(see [9]), one easily checks that $\phi$ is in $H^2(\partial B^2)$. Proposition 4.2 thus shows that $\text{Ext}_G(H^2(\partial B^2) \otimes N, H^2(\partial B^3))$ is zero. The proof is complete.

For a proper Hardy submodule $N$ of finite codimension in $H^2(\partial B^2)$, since $\text{Ext}_G(H^2(\partial B^2) \otimes N, N)$ is never zero, it follows that $N$ is never similar to $H^2(\partial B^2)$ by Proposition 4.4. We refer the reader to [3] for a further consideration of the rigidity of Hardy submodules over the ball algebra.

**Remark 4.5.** The main results of the present paper are also valid for strongly pseudoconvex domains with smooth boundary by [5, 10].

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**References**


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**Canonical functional extensions on von Neumann algebras**

by **Carlo Cecchini** (Udine)

**Abstract.** The topology and the structure of the set of the canonical extensions of positive, weakly continuous functionals from a von Neumann subalgebra $M_0$ to a von Neumann algebra $M$ are described.

1. **Introduction.** The aim of this paper is to give some results about the structure of the set $R(M, M_0)$ of the canonical extensions of positive, weakly continuous functionals (called canonical functional extensions, c.f.e.) from a von Neumann subalgebra $M_0$ to a von Neumann algebra $M$ (cf. [3–5]). After the necessary preliminaries in Section 2, Section 3 is devoted to the introduction of a set $V(\omega_0)$ of vectors in the Hilbert space of the standard representation for $M$, canonically associated with $R(M, M_0)$ in the framework of the modular theory of von Neumann algebras. Section 4 contains some topological results on $R(M, M_0)$ and $V(\omega_0)$. In Section 5 structural properties for different c.f.e. are compared and the possibility of defining Radon–Nikodym derivatives for c.f.e. in the spirit of Connes’ type cocycles for conditional expectations on von Neumann algebras (cf. [6]) is considered. In Section 6 we consider the special situation in which a c.f.e. is dominated (i.e. majorized by some multiple of another) in order to obtain some further comparison results, and we conclude by giving a sufficient condition for a given c.f.e. to dominate no other c.f.e.

2. **Preliminaries and notations.** Let $M$ be a von Neumann algebra acting on a Hilbert space $H$. We denote by $S(M)$ (resp. $S_f(M)$) the set of normal (resp. normal faithful) states on $M$. For $\xi \in H$ and $a$ in $M$ we set $\omega_\xi(a) = \langle \xi, a\xi \rangle$. Let $\phi$ and $\omega$ be in $(M^*)^+$. We say that $\phi$ is dominated by $\omega$ (and denote by $m(\omega)$ the set of such functionals) if it is majorized by some positive multiple of $\omega$. If $M_0$ is a von Neumann subalgebra of $M$ we set $\omega_0 = \omega|M_0$ for all $\omega$ in $M_*$. For $\omega$ in $S(M)$ we denote by $[e(\omega)]$ the $\omega$ conditional expectation from $M$ to $M_0$ introduced in [1] (see also [4], [5]).

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For $\omega$ in $(M_r)^+$, $[\sigma(\omega)]^*\otimes\omega$ denotes the modular automorphism group for $\omega$ on $M$, and if also $\phi$ is in $(M_r)^+$ and its support is contained in the support of $\omega$ then $(D\Phi : D\omega)|_{i/2}$ will denote the Connes cocycle for $\omega$ and $\phi$ in $M$. This cocycle admits $(|7|)$ a continuous extension $(D\Phi : D\omega)|_{i/2}$ as a linear, densely defined, in general not closable operator which commutes with $M'$ and whose domain coincides with $D(H,\omega) = \{\xi \in H : \omega \xi \in \tau M(\omega)\}$. We denote the linear operator which coincides with $(D\Phi : D\omega)|_{i/2}$ on $supp \omega$ and is zero on its orthogonal space by $\phi(\omega)$. If in particular $\phi$ itself is in $\tau M(\omega)$, then $(\phi/\omega)$ admits a bounded closure $[\phi/\omega]|_M$ in $M$.

Let now $\xi$ be a vector in $H$, $H(M,\xi)$ the closure of $\{a\xi : a \in M\}$ and $E(M,\xi)$ the orthogonal projection from $H$ to $H(M,\xi)$. A selfdual positive cone for $M$ in $H$ is any cone $V$ in $H$ containing a separating vector $\xi$ for $M$ in $H$ such that $V$ is the selfdual positive cone for $E(M,\xi)M$ in $H(M,\xi)$ in the sense of the modular theory of von Neumann algebras. We denote the associated isometric involution by $J(M,V)$ (or, briefly, by $J(V)$ when there is no danger of confusion); in the above situation we also write $E(\xi)$ for $E(M,\xi)$. Each $\phi$ in $(M_r)^+$ is implemented by the unique vector $\{\phi/\omega\}\xi$ in $V$.

Let $V_1, V_2$ be two selfdual positive cones for $M$ in $H$, and for all $\omega$ in $S(M)$ let $E_1(\omega) \subset V_i$ $(i = 1, 2)$. There is then a unique partial isometry $u(\xi_1, V_2 \xi_1)$ in $M'$ with initial projection $E(\xi_1)$ and final projection $E(\xi_2)$ such that $u(\xi_1, V_2 \xi_1)$ satisfies $\phi(\omega) = E(\xi_2)$ for all $\omega$ in $S(M)$.

We use the following standard fact from the modular theory of von Neumann algebras [2]:

2.1. Theorem. Let $\xi$ and $\eta$ be in $V$. Then

$$|\xi - \eta|^2 \leq |\xi - \eta||\xi - \eta|.$$

We now reformulate our key definition introduced in [3]-[5]:

2.2. Definition. Let $M$ be a von Neumann algebra with a von Neumann subalgebra $M_0$. A mapping $\gamma : \{(M_0)^+\} \rightarrow \{M^0\}$ is called a (V implemented) canonical functional extension (c.f.e.) if there is a faithful representation $\pi(M)$ of $M$ on a Hilbert space $H$ and a selfdual positive cone $V_0$ for $\pi(M_0)$ in $H$ such that $\gamma(\omega \circ \pi(M_0)) = \omega \circ \pi$ for all vectors $\xi$ in $V_0$.

We denote the set of all (c.f.e.) for the couple $(M, M_0)$ by $R(M, M_0)$.

If $\xi$ is a norm one projection from $M$ to $M_0$ then the mapping $\gamma_0 : \{(M_0)^+\} \rightarrow \{M^0\}$ is in $R(M, M_0)$. A V implemented canonical functional extension is of this form if $V_0$ is contained in some selfdual positive cone $V$ for $M$ in $H$ (cf. [3]), and will be called regular.

In the following we simplify our notation by assuming, since this is not restrictive for our purposes, $M$ to act standardly on the Hilbert space $H$. We also fix a reference selfdual positive cone $V$ for $M$ in $H$ together with a reference $\phi_0$ in $S_f(M_0)$, denote by $\xi(\phi)$ the unique vector in $V$ implementing $\phi$ in $(M_r)^+$ and shorten $J(V)$ to $J$. All the above recalled objects from the modular theory of von Neumann algebras will be endowed with a subscript "0" when referring to the von Neumann subalgebra $M_0$.

3. The set $V(\omega_0)$. Any $\phi$ in $(M_r)^+$ can be written (uniquely if its restriction $\phi_0$ to $M_0$ is faithful) as $\phi = q(\phi_0)$ with $q$ in $R(M, M_0)$ (cf. [4], [5]); conversely, for all $q$ in $R(M, M_0)$ and $\phi_0$ in $(M_0)^+$ there is a unique $\phi$ in $(M_r)^+$ satisfying $\phi = q(\phi_0)$. A natural object to be associated with the resulting parametrization of $(M_r)^+$ through $R(M, M_0)$ and $(M_0)^+$ in the framework of the modular theory of von Neumann algebras is the set $V(\omega_0)$ defined below.

3.1. Definition. For all $q$ in $R(M, M_0)$ we let $V(\omega_0, q)$ be the selfdual positive cone for $M_0$ in $H$ containing $\xi(q(\omega_0))$ (which therefore implements $\xi(q)$), and $V(\omega_0) = \bigcup_{q \in R(M, M_0)} V(\omega_0, q)$.

It is an immediate consequence of our construction that for any $\phi$ in $(M_r)^+$ there are representative vectors $\{\xi(q(\omega_0))\}(\phi)$ (or, briefly, $\xi(q(\phi))$ when no confusion arises) in $V(\omega_0)$. The uniqueness is guaranteed if $\phi_0$ is faithful. However, if $\phi = q(\phi_0)$ we denote by $\xi(q(\phi_0))$ the representative vector of $\phi$ in $V(\omega_0, q)$, which is unique. If $V_1$ and $V_2$ are selfdual positive cones for $M$ in $H$, we have $V_2(\omega_0) = u(V_1, V_2) V_1(\omega_0)$. We also remark (cf. Prop. 4.1 in the following) that $V(\omega_0)$ does depend on our choice of $\omega_0$.

3.2. Example (the abelian case). Let $M \cong L^\infty(\Omega, \Sigma, \mu)$ and $M_0 \cong L^\infty(\Omega, \Sigma_0, \mu_0)$ with $\Sigma_0$ a subsigma algebra of $\Sigma$, $\mu(\Omega) = 1$ and $\mu_0 = \mu|\Sigma_0$. Then $\xi(q)$ is the same as $V(\omega_0)$ and corresponds to the positive functions in $L^2(\Omega, \Sigma, \mu)$, there is a bijection between $R(M, M_0)$ and the set of positive functions in $L^1(\Omega, \Sigma, \mu)$ with conditional expectation $L^1(\Omega, \Sigma_0, \mu_0)$ the identity function, and $V(\omega_0, q)$ is the set of the positive functions in $L^2(\Omega, \Sigma, \mu)$ which can be written as the product of the square root of the function in $L^2(\Omega, \Sigma, \mu)$ which corresponds under the above mentioned bijection to $q$ and a positive function in $L^2(\Omega, \Sigma_0, \mu_0)$.

We can associate $(|4|, [5])$ with each couple $(\phi, \phi_0)$ in $R(M, M_0) \times (M_0)^+$ the unique partial isometry $u(\phi, \phi_0)$ in $M$ satisfying $u(\phi, \phi_0) = J(\omega(\phi), \omega_0) J(\omega(\phi_0), \omega_0)$, where $W$ is any selfdual positive cone for $M$ in $H$ containing $\xi(\phi)$ in $V(\omega_0, q)$.

The proof of the following proposition is straightforward.

3.3. Proposition. Let $\psi_0$ be in $S_f(M_0)$ and for all $\phi_0$ in $(M_0)^+$ denote by $V(\phi, \phi_0)$ the partial isometry obtained by taking in our construction $\phi_0$ instead of $\omega_0$ as our reference state in $S_f(M_0)$, then $u(\phi, \phi_0) = u(\psi, \psi_0)^*$, and, for all $\phi_0$ in $(M_0)^+$, $V(\phi, \phi_0) = V(\psi_0, \phi_0)$.
It is immediate to check that \( u(\varphi, \varphi_0) \) does not depend on the cone \( V \) we start with. For a fixed \( \varphi \) in \( R(M, M_0) \) the mapping \( \varphi_0 - u(\varphi, \varphi_0) \) (\( \varphi_0 \) in \( ((M_0)_*)^+ \)) tells us how far \( \varphi \) is from a regular canonical functional extension. In this case (and only then) \( u(\varphi, \varphi_0) \) coincides with \( \text{supp}\( g(\varphi_0) \) for all \( \varphi_0 \) in \( ((M_0)_*)^+ \) (cf. [3]). The partial isometry \( u(\varphi, \varphi_0) \) is the one occurring in \([3][5]\) in the comparison of the \( \omega \)-conditional expectations \( [e(\varphi(\varphi_0))] \) and \( [e(\varphi(\varphi_0))] \) from \( M \) to \( M_0 \).

4. Topological results

4.1. Definition. We say a sequence \( \varphi_n \) converges to \( \varphi \) in \( R(M, M_0) \) when \( \varphi_n(\varphi_0) \to \varphi(\varphi_0) \) (in norm) for all \( \varphi_0 \) in \( ((M_0)_*)^+ \).

We recall that by \([5]\) a sufficient condition for \( \varphi_n \) to converge to \( \varphi \) in \( R(M, M_0) \) is that \( \varphi_n(\varphi_0) \to \varphi(\varphi_0) \) in norm.

4.2. Theorem. Let \( \varphi_n \) and \( \varphi \) be in \( R(M, M_0) \), \( \varphi_n \) be in \( ((M_0)_*)^+ \) and \( \varphi_0 \) be in \( ((M_0)_*)^+ \) and faithful. Then \( \varphi_n \to \varphi \) in \( R(M, M_0) \) and \( (\varphi_n) \to \varphi \) in norm iff \( \varphi_n(\varphi_0) \to \varphi(\varphi_0) \) in norm.

Proof. With no loss of generality we can assume \( \varphi_0 = \omega_0 \). If \( \varphi_n(\varphi_0) \to \varphi(\omega_0) \) in norm then trivially \( \varphi_n(\omega_0) \to \omega_0 \) in norm. For all natural \( n \) let \( (u) \) be a partial isometry in \( M_0 \) such that \( \xi(\varphi_n(\omega_0)) = (u) \xi(\varphi(\omega_0)) \); then,

\[
\xi(\varphi_n(\omega_0)) = \{(\varphi_n(\omega_0))\} = (u) \xi(\varphi(\omega_0)) = (u) \xi(\varphi(\omega_0))
\]

and we get

\[
\|\xi(\varphi_n(\omega_0)) - \xi(\varphi(\omega_0))\| = \|u\| \xi(\varphi(\omega_0)) - \xi(\varphi(\omega_0))\| = 0
\]

by Theorem 2.1, since \( \xi(\varphi(\omega_0)) = \xi(\varphi(\omega_0)) \) (resp. \( \xi(\varphi(\omega_0)) \)) is the representative vector of \( \omega_0 \) (resp. \( \omega_0 \)) in \( V(\omega_0, \varphi) \) and \( (\varphi_n) \to \omega_0 \) (in norm).

Again by Theorem 2.1,

\[
\|\varphi_n(\omega_0) - \varphi(\omega_0)\| \leq \|\varphi_n(\omega_0) - \varphi(\omega_0)\| + \|\varphi_n(\omega_0) - \varphi(\omega_0)\| = 0
\]

which implies \( \varphi_n(\omega_0) \to \varphi(\omega_0) \) in norm.

The proof of our first implication is now completed by noticing that \( \|\varphi_n(\omega_0) - \varphi(\omega_0)\| \leq \|\varphi_n(\omega_0) - \varphi_n(\omega_0)\| + \|\varphi_n(\omega_0) - \varphi(\omega_0)\|\), and that both the latter summands converge to zero.

The converse implication has a similar proof:

\[
\|\varphi_n(\omega_0) - \varphi(\omega_0)\| \leq \|\varphi_n(\omega_0) - \varphi_n(\omega_0)\| + \|\varphi_n(\omega_0) - \varphi(\omega_0)\|
\]

the first summand tends to zero by Theorem 2.1 since \( (\varphi_n) \to \omega_0 \) as earlier, and the last because \( \varphi_n \to \varphi \) in \( R(M, M_0) \) implies \( \varphi_n(\omega_0) \to \varphi(\omega_0) \) in norm.

A slight generalization of the preceding proof shows that the "only if" part of the above statement also holds when \( \varphi_0 \) is no longer faithful.

4.3. Theorem. Let \( \varphi_n \) and \( \varphi \) be in \( R(M, M_0) \). Then \( \varphi_n \to \varphi \) in \( R(M, M_0) \) if \( \varphi(\omega_0, \varphi_n) \to \varphi(\omega_0, \varphi) \) pointwise in norm.

Proof. Let \( \varphi_n \to \varphi \) in \( R(M, M_0) \). We must prove that \( \xi(\varphi_n(\varphi_0)) \to \xi(\varphi(\varphi_0)) \) for all \( \varphi_0 \) in \( ((M_0)_*)^+ \). Let \( \varepsilon > 0 \). As \( \omega_0 \) is norm dense in \( ((M_0)_*)^+ \) there is a \( \varphi_0 \) in \( m(\omega_0) \) such that \( \|\varphi_0 - \varphi_0\| \leq \varepsilon \). By Theorem 2.1 we have

\[
\|\xi(\varphi(\varphi_0)) - \xi(\varphi(\varphi_0))\| \leq \varepsilon^{1/2}
\]

and, for all \( n \),

\[
\|\xi(\varphi_n(\varphi_0)) - \xi(\varphi_n(\varphi_0))\| \leq \varepsilon^{1/2}
\]

since \( \xi(\varphi(\varphi_0)) \) (resp. \( \xi(\varphi_n(\varphi_0)) \)) and \( \xi(\varphi(\varphi_0)) \) (resp. \( \xi(\varphi_n(\varphi_0)) \)) are the representative vectors of \( \varphi_0 \) and \( \varphi_0 \) in the same selfdual positive cone \( V(\omega_0, \varphi) \) (resp. \( V(\omega_0, \varphi_n) \)) for \( M_0 \) in \( H \). Similarly,

\[
\|\xi(\varphi_n(\omega_0)) - \xi(\varphi(\omega_0))\| \leq \|\varphi_n(\omega_0) - \varphi(\omega_0)\|^{1/2}
\]

Then

\[
\|\xi(\varphi_n(\omega_0)) - \xi(\varphi(\omega_0))\|
\]

\[
\leq \|\xi(\varphi_n(\omega_0)) - \xi(\varphi_n(\omega_0))\| + \|\xi(\varphi_n(\omega_0)) - \varphi(\omega_0))\|
\]

\[
\leq \|\varepsilon(\varphi(\omega_0)) - \varepsilon(\varphi(\omega_0))\|^{1/2}
\]

and our claim is proved, since \( \varphi_n(\omega_0) - \varphi(\omega_0)\) \to \( \varphi(\omega_0) \to \varphi(\omega_0) \) in norm.

Conversely, \( \|\xi(\varphi_n(\omega_0)) - \xi(\varphi(\omega_0))\| \to 0 \) implies, by Theorem 2.1 again, \( \varphi_n(\omega_0) \to \varphi(\omega_0) \) in norm and therefore \( \varphi_n \to \varphi \) in \( R(M, M_0) \).

4.4. Corollary. Let \( \varphi_n \) and \( \varphi \) be in \( R(M, M_0) \), \( \varphi_0 \) and \( (\varphi_n) \) be in \( ((M_0)_*)^+ \), and let \( \varphi_0 \) be faithful. Then \( \varphi_n(\varphi_n(\omega_0)) \to \varphi(\omega_0) \) in norm iff \( \xi(\varphi_n(\omega_0)) \to \xi(\varphi(\omega_0)) \) in norm.

Proof. By Theorem 4.2, \( \varphi_n(\varphi_n(\omega_0)) \to \varphi(\omega_0) \) in norm iff \( \varphi_n \to \varphi \) in \( R(M, M_0) \) and \( (\varphi_n) \to \varphi_0 \) in norm. If this is the case, then

\[
\|\xi(\varphi_n(\varphi_n(\omega_0))) - \xi(\varphi(\omega_0))\| \leq \|\xi(\varphi_n(\varphi_n(\omega_0))) - \xi(\varphi_n(\omega_0))\| + \|\xi(\varphi_n(\omega_0)) - \xi(\varphi(\omega_0))\|
\]

the first summand tends to zero by Theorem 4.3 and the second by Theorem 2.1.
The converse is trivial (and true even when φ₀ is no longer faithful).

4.5. THEOREM. The mapping \((ρ, φ₀) \mapsto u(ρ, φ₀)\) is continuous from the product of the norm topology on \(((M₀)⁺)\) and the \(R(M₀)\) topology to the strong operator topology.

Proof. Let \(ρₙ \to ρ\) in \(R(M₀, M₀)\) and \((φ₀ₙ) \to φ₀\) in norm, and let us shorten \(u(ρₙ, φ₀ₙ)\) to \(uₙ\) and \(u(ρ, φ₀)\) to \(u\). For \(aₙ\) in \(M\), using once more Theorem 2.1, we have

\[
\|J(uₙ - u)Ja(φ₀φ₀)\| \leq \|aₙ\| \|Jₙuₙ(φ₀φ₀) - ρ₀(φ₀φ₀)\| \xi(φ₀φ₀)\| \xi(φ₀φ₀)\|
\]

\[
\leq \|a\| \|Jₙuₙ(φ₀φ₀) - Jₙuₙ(φ₀φ₀)\| + \|Jₙuₙ(φ₀φ₀) - φ₀Æ(φ₀φ₀)\| \xi(φ₀φ₀)\| \xi(φ₀φ₀)\|
\]

\[
\leq \|a₀\| \xi(φ₀φ₀) - \xi(φ₀φ₀)\| \xi(φ₀φ₀)\| \xi(φ₀φ₀)\|
\]

and by our hypothesis and Theorem 4.4 both summands in parenthesis tend to zero. This implies \(\|uₙ - u\| \to 0\) for \(ξ\) in supp \(ρ(φ₀)\), and therefore our claim, since supp \(ρ(φ₀ₙ)\) \to supp \(ρ(φ₀)\) strongly.

5. Comparing the structure of canonical functional extensions

5.1. LEMMA. Let \(φ₀\) be in \(((M₀)⁺)\) and \(R\) be a subset of \(R(M₀, M₀)\). There is a selfdual positive cone \(U\) for \(M\) in \(H\) containing \(ξ₀(φ₀φ₀)\) for all \(a\) in \(R\) iff there is a partial isometry \(u\) in \(M\) with initial projection the identity such that \(u(ρ, φ₀) = u\) supp \(ρ(φ₀)\) for all \(a\) in \(R\).

Proof. Let \(u\) be as above. Then \(U = \{Ju(φ₀)[ξ(V)]ψ : ψ ∈ (M⁺)\}⁺\) is a selfdual positive cone for \(M\) in \(H\) which clearly satisfies our requirements.

Conversely, if \(U\) is as above then the partial isometry \(u(φ₀)\) in \(M\) defined by setting \(Ju(φ₀)[ξ(V)]ψ = ξ(U)ψ\) for a fixed faithful \(ψ\) in \((M⁺)\) is unique and for all \(a\) in \(R\), \(u(ρ, φ₀) = u\) supp \(ρ(φ₀)\).

5.2. REMARK. If \(R = R(M₀, M₀)\) our first condition above is the same as requiring the \(φ₀\) section of \(V(φ₀)\) to be contained in some selfdual positive cone for \(M\) in \(H\).

5.3. PROPOSITION. Let \(φ₀\) be in \(((M₀)⁺)\)⁺, \(q₀, q₂\) in \((R(M₀, M₀)\) and supp \(q₂(φ₀) ≤ \) supp \(q₂(φ₀)\). Then \(u(q₁, φ₀) = u(q₂, φ₀)\) if \(u(q₁, φ₀) = \) supp \(q₁(φ₀)\) = \(supp q₂(φ₀)\) and \((φ₀/φ₀)\{q₁(φ₀)/q₂(φ₀)\} = (q₁(φ₀)/q₂(φ₀))\{φ₀/φ₀\}\{φ₀/φ₀\}.

Proof. Let \(u(q₁, φ₀) = u(q₂, φ₀)\); then supp \(q₁(φ₀) = \) supp \(q₂(φ₀)\) since the former is the initial projection of \(u(q₁, φ₀)\) and the latter the initial projection of \(u(q₂, φ₀)\). So \(\{q₁(φ₀)/q₂(φ₀)\})\) is well defined, as are \(\{φ₀/φ₀\} \circ \{q₁(φ₀)/q₂(φ₀)\}\{φ₀/φ₀\}\{φ₀/φ₀\}, whose domains coincide

with \(\{a(φ₀/φ₀) : a \in M\}⁺\). By Lemma 5.1 there is a selfdual positive cone \(U\) for \(M\) in \(H\) to which both the vectors

\[ξ₀(q₁(φ₀)) = \{φ₀/φ₀\} \{q₁(φ₀)/q₂(φ₀)\} \xi(φ₀/φ₀)\]

and

\[ξ₀(q₂(φ₀)) = \{φ₀/φ₀\} \xi(φ₀/φ₀),\]

and therefore \(\{q₁(φ₀)/q₂(φ₀)\} \{φ₀/φ₀\}\xi(φ₀/φ₀)\), belong. The state \(q₁(φ₀)\) on \(M\) is implemented by the vector \(\{φ₀/φ₀\} \{q₁(φ₀)/q₂(φ₀)\} \xi(φ₀/φ₀)\) and the vector \(\{q₁(φ₀)/q₂(φ₀)\} \{φ₀/φ₀\}\xi(φ₀/φ₀)\), which are in the same selfdual positive cone for \(M\) in \(H\); so they must coincide. We can now conclude by noticing that both the operators \(\{φ₀/φ₀\} \{q₁(φ₀)/q₂(φ₀)\}\) and \(\{q₁(φ₀)/q₂(φ₀)\} \{φ₀/φ₀\}\) commute with \(M\).

Conversely, supp \(q₁(φ₀) = \) supp \(q₂(φ₀)\) guarantees that the vector \(\{φ₀/φ₀\}\xi(φ₀/φ₀) = \xi₀(q₂(φ₀))\) belongs to any selfdual positive cone for \(M\) in \(H\) to which \(\{q₁(φ₀)/q₂(φ₀)\} \{φ₀/φ₀\}\xi(φ₀/φ₀)\) belongs. This last vector coincides by our hypothesis with \(\{φ₀/φ₀\} \{q₁(φ₀)/q₂(φ₀)\}\xi(φ₀/φ₀)\) \(\xi(φ₀/φ₀)\) = \(ξ₀(q₁(φ₀))\). Now Lemma 5.1 and supp \(q₁(φ₀) = \) supp \(q₂(φ₀)\) imply our claim.

5.4. DEFINITION. Let \(q₁\) and \(q₂\) be in \(R(M₀, M₀)\), and \(S₀\) be a subset of \(((M₀)⁺)⁺\). We denote by \(\{q₁(S₀)/q₂(S₀)\}\) when it exists, the linear (not necessarily bounded or even closable) operator which coincides, for all \(φ₀\) in \(S₀\), with \((Dq₁(φ₀) : Dq₂(φ₀))/₂\) on \(D(H, q₂(φ₀))\) and is zero on the linear subspace of \(H\) orthogonal to the linear span of \(\bigcup_{φ₀∈S₀} supp q₂(φ₀)\).

If \(\{q₁(S₀)/q₂(S₀)\}\) is bounded we denote its closure by \(\{q₁(S₀)/q₂(S₀)\}\); if \(S₀ = \{(M₀)⁺\}⁺\) we shorten \(\{q₁(S₀)/q₂(S₀)\}\) and \(\{q₁(S₀)/q₂(S₀)\}\) to \(\{q₁/q₂\}\) and \(\{q₁/q₂\}\).

Our next proposition shows that the existence of \(\{q₁/q₂\}\) is a rather restrictive condition.

5.5. PROPOSITION. Let \(q₁\) and \(q₂\) be in \(R(M₀, M₀)\), \(S₀\) be a subset of \(S(M₀)\) and assume \(\{q₁(S₀)/q₂(S₀)\}\) exists. Then for all \(φ₀\) in \(S₀\) we have

\[ξₐ(φ₀)\xiₐ(q₂(S₀)) \{q₁(S₀)/q₂(S₀)\} \xiₐ(q₂(S₀)) = \xiₐ(q₂(S₀))\]

\[= \{q₁(q₂(q₂)/q₂(φ₀)) \{q₁(φ₀)/q₂(φ₀)\}\xi(q₂(φ₀))\}
\]

\[= \{q₁(q₂(q₂)/q₂(φ₀)) \xi(q₂(φ₀))\}, ξ₁(q₂(φ₀))\}
\]

so \(ξ₂(q₂(φ₀)) \{q₁(S₀)/q₂(S₀)\} \xi₂(q₂(φ₀)) = ξ₂(q₂(φ₀)), which implies our claim.

Proof. For all \(φᵦ\) in \(M₀\) we have

\[\{q₁(q₂(q₂)/q₂(φ₀)) \{q₁(φ₀)/q₂(φ₀)\}\xi(q₂(φ₀))\}
\]

\[= \{q₁(q₂(q₂)/q₂(φ₀)) \xi(q₂(φ₀))\}, ξ₁(q₂(φ₀))\}
\]

so \(ξ₂(q₂(φ₀)) \{q₁(S₀)/q₂(S₀)\} \xi₂(q₂(φ₀)) = ξ₂(q₂(φ₀)), which implies our claim.
5.6. THEOREM. Let \( g_1 \) and \( g_2 \) be in \( R(M, M_0) \) and assume \( \{g_1, g_2\} \) exists. Then \( u(g_1, \phi_0) = u(g_2, \phi_0) \) for all \( \phi_0 \) in \( ((M_0)_+)^\ast \) iff for all \( \phi_0 \) in \( m(\phi_0) \) the operators \( \{\phi_0/\omega_0\} \) and \( \{g_1, g_2\} \) commute and supp \( g_1(\phi_0) = \text{supp } g_2(\phi_0) \).

Proof. The "if" part is proved by noticing first that our hypothesis implies by Proposition 5.3 that \( u(g_1, \phi_0) = u(g_2, \phi_0) \) for all \( \phi_0 \) in \( m(\phi_0) \). As \( m(\phi_0) \) is norm dense in \( ((M_0)_+)^\ast \), Theorem 4.2 yields our claim.

The converse implication follows immediately from Proposition 5.3.

5.7. THEOREM. Let \( g_1 \) and \( g_2 \) in \( R(M, M_0) \) be such that \( \text{supp } g_1(\omega_0) = \text{supp } g_2(\omega_0) \), and assume there is a linear extension \( T \) of \( \{g_1(\omega_0)/g_2(\omega_0)\} \) whose domain is the linear span of the union of all \( D(H, \omega_2(\phi_0)) \) with \( \phi_0 \) in \( ((M_0)_+)^\ast \) and which commutes with \( M' \) and with \( \{\phi_0/\omega_0\} \) for all \( \phi_0 \) in \( ((M_0)_+)^\ast \). If \( u(g_1, \phi_0) = u(g_2, \phi_0) \) for all \( \phi_0 \) in \( ((M_0)_+)^\ast \) then \( \{g_1, g_2\} \) exists and coincides with \( T \).

Proof. It is enough to prove that for all \( a' \) in \( M' \) and all \( \phi_0 \) in \( ((M_0)_+)^\ast \) we have

\[
Ta'g_1(\phi_0) = Ta'g_2(\phi_0) = g_1(\phi_0/a_2(\phi_0))a'g_2(\phi_0).
\]

By hypothesis and Proposition 5.3 we get

\[
Ta'g_1(\phi_0) = Ta'g_2(\phi_0) = a'g_1(\phi_0/\omega_0)Tg_2(\phi_0) = a'g_1(\phi_0/\omega_0)g_2(\phi_0) = a'g_1(\phi_0/\omega_0)g_2(\phi_0) = g_1(\phi_0/\omega_0)g_2(\phi_0).
\]

6. Dominated canonical functional extensions

6.1. Definition. Let \( g_1 \) and \( \phi \) be in \( R(M, M_0) \). We say that \( g_1 \) is dominated by \( \phi \) if \( g_1(\omega_0) \) is dominated by \( \phi(\omega_0) \) for all \( \phi_0 \) in \( ((M_0)_+)^\ast \).

We recall that from [5] it follows that if \( g_1(\omega_0) \) in \( S(M) \) is dominated by \( \phi(\omega_0) \) then \( g_1 \) is dominated by \( \phi \).

6.2. Theorem. Let \( g_1 \) and \( \phi \) be in \( R(M, M_0) \), \( g_1(\omega_0) \) be in \( (M_+)^\ast \) and faithful, and \( g_1 \) be dominated by \( \phi \). Then for all \( \phi_0 \) in \( ((M_0)_+)^\ast \),

\[
u(g_1, \phi_0)[g_1(\phi_0)/\phi(\phi_0)] = g_1(\phi_0/\phi_0)|u(\phi, \phi_0)|.
\]

Proof. We have

\[
u(g_1, \phi_0)[g_1(\phi_0)/\phi(\phi_0)] = \phi(\phi_0)|g_1(\phi_0)/\phi(\phi_0)|\phi(\phi_0) = J(\phi_0/\phi_0)J(\phi_0/\phi(\phi_0)) = g_1(\phi_0/\phi_0)|u(\phi, \phi_0)|\phi(\phi_0).
\]

and our claim follows.

6.3. COROLLARY. Under the above hypothesis,

\[
||u(g_1(\phi_0)/\phi(\phi_0)) = u(g_1(\phi_0)/\phi(\phi_0))||^2 = u(g_1(\phi_0)/\phi(\phi_0)) = u(g_1(\phi_0)/\phi(\phi_0)).
\]

and for all \( a \) in \( M_+^\ast \),

\[
|u(g_1(\phi_0)/\phi(\phi_0))| = |u(g_1(\phi_0)/\phi(\phi_0))||a|^2 = |u(g_1(\phi_0)/\phi(\phi_0))||a|^2|a|^2 = |u(g_1(\phi_0)/\phi(\phi_0))||a|^2.
\]

Proof. The first statement is an immediate consequence of Theorem 6.2; we also have for \( a \) in \( M_+^\ast \),

\[
|u(g_1(\phi_0)/\phi(\phi_0))||a|^2 = |u(g_1(\phi_0)/\phi(\phi_0))||a|^2 = |u(g_1(\phi_0)/\phi(\phi_0))||a|^2 = |u(g_1(\phi_0)/\phi(\phi_0))||a|^2.
\]

By combining Theorems 5.6, 5.7, and 6.2 we get an immediate consequence:

6.4. COROLLARY. Under the above hypothesis consider the following statements:

(a) \( u(g_1, \phi_0) = u(g, \phi_0) \) for all \( \phi_0 \) in \( ((M_0)_+)^\ast \).

(b) \( g_1(\phi_0) \) exists.

(c) \( [g_1(\phi_0)/\phi(\phi_0)] \) commutes with \( u(g, \phi_0) \) for all \( \phi_0 \) in \( ((M_0)_+)^\ast \).

(d) For all \( \phi_0 \) in \( m(\phi_0) \) the operators \( [g_1(\phi_0)/\phi(\phi_0)] \) and \( [\phi_0/\omega_0] \) commute, and supp \( g_1(\phi_0) = \text{supp } \phi(\phi_0) \).

The following implications hold: (a)\&(b) \( \Rightarrow \) (c)\&(d); (a)\&(d) \( \Rightarrow \) (b)\&(c);

(b)\&(d) \( \Rightarrow \) (a)\&(c).

6.5. The particular case of norm one projections (cf. also [6]). Let \( g_1 \) and \( \phi \) be in \( R(M, M_0) \), with \( g_1 \) dominated by \( \phi \), supp \( g_1(\omega_0) = \text{supp } \phi(\omega_0) = I \), and \( \phi \) a norm one projection from \( M \) to \( M_0 \) such that \( \phi(\omega_0) = \omega_0 \circ \varepsilon \) for all \( \omega_0 \) in \( ((M_0)_+)^\ast \). Obviously in this situation (a) of Corollary 6.4 is equivalent to the existence of a norm one projection \( \varepsilon_1 \) from \( M_0 \) to \( M_0 \) such that \( \phi(\omega_0) = \omega_0 \circ \varepsilon_1 \) for all \( \omega_0 \) in \( ((M_0)_+)^\ast \). In this case we also have supp \( g_1(\phi_0) = \text{supp } \phi(\phi_0) \) for all \( \phi_0 \) in \( (M_0)_+^\ast \).

(a) In our situation (a) of Corollary 6.4 implies \( [g_1(\omega_0)/\phi(\phi_0)] \) and \( [\phi_0/\omega_0] \) commute for all \( \phi_0 \) in \( ((M_0)_+)^\ast \) and therefore (d); indeed, since \( [g_1(\phi_0)/\phi_0(\phi_0)] = [\phi_0/\omega_0] \), we get
\[
\{\phi_0/\omega_0\}[\phi_1(\omega_0)/\rho(\omega_0)]\xi(\rho(\omega_0))
\]
\[
= \{\phi_1(\phi_0)/\phi_1(\omega_0)\}\{\phi_1(\omega_0)/\rho(\omega_0)\}\xi(\rho(\omega_0)) = \{\phi_1(\phi_0)/\phi_1(\omega_0)\}\xi(\rho(\omega_0))
\]
\[
= J\{\phi_1(\phi_0)/\phi_1(\omega_0)\}\xi(\rho(\omega_0)) = J\{\phi_1(\phi_0)/\phi_1(\omega_0)\}J\{\phi_1(\omega_0)/\rho(\omega_0)\}\xi(\rho(\omega_0))
\]
\[
= [\phi_1(\omega_0)/\rho(\omega_0)]J\{\phi_1(\phi_0)/\phi_1(\omega_0)\}\xi(\rho(\omega_0))
\]
\[
= [\phi_1(\omega_0)/\rho(\omega_0)]\{\phi_1(\phi_0)/\phi_1(\omega_0)\}\xi(\rho(\omega_0))
\]
\[
= [\phi_1(\omega_0)/\rho(\omega_0)]\{\phi_1(\phi_0)/\rho(\omega_0)\}\xi(\rho(\omega_0)).
\]

By Corollary 6.4, \([\phi_1/\rho]\) exists and coincides with \([\phi_1(\omega_0)/\rho(\omega_0)]\). Let \(\Delta\) (resp. \(\Delta_0\)) be the modular operator associated with \(M\) and \(\xi(\rho(\omega_0))\) (resp. \(M_0\) and \(\xi(\rho(\omega_0))\)). For all positive \(\alpha_0 \in M_0\) there is some \(\phi_0\) in \(M_0\) such that \(\Delta^{1/4}\Delta_0^{1/4} = \Delta_0^{1/4}\Delta^{1/4}\) is the closure of \([\phi_0/\omega_0]\) (cf. [2]); let also \(\alpha\) be the positive operator in \(M\) which is the closure of \(\Delta^{1/4}\Delta_0^{1/4}\). We then have
\[
\alpha_0\alpha\xi(\rho(\omega_0)) = \Delta^{1/4}\Delta_0^{1/4}\alpha_0\alpha\xi(\rho(\omega_0)) = \alpha_0\alpha\xi(\rho(\omega_0)),
\]
which implies that \(\alpha_0\) is in \((M_0)^+\) and therefore in the relative commutant of \(M_0\) in \(M\). Since there is a norm one projection from \(M\) to the relative commutant of \(M_0\) in \(M\), also \([\phi_1/\rho]\) belongs to it.

(b) Assume \([\phi_1/\rho]\) exists and let \(\phi_0\) be in \(m(\omega_0)\). Then
\[
[\phi_0/\omega_0][\phi_1/\rho]\xi(\rho(\omega_0)) = [\phi_0/\omega_0][\phi_1(\omega_0)/\rho(\omega_0)]\xi(\rho(\omega_0))
\]
\[
= [\phi_0/\omega_0]J[\phi_1(\omega_0)/\rho(\omega_0)]\xi(\rho(\omega_0))
\]
\[
= J[\phi_1(\omega_0)/\rho(\omega_0)]J[\phi_0/\omega_0]\xi(\rho(\omega_0))
\]
\[
= J[\phi_1(\omega_0)/\rho(\omega_0)]J[\phi_0/\omega_0]\xi(\rho(\omega_0))
\]
\[
= [\phi_1(\omega_0)/\rho(\omega_0)]\{\phi_1(\phi_0)/\rho(\omega_0)\}\xi(\rho(\omega_0))
\]
\[
= [\phi_1(\omega_0)/\rho(\omega_0)]\xi(\rho(\omega_0)).
\]

Proof. The equality
\[
u(\phi_1, \phi_0)[(\phi_1(\phi_0)/\rho(\omega_0)]\xi(\rho(\omega_0)) = \{\phi_1(\omega_0)/\rho(\omega_0)\}u(\phi_0, \phi_0)\xi(\rho(\omega_0))
\]
can be proved as in Theorem 6.2, since
\[
J[\phi_0/\omega_0][(\phi_1(\omega_0)/\rho(\omega_0)]\xi(\rho(\omega_0)) = J[\phi_0/\omega_0]J[(\phi_1(\omega_0)/\rho(\omega_0)]\xi(\rho(\omega_0))
\]
\[
= \{\phi_1(\omega_0)/\rho(\omega_0)\}J[(\phi_1(\phi_0)/\rho(\omega_0)]\xi(\rho(\omega_0)),
\]
as \(\{\phi_1(\omega_0)/\rho(\omega_0)\}\) commutes with \(M'\) to which \(J[\phi_0/\omega_0]J\) belongs. If \(\alpha'\) is in \(M'\) we get
\[
u(\phi_1, \phi_0)[(\phi_1(\phi_0)/\rho(\omega_0)]\alpha'\xi(\rho(\omega_0)) = \{\phi_1(\omega_0)/\rho(\omega_0)\}u(\phi_0, \phi_0)\alpha'\xi(\rho(\omega_0)),
\]
which is our claim.

For a normal faithful state \(\omega\) on \(M\), set \(L(\omega) = \{a \in M : \{\epsilon(\omega)\}|a|^2 = \{\epsilon(\omega)|a|^2\}\}\), \(R(\omega) = \{a \in M : \{\epsilon(\omega)|a|^2 = \{\epsilon(\omega)|a|^2\}\}\), and \(LR(\omega) = L(\omega) \cap R(\omega)\).

6.7. Lemma. Let \(\phi\) be a normal state in \(m(\omega)\) such that \([\sigma(\omega)]^t[\phi/\omega]\) is in \(L(\omega)\) for all real \(t\). Then \([\sigma(\omega)]^t[\phi/\omega]\) is in \(L(\omega)\) for all real \(t\).

Proof. Since \([\sigma(\omega)]^t[\phi/\omega]\) is \([\sigma(\omega)]^t[\phi/\omega]\) with \([\sigma(\omega)]^t[\phi/\omega]\) is \([\sigma(\omega)]^t[\phi/\omega]\) and \([\sigma(\omega)]^t[\phi/\omega]\) is \([\sigma(\omega)]^t[\phi/\omega]\) is \([\sigma(\omega)]^t[\phi/\omega]\) it is enough to show our claim for \(t = 0\). If \(\Delta\) is the modular operator associated with \(M\) and \(\xi(\omega)\), our hypothesis is equivalent to
\[
J_0EJ^\Delta[\phi/\omega]\xi(\omega) = \epsilon(\omega)[(\sigma(\omega)(\phi/\omega))\xi(\omega)
\]
\[
= J_0EJ^\Delta[\phi/\omega]\xi(\omega) = J_0EJ^\Delta[\phi/\omega]\xi(\omega)
\]
for all real \(t\). So \(\Delta^t[\phi/\omega]\xi(\omega)\) is in \(H_0\) for all real \(t\). This implies that \(\Delta^{1/2}[\phi/\omega]\xi(\omega)\) is in \(H_0\), \(\Delta^{1/2}[\phi/\omega]\xi(\omega)\) and \(\xi(\omega)\) in \(L(\omega)\), and our claim follows.

6.8. Proposition. Let \(\phi_1 \neq \rho\) be in \((R(M,M_0), \rho)\), \(\rho_1\) be dominated by \(\phi\) and \(\rho(\omega_0)\) be faithful. Then \(\rho_1 = \rho\) iff \([\sigma(\omega)]^t[\rho(\omega_0)]\) is in \(L(\omega)\) for all real \(t\).

Proof. The “if” part can be proved by noticing that by Lemma 6.7, \([\sigma(\omega)]^t[\rho(\omega_0)]\) is in \(LR(\omega)\) and therefore so is \([\sigma(\omega)]^t[\rho(\omega_0)]\). As in the proof of Proposition 5.5, \([\sigma(\omega)]^t[\rho(\omega_0)]\) is \([\sigma(\omega)]^t[\rho(\omega_0)]\). So
\[
J_0EJ^\Delta[\phi/\omega]\xi(\omega) = \epsilon(\omega)\xi(\omega),
\]
which implies that \([\sigma(\omega)]^t[\rho(\omega_0)]\) is the identity, \(\rho_1 = \rho\) and finally \(\rho_1 = \rho\).

The converse implication is trivial.

6.9. Theorem. Let \(\xi(\rho(\omega_0))\) be cyclic and separating for both \(M\) and \(M_0\). Then there is no canonical \(\phi_1\) in \((R(M,M_0), \rho)\) dominated by \(\phi\) except \(\phi\) itself.

Proof. In this case \(E\) is the identity, \(LR(\omega) = M\) and therefore any \(\rho_1\) in \((R(M,M_0), \rho)\) dominated by \(\phi\) satisfies the hypothesis in Proposition 6.8.
6.10. Remark. In general a necessary and sufficient condition for $\xi(\omega)$ to be cyclic and separating for both $M$ and $M_0$ is that $LR(\omega) = M$.

Theorem 6.9 shows how far the subset of $R(M, M_0)$ of its elements dominated by $\varphi$ can be from being dense in $R(M, M_0)$, while this is the case for $m(\omega)$ in $(M_\ast)^+$. 

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Interpolation of the measure of non-compactness by the real method

by

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Dedicated to Professor David E. Edmunds on the occasion of his 65th birthday

Abstract. We investigate the behaviour of the measure of non-compactness of an operator under real interpolation. Our results refer to general Banach couples. An application to the essential spectral radius of interpolated operators is also given.

Introduction. In 1960 Krasnosel’skiǐ [11] proved that compactness of an operator can be interpolated between $L_p$-spaces. A motivation for this result might have been a remark by S. G. Kreĭn on the interpolation character that certain compactness results for integral operators between $L_p$-spaces established by Kantorovich in 1956 seemed to have (see [12], p. 118).

At the beginning of the sixties, with the foundation of abstract interpolation theory, Krasnosel’skiǐ’s result led to the investigation of interpolation properties of compact operators between abstract Banach spaces. The main contributions during that period are due to J. L. Lions, J. Peetre, E. Gagliardo, A. Calderón, A. Persson, S. G. Kreĭn, Yu. I. Petunin and K. Hayakawa (see [2] and [16] for precise references).

More recently, the paper [4] by Cobos, Edmunds and Potter opened a new research period in this area, and in 1992, culminating the efforts of several authors (see the paper [6] by Cobos and Peetre for references) M. Cwikel [9] proved that compactness of an operator can be interpolated between any Banach couples by the real method.

In the present paper we investigate the behaviour under real interpolation of the measure of non-compactness, a concept that means more than only continuity but not so much as compactness. Previous results on this

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