenfunction corresponding to $\lambda_{1,n}$ is nonnegative.

(b) Theorems 2.2 and 2.3 depend only on the weak continuity of $\int_\Omega V^{+u^2} \, dx$ and on the weak lower semicontinuity of $\int_\Omega V^{-u^2} \, dx$. It is easy to formulate an abstract version of these results.

(c) Necessary and sufficient conditions for the weak continuity of $\int_\Omega V^{+u^2} \, dx$, in terms of capacities, may be found in [13, Section 2.4.2]. We would like to thank A. Laptev for bringing the reference [13] to our attention.

In order to prove the simplicity of $\lambda_1$ which we mentioned in the introduction, we need the following additional assumption:

(H1) There exists $p > N/2$ and a closed subset $S$ of measure 0 in $\mathbb{R}^N$ such that $\Omega \setminus S$ is connected and $V \in L^p_{\text{loc}}(\Omega \setminus S)$.

THEOREM 2.5. Under assumptions (H) and (H1), $\lambda_1$ is a simple eigenvalue of (3).

Proof. Let $u$ be an eigenfunction corresponding to $\lambda_1$ such that $\int_\Omega V u^2 \, dx = 1$. Since $|u|$ is a solution of (P1), $|u|$ is also an eigenfunction. Hence $u^+$ and $u^-$ are eigenfunctions.

By regularity theory (see [12, Theorem 11.7]), any eigenfunction belongs to $W^{2,q}_{\text{loc}}(\Omega \setminus S) \cap C^{0,\alpha}_{\text{loc}}(\Omega \setminus S)$, $q > N/(N-2)$, $0 < \alpha < 2 - N/p$. The unique continuation theorem of Jerison and Kenig [11] implies that $u = u^+$ or $u = -u^-$. It follows immediately that $\lambda_1$ is simple.

3. Examples and counterexamples. We assume in this section that $\Omega = \mathbb{R}^N$. The following result, due to Tertikas, is contained in Proposition 4.5 of [18]:

THEOREM 3.1. Let $V \in L^p_{\text{loc}}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N \setminus \{0\})$. If $u$ is an eigenfunction of (3), then

$$\int_{\mathbb{R}^N} (2V(x) + x \cdot \nabla V(x)) u^2(x) \, dx = 0.$$

REMARK 3.2. Theorem 3.1 has a simple formal explanation. An eigenvalue of (3) is a stationary point of $\varphi/\psi$. If $T(q) u(x) := u(x/q)$, then

$$\frac{d}{dq} \bigg|_{q=1} \frac{\varphi(T(q) u)}{\psi(T(q) u)} = 0$$

implies (9) (see [20, Appendix B]).

EXAMPLE 3.3. As observed by Tertikas, if $W_1(x) := 1/(1 + |x|^2)$, then for all $x \in \mathbb{R}^N, 2W_1(x) + x \cdot \nabla W_1(x) > 0$, and if $W_2(x) := 1/((|x|^2 + 1) |x|^2)$, then for all $x \in \mathbb{R}^N \setminus \{0\}, 2W_2(x) + x \cdot \nabla W_2(x) < 0$. By Theorem 3.1, (3) has no eigenvalue if $V = W_1$ or $V = W_2$.

Now observe that $W_1 \in L^q(\mathbb{R}^N)$ for all $q > N/2$, $W_2 \in L^q(\mathbb{R}^N)$ for all $q \in (N/4, N/2)$ but neither $W_1$ nor $W_2$ is in $L^{N/2}(\mathbb{R}^N)$.

EXAMPLE 3.4. Define

$$W_3(x) := \frac{1}{(1 + |x|^2) \log(2 + |x|^2)^{2/N}},$$

$$W_4(x) := \frac{1}{|x|^2(1 + |x|^2) \log(2 + 1/|x|^2)^{2/N}}.$$
By Theorem 2.3, (3) has infinitely many positive eigenvalues if \( V = W_3 \) or \( W_4 \) although \( W_3, W_4 \) are not in \( L^{N/2}(\mathbb{R}^N) \) \((W_3, W_4 \text{ are in the same } L^p\text{-spaces as respectively } W_1 \text{ and } W_2)\).

**Theorem 3.5.** If \( |x|^p V(x) \to \infty \text{ as } |x| \to \infty \) or \( |x-y|^p V(x) \to \infty \) as \( x \to y \) for some \( y \), then the infimum in \((P_1)\) is 0 and (is not achieved).

**Proof.** We only consider the case of \( |x|^p V(x) \to \infty \) as \( x \to 0 \), the other cases being similar. Let \( u \in \mathcal{D}(\mathbb{R}^N) \) and set \( u_r(x) := u(x/r) \). Then
\[
\lim_{r \to 0} \frac{\int_{\mathbb{R}^N} |\nabla u_r(x)|^p dx}{\int_{\mathbb{R}^N} V(x) u_r(x)^2 dx} = \lim_{r \to 0} \frac{\int_{\mathbb{R}^N} |\nabla u(x)|^p dx}{\int_{\mathbb{R}^N} r^N |V(x) u(x)|^p dx}.
\]
Since \( u \) has compact support and \( u^2/|x|^p \in L^1(\mathbb{R}^N) \), it follows easily that the right-hand side above tends to 0 as \( r \to 0 \).

In the case of \( |x| \to \infty \) the function \( u \in \mathcal{D}(\mathbb{R}^N) \) should be chosen so that 0 \( \not\in \text{ supp } u \).

4. The \( p \)-Laplacian. Our purpose here is to extend the results of Section 2 to the nonlinear eigenvalue problem
\[
(10) \quad -\Delta_p u = \lambda V(x)|u|^{p-2} u, \quad u \in D_0^{1,p}(\Omega),
\]
where \( \Delta_p u := \text{div}(|\nabla u|^{p-2} \nabla u) \) is the \( p \)-Laplacian with \( 1 < p < N \) and \( \Omega \) is an open, in general unbounded, subset of \( \mathbb{R}^N \). The assumption \((H_1)\) of Section 2 now reads:

\[(H_p) \quad V \in L^{N/p}_0(\Omega), \quad V^+ = V_1 + V_2 \neq 0, \quad V_1 \in L^{N/p}(\Omega),
\]
\[
\lim_{|x| \to \infty} |x-y|^p V_2(x) = 0 \quad \text{for every } y \in \bar{\Omega}, \quad \lim_{|x| \to \infty} |x|^p V_2(x) = 0.
\]

Consider the problem
\[(Q_1) \quad \text{minimize } \int_{\Omega} |\nabla u|^p dx, \quad u \in D_0^{1,p}(\Omega), \quad \int_{\Omega} V|u|^p dx = 1.
\]

It is easy to show that \( \int_{\Omega} V^+ |u|^p dx \) is weakly continuous in \( D_0^{1,p}(\Omega) \). The proof parallels that of Lemma 2.1 except that now we use the Hardy inequality
\[
\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx \leq \left( \frac{p}{N-p} \right)^p \int_{\mathbb{R}^N} |\nabla u|^p dx, \quad u \in D_0^{1,p}(\mathbb{R}^N)
\]
(see [9] for a simple proof).

**Theorem 4.1.** Under assumption \((H_p)\), problem \((Q_1)\) has a solution \( e_1 \geq 0 \). Moreover, \( e_1 \) is an eigenfunction of \((10)\) corresponding to the eigenvalue \( \lambda_1 := \int_{\Omega} |\nabla e_1|^p dx \).

**Proof.** Repeat the argument of Theorem 2.2.

Since equation \((10)\) is nonlinear (unless \( p = 2 \)), it is not possible to obtain higher eigenvalues by the method of Section 2. Instead we shall use critical point theory. Let
\[
\varphi(u) := \int_{\Omega} |\nabla u|^p dx \quad \text{and} \quad \psi(u) := \int_{\Omega} V|u|^p dx.
\]
Since the set \( \{ u \in D_0^{1,p}(\Omega) : \psi(u) = 1 \} \) is not a manifold unless further assumptions are made on \( V^- \), we introduce a new space \( X := \{ u \in D_0^{1,p}(\Omega) : ||u||_X < \infty \} \), where
\[
||u||_X^p := \int_{\Omega} |\nabla u|^p dx + \int_{\Omega} V^- |u|^p dx.
\]

Then \( M := \{ u \in X : \varphi(u) = 1 \} \) is a \( C^1 \)-manifold, critical points of \( \varphi|_M \) are eigenfunctions and the corresponding critical values are eigenvalues of \((10)\).

Let \( \psi := \int_{\Omega} V^+ |u|^p dx \).

**Lemma 4.2.** If \( V \) satisfies \((H_p)\), then:

(i) The Fréchet derivative of \( \varphi_+ \) is completely continuous as a mapping from \( X \) to \( X^* \).

(ii) \( \psi_+ \leq c \psi(u) \) for some \( c > 0 \) and all \( u \in X \).

**Proof.** (i) Let \( u_n \rightharpoonup u \). By the Hölder and Sobolev inequalities,
\[
\int_{\Omega} V_1(|u_n|^{p-2} u_n - |u|^{p-2} u) v dx
\]
\[
\leq \int_{\Omega} V_1 |u_n|^{p-2} u_n - |u|^{p-2} u| |v|^{(p-1)/p} dx \left( \int_{\Omega} V_1 |v|^p dx \right)^{1/p}
\]
\[
\leq d_1 ||v||_X \left( \int_{\Omega} V_1 |u_n|^{p-2} u_n - |u|^{p-2} u| |v|^{(p-1)/p} dx \right)^{(p-1)/p}
\]
It is easy to see that \(|u_n|^{p-2} u_n - |u|^{p-2} u| \rightharpoonup 0 \) in \( L^{N/(N-p)}(\Omega) \) (indeed, otherwise there would exist a subsequence going weakly to some \( v \neq 0 \) and a.e. to 0, a contradiction to [19, Theorem 10.36]). Since \( V_1 \in L^{N/p}(\Omega) \), the right-hand side above tends to 0 uniformly for \( ||v||_X \leq 1 \). This shows the complete continuity of the \( V_1 \)-part.

Using the notation of Lemma 2.1 and the Hölder, Hardy and Sobolev inequalities, we see that
\[
\int_{\Omega} V_2(|u_n|^{p-2} u_n - |u|^{p-2} u) v dx \leq d_2 \varepsilon ||v||_X (||u_n||_X^{-1} + ||u||_X^{-1}) \leq d_2 \varepsilon ||v||_X.
\]

Similarly, the above integral taken over \( A \) is \( \leq d_4 \varepsilon ||v||_X \) (the \( d_i \)'s are independent of \( \epsilon \)). Since \( \Omega_2 \setminus A \) is bounded and \( V_2 \in L^{\infty}(\Omega_2 \setminus A) \), it follows from the
continuity of the superposition operator \([14, 20]\) that \(|u_n|^{p-2}u_n \rightharpoonup |u|^{p-2}u\) in \(L^{p/(p-1)}(\Omega_2 \setminus A)\)
and
\[\int_{\Omega_2 \setminus A} V_2(|u_n|^{p-2}u_n - |u|^{p-2}u)\,dx \to 0.\]

(ii) By the Hölder and Sobolev inequalities,
\[\int_{\Omega_2} V_2|u|^p\,dx \leq d_2 \int_{\Omega} |\nabla u|^p\,dx.\]

Fixing some \(\varepsilon > 0\) and using the Hölder, Hardy and Sobolev inequalities again, it is easy to see that
\[\int_{\Omega_2} V_2|u|^p\,dx \leq d_2 \\varepsilon \int_{\Omega} |\nabla u|^p\,dx,\]
and similar inequalities hold on \(A\) and \(\Omega_2 \setminus A\). The conclusion now follows by recalling the definitions of \(\psi^+\) and \(\varphi^+\).

Let \(\mu > 0\) and let \(A_\mu : X \to X^*\) be the operator given by
\[\langle A_\mu(u), v \rangle = \int_{\Omega} |\nabla u|^{p-2}(\nabla u \cdot \nabla v)\,dx + \mu \int_{\Omega} |u|^{p-2}uv\,dx\]
\((\cdot, \cdot)\) denotes the duality pairing.

**Lemma 4.3.** If \(u_n \rightharpoonup u\) and \(\langle A_\mu(u_n), u_n - u \rangle \to 0\), then \(u_n \rightharpoonup u\) in \(X\).

**Proof.** Our argument is borrowed from [6] where it appears in the proof of Lemma 3.3. Clearly, \(A_\mu(u_n) - A_\mu(u), u_n - u \to 0\). By the Hölder inequality,
\[\int_{\Omega} V^{-}(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\]
\[= \int_{\Omega} V^{-}(|u_n|^p + |u|^p - |u_n|^{p-2}u_n + |u|^{p-2}u_n)\]
\[\geq \int_{\Omega} V^{-}(|u_n|^p + |u|^p) - \left( \int_{\Omega} V^{-}|u_n|^p \right)^{p/(p-1)} \left( \int_{\Omega} V^{-}|u|^p \right)^{1/p}\]
\[= [\left( \int_{\Omega} V^{-}|u_n|^p \right)^{(p-1)/p} - \left( \int_{\Omega} V^{-}|u|^p \right)^{(p-1)/p}]\]
\[\times \left[ \left( \int_{\Omega} V^{-}|u_n|^p \right)^{1/p} - \left( \int_{\Omega} V^{-}|u|^p \right)^{1/p} \right] \geq 0.\]

Since the left-hand side above tends to 0, \(\int_{\Omega} V^{-}|u_n|^p\,dx \to \int_{\Omega} V^{-}|u|^p\,dx\). Similarly, \(\int_{\Omega} |\nabla u_n|^p\,dx \to \int_{\Omega} |\nabla u|^p\,dx\), hence \(\|u_n\|_X \to \|u\|_X\) and therefore \(u_n \rightharpoonup u\) in \(X\).

Let \(A\) be a closed subset of \(M\) such that \(A = -A\). Recall [14, 16] that the *Kra"osnessel'skii genus* \(\gamma(A)\) is by definition the smallest integer \(k\) for which there exists an odd mapping \(A \to \mathbb{R}^k \setminus \{0\}\). If there is no such mapping for any \(k\), then \(\gamma(A) := +\infty\). Moreover, \(\gamma(\emptyset) := 0\). Let
\[\lambda_n := \inf_{\gamma(A) \geq n} \sup_{u \in A} \varphi(u), \quad n = 1, 2, \ldots\]

Since \(\{x \in \mathbb{R}^N : V(x) > 0\}\) has positive measure, for each \(n\) there is a set \(A \subset M\) which is homeomorphic to the unit sphere \(S^{n-1} \subset \mathbb{R}^n\) by an odd homeomorphism. Since \(\gamma(S^{n-1}) = n\), there exist sets of arbitrarily large genus and all \(\lambda_n\) are well defined. Moreover, \(\lambda_1 = \inf_{u \in M} \varphi(u)\). Hence \(\lambda_3\) coincides with the first eigenvalue obtained in Theorem 4.1 and \(\lambda_n \geq \lambda_1 > 0\) for all \(n\). If \(M\) is of class \(C^2\) (which is the case for \(p \geq 2\)) and \(\varphi|_M\) satisfies the Palais–Smale condition, then classical critical point theory [16, Section II.5] implies that the \(\lambda_n\)'s are critical values. If \(1 < p < 2\), then \(M\) is only of class \(C^1\); however, the same conclusion remains valid as follows from the results contained in [5] and [17].

As \(\lambda_n\) is a critical value of \(\varphi|_M\), there exists a critical point \(e_n\) with \(\varphi(e_n) = \lambda_n\). Hence \(\varphi^+(e_n) = \mu \varphi^+(e_n)\), where \(\mu\) is a Lagrange multiplier, and (2) is satisfied with \(u = e_n\) and \(\lambda = \mu\). Since \(\varphi^+(e_n) = \varphi(e_n) = \mu \varphi(e_n) = \mu(e_n)\), we have \(\mu = \varphi(e_n) = \lambda_n\), so \(\lambda_n\) is an eigenvalue and \(e_n\) is a corresponding eigenfunction.

**Theorem 4.4.** Under assumption (\(H_p\)), \(\varphi|_M\) has a sequence of critical points \((e_n)\) with corresponding critical values \(\lambda_n = \int_{\Omega} V e_n^p\,dx\). Moreover, each \(e_n\) is an eigenfunction of (10), \(\lambda_n\) is an associated eigenvalue, and \(\lambda_n \to \infty\) as \(n \to \infty\).

**Proof.** Let \((u_k)\) be a Palais–Smale sequence. Then there exist \(\mu_k \in \mathbb{R}\) such that
\[\varphi'(u_k) - \mu_k \varphi(u_k) \to 0\]
(cf. [20, Proposition 5.12]). Since \(\varphi(u_k)\) is bounded, so is \(\psi^+(u_k)\) according to Lemma 4.2(ii), and therefore also
\[\psi^-(u_k) = \psi^+(u_k) - 1\]
is bounded. Hence \(\|u_k\|_X^p \equiv \varphi(u_k) + \psi^-(u_k)\) is bounded and we may assume passing to a subsequence that \(u_k \rightharpoonup u\). Since \(\psi^+(\cdot)\) is completely continuous, \(\psi^+(u_k) \to \psi^+(u)\) and it follows from (12) that \(u \neq 0\). By (11),
\[p(\varphi(u_k) - \mu_k) = \langle \varphi'(u_k), u_k \rangle - \mu_k \varphi(u_k) \to 0.\]
Therefore $(\mu_k)$ is bounded and we may assume $\mu_k \to \mu$. Moreover, taking the limit above we obtain $0 < \varphi(u) \leq \mu$, so $\mu > 0$. We may rewrite (11) as

$$A_{\mu_k}(u_k) - \lambda_k(\psi^+)'(u_k) \to 0.$$  

Since $A_{\mu_k}(u_k) - A_{\mu_k}(u_k) \to 0$ as is easily seen from the definition of $A_{\mu}$ and since $(\psi^+)'(u_k) \to (\psi^+)'(u)$, it follows that $A_{\mu_k}(u_k)$ is strongly convergent. So $(A_{\mu_k}(u_k), u_k - u)$ to 0 and $u_k \to u$ according to Lemma 4.3.

We have shown that $\varphi|_{\mathcal{M}}$ satisfies the Palais–Smale condition. It follows from our earlier discussion that each $\lambda_n$ is a critical value of $\varphi|_{\mathcal{M}}$ and an eigenvalue of the problem (10). Moreover, if $\lambda_n = \ldots = \lambda_{m-m}$ for some $m \geq 1$, then the set of critical points corresponding to $\lambda_n$ has genus $\geq m+1$ [16, Lemma II.5.6] and is therefore infinite. Hence, the eigenfunctions $\varphi_n$ may be chosen so that $\varphi_n \neq \varphi_j$ whenever $n \neq j$. Finally, a well known argument [14, Proposition 9.3.3] shows that the critical values $\lambda_n$ must necessarily tend to infinity.

**Remark.** 4.5. It was shown in [7] that if $\Omega = \mathbb{R}^N$ and $V \in L^{N/p}((\mathbb{R}^N) \cap L^{(N+2)/p}(\mathbb{R}^N))$ for some $\delta > 0$, then the principal eigenvalue $\lambda_1$ of (10) is simple.

In [15] Rosenblum and Solomyak studied the existence of the principal eigenvalue of (1) in $\mathbb{R}^N$ under weak conditions on $V$. While our hypotheses (on $V$) were formulated in terms of pointwise limits, those in [15] involved capacities and conditions on integrals. We would like to thank the referee for pointing out this reference.

**References**


