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References


Recall some notations and terminology from [4].

For closed subspaces $M, L$ of a Banach space $X$ we write $M \subset L$ ($M$ is essentially contained in $L$) if there is a finite-dimensional subspace $F \subset X$ such that $M \subset L + F$. Equivalently, $\dim M/\langle M \cap L \rangle = \dim (M + L)/L < \infty$. Similarly we write $M \subsetneq L$ if $M \subset L$ and $L \not\subsetneq M$.

For a (bounded linear) operator $T \in \mathcal{L}(X)$ write $R^\infty(T) = \bigcap_{m=0}^\infty R(T^m)$ and $N^\infty(T) = \bigcup_{m=0}^\infty N(T^m)$.

An operator $T \in \mathcal{L}(X)$ is called semiregular (essentially semiregular) if $R(T)$ is closed and $N(T) \subset R^\infty(T)$ ($N(T) \subset R^\infty(T)$, respectively). Further, $T$ is called quasi-Fredholm if there exists $d \geq 0$ such that $R(T^{d+1})$ is closed and $R(T^d + N(T)) = R(T) + N^\infty(T)$ (equivalently, $N(T) \cap R(T^d) = N(T) \cap R^\infty(T)$).

The proof of Theorem 15 of [4] relies on the following statement (where $d$ is the integer whose existence is postulated in the definition of quasi-Fredholm operators):

If $T$ is quasi-Fredholm and $F$ of rank $1$ then $N(T) \cap R(T^d) \subset R^\infty(T + F)$.

This, however, need not be satisfied.

Counterexample. Let $H$ be the Hilbert space with an orthonormal basis $\{e_1, e_2, \ldots\}$. Define $T, F \in \mathcal{L}(H)$ by

$Te_1 = 0, \quad Te_n = e_{n-1} \quad (n \geq 2), \quad Fe_2 = -e_1, \quad Fe_n = 0 \quad (n \neq 2)$. 

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Then $T$ is quasi-Fredholm (with $d = 0$) and is surjective, $F$ has rank 1, and $T + F$ is given by

$$(T + F)e_1 = (T + F)e_2 = 0, \quad (T + F)e_n = e_{n-1} \quad (n \geq 3).$$

It follows that $R^\infty(T + F) = R(T + F)$ is equal to the linear span of $(e_2, e_3, \ldots)$, and $N(T)$ to the one-dimensional space spanned by $e_1$. Thus $N(T) \subset R^\infty(T + F)$.

We now proceed to give a correct proof of Theorem 15 of [4].

**Theorem.** Let $T \in \mathcal{L}(X)$ be a quasi-Fredholm operator and let $F \in \mathcal{L}(X)$ be a finite-rank operator. Then $T + F$ is also quasi-Fredholm.

**Proof.** Clearly it is sufficient to consider only the case of $\dim R(T) = 1$. Thus there exist $x \in X$ and $\varphi \in X^*$ such that $Fx = \varphi(x) x$ ($x \in X$).

Since $R((T + F)^n) \subset R((T^m)^n)$ for all $n$ by Observation 8 following Table 1 in [4], $R((T + F)^n)$ is closed if and only if $R(T^n)$ is closed, and it is sufficient to show only the algebraic condition in the definition of quasi-Fredholm operators for $T + F$.

Since $T$ is quasi-Fredholm, there exists $d \geq 0$ such that $N(T) \cap R(T^d) \subset R^\infty(T)$ and $R(T^d), R(T^{d+1})$ are closed. Set $M = R(T^d)$ and $T_1 = T|M$. Then $N(T_1) = N(T) \cap R(T^d) \subset R^\infty(T)$ and the range $R(T_1) = R(T^{d+1})$ is closed. Thus $T_1$ is semisimple.

It is sufficient to show that $N(T_1) \subset R^\infty(T + F)$. Indeed, then we have

$$N(T + F) \cap R((T + F)^d) = N(T) \cap R(T^d) = N(T_1) \subset R^\infty(T + F)$$

so that $N(T + F) \cap R((T + F)^d) = N(T + F) \cap R^\infty(T + F)$.

This means that $N(T + F) \cap R((T + F)^n) = N(T + F) \cap R^\infty(T + F)$ for some $n \geq d$ and $T + F$ is quasi-Fredholm.

To prove $N(T_1) \subset R^\infty(T + F)$ we distinguish two cases:

A. $N(T_1) \subset \ker \varphi$. Let $x_0 \in N(T_1)$. Since $T_1$ is semisimple, there exist vectors $x_1, x_2, \ldots \in R^\infty(T_1)$ such that $Tx_i = x_{i-1}$ for all $i$. By the assumption $\varphi(x_i) = 0$, so that $Fx_i = 0$ for all $i$. Then for $n \in N$ we have

$$(T + F)^nx_0 = (T + F)^{n-1}x_{n-1} = \ldots = (T + F)x_1 = x_0,$$

so that $x_0 \in R((T + F)^n)$. Since $x_0$ and $n$ were arbitrary, we have $N(T_1) \subset R^\infty(T + F)$.

B. $N(T_1) \not\subset \ker \varphi$. There exists $k \geq 1$ such that $N(T_1^k) \not\subset \ker \varphi$. Choose the minimal $k$ with this property so that $N(T_1^{k-1}) \subset \ker \varphi$ and there exists $u \in N(T_1^k)$ with $\varphi(u) = 1$.

Set

$$Y = \{x \in N(T_1) : \text{there is } y \in M \text{ with } T^{k-1}y = x \text{ and } T^iy \in \ker \varphi \text{ (} i = 0, \ldots, k - 1 \text{)}.\}
$$

We show that $\dim N(T_1)/Y \leq k$. Indeed, let $x^{(1)}, \ldots, x^{(k+1)} \in N(T_1)$. Since $T_1$ is semisimple, there are $y^{(1)}, \ldots, y^{(k+1)} \in M$ such that $T^{k-1}y^{(i)} = x^{(i)}$ ($i = 1, \ldots, k + 1$). Then there exists a nontrivial linear combination

$$y = \sum_{j=1}^{k+1} \alpha_j y^{(j)}$$

such that $T^i y \in \ker \varphi$ for all $i = 0, \ldots, k - 1$. Consequently,

$$\sum_{j=1}^{k+1} \alpha_j x^{(j)} \in Y \text{ and } \dim N(T_1)/Y \leq k.$$ 

Hence $Y \cong N(T_1)$ and it is sufficient to show $Y \subset R^\infty(T + F)$.

Let $x \in Y$. We prove by induction on $n$ the following statement:

(1) There exists $x_n \in M$ such that

$$T^nx_n = x \quad \text{and} \quad T^nx_n \in \ker \varphi \quad (i = 0, \ldots, n).$$

Clearly (1) for $n = 0, \ldots, k - 1$ follows from the definition of $Y$.

Suppose that (1) is true for some $n \geq k - 1$, i.e., there is $x_n \in M$ such that $T^nx_n = x$ and $T^nx_n \in \ker \varphi$ ($i = 0, \ldots, n$). Since $T_1$ is semisimple, we can find $x'_{n+1} \in M$ such that $T_1x_{n+1} = x_n$. Set $x_{n+1} = x'_{n+1} - \varphi(x'_{n+1})u$.

Then

$$T^{n+1}x_{n+1} = T^nx_n - \varphi(x'_n)T^nu = x.$$

Clearly $\varphi(x'_{n+1}) = 0$. For $1 \leq i \leq k - 1$ we have $\varphi(T_1x_{n+1}) = \varphi(x'_{n+1}) = \varphi(T^iu) = 0$ since $T^iu \in N(T_1^{k-1}) \subset \ker \varphi$. For $k \leq i \leq n$ we have $T^iu = 0$ so that $\varphi(T_1x_{n+1}) = \varphi(T^iu) = 0$ by the induction assumption.

Thus (1) is true for all $n$ and $(T + F)^nx_n = (T + F)^{n-1}T_1x_n = \ldots = T^nx_n = x$. Thus $x \in R((T + F)^n)$ for all $n$ and consequently $Y \subset R^\infty(T + F)$.

This finishes the proof of the theorem.

As a corollary we obtain the corresponding result for essentially semisimple operators (see [2]). Recall the numbers $k_n(T)$ defined for an operator $T \in \mathcal{L}(X)$ and $n \geq 0$ by

$$k_n(T) = \dim[R(T) + N(T^{n+1})]/[R(T) + N(T^n)]$$

(see [4] and [1]).

**Corollary.** If $T, F \in \mathcal{L}(X)$, $T$ is essentially semisimilar and $F$ of finite rank then $T + F$ is essentially semisimilar.

**Proof.** By the previous theorem $T + F$ is quasi-Fredholm so $k_i(T + F) = 0$ for all $i$ sufficiently large. Also $k_i(T) \leq \infty$ implies $k_i(T + F) \leq \infty$ so all $i$. Thus $T + F$ is essentially semisimilar.

This finishes the "corrigendum" part of the paper. For the "addendum" part, we give counterexamples that will complete Table 2 of [4] answering thus some questions posed in that paper.
Recall the classes defined in [4]:
\[ R_{11} = \{ T \in \mathcal{L}(X) : T \text{ is semiregular} \}, \]
\[ R_{12} = \{ T \in \mathcal{L}(X) : T \text{ is essentially semiregular} \}, \]
\[ R_{13} = \{ T \in \mathcal{L}(X) : R(T) \text{ is closed and } k_n(T) < \infty \text{ for all } n \in \mathbb{N} \}, \]
\[ R_{14} = \{ T \in \mathcal{L}(X) : T \text{ is quasi-Fredholm} \}, \]
\[ R_{15} = \{ T \in \mathcal{L}(X) : \text{there is } d \in \mathbb{N} \text{ with } R(T^{d+1}) \text{ closed and } k_n(T) < \infty (n \geq d) \}. \]

Further, for \( i = 11, \ldots, 15 \), set \( \sigma_i(T) = \{ \lambda \in \mathbb{C} : \lambda \notin R_i \} \).

**EXAMPLE 1.** In general, \( \sigma_{13} \) and \( \sigma_{15} \) are not closed. Consequently, \( R_{13} \) is not stable under small commuting perturbations:

Consider the operator defined in Example 14 of [4],
\[ S = \bigoplus_{n=1}^{\infty} S_n, \]
where \( S_n \in \mathcal{L}(H_n), H_n \) is an \( n \)-dimensional Hilbert space with an orthonormal basis \( e_{n1}, \ldots, e_{nn} \) and \( S_n \) is the shift operator, that is, \( S_ne_{ni} = 0, S_ne_{ni} = e_{n,i+1} (2 \leq i \leq n) \). Then \( S \in R_{13} \subset R_{15} \) (see Example 14 of [4]).

Let \( \epsilon \neq 0, |\lambda| < 1 \). Then \( S_n - \epsilon \) is invertible for all \( n \in \mathbb{N} \) so that \( S - \epsilon \) is injective.

For \( n \in \mathbb{N} \) set \( x_n = \sum_{i=1}^{n} \epsilon^{n-i} e_{ni} \). Then \( ||x_n|| \geq 1 \) and
\[ ||(S - \epsilon)x_n|| = ||\epsilon^n e_{nn}|| = |\epsilon^n|. \]
Thus \( S - \epsilon \) is not bounded below and \( R(S - \epsilon) \) is not closed. Hence \( S - \epsilon \notin R_{13} \) and \( \sigma_{13}(S) \) is not closed.

Further, for each \( k \in \mathbb{N} \), we have
\[ ||(S - \epsilon)^k x_n|| = ||\epsilon^n|| \cdot ||(S - \epsilon)^{k-1} e_{nn}|| \leq ||\epsilon^n|| \cdot ||(S - \epsilon)^{k-1}|| \]
\[ \leq ||\epsilon^n|| \cdot (1 + |\epsilon|)^{k-1} \]
so that \( \lim_{n \to \infty} ||(S - \epsilon)^k x_n|| = 0 \) for all \( k \in \mathbb{N} \) and \( R((S - \epsilon)^k) \) is not closed. Consequently, \( S - \epsilon \notin R_{15} \) and \( \sigma_{15}(S) \) is not closed.

**EXAMPLE 2.** The class \( R_{13} \) is not stable under commuting compact perturbations:

Consider the operator \( S \) from Example 1 and let \( K = \bigoplus_{n=1}^{\infty} (1/n)I_n \), where \( I_n \) denotes the identity operator on \( H_n \). Clearly \( K \) is compact, \( KS = SK, S + K \) is injective and, as above, \( S + K \) is not bounded below. Thus \( R(S + K) \) is not closed and \( S + K \notin R_{13} \).

**EXAMPLE 3.** \( R_{13} \) is not stable under commuting quasinilpotent perturbations:

For \( k \in \mathbb{N} \) let \( H^{(k)} \) be the Hilbert space with an orthonormal basis \( e_{ni}^{(k)} \) \((n \in \mathbb{N}, i = 1, \ldots, \max\{k, n\})\). Let \( S^{(k)} \in \mathcal{L}(H^{(k)}) \) be the shift to the left,
\[ S^{(k)} e_{ni}^{(k)} = \begin{cases} e_{n,i-1}^{(k)} & (i \geq 2), \\ 0 & (i = 1). \end{cases} \]
Set \( S = \bigoplus_{k=1}^{\infty} S^{(k)} \). Clearly \( S \) is a direct sum of finite-dimensional shifts where an \( n \)-dimensional shift appears \( 2n - 1 \) times (once in each \( S^{(1)}, \ldots, S^{(n-1)} \) and \( n \) times in \( S^{(n)} \)). Thus \( S \in R_{13} \).

Define \( Q^{(k)} \in \mathcal{L}(H^{(k)}) \) by \( Q^{(k)} e_{ni}^{(k)} = (1/n) e_{n+1, i}^{(k)} \) for all \( n, i \). Let \( Q = \bigoplus_{k=1}^{\infty} Q^{(k)} \). Clearly \( SQ \) and \( Q \) is quasinilpotent since \( ||Q||^{1/2} = (1/j)!^{1/2} \to 0 \).

We prove that \( S - Q \notin R_{13} \). Set
\[ x^{(k)} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} e_{n,n}^{(k)} \in H^{(k)}. \]
Then
\[ (S - Q)x^{(k)} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} e_{n,n-1}^{(k)} - \sum_{n=1}^{\infty} \frac{1}{n!} e_{n+1,n}^{(k)} = 0. \]
Further \( x^{(k)} \notin R(S^{(k)}) + R(Q^{(k)}) \) so that \( x^{(k)} \notin R(S^{(k)} - Q^{(k)}) \). It is easy to see that each linear combination of \( x^{(k)} \)'s has the same property with respect to \( S \) and \( Q \) so that these vectors are linearly independent modulo \( R(S - Q) \). Thus
\[ k_0(S - Q) = \dim N(S - Q)/(N(S - Q) \cap R(S - Q)) = \infty \]
and \( S - Q \notin R_{13} \).

Consequently, the complete version of Table 2 of [4] is:

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<tr>
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References


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This is a continuation of Functional Analysis and Operator Theory (Banach Center Publications, Volume 30, 1994) showing yet other aspects of current research. The 27 invited papers point out some intriguing open problems along with the corresponding motivation, background, and relevant references.

About 40% of the volume is devoted to semigroups (in particular, powers) of operators, their various means, spectral and resolvent conditions (G. R. Allan, J. A. van Casteren, T. A. Gillespie, Yu. Lyubich, O. Nevanlinna, H. C. Rönnefarth, J. C. Strikwerda and B. A. Wade, A. Święch, Vũ Quốc Phong), their role in the differential problems (R. deLaubenfels, G. Lumer, N. Sauer), and the existence of common invariant subspaces (H. Radjavi).

A 30-page survey of moment problems and their connections with subnormal operators is given by R. E. Curto. Operators on function spaces, operator algebras and complex analysis methods appear throughout (dominating the contributions by A. Böttcher and H. Wolk, B. Magajna, K. Rudol, K. Stroethoff, H. Upmeier). Spectral decompositions are considered by M. Putinar, and unitary extensions of isometries by R. Aroczna.

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