Averages of uniformly continuous retractions

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Abstract. Let $X$ be an infinite-dimensional complex normed space, and let $B$ and $S$ be its closed unit ball and unit sphere, respectively. We prove that the identity map on $B$ can be expressed as an average of three uniformly continuous retractions of $B$ onto $S$. Moreover, for every $0 \leq r < 1$, the three retractions are Lipschitz on $rB$. We also show that a stronger version where the retractions are required to be Lipschitz does not hold.

1. Introduction. Let $Y$ be a strictly convex infinite-dimensional normed space, and let $T$ be a topological space. Let $C = C(T, Y)$ be the normed space of continuous bounded functions from $T$ into $Y$, with the usual uniform norm. Let $B_Y$ and $B_C$ be the closed unit balls of $Y$ and $C$, respectively, and let $S_Y$ be the unit sphere of $Y$. Note that $f$ is an extreme point of $B_C$ if and only if $f$ maps into $S_Y$. Finally, for every metric space $M$ denote the identity map on $M$ by $I_M$.

Peck [8] proved that if $T$ is a compact Hausdorff space, then $B_C$ is the convex hull of its extreme points. In [2] it was proved that every $f \in B_C$ can be expressed as an average of four extreme points of $B_C$, a fact which implies that $I_{B_C}$ can be expressed as an average of four retractions of $B_Y$ onto $S_Y$. Cantwell [3] conjectured that the number of retractions can be reduced. Indeed, the number of retractions was reduced in [6] to three, the lowest possible number, and in [4] it was proved that this result holds in every infinite-dimensional complex normed space.

In this paper we focus on two subspaces of $C(M, X)$, where $M$ is a metric space and $X$ is an infinite-dimensional complex normed space. Namely, we consider the subspace $U = U(M, X)$ of uniformly continuous functions, and its subspace $L = L(M, X)$ of Lipschitz functions.

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In Section 2 we prove the main theorem (2.1) of this paper: Let $X$ be an infinite-dimensional complex normed space. Then $I_B$ can be expressed as an average of three uniformly continuous retractions of $B$ onto $S$. Moreover, for every $0 \leq r < 1$, the three retractions are Lipschitz on $rB = \{rx : x \in B\}$. We do not know if this is true for every infinite-dimensional real normed space.

In Section 3 we show (Lemma 3.4) that if $X$ is a Hilbert space, then $I_B$ cannot be expressed as an average of any finite number of retractions of $B$ onto $S$ which are Lipschitz (or even Hölder with exponent $p > 1/2$). We do not know if this is true for every normed space.

We also show (Corollary 3.3) that if $X$ is strictly convex then the convex hull of the extreme points of $B$ (respectively, $B_{L}$) is equal to $B$ (respectively, contains $B_{L}$).

2. Main theorem. Let $X$ be an infinite-dimensional complex normed space, and let $B$ and $S$ be its closed unit ball and unit sphere, respectively.

In this section we prove the following theorem:

**Theorem 2.1.** Let $a_1, a_2, a_3 \in (0, 1/2)$ be such that $\sum_{i=1}^{3} a_i = 1$. Then there are three uniformly continuous retractions $f_1, f_2, f_3 : B \to S$ such that $I_B \equiv \sum_{i=1}^{3} a_i f_i$. Moreover, the restrictions $f_i|_{rB}$ are Lipschitz for every $0 \leq r < 1$.

We use the following theorem which was first proved by Nowak [7] for some Banach spaces, and later by Benyamini and Sternfeld [1] for arbitrary normed spaces (see also [5]).

**Theorem 2.2.** For every infinite-dimensional normed space $Z$, there exists a Lipschitz retraction from $B$ onto $S$.

We also need the following three lemmas. The first one (and its proof) also holds for real spaces.

**Lemma 2.3.** Let $\alpha \in [0, 1/2)$. Then there are two Lipschitz functions $g : B \to S$ and $h : B \to B$ such that $g|_{S} \equiv h|_{S} \equiv I_S$ and $I_B \equiv \alpha g + (1 - \alpha) h$. Moreover, $h|_{rB} \subseteq rB$ for every $1/2(1 - \alpha) \leq r \leq 1$.

**Proof.** Let $\delta := (1 - 2\alpha)/2$ and note that $0 < \delta \leq 1/2$. By Theorem 2.2 there is a Lipschitz retraction $f : \delta B \to \delta S$.

Define the two required functions $g : B \to S$ and $h : B \to B$ by

$$g(x) = \begin{cases} \frac{x}{\|x\|} & \text{if } \|x\| \geq \delta, \\ \frac{f(x)}{\delta} & \text{if } \|x\| \leq \delta, \end{cases} \quad h(x) = \frac{x - \alpha g(x)}{1 - \alpha}.$$

Clearly, $g$ and $h$ are Lipschitz, $I_B \equiv \alpha g + (1 - \alpha) h$, and $g|_{S} \equiv h|_{S} \equiv I_S$. To see that $h$ maps into $B$ and the second property in the theorem holds, let $x \in B$ and let $1/(2(1 - \alpha)) \leq r \leq 1$. Note that $2\alpha \leq r$. Hence, for every $0 \leq t \leq r$.

$$|t - \alpha| \leq r - \alpha$$

Consider three cases.

1. If $0 \leq \|x\| \leq \delta$, then

$$\|h(x)\| \leq \frac{\|x\| + \alpha \|g(x)\|}{1 - \alpha} \leq \frac{\delta + \alpha}{1 - \alpha} \leq \frac{1}{2(1 - \alpha)} \leq \|x\|.$$

2. If $\delta \leq \|x\| \leq r$, then

$$\|h(x)\| = \frac{\|x - \alpha g(x)\|}{1 - \alpha} = \frac{\|x\| - \alpha}{1 - \alpha} \leq \frac{r - \alpha}{1 - \alpha} \leq r \quad \text{by (x)}.$$

3. If $r \leq \|x\| \leq 1$, then

$$\|h(x)\| = \frac{\|x\| - \alpha}{1 - \alpha} \leq \frac{\|x\| - \alpha}{1 - \alpha} \leq 1.$$}

**Notation.** For every $0 \leq r_1 \leq r_2$, define $R(r_1, r_2) = \{x \in X : r_1 \leq \|x\| \leq r_2\}$.

The next lemma is the only place in the proof of the theorem where we use the fact that $X$ is a complex space.

**Lemma 2.4.** Let $\alpha \in (0, 1)$ and let $\beta \in [1/2 - \alpha, 1/2, 1/2)$. Then there are two uniformly continuous retractions $\varphi_1, \varphi_2 : R(\alpha, 1) \to S$ such that $I_{R(\alpha, 1)} \equiv \beta \varphi_1 + (1 - \beta) \varphi_2$. Moreover, the restrictions $\varphi_i|_{R(\alpha, r)}$ are Lipschitz for every $0 \leq r < 1$.

**Proof.** Define a Lipschitz function $G : [a, 1] \to [-1, 1]$ by

$$G(t) = \frac{t^2 + 2\beta t - 1}{2\beta}.$$

Define two functions $z_1, z_2 : [\alpha, 1) \to S$ by

$$z_1(t) = G(t) + i\sqrt{1 - (G(t))^2} \quad \text{and} \quad z_2(t) = \frac{t - \beta z_1(t)}{1 - \beta}.$$

Then $z_1(1) = z_2(1) = 1, t = \beta z_1(t) + (1 - \beta) z_2(t)$ for every $t \in [\alpha, 1]$, $z_1$ and $z_2$ are uniformly continuous, and $z_1|_{R(\alpha, r)}$ and $z_2|_{R(\alpha, r)}$ are Lipschitz for every $0 \leq r < 1$.

Define the required retractions $\varphi_1, \varphi_2 : R(\alpha, 1) \to S$ by

$$\varphi_1(x) = z_1(\|x\|) \frac{x}{\|x\|} \quad \text{and} \quad \varphi_2(x) = z_2(\|x\|) \frac{x}{\|x\|}.$$
that $g|_S \equiv h|_S \equiv I_S$. Then there are two uniformly continuous retractions $f_1, f_2 : B \to S$ such that:

1. $ag + dh \equiv cf_1 + bf_2$ (as functions from $B$ into $X$).
2. For every $0 \leq r < 1$, if $g|_{rB}$ and $h|_{rB}$ are Lipschitz and $h[rB] \subseteq rB$, then $f_1|_{rB}$ are Lipschitz.

**Proof.** Let

$$
\alpha := \frac{a-d}{a+d} \quad \text{and} \quad \beta := \frac{c}{a+d}.
$$

Then $\alpha \in (0,1)$ and

$$
\frac{1}{2} \geq \frac{c}{b+c} = \frac{c}{a+d} = \beta \geq \frac{d}{a+d} = \frac{1}{2} - \alpha.
$$

Let $\varphi_1$ and $\varphi_2$ be the retractions from Lemma 2.4 with respect to $\alpha$ and $\beta$.

Define a uniformly continuous function $f : B \to X$ by

$$
f(x) = \frac{ag(x) + dh(x)}{a+d}.
$$

Then $f$ maps into $R(\alpha,1)$ because for every $x \in B$,

$$
1 \geq \|f(x)\| \geq \frac{\|a\|g(x)| - d\|h(x)|}{a+d} = \frac{|a-d\|h(x)|}{a+d} = \frac{\alpha - d\|h(x)|}{a+d} = \alpha.
$$

Also, $f|_S \equiv I_S$. Hence, we can define the two required uniformly continuous retractions by $f_i \equiv \varphi_i \circ f : B \to S$.

1 holds because for every $x \in B$, by Lemma 2.4,

$$
\frac{ag(x) + dh(x)}{a+d} = f(x) = \beta \varphi_1(f(x)) + (1-\beta)\varphi_2(f(x)) = \frac{c}{a+d}\varphi_1(f(x)) + \frac{b}{a+d}\varphi_2(f(x)) = \frac{c\varphi_1(f(x)) + b\varphi_2(f(x))}{a+d}.
$$

2 holds because if we let $0 \leq r < 1$ and let $g$ and $h$ be as in (3), then $f_1|_{rB}$ is Lipschitz. Let

$$
t := \frac{a+dr}{a+d}.
$$

Then $\alpha \leq t < 1$. By Lemma 2.4, $\varphi_1|_{R(\alpha,t)}$ are Lipschitz.

Since $h[rB] \subseteq rB$, for every $x \in rB$ we have

$$
\alpha \leq \|f(x)\| \leq \frac{\|a\|g(x)| + d\|h(x)|}{a+d} \leq \frac{\|a\|g(x)| + dr}{a+d} = t.
$$

Hence, $f_1|_{rB}$ maps into $R(\alpha,t)$. Therefore $f_1|_{rB} \equiv (\varphi_1 \circ f)|_{rB} \equiv \varphi_1|_{R(\alpha,t)} \circ f|_{rB}$. Thus, $f_1|_{rB}$ are Lipschitz functions as compositions of two Lipschitz functions.

**Proof of Theorem 2.1.** Let $\alpha_1, \alpha_2, \alpha_3 \in (0,1/2)$ be such that $\sum_{i=1}^{3} \alpha_i = 1$. Assume that $\alpha_0 \leq \alpha_2 < \alpha_1$. Choose $\alpha_0$ such that $\alpha_1 < \alpha_0 < 1/2$. Let $g : B \to S$ and $h : B \to B$ be the Lipschitz functions from Lemma 2.3 with respect to $\alpha_0$. Then

$$
I_B \equiv \alpha_0g + (1-\alpha_0)h \equiv \alpha_0g + \alpha_2h + (1-\alpha_2-\alpha_0)h.
$$

Applying Lemma 2.5 with respect to $0 \leq \alpha_0 \leq \alpha_2 < \alpha_0 \leq \alpha_3, g,$ and $h,$ we obtain two uniformly continuous retractions $f_2, f_3 : B \to S$ such that

$$
\alpha_0g + \alpha_3h \equiv \alpha_3f_3 + \alpha_1f_1.
$$

Applying Lemma 2.5 again but with respect to $0 < \alpha_1 + \alpha_3 - \alpha_0 \leq \alpha_3 \leq \alpha_1 < \alpha_0, h,$ and $f_0,$ we obtain two uniformly continuous retractions $f_3, f_1 : B \to S$ such that

$$
\alpha_0f_0 + (\alpha_1 + \alpha_3 - \alpha_0)h \equiv \alpha_3f_3 + \alpha_1f_1.
$$

Combining (1), (2), and (3), we get $I_B \equiv \sum_{i=1}^{3} \alpha_i f_i$.

To prove the second property in the theorem, let $0 \leq r < 1$. Choose $r_0$ such that $\max\{r,1/(2(1-\alpha_0))\} \leq r_0 < 1$. By Lemma 2.3, $h[r_0B] \subseteq r_0B$. Therefore, by Lemma 2.5, $f_2|_{r_0B}$ and $f_0|_{r_0B}$ are Lipschitz. Again by Lemma 2.5, $f_3|_{r_0B}$ and $f_1|_{r_0B}$ are Lipschitz. Thus, $f_1|_{rB}$ are Lipschitz since $r \leq r_0$.

**3. Observations.** First, we have two immediate corollaries of Theorem 2.1.

**Corollary 3.1.** There are three uniformly continuous retractions $f_1, f_2, f_3 : B \to S$ such that $I_B \equiv f_1 + f_2 + f_3$.

**Corollary 3.2.** Let $n \geq 3$ and let $0 < \alpha_1 \leq \ldots \leq \alpha_n < 1/2$ be such that $\sum_{i=1}^{n} \alpha_i = 1$. Then there are uniformly continuous retractions $f_1, \ldots, f_n : B \to S$ such that $I_B \equiv \sum_{i=1}^{n} \alpha_i f_i$.

**Remarks.** 1. Each retraction $f_i : B \to S$ in Theorem 2.1 is a uniform limit of the Lipschitz functions $f^n_i : B \to S$ defined by $f^n_i(x) = f_i((1-1/n)x)$.

2. Corollary 3.2 fails if $\alpha_n > 1/2$ since otherwise it follows from $0_X = \sum_{i=1}^{n} \alpha_i f_i(0_X)$ that

$$
1 = \|f_n(0_X)\| = \left\|\sum_{i=1}^{n-1} \frac{\alpha_i}{\alpha_n} f_i(0_X)\right\| \leq \sum_{i=1}^{n-1} \frac{\alpha_i}{\alpha_n} \|f_i(0_X)\| = 1 - \frac{\alpha_n}{\alpha_n} < 1.
$$

This argument also shows that for strictly convex spaces the corollary fails if $\alpha_n = 1/2$.

Let $U$, $L$, $B_U$, and $B_L$ be as in the Introduction, and let $E_U$ and $E_L$ be the extreme points of $B_U$ and $B_L$, respectively.
Corollary 3.3. Assume that \( X \) is strictly convex. Let \( \alpha_1, \alpha_2, \alpha_3 \in (0, 1/2) \) be such that \( \sum_{i=1}^{3} \alpha_i = 1 \). Then there exists a uniformly continuous function \( F : B_H \to E_U \times E_U \times E_U \) such that for every \( g \in B_U \), \( \sum_{i=1}^{3} \alpha_i(F(g))_i = g \). Moreover, for every \( 0 \leq r < 1 \), \( F|_{rB_U} \) is a Lipschitz function into \( E_U \times E_U \times E_U \). Hence, \( B_U = \frac{1}{2}(E_U + E_U + E_U) \) and \( B_L \setminus S_L \subseteq \frac{1}{2}(E_U + E_U + E_U) \).

Proof. Let \( f_i \) be the retractions from Theorem 2.1. We leave it to the reader to check that \( F(g) := (f_1 \circ g, f_2 \circ g, f_3 \circ g) \) is as required.\[\blacksquare\]

Lemma 3.4. Let \( H \) be an infinite-dimensional real Hilbert space. Let \( n \geq 3 \) and let \( \alpha_1, \ldots, \alpha_n \in (0, 1/2) \) be such that \( \sum_{i=1}^{n} \alpha_i = 1 \). Let \( f_1, \ldots, f_n \) be retractions of \( B_H \) into \( S_H \) such that \( I_{B_H} = \sum_{i=1}^{n} \alpha_i f_i \). Then there is \( 1 \leq j \leq n \) such that \( f_j \) is not a locally \( p \)-Hölder function for any \( p > 1/2 \); in particular, \( f_j \) is not a locally Lipschitz function.

Proof. Let \( x \in S_H \). Assume, for contradiction, that for every \( 1 \leq i \leq n \) there is a neighborhood \( N \) of \( x \) and there are \( 1/2 < p_i \leq 1 \) such that \( f_i|_N \) is a \( p_i \)-Hölder function with constant \( k_i \geq 0 \). Let \( p := \min\{p_1, \ldots, p_n\} \) and \( k := \max\{k_1, \ldots, k_n\} \).

Let \( 0 < t < 1 \) be large enough so that

\[
\frac{\sqrt{2(1-t)}}{(1-t)^p} > k \quad \text{and} \quad tx \in N.
\]

Then there is \( 1 \leq j \leq n \) such that \( \langle x, f_j(tx) \rangle \leq t \), since otherwise

\[
t = \langle x, tx \rangle = \left\langle x, \sum_{i=1}^{n} \alpha_i f_i(tx) \right\rangle = \sum_{i=1}^{n} \alpha_i \langle x, f_i(tx) \rangle > \sum_{i=1}^{n} \alpha_i t = t.
\]

Therefore,

\[
\|f_j(x) - f_j(tx)\| = \sqrt{\|f_j(x)\|^2 + \|f_j(tx)\|^2 - 2\langle f_j(x), f_j(tx) \rangle} = \sqrt{1 + 1 - 2\langle x, f_j(tx) \rangle} = \sqrt{2(1 - \langle x, f_j(tx) \rangle)} \geq \sqrt{2(1-t)} > k(1-t)^p = k\|x - tx\|^p \geq k_j\|x - tx\|^{p_j},
\]

contradicting the fact that \( f_j|_N \) is a \( p_j \)-Hölder function with constant \( k_j \).\[\blacksquare\]

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