Commutativity of compact selfadjoint operators

by

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Abstract. The relationship between the joint spectrum \( \gamma(A) \) of an \( n \)-tuple \( A = (A_1, \ldots, A_n) \) of selfadjoint operators and the support of the corresponding Weyl calculus \( T(A) : f \mapsto f(A) \) is discussed. It is shown that one always has \( \gamma(A) \subseteq \text{supp}(T(A)) \). Moreover, when the operators are compact, equality occurs if and only if the operators \( A_j \) mutually commute. In the non-commuting case the equality fails badly: While \( \gamma(A) \) is countable, \( \text{supp}(T(A)) \) has to be an uncountable set. An example is given showing that, for non-compact operators, coincidence of \( \gamma(A) \) and \( \text{supp}(T(A)) \) no longer implies commutativity of the set \( \{A_i\} \).

Introduction. A notion of joint spectrum \( \gamma(A) \) for a commuting \( n \)-tuple of bounded linear operators \( A = (A_1, \ldots, A_n) \) in a Banach space \( X \) was introduced by McIntosh and Pryde in [5]. Namely

\[
\gamma(A) = \left\{ \lambda \in \mathbb{R}^n : 0 \in \sigma \left( \sum_{j=1}^{n} (A_j - \lambda_j I)^2 \right) \right\},
\]

where \( \sigma(B) \) is the usual spectrum of a single operator \( B \). For \( n \)-tuples \( A \) satisfying \( \sigma(A_j) \subset \mathbb{R}, \ 1 \leq j \leq n \), this particular joint spectrum \( \gamma(A) \) coincides with most other known joint spectra [6], and has proved to be effective in the solution of certain linear systems of operator equations [5, 7].

For commuting \( n \)-tuples \( A \) satisfying an estimate of the form

\[
\| e^{i\xi \cdot A} \| \leq C (1 + |\xi|^s), \quad \xi \in \mathbb{R}^n,
\]

for some positive constants \( C \) and \( s \) (where \( \langle \xi, A \rangle = \sum_{j=1}^{n} \xi_j A_j \) and \( | \cdot | \) denotes the usual Euclidean norm in \( \mathbb{R}^n \)) it turns out that \( \gamma(A) \) is precisely the support, \( \text{supp}(T(A)) \), of a certain functional calculus \( T(A) : A \mapsto \mathcal{L}(X) \), that is,

\[
\text{supp}(T(A)) = \gamma(A)
\]

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the commutativity of \( A \) is equivalent to the equality of a purely algebraic notion (namely, the set \( \gamma(A) \)) with a purely analytic notion (namely, the set \( \text{supp}(T(A)) \)).

The main ingredients in the proofs of the above results are the notion of the maximal abelian subspace of \( A \) (introduced in [3]), Theorem 1 below which states that particular kinds of isolated points of \( \text{supp}(T(A)) \) (called hyperisolated) are joint eigenvalues of \( A \), and the fact (cf. Proposition 4) that every isolated point of \( \text{supp}(T(A)) \) is hyperisolated whenever \( \text{supp}(T(A)) \) is a countable set.

Since any compact subset of \( \mathbb{R}^n \) is the support of some (even commuting) \( n \)-tuple of bounded selfadjoint operators \([1, p. 255]\), it cannot be expected that Theorem 4 has a larger range of applicability. Indeed, we exhibit a pair \( A = (A_1, A_2) \) of bounded selfadjoint (but not compact) operators \( A_1 \) and \( A_2 \) in an infinite-dimensional Hilbert space for which equality in (2) does hold, but such that \( A_1 A_2 \neq A_2 A_1 \); see Example 1.

In the final section of the paper a study is made, for pairs \( A = (A_1, A_2) \) of compact selfadjoint operators \( A_1 \) and \( A_2 \) of the connection between the sets \( \gamma(A), \text{supp}(T(A)) \) and \( \sigma(A_1 A_2) \) with the aim of extending Proposition 10 of [3] from 2-dimensional spaces to finite-dimensional spaces.

1. **Basic properties of \( \gamma(A) \) and \( \text{supp}(T(A)) \).** In this section we collect together some basic facts about the sets \( \gamma(A) \) and \( \text{supp}(T(A)) \) which are needed in the sequel. We begin with a simple but useful result.

**Lemma 1.** Let \( A = (A_1, \ldots, A_n) \) be an \( n \)-tuple of bounded selfadjoint operators in a Hilbert space \( \mathcal{H} \) and \( M \) be a closed linear subspace of \( \mathcal{H} \) which is invariant for \( A \) (i.e., invariant for each operator \( A_j, j = 1, \ldots, n \)).

(i) The orthogonal complement \( M^\perp \) is invariant for each operator \( A_j \), \( 1 \leq j \leq n \).

(ii) If \( A_M \) (respectively, \( A_{M^\perp} \)) denotes the selfadjoint \( n \)-tuple in the Hilbert space \( M \) (respectively, \( M^\perp \)) consisting of the restrictions of \( A_j \), \( 1 \leq j \leq n \), to \( M \) (respectively, \( M^\perp \)), then

(a) \( \text{supp}(T(A_M)) \cap \text{supp}(T(A_{M^\perp})) \), and

(b) \( \gamma(A) = \gamma(A_M) \cup \gamma(A_{M^\perp}) \).

**Proof.** (i) follows from \( A_j^* (M^\perp) \subset M^\perp \) and the selfadjointness of each \( A_j, 1 \leq j \leq n \).

(ii) We have \( H = M \oplus M^\perp \) and \( A_j = (A_j)_M \oplus (A_j)_{M^\perp} \) for each \( j = 1, \ldots, n \).

(a) It follows that \( (i(\xi, A_j)^* = (i(\xi, A_M)^*) \oplus (i(\xi, A_{M^\perp})^*) \), \( \xi \in \mathbb{R}^n \), \( r \in \mathbb{N} \), and hence, via the power series expansion of the exponential function, that

\[ e^{i(x, A)} = e^{i(x, A_M)} \oplus e^{i(x, A_{M^\perp})}, \quad \xi \in \mathbb{R}^n. \]
It is then clear from the definition of $T(A)f$ as a Bochner integral with respect to the uniform operator topology of $L(H)$ (see (3)) that

$$T(A)f = T(A(M)f) \oplus T(A(M^\perp)f), \quad f \in \mathcal{S}({\mathbb{R}}^n),$$

from which (a) follows.

(b) follows from the formulae

$$\sum_{j=1}^n (\lambda_j I - A_j)^2 = \sum_{j=1}^n (\lambda_j I - (A_j)M)^2 \oplus \sum_{j=1}^n (\lambda_j I - (A_j)M^\perp)^2, \quad \lambda \in \mathbb{R}^n,$$

together with the fact that $U \oplus V$ is invertible in $H = M \oplus M^\perp$ if and only if $U$ is invertible in $M$ and $V$ is invertible in $M^\perp$. ■

We recall that $\lambda \in \mathbb{C}^n$ is called a joint eigenvalue of an $n$-tuple of bounded selfadjoint operators $A = (A_1, \ldots, A_n)$ if there exists a non-zero vector $x \in H$ such that $A_ix = \lambda_ix$ for each $i = 1, \ldots, n$. The vector $x$ is then called a joint eigenvector of $A$ corresponding to $\lambda$.

**Lemma 2.** Let $A = (A_1, \ldots, A_n)$ be an $n$-tuple of bounded selfadjoint operators in a Hilbert space $H$ and $\lambda \in \mathbb{R}^n$ be a joint eigenvalue of $A$. Then $\lambda \in \gamma(A) \cap \text{supp}(T(A))$.

**Proof.** Let $x \neq 0$ be a joint eigenvector of $A$ corresponding to $\lambda$. A simple calculation (using power series expansion) shows that $e^{i\xi(A)x} = e^{i(\xi, \lambda)x}$, $\xi \in \mathbb{R}^n$. It then follows from (3) and the Fourier inversion theorem that

$$[T(A)f]x = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(\xi, \lambda)} \widehat{f}(\xi) \, d\xi, \quad f \in \mathcal{S}({\mathbb{R}}^n),$$

for every $f \in \mathcal{S}({\mathbb{R}}^n)$. So, given any neighbourhood $U$ of $\lambda$ in $\mathbb{R}^n$ choose $f \in C_c^\infty(\mathbb{R}^n)$ satisfying $\text{supp}(f) \subset U$ and $f(\lambda) = 1$. Then $[T(A)f]x = x \neq 0$, that is, $T(A)f \neq 0$. Accordingly, $\lambda \in \text{supp}(T(A))$.

Since joint eigenvalues of $A$ are also joint approximate eigenvalues, it follows from [3, Proposition 2] that $\lambda \in \gamma(A)$. ■

**Lemma 3.** Let $A = (A_1, \ldots, A_n)$ be an $n$-tuple of bounded selfadjoint operators in a Hilbert space $H$. Then $\gamma(A) \subset \text{supp}(T(A))$.

**Proof.** Let $\lambda \in \gamma(A)$. Then $\lambda$ is a joint approximate eigenvalue of $A$ by [3, Proposition 2]. Choose vectors $x_n \in H$ satisfying $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and such that $\lim_{n \to \infty} \|A_jx_n - \lambda_jx_n\| = 0$ for $j = 1, \ldots, n$. Let $U$ be a free ultrafilter on $\mathbb{N}$ and $H_U = L^2(H) \otimes U(H)$ denote the $U$-product of $H$ (where $L^2(H)$ is the space of all bounded sequences converging to 0 along $U$; see [11, V.1]). Furthermore, let $(A_j)U$ be the canonical extension of $A_j$. Then $A_U = (A_1)_U, \ldots, (A_n)_U)$ is an $n$-tuple of selfadjoint operators on the Hilbert space $H_U$ and $x_U = (x_n) + c_U(H) \in H_U$ is an eigenvector of $(A_j)_U$ corresponding to $\lambda_j$, for each

$1 \leq j \leq n$. That is, $x_U$ is a joint eigenvector of $A_U$ corresponding to $\lambda$. From Lemma 2 we conclude that $\lambda \in \text{supp}(T(A_U))$. So, it remains to show that $\text{supp}(T(A_U)) = \text{supp}(T(A))$. This will follow from the identity

$$[T(A)f]u = T(A)fu, \quad f \in \mathcal{S}({\mathbb{R}}^n).$$

To establish this identity we note that the mapping $B \to B_U$ is an isometric homomorphism of the Banach algebra $L(H)$ into $L(H_U)$; see [11, V.1.2]. Thus we have $(\eta, A_U) = (\eta, A_U)$ for $\eta \in \mathbb{R}^n$. Then (by power series expansion) it follows that $e^{i(\eta, A)}u = e^{i(\eta, A)}u$ and finally, for $f \in \mathcal{S}({\mathbb{R}}^n)$, we have (by approximating the integral via Riemann sums)

$$\int_{\mathbb{R}^n} e^{i(\eta, A)}u \, d\eta = \int_{\mathbb{R}^n} e^{i(\eta, A)}u \, d\eta = (2\pi)^{n/2}T(A)f.$$
commute. By definition of $M[A]$ it follows that the closed subspace of $H$ generated by $\{H_\lambda(A) : \lambda \in \gamma(A)\}$ is contained in $M[A]$. On the other hand, the restrictions $(A_j)_{M[A]} : 1 \leq j \leq n$, form a mutually commuting family of compact selfadjoint operators in the Hilbert space $M[A]$. Accordingly, there exists an orthonormal basis of $M[A]$ consisting of joint eigenvectors of $(A_j)_{M[A]} : 1 \leq j \leq n$. Each such joint eigenvector $x \in M[A]$ of $A_{M[A]}$ is also a joint eigenvector of $A$ with the same joint eigenvalue $\mu$ as for $A_{M[A]}$. Lemma 2 implies that $\mu \in \gamma(A)$ and hence, $M[A]$ is contained in the closed subspace of $H$ generated by $\{H_\lambda(A) : \lambda \in \gamma(A)\}$.

The next result shows that for compact $n$-tuples $A$ the Weyl calculus $T(A)$ almost takes its values in the compact operators on $H$.

**Proposition 2.** Let $A = (A_1, \ldots, A_n)$ be an $n$-tuple of compact self-adjoint operators in a Hilbert space $H$. Then $T(A)f - f(0)I$ is a compact operator for every $f \in S(\mathbb{R}^n)$.

**Proof.** For $\xi \in \mathbb{R}^n$ fixed, a consideration of the power series expansion of $e^{i\langle \xi, A \rangle}$, together with the fact that each operator $(i\langle \xi, A \rangle)^r$, $r = 1, 2, \ldots$, is compact, shows that $e^{i\langle \xi, A \rangle} - I$ is compact. Let $B_N = \{x \in \mathbb{R}^n : |x| < N\}$ for each $N = 1, 2, \ldots$ and fix $f \in S(\mathbb{R}^n)$. Since the map $\xi \mapsto e^{i\langle \xi, A \rangle}$, $\xi \in \mathbb{R}^n$, is continuous for the operator norm topology in $L(H)$, the integral

$$K_N(f) = \int_{B_N} (e^{i\langle \xi, A \rangle} - I)\tilde{f}(\xi) \, d\xi$$

exists as the operator norm limit of Riemann sums and hence, is a compact operator. The conclusion follows from the identities

$$(2\pi)^{n/2}(T(A)f - f(0)I) = \int_{B_N} (e^{i\langle \xi, A \rangle} - I)\tilde{f}(\xi) \, d\xi + \int_{\mathbb{R}^n \setminus B_N} (e^{i\langle \xi, A \rangle} - I)\tilde{f}(\xi) \, d\xi,$$

with the estimates

$$\left\| \int_{\mathbb{R}^n \setminus B_N} (e^{i\langle \xi, A \rangle} - I)\tilde{f}(\xi) \, d\xi \right\| \leq 2 \int_{\mathbb{R}^n \setminus B_N} |\tilde{f}(\xi)| \, d\xi,$$

valid for $N = 1, 2, \ldots$, which show that $K_N(f) \to T(A)f - f(0)I$ as $N \to \infty$ in the operator norm topology.

Given a function $f : \mathbb{R}^n \to \mathbb{C}$ and $\nu \in \mathbb{R}^n$ define the $\nu$-translate $f_\nu : \mathbb{R}^n \to \mathbb{C}$ of $f$ by $f_\nu(x) = f(x - \nu)$ for $x \in \mathbb{R}^n$. For a subset $K \subset \mathbb{R}^n$ let $K - \nu = \{x - \nu : x \in K\}$. Finally, if $A = (A_1, \ldots, A_n)$ is an $n$-tuple of elements from $L(H)$ denote the $n$-tuple $(A_1 - \nu_1 I, \ldots, A_n - \nu_n I)$ by $A - \nu I$.

**Lemma 4.** Let $A = (A_1, \ldots, A_n)$ be an $n$-tuple of bounded selfadjoint operators in a Hilbert space $H$ and $\lambda \in \mathbb{R}^n$. Then

(i) $T(A)f_\lambda = T(A - \lambda I)f$ for every $f \in S(\mathbb{R}^n)$,

(ii) $\text{supp}(T(A - \lambda I)) = \text{supp}(T(A)) - \lambda$, and

(iii) $\gamma(A - \lambda I) = \gamma(A) - \lambda$.

**Proof.** (i) follows from the definition of $T(A)f_\lambda$, the fact that $\tilde{f}_\lambda = \mathcal{e}^{-i\langle \cdot, \lambda \rangle}\tilde{f}$ and the observation that

$$\mathcal{e}^{-i\langle \xi, \lambda \rangle}\mathcal{e}^{i\langle \xi, A \rangle} = \mathcal{e}^{-i\langle \xi, \lambda A \rangle} = \mathcal{e}^{i\langle \xi, A - \lambda I \rangle}, \quad \xi \in \mathbb{R}^n,$$

since the operators $(\xi, \lambda)$ and $(\xi, A)$ commute.

(ii) follows from (i), the definition of the support of a distribution, and the fact that $\text{supp}(f_\lambda) = \lambda + \text{supp}(f)$ for every $f \in C_c^\infty(\mathbb{R}^n)$.

(iii) follows from the definition of the sets involved.

We conclude this section with a topological result needed later.

**Proposition 3.** Let $K$ be a subset of $\mathbb{R}^n$ which is compact, infinite and countable. Let $P$ denote the set of all isolated points of $K$. Then

(i) $P$ is an infinite set, and

(ii) $K = \overline{P}$ (the bar denoting closure).

**Proof.** (i) The set $K = \bigcup_{\lambda \in K} \{\lambda\}$ is a countable union. By Baire's Theorem at least one set $\{\lambda\}$ has non-empty interior, that is, $\lambda \in P$. Choose any $\lambda \in P$. Then $K \setminus \{\lambda\}$ is again compact, infinite and countable and hence, also has isolated points. Continuing this argument inductively it follows that $P$ is infinite.

(ii) Suppose $\overline{P} \neq K$. Then $M = K \setminus \overline{P}$ is a non-empty, open subset of the compact space $K$. The set $M = \bigcup_{\lambda \in M} \{\lambda\}$ is a countable union, hence by Baire's Theorem at least one set $\{\lambda\}$, $\lambda \in M$, is open in $M$. Since $M$ is open (in $K$), $\{\lambda\}$ is open in $K$, that is, $\lambda \in P$, a contradiction.

2. Commutativity criteria. The purpose of this section is to present some criteria which characterize commutativity of $n$-tuples $A = (A_1, \ldots, A_n)$ of compact selfadjoint operators. These results are consequences of the following important fact concerning the nature of particular kinds of isolated points of $\text{supp}(T(A))$. First we require a new notion.

**Definition 2.** Let $M$ be a compact subset of $\mathbb{R}^n$. A point $\lambda \in M$ is called hyperisolated if it is isolated and there is a hyperplane (i.e. a maximal proper affine subspace of $\mathbb{R}^n$), say $L$, such that $L \cap M = \{\lambda\}$.

Analytically this means that there is a (necessarily non-zero) $\eta \in \mathbb{R}^n$ and $\varepsilon > 0$ such that $|\langle \lambda - \mu, \eta \rangle| \geq \varepsilon$ for every $\mu \in M$ with $\mu \neq \lambda$. 
Remark 1. \( \lambda \) is hyperisolated in \( M \) if and only if there exists a direction \( \eta \) and \( \varepsilon > 0 \) such that the \( n \)-dimensional strip
\[
S(\lambda, \eta, \varepsilon) = \lambda + \{ x \in \mathbb{R}^n : ||x, \eta|| < \varepsilon \}
\]
intersects \( M \) only at the point \( \lambda \).

Theorem 1. Let \( A = (A_1, \ldots, A_n) \) be an \( n \)-tuple of bounded selfadjoint operators in a Hilbert space \( H \) and \( \lambda \in \text{supp}(T(A)) \) be hyperisolated. Then \( \lambda \) is a joint eigenvalue of \( A \). Moreover, the decomposition \( H = H_\lambda(A) \oplus H_\lambda(A)^\perp \) reduces the \( n \)-tuple \( A \) (that is, \( A = A_{H_\lambda} \oplus A_{H_\lambda^\perp} \)) to the above form, and \( \text{supp}(T(A_{H_\lambda^\perp})) = \text{supp}(T(A)) \setminus \{ \lambda \} \).

Remark 2. (a) It will be shown in the course of the proof of Theorem 1 that the corresponding eigenprojection \( E_\lambda(A) \) equals \( T(A) \phi \), where \( \phi \in C_0^\infty(\mathbb{R}^n) \) is supported in a neighbourhood \( U_\lambda \) of \( \lambda \) with \( U_\lambda \cap \text{supp}(T(A)) = \{ \lambda \} \) and \( \phi \) is constant equal to \( 1 \) in a smaller neighbourhood of \( \lambda \). It follows from Proposition 2 that \( E_\lambda(A) \) is a finite rank projection whenever the operators \( A_j, 1 \leq j \leq n \), are compact and \( \lambda \neq 0 \).

(b) As a consequence of Theorem 1 we obtain \( \text{supp}(T(A)) = \{ \lambda \} \) if and only if \( A = \lambda I \). Furthermore, if \( \text{supp}(T(A)) \) is a finite set, then \( \text{supp}(T(A)) = \{ \lambda(1), \ldots, \lambda(m) \} \), and we can successively split off the joint eigenspaces. After \( m \) steps we end up with the following representation of \( A \): there exist zero-orthogonal projections \( P_1, \ldots, P_m \) satisfying \( \sum_{j=1}^m P_j = I \) and \( P_k P_j = P_j P_k = 0 \) for \( k \neq j \) such that \( A = \sum_{j=1}^m \lambda(j) P_j \). In particular, \( A_r = \sum_{j=1}^r \lambda(j) P_j \) for each \( 1 \leq r \leq n \), where \( \lambda(j) = (\lambda(1), \ldots, \lambda(j)) \).

For ease of reading we postpone the proof of Theorem 1 to the end of this section. We prefer first to establish some consequences. We begin with a finite-dimensional result.

Theorem 2. Let \( H \) be a Hilbert space of finite dimension \( k \geq 1 \) and \( A = (A_1, \ldots, A_n) \) be an \( n \)-tuple of selfadjoint operators in \( H \). The following statements are equivalent.

(i) The operators \( A_j, 1 \leq j \leq n \), mutually commute.
(ii) \( \text{supp}(T(A)) \) is a finite subset of \( \mathbb{R}^n \).
(iii) \( \text{supp}(T(A)) \) has at most \( k \) elements.
(iv) \( \gamma(A) = \text{supp}(T(A)) \).

Proof. (i)\(\Rightarrow\)(ii) follows from the main theorem in [10]; see also Remark 2(b). The implication (i)\(\Rightarrow\)(iv) is well known (cf. Introduction) and (iv)\(\Rightarrow\)(i) is obvious (see [3, Proposition 2]). Clearly (iii)\(\Rightarrow\)(ii). So, it remains to establish (ii)\(\Rightarrow\)(iii). Since each point of a finite set is hyperisolated it follows from Theorem 1 that each point of \( \text{supp}(T(A)) \) is a joint eigenvalue of \( A \). Since joint eigenvectors corresponding to distinct joint eigenvalues are eigenvectors of some \( A_j, 1 \leq j \leq n \), corresponding to distinct eigenvalues of \( A_j \), those joint eigenvectors are necessarily orthogonal in \( H \). So, there can be at most \( k \) points in \( \text{supp}(T(A)) \).

The next result illustrates "how different" the set \( \text{supp}(T(A)) \) is when the \( n \)-tuple \( A \) does not commute.

Theorem 3. Let \( H \) be a Hilbert space of finite dimension \( k \geq 1 \) and \( A = (A_1, \ldots, A_n) \) be an \( n \)-tuple of selfadjoint operators in \( H \). Then \( \text{supp}(T(A)) \) is either a set with at most \( k \) elements (in which case \( A \) commutes), or \( \text{supp}(T(A)) \) is an uncountable set (in which case \( A \) is not commutative).

Proof. Suppose that \( \text{supp}(T(A)) \) has more than \( k \) elements, in which case it is an infinite set by Theorem 2. Suppose that it is a countable set. Then Proposition 3 implies that the set \( P \) of all isolated points of \( \text{supp}(T(A)) \) is infinite as well. Since each point of \( P \) is hyperisolated (see the following Proposition 4) each such point is a joint eigenvalue of \( A \) by Theorem 1. This is impossible as \( H \) is finite-dimensional and joint eigenvectors of \( A \) corresponding to distinct joint eigenvalues are orthogonal. Accordingly, \( \text{supp}(T(A)) \) is an uncountable subset of \( \mathbb{R}^n \).

Proposition 4. Let \( A = (A_1, \ldots, A_n) \) be an \( n \)-tuple of bounded selfadjoint operators in a Hilbert space \( H \). If \( \text{supp}(T(A)) \) is a countable subset of \( \mathbb{R}^n \), then

(i) each isolated point of \( \text{supp}(T(A)) \) is hyperisolated, and
(ii) \( \gamma(A) = \text{supp}(T(A)) \).

Proof. (i) By a suitable translation it suffices to consider the special case of 0 being an isolated point of \( \text{supp}(T(A)) \); see Lemma 4. Since the countable union of hyperplanes \( V = \bigcup \{ \text{ker}(\lambda) : \lambda \in \text{supp}(T(A)) \setminus \{0\} \} \) cannot be all of \( \mathbb{R}^n \) (hyperplanes have Lebesgue measure 0) there must exist a point \( \eta \in \mathbb{R}^n \setminus V \). Then the hyperplane \( \text{ker}(\lambda, \eta) \) intersects \( \text{supp}(T(A)) \) only in 0. Here \( \langle \cdot, \lambda \rangle \) denotes the linear functional \( x \mapsto \langle x, \lambda \rangle, x \in \mathbb{R}^n \).

(ii) By Theorem 1 and part (i) all isolated points of \( \text{supp}(T(A)) \) are joint eigenvalues. By Lemma 2 they belong to \( \gamma(A) \). By Proposition 3, \( \text{supp}(T(A)) \) is the closure of its isolated points. Since \( \gamma(A) \) is a closed set [3, Proposition 1], it follows that \( \text{supp}(T(A)) \subset \gamma(A) \). The converse inclusion is just Lemma 3.

The following result may be viewed as a natural extension of Theorem 2 to a class of operators in infinite-dimensional spaces.

Theorem 4. Let \( A = (A_1, \ldots, A_n) \) be an \( n \)-tuple of compact selfadjoint operators in a Hilbert space \( H \). The following statements are equivalent.

(i) The operators \( A_j, 1 \leq j \leq n \), mutually commute.
(ii) \( \text{supp}(T(A)) \) is a countable subset of \( \mathbb{R}^n \).
(iii) $\text{supp}(T(A))$ is a countable subset of $\mathbb{R}^n$ with 0 as only possible limit point.

(iv) $\gamma(A) = \text{supp}(T(A))$.

Proof. (i) $\Rightarrow$ (iv) is well known (cf. Introduction) and (iv) $\Rightarrow$ (iii) by Corollary 3.1 of [3]. The implication (iii) $\Rightarrow$ (ii) is obvious and (ii) $\Rightarrow$ (iv) follows from Proposition 4.

So, it remains to establish (iv) $\Rightarrow$ (i). Let $M = M[A]$ be the maximal abelian subspace of $A_0$, in which case $H = M \oplus M^\perp$. By Lemma 1 it follows that

$$\text{supp}(T(A_{M[A]})) \subseteq \text{supp}(T(A)) = \gamma(A)$$

and hence, $\text{supp}(T(A_{M[A]}))$ is a countable set. Suppose that $M^\perp \neq \{0\}$. Then $\text{supp}(T(A_{M[A]}))$ is a nonempty, countable, compact set, hence it has an isolated point $\lambda$ (by Proposition 3). By Proposition 4(i) it is hyperisolated and then by Theorem 1 it is a joint eigenvalue of $A_{M[x]}$. Clearly a corresponding joint eigenvector $x \in M^\perp$ of $A_{M[x]}$ is also a joint eigenvector of $A$. This is a contradiction since, by Proposition 1(ii), joint eigenvectors belong to $M$.

Remark 3. Slightly more is true than proved in Theorem 4. Namely, let $A = (A_1, \ldots, A_n)$ be an $n$-tuple of bounded selfadjoint operators. It is not assumed that the operators $A_j$, $1 \leq j \leq n$, are compact. If $\text{supp}(T(A))$ is a countable set, then the operators $A_j$, $1 \leq j \leq n$, mutually commute. This follows from the same argument as used to establish (iv) $\Rightarrow$ (i) in the proof of Theorem 4, after noting that Proposition 4 implies $\gamma(A) = \text{supp}(T(A))$.

Theorem 4 shows that the spectral set $\gamma(A)$, originally introduced for commuting $n$-tuples $A$ actually characterizes commutativity of $A$ for the case of $n$-tuples of compact selfadjoint operators. The following example shows that the hypothesis of the operators $A_j$, $1 \leq j \leq n$, being compact cannot be omitted.

Example 1. Let $B = (B_1, B_2)$ where $B_1 = (\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix})$ and $B_2 = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$ are considered as selfadjoint operators in $\mathbb{C}^2$. It will be shown in Example 2 of Section 3 that $\text{supp}(T(B)) = \mathbb{D}$, where $\mathbb{D} = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \}$. Since $B_1B_2 \neq B_2B_1$, it follows from [3, Proposition 7] that $\gamma(B) = \emptyset$. Let $\{ (\lambda^k)_{k=1}^\infty \}$ be a countable dense subset of $\mathbb{D}$ and let $T$ be the multiplication operator on $\mathbb{C}^2$ with the (bounded) sequence $\{ (\lambda^k)_{k=1}^\infty \}$. Then $T$ is normal, hence $C_1 = \mathbb{R}(T) = \frac{1}{2}(T + T^*)$ and $C_2 = \mathbb{S}(T) = \frac{1}{2i}(T - T^*)$ commute. Thus, for the pair $C = (C_1, C_2)$, we have $\gamma(C) = \text{supp}(T(C))$. Each $\lambda^k$ is a joint eigenvalue of $C$ (the $k$th unit vector is a corresponding eigenvector) and so, by the closedness of $\gamma(C)$, it follows that $\mathbb{D} \subseteq \gamma(C)$. Since $T$ is a contraction, for each $\nu = \nu_1 + i\nu_2$, $\nu \notin \mathbb{D}$, the operator

$$(\nu_1I - C_1)^2 + (\nu_2I - C_2)^2 = (\nu I - T)(\nu I - T)^*$$

is invertible, that is, $(\nu_1, \nu_2) \notin \gamma(C)$. Accordingly, $\gamma(C) = \text{supp}(T(C)) = \mathbb{D}$. Let $A_j = B_j \oplus C_j$, for $j \in \{1, 2\}$, act in the Hilbert space $H = \mathbb{C}^2 \oplus \mathbb{C}^2$ (ident $\mathbb{D}$). Then Lemma 1 implies that

$$\text{supp}(T(A)) = \text{supp}(T(B)) \cup \text{supp}(T(C)) = \mathbb{D} \cup \mathbb{D} = \mathbb{D}$$

and also that

$$\gamma(A) = \gamma(B) \cup \gamma(C) = \emptyset \cup \emptyset = \mathbb{D}.$$ 

However, $A_1A_2 \neq A_2A_1$, since $B_1B_2 \neq B_2B_1$.

Proof of Theorem 1. By translating to the origin (cf. Lemma 4) it suffices to prove the result for the case when 0 is a hyperisolated point of $\text{supp}(T(A))$.

Choose a non-negative function $\varphi \in C^\infty_c(\mathbb{R}^n)$ which is supported inside a disc $B_\varepsilon = \{ x \in \mathbb{R}^n : |x| < \varepsilon \}$, for some $\varepsilon > 0$, such that $\varphi$ is constantly 1 near 0 (say, in $B_{\varepsilon/2}$ and $\mathbb{R} \setminus \text{supp}(T(A)) = \{0\}$).

Define a distribution $U : \mathcal{S}(\mathbb{R}^n) \to \mathcal{L}(H)$ by

$$U(f) = T(A)(\varphi f), \quad f \in \mathcal{S}(\mathbb{R}^n).$$

Then $\text{supp}(U) = \{0\}$ and $U$ is of finite order, say $N$ (with $N$ not exceeding the finite order of $T(A)$ [1, Lemma 3.8]). So, there exist bounded operators $R_\alpha$ for $|\alpha| \leq N$ (multi-index notation) such that

$$U(f) = \sum_{|\alpha| \leq N} (D^\alpha f)(0)R_\alpha, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

Being compactly supported, this distribution has a (unique) extension to $C^\infty_c(\mathbb{R}^n)$. For fixed $\xi \in \mathbb{R}^n$, let $e_\xi(x) = e^{i\langle \xi, x \rangle}$, $x \in \mathbb{R}^n$, in which case

$$T(A)(\varphi e_\xi) = \sum_{|\alpha| \leq N} \xi^\alpha R_\alpha e_\xi.$$

On the other hand, since $\langle \varphi e_\xi \rangle = (\hat{\varphi}(\xi))$ it follows that

$$T(A)(\varphi e_\xi) = (2\pi)^{-n/2} \int e^{i\langle \xi, A \rangle} \hat{\varphi}(y - \xi) dy = (2\pi)^{-n/2} \int e^{i\langle u + \xi, A \rangle} \hat{\varphi}(u) du$$

and hence, since $\|e^{i\langle u + \xi, A \rangle}\| = 1$, we have

$$\|T(A)(\varphi e_\xi)\| \leq (2\pi)^{-n/2} \int \|\hat{\varphi}(u)\| du < \infty$$

for all $\xi \in \mathbb{R}^n$. It follows from (5) that $R_0 = 0$ whenever $|\alpha| > 0$. Denoting $R_0$ simply by $R$ gives

$$T(A)(\varphi f) = f(0)R, \quad f \in C^\infty_c(\mathbb{R}^n).$$

Fact 1. The operator $R$ is non-zero, selfadjoint and coincides with $T(A)\varphi$. 

Proof. Substitute \( f = \varphi \) into (6) and use the fact that \( \varphi^2 = \varphi \) in a neighbourhood of \( \supp(T(A)) \) yields \( R = T(A)\varphi \). That \( R \) is selfadjoint then follows from [1, Theorem 2.9]. To see that \( R \neq 0 \), let \( f_k \in C^\infty_\text{c}(\mathbb{R}^n) \) be any function supported in \( B_\delta \) such that \( T(A)f_k \neq 0 \); since \( 0 \in \supp(T(A)) \) such a function \( f_k \) exists. Then \( f_k \varphi \) coincides with \( f_k \) in a neighbourhood of \( \supp(T(A)) \) and so \( T(A)(f_k \varphi) = T(A)f_k \neq 0 \). But \( T(A)(f_k \varphi) = f_k(0)R \) by (6). Accordingly, \( f_k(0)R \neq 0 \) and hence, also \( R \neq 0 \).

Since \( 0 \) is hyperisolated there is \( \eta \in \mathbb{R}^n \) with \( |\eta| = 1 \) and \( \delta > 0 \) such that the strip \( S(\eta, \delta) = \{ x \in \mathbb{R}^n : ||x, \eta|| < \delta \} \) intersects \( \supp(T(A)) \) only in \( 0 \). Let \( M_{\eta} = (m_{\eta j})_{1 \leq j \leq n} \) be an orthogonal \((n \times n)\)-matrix which maps the unit vector \( \eta \) onto the unit vector \( e_1 = (1, 0, \ldots, 0) \), i.e. \( M_{\eta} \eta = e_1 \). Let \( M_{\eta}A \) be the \( n \)-tuple of selfadjoint operators given by \((M_{\eta}A)_j = \sum_{k=1}^n m_{\eta j} A_k \). By [1, Theorem 2.9(a)] we have

\[
T(M_{\eta}A)f = T(A)(f \circ M_{\eta}), \quad f \in S(\mathbb{R}^n).
\]

Since both distributions \( T(A) \) and \( T(M_{\eta}A) \) have compact support, identity (7) also holds for \( f \in C^\infty(\mathbb{R}^n) \). We choose a non-zero \( C^\infty \)-function \( \varphi_1 : \mathbb{R} \to [0, 1] \) which is constantly 1 on a neighbourhood of 0 and vanishes on \( \{ t \in \mathbb{R} : |t| \geq \delta \} \). Define \( \tilde{\varphi} \in C^\infty(\mathbb{R}^n) \) by \( \tilde{\varphi}(x) = \varphi_1(x, \eta) \). Then \( \tilde{\varphi} \) coincides with \( \varphi \) on a neighbourhood of \( \supp(T(A)) \) and hence,

\[
R = T(A)\varphi = T(A)\tilde{\varphi} = T(M_{\eta}A)(\tilde{\varphi} \circ M_{\eta}^t),
\]

where \( M_{\eta}^t \) denotes the transpose of \( M_{\eta} \). We have

\[
(\tilde{\varphi} \circ M_{\eta}^t)(x) = \varphi_1((M_{\eta}^t x, \eta)) = \varphi_1((x, M_{\eta} \eta)) = \varphi_1(x_1).
\]

That is, \( \tilde{\varphi} \circ M_{\eta}^t \) is a function depending on just one of the variables. It follows from Theorem 2.9(b) of [1] that

\[
R = T(M_{\eta}A)(\tilde{\varphi} \circ M_{\eta}^t) = T((M_{\eta}A)^t)\varphi_1.
\]

Note that \( M_{\eta} \eta = e_1 \) implies \( M_{\eta}^t e_1 = \eta \), that is, \( m_{1j} = \eta_j \) for \( j = 1, \ldots, n \). It follows that \((M_{\eta}A)_1 = \eta_1 A_1 = \sum_{j=1}^n m_{1j} A_j = (\eta, A) \). Thus we have

\[
R = T(M_{\eta}A)(\tilde{\varphi} \circ M_{\eta}^t) = T((\eta, A)\varphi_1) = (2\pi)^{-1/2} \int_0^\infty e^{it(\eta, A)} \tilde{\varphi}_1(t) \, dt.
\]

By multiplicativity of the Weyl calculus for a single operator [1, Lemma 3.1] and the fact that \( \tilde{\varphi}^2 = \tilde{\varphi} \) on a neighbourhood of \( \supp(T(A)) \) we have

\[
R = T(T(A)\varphi^2) = T((M_{\eta}A)_1)(\varphi^2) = (T((\eta, A)\varphi_1))^2 = R^2.
\]

Thus we have established

**Fact 2.** \( R \) is a projection.
3. Pairs of selfadjoint operators. The aim of this section is to extend Proposition 10 of [3], formulated for 2-dimensional Hilbert spaces, to arbitrary finite-dimensional Hilbert spaces. First a preliminary result is needed.

PROPOSITION 5. Let \( A = (A_1, A_2) \) be a pair of bounded selfadjoint operators in a Hilbert space \( H \). Then

(i) \( \gamma(A) \subset \sigma(A_1 + iA_2) \).

Suppose, in addition, that \( A_1 \) and \( A_2 \) are compact.

(ii) If \( A_1 A_2 = A_2 A_1 \), then \( \gamma(A) = \sigma(A_1 + iA_2) \).

Proof. (i) Choose \( \lambda \in \gamma(A) \). Then \( 0 \not\in \sigma(S) \), where \( S = (A_1 - \lambda I)^2 + (A_2 - \lambda I)^2 \). Since \( S \) is selfadjoint, there are unit vectors \( x_n \) such that \( Sx_n \to 0 \) in \( H \) as \( n \to \infty \). Then also \( (Sx_n, x_n) \to 0 \) and hence, \( (A_j - \lambda I)x_n \to 0 \) as \( n \to \infty \), for each \( j \in \{1, 2\} \), from which the result follows.

(ii) Since \( A_1 + iA_2 \) is a compact (normal) operator its spectrum is a countable set with 0 as only possible limit point. Suppose that \( \lambda \in \sigma(A_1 + iA_2) \setminus \{0\} \), in which case \( \lambda \) is an eigenvalue of \( A_1 + iA_2 \). So, there is \( x \neq 0 \) such that \( (A_1 + iA_2)x = (\lambda + i\lambda_2)x \), where \( \lambda = \lambda_1 + i\lambda_2 \). That is, \( [(A_1 - \lambda_1 I) + i(A_2 - \lambda_2 I)]x = 0 \) and hence, also

\[
[(A_1 - \lambda_1 I) - i(A_2 - \lambda_2 I)](A_1 - \lambda_1 I) + i(A_2 - \lambda_2 I)x = 0.
\]

Expanding this identity and using \( A_1 A_2 = A_2 A_1 \) gives

\[
[(A_1 - \lambda_1 I)^2 + (A_2 - \lambda_2 I)^2]x = 0.
\]

Since \( x \neq 0 \) it follows that \( \lambda = (\lambda_1, \lambda_2) \) belongs to \( \gamma(A) \). This shows that \( \sigma(A_1 + iA_2) \setminus \{0\} \subset \gamma(A) \). If \( 0 \) is also an eigenvalue of \( A_1 + iA_2 \), then the same argument shows that \( 0 \in \gamma(A) \). Otherwise, \( 0 \) is a limit point of \( \sigma(A_1 + iA_2) \), in which case the closedness of \( \gamma(A) \) ensures that \( 0 \in \gamma(A) \). Hence, \( \sigma(A_1 + iA_2) \subset \gamma(A) \) and so, by part (i), \( \sigma(A_1 + iA_2) = \gamma(A) \).

PROPOSITION 6. Let \( H \) be a Hilbert space of dimension \( k < \infty \) and \( A = (A_1, A_2) \) be a pair of selfadjoint operators in \( H \). The following statements are equivalent:

(i) \( A_1 A_2 = A_2 A_1 \).

(ii) \( A_1 + iA_2 \) is a normal operator.

(iii) The standard (polynomial) functional calculus \( S(A_1 + iA_2) : C^k(\mathbb{R}^2) \to \mathcal{L}(H) \) of the single operator \( A_1 + iA_2 \), when restricted to \( C^\infty(\mathbb{R}^2) \), agrees with the extension of the Weyl calculus \( T(A) : S(\mathbb{R}^2) \to \mathcal{L}(H) \) to \( C^\infty(\mathbb{R}^2) \).

(iv) \( S(A_1 + iA_2)\overline{x} = T(A)\overline{x} \), where \( \overline{x} = x - iy \).

(v) The Weyl calculus \( T(A) \) is multiplicative in \( C^\infty(\mathbb{R}^2) \).

(vi) \( \text{supp}(T(A)) \) is a finite subset of \( \mathbb{R}^2 \).

(vii) \( T(A) \) has order zero as a distribution.

(viii) \( \text{supp}(T(A)) = \gamma(A) \).

(ix) \( \text{supp}(T(A)) = \sigma(A_1 + iA_2) \), where \( \lambda_1 + i\lambda_2 \in C \) is identified with \( (\lambda_1, \lambda_2) \in \mathbb{R}^2 \).

Proof. The mutual equivalence of the first five statements follows from [3, Proposition 9]. The equivalence of (i) with both (vi) and (vii) is the main Theorem in [10]; see also [4] for the case \( k = 2 \). Theorem 2 implies that (i) \( \Leftrightarrow \) (viii). Clearly (ix) \( \Rightarrow \) (vi). Finally, (i) \( \Rightarrow \) (viii) by Proposition 5 and the implication (i) \( \Rightarrow \) (viii).

Remark 4. In Proposition 10 of [3] it is shown (for 2-dimensional Hilbert spaces) that each of the statements in Proposition 6 is equivalent to

\( (x) \) \( \gamma(A) \neq \emptyset \).

\( (x) \) \( \sigma(A_1 + iA_2) = \gamma(A) \), where \( \lambda_1 + i\lambda_2 \in C \) is identified with \( (\lambda_1, \lambda_2) \in \mathbb{R}^2 \).

It is shown in Remark 2 of [3] that the statements of Proposition 6 are not equivalent to statement (x) for \( \dim(H) > 2 \). We conclude with an example which shows that the statements of Proposition 6 are also not equivalent to statement (x) for \( 2 < \dim(H) < \infty \). Indeed, it is shown that \( \gamma(A) = \sigma(A_1 + iA_2) \) is a proper subset of \( \text{supp}(T(A)) \).

EXAMPLE 2. Let \( B_1 \) and \( B_2 \) be the \( (2 \times 2) \)-selfadjoint matrices given in Example 1 and \( u \in \mathbb{R}^2 \). Let \( A_j = (B_j^0, 0) \) for \( j \in \{1, 2\} \), considered as operators in the Hilbert space \( H = C^4 \). Since \( B_1 + iB_2 \) is nilpotent (of order 2) it follows that \( \sigma(B_1 + iB_2) = \{0\} \). Since \( A_1 + iA_2 \) equals the direct sum \( (B_1 + iB_2) \oplus (u_1 + iu_2)I \) in \( C^3 \) \( \oplus \mathbb{C} \) it follows that

\[
\sigma(A_1 + iA_2) = \sigma(B_1 + iB_2) \cup \sigma((u_1 + iu_2)I) = \{0\} \cup \{u_1 + iu_2\},
\]

It was shown in Remark 2 of [3] that \( \gamma(A) = \{u_1, u_2\} \), where \( A = (A_1, A_2) \). Moreover, Lemma 1 implies that

\[
\text{supp}(T(A)) = \text{supp}(T(B)) \cup \{u_1, u_2\}.
\]

Putting \( u_1 = u_2 = 0 \) shows that \( \gamma(A) = \sigma(A_1 + iA_2) \), even though \( A_1 A_2 \neq A_2 A_1 \). However, as must be the case, \( \sigma(A_1 + iA_2) = \gamma(A) \) is a proper subset of \( \text{supp}(T(A)) \).

It remains to show that \( \text{supp}(T(B)) = D \). Let \( S^2 = \{x \in \mathbb{R}^3 : |x| = 1\} \) be the 2-dimensional sphere and \( m \) denote normalized surface measure on \( S^2 \). For \( f \in S(\mathbb{R}^2) \), the functions

\[
x = (x_1, x_2) \mapsto f(x_1, x_2) \pm \partial_1 f(x_1, x_2) + (x_1 + x_2) \partial_2 f(x_1, x_2), \quad x \in S^2,
\]

are denoted by \( f \pm \partial_1 f + (x_1 + x_2) \partial_2 f \). It follows from Theorem 4.1 of [1] that
(8) \[ T(B)f = \left( \int_{S^2} (f + \partial_1 f + (x_1 + x_2) \partial_2 f) \, dm(x) \right) \int_{S^2} \partial_3 f \, dm(x) \]
\[ + \int_{S^2} (f - \partial_1 f + (x_1 + x_2) \partial_2 f) \, dm(x) \]
for every \( f \in S(\mathbb{R}^2) \); see [4] for the details. It is clear from (8) that \( \text{supp}(T(B)) \subset \mathbb{D} \). Since \( \text{supp}(T(B)) \) is equal to the union of the supports of the four distributions forming the entries of the right-hand side of (8) it suffices to show that the support of the \( \mathbb{C} \)-valued distribution
\[ V : f \mapsto \int_{S^2} \partial_3 f \, dm(x), \quad f \in S(\mathbb{R}^2), \]
contains \( \mathbb{D} \). But \( Vf = 2 \int_{S^2_+} \partial_3 f \, dm(x) \) where \( S^2_+ = \{ x \in S^2 : x_3 \geq 0 \} \) and so the problem reduces to showing that the support of the distribution
\[ U : f \mapsto \int_{S^2_+} \partial_3 f \, dm(x), \quad f \in S(\mathbb{R}^2), \]
contains \( \mathbb{D} \). Let \( \psi(u, v) = (1 - u^2 - v^2)^{-1/2} \) for \( u^2 + v^2 < 1 \). Then a transformation of measure shows that \( Uf = \int_0^1 \psi(u,v) \partial_3 f(u,v) \, du \, dv \). By considering functions of the form \( f(u,v) = g(u)h(v) \), for suitable \( g \) and \( h \), it can be shown that all interior points of \( \mathbb{D} \) belong to \( \text{supp}(U) \) and hence, \( \mathbb{D} \subset \text{supp}(U) \). □

References