Ambiguous loci of the farthest distance mapping from compact convex sets

by

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Abstract. Let $E$ be a strictly convex separable Banach space of dimension at least 2. Let $K(E)$ be the space of all nonempty compact convex subsets of $E$ endowed with the Hausdorff distance. Denote by $K^0$ the set of all $X \in K(E)$ such that the farthest distance mapping $a \mapsto M_X(a)$ is multivalued on a dense subset of $E$. It is proved that $K^0$ is a residual dense subset of $K(E)$.

1. Introduction and preliminaries. Throughout the present paper $E$ denotes a strictly convex separable Banach space of dimension at least 2, and $K(E)$ (resp. $B(E)$) the family of all nonempty compact convex (resp. closed bounded) subsets of $E$. The spaces $K(E)$ and $B(E)$ are equipped with the Hausdorff distance $h$ under which, as is well known, both are complete. For $X \in B(E)$ and $a \in E$ we set

$$e_X(a) = \sup \{\|x-a\| \mid x \in X\}.$$ 

Given $X \in B(E)$ and $a \in E$, let us consider the maximization problem, denoted $\max(a, X)$, which consists in finding some point $x \in X$ such that $\|x-a\| = e_X(a)$. Any such $x$ is said a solution of $\max(a, X)$ and any sequence $\{x_n\} \subset X$ satisfying $\lim_{n \to \infty} \|x_n - a\| = e_X(a)$ is called a maximizing sequence of $\max(a, X)$.

In a metric space $Z$, $B_Z(z, r)$ (resp. $\bar{B}_Z(z, r)$) is an open (resp. closed) ball with center $z \in Z$ and radius $r > 0$ (resp. $r \geq 0$). For any $X \subset Z$, $\bar{X}$ and diam $X$ ($X \neq \emptyset$) stand for the closure of $X$ and the diameter of $X$, respectively.

A set $X \subset Z$ is called everywhere uncountable in $Z$ if for every $z \in Z$ and $r > 0$ the set $X \cap B_Z(z, r)$ is nonempty and uncountable.

For $X \in K(E)$ we denote by $M_X : E \to K(E)$ the farthest distance mapping, defined by

$$M_X(a) = \{x \in X \mid \|x-a\| = e_X(a)\}.$$
We call \( M_X(a) \) the solution set of the maximization problem \( \max(a, X) \). Moreover, the set
\[
A(M_X) = \{ a \in E \mid M_X(a) \text{ contains at least 2 points} \}
\]
is called the ambiguous locus of \( M_X \).

In this note we consider approximation problems for the mapping \( e_X \) from sets \( X \in K(E) \). It is known that, if \( E \) is also uniformly convex, then the ambiguous locus of any set \( X \in K(E) \) is \( \sigma \)-porous, thus of the first Baire category and of Lebesgue measure zero if \( E = \mathbb{R}^n \) (see [4] and, for similar results, Bartkić and Berens [2] and Zajićev [13]). However, the set \( A(M_X) \), though small from the category and the measure point of view, can be unexpectedly rich in points scattered all over \( E \). More precisely, we show that in every strictly convex separable Banach space \( E \) of dimension at least 2 there exists a nonempty compact convex set \( X \) for which the ambiguous locus \( A(M_X) \) is everywhere uncountable in \( E \). Actually we prove more, namely that such a property of \( X \) is shared by most compact convex sets in \( K(E) \), in the Baire category sense.

For \( a \in E \) and \( X \in B(E) \) the set \( M_X(a) \) can be empty (see Miyajima and Wada [11] for some examples). Under suitable assumptions on \( E \) and \( X \), Asplund [1] and Lau [9] (see also Edelman [7], Panda and Dwivedi [12], Deville and Zizler [5]) have proved that the set of all \( a \in E \) for which \( M_X(a) \) is empty is of the Baire first category in \( E \). The question whether this set can be locally rich in points seems not yet settled.

Our approach is based on the Baire theorem. This has proven to be a useful tool in order to get existence results in several problems of geometry, starting with the classical work of Klee [10]. Developments of such ideas can be found in Gruber [8] and Zamfirescu [14, 15].

2. Lemmas

**Lemma 2.1.** Let \( a, x_1, x_2 \in E \), \( x_1 \neq x_2 \), be such that \( \|x_1 - a\| = \|x_2 - a\| \). For \( \theta \in \Delta = \{d_1, d_2\}, 0 < d_1 \leq d_2 \leq 1 \), set \( a_i(\theta) = a + \theta(x_i - a), \ i = 1, 2 \). Then there exists an \( \varepsilon_0 > 0 \) such that, for every \( \theta \in \Delta \),
\[
(2.1) \quad \|x_2 - a_1(\theta)\| > \|x_1 - a_1(\theta)\| + \varepsilon_0,
\]
\[
(2.2) \quad \|x_1 - a_2(\theta)\| > \|x_2 - a_2(\theta)\| + \varepsilon_0.
\]

**Proof.** It suffices to prove (2.1) (the proof of (2.2) is analogous). If the statement is not true, there exists a \( \delta \in \Delta \) such that \( \|x_2 - a_1(\delta)\| \leq \|x_1 - a_1(\delta)\| \). Furthermore,
\[
\|x_1 - a\| = \|a_1(\delta) - a\| + \|x_1 - a_1(\delta)\| \geq \|a_1(\delta) - a\| + \|x_2 - a_1(\delta)\|
\]
\[
\geq \|a_1(\delta) - a\| + \|x_2 - a\| - \|a_2(\delta) - a\| = \|x_2 - a\|,
\]
which implies that
\[
\|x_2 - a_1(\delta)\| + \|a_2(\delta) - a\| = \|x_2 - a_1(\delta)\| + \|a_1(\delta) - a\|.
\]
Since \( E \) is strictly convex, for some \( \beta > 0 \) we have \( x_2 - a_1(\delta) = \beta(a_1(\delta) - a) \). Hence \( x_2 - a = (1 + \beta)(a_1(\delta) - a) = (1 + \beta)(x_1 - a_1(\delta)) \), which yields \( x_2 = x_1 \), a contradiction. This completes the proof.

**Lemma 2.2.** In addition to the assumptions of Lemma 2.1, set \( b_\theta(t) = (1 - t)a_1(\theta) + ta_2(\theta), \ t \in [0, 1] \). Then there exists an \( \varepsilon > 0 \) such that, for every \( \theta \in \Delta \) and every \( C_1 \subset B_E(x_1, \varepsilon) \), \( C_2 \subset B_E(x_2, \varepsilon) \) with \( C_1, C_2 \neq \emptyset \),
\[
(2.3) \quad e_{C_1}(a_1(\theta)) > e_{C_1}(a_1(\theta)),
\]
\[
(2.4) \quad e_{C_1}(a_2(\theta)) > e_{C_2}(a_2(\theta))
\]
Moreover, there exists a \( t = t(\theta, C_1, C_2) \in [0, 1] \) such that
\[
(2.5) \quad e_{C_1}(b_\theta(t)) = e_{C_2}(b_\theta(t)).
\]

**Proof.** By Lemma 2.1 there exists an \( \varepsilon_0 > 0 \) such that for every \( \theta \in \Delta \),
\[
(2.1) \quad \|x_2 - a_1(\theta)\| > \|x_1 - a_1(\theta)\| + \varepsilon_0,
\]
\[
(2.2) \quad \|x_1 - a_2(\theta)\| > \|x_2 - a_2(\theta)\| + \varepsilon_0.
\]
Hence \( (2.3) \) is proved. The proof of \( (2.4) \) is analogous. Furthermore, the function \( \theta \mapsto e_{C_1}(b_\theta(t)) - e_{C_2}(b_\theta(t)) \) is continuous on \([0, 1]\) and, by \( (2.3) \) and \( (2.4) \), assumes values of opposite signs at the end points of \([0, 1]\). Thus there exists a \( t = t(\theta, C_1, C_2) \in [0, 1] \) for which \( (2.5) \) is satisfied. This completes the proof.

**Lemma 2.3.** Let \( a \in E \) and \( 0 < r < R \) and \( x_1, x_2 \in E \), \( x_1 \neq x_2 \), be such that \( \|x_1 - a\| = \|x_2 - a\| = R \). Let \( X \subset B_E(a, r) \) with \( X \in K(E) \). Set \( \Delta = [d/8, d/4] \), where \( d = (R - r)/R \). Define
\[
Z = \overline{c}(X \cup \{x_1, x_2\})
\]
and let \( b_\theta(t) \) and \( a_\theta(\theta) \) be defined as in the previous lemmas. Then:
(i) For \( \theta \in \Delta \) and \( t \in [0, 1] \), the maximization problem \( \max(b_\theta(t), Z) \) has solution set \( M_Z(b_\theta(t)) \) satisfying
\[
(2.6) \quad M_Z(b_\theta(t)) \subset \{x_1, x_2\},
\]
Moreover, \( M_Z(b_\theta(0)) = x_2 \) and \( M_Z(b_\theta(1)) = x_1 \).
(ii) For \( \theta \in \Delta \) and \( t \in [0, 1] \), every maximizing sequence \( \{x_n\} \) of \( \max(b_\theta(t), Z) \) has a subsequence which converges to a point \( z \in \{x_1, x_2\} \).
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\textbf{Proof.} For \( \theta \in \Delta \) and \( t \in [0,1] \), define \( \varphi_{b(t)} : Z \to \mathbb{R} \) by \( \varphi_{b(t)}(x) = \|x - b(t)\| \). As the function \( \varphi_{b(t)} \) is continuous on the compact set \( Z \), \( \varphi_{b(t)} \) attains its supremum at some point, say \( \bar{x} \in \bar{Z} \). Set

\[ E = \{ z \in Z \mid \varphi_{b(t)}(z) = \varphi_{b(t)}(\bar{x}) \} \]

and observe that \( E \neq \emptyset \). We claim that \( E \subset \{ \varphi_1, \varphi_2 \} \).

Indeed, as \( \varphi_{b(t)} \) is strictly convex on \( Z \), a convex set, we have \( E \subset \text{ext} \bar{Z} \), where \( \text{ext} \bar{Z} \) denotes the set of the extreme points of \( Z \). Moreover, by Krein–Milman’s theorem [6], \( \text{ext} \bar{Z} \subset X \cup \{ \varphi_1, \varphi_2 \} \), and thus \( E \subset X \cup \{ \varphi_1, \varphi_2 \} \). To prove the claim it suffices to show that \( E \cap X = \emptyset \). Suppose otherwise, and let \( u \in E \cap X \). Then

\[ \varphi_{b(t)}(u) = \|u - b(t)\| \leq \|u - a\| + \|a - b(t)\| \leq r + \frac{R - r}{4} \]

since, by a simple calculation, \( \|a - b(t)\| \leq \theta R \leq (d/4)R = (R - r)/4 \). On the other hand, for \( i = 1, 2 \), we have

\[ \varphi_{b(t)}(x_i) = \|x_i - b(t)\| \geq \|x_i - a\| - \|a - b(t)\| \geq R - \frac{R - r}{4} \]

Hence \( \varphi_{b(t)}(u) < \varphi_{b(t)}(x_i) \), \( i = 1, 2 \), which implies that \( u \notin E \), a contradiction. Thus \( E \subset \{ \varphi_1, \varphi_2 \} \). Since \( E = M_Z(b_b(t)) \), (2.6) is proved. Moreover, by Lemma 2.1, we have

\[ \varphi_{b(t)}(x_2) = \|x_2 - a(t)\| > \|x_1 - a(t)\| = \varphi_{b(t)}(x_1) \]

which implies that \( M_Z(b_b(t)) = x_2 \). Similarly one can show that \( M_Z(b_b(1)) = x_1 \), and so (i) is proved.

To prove (ii), for given \( \theta \in \Delta \) and \( t \in [0,1] \), let \( \{ \varphi_n \} \subset Z \) be a maximizing sequence of \( \max(b_b(t), Z) \). As \( Z \) is compact, passing to a subsequence we can assume that \( \lim_{n \to \infty} \varphi_n = \bar{x} \) for some \( \bar{x} \in Z \). This implies that \( \bar{x} \in M_Z(b_b(t)) \) and so, by (i), \( \bar{x} \in \{ \varphi_1, \varphi_2 \} \). This completes the proof.

\textbf{Lemma 2.4.} Under the assumptions of Lemma 2.3, for every \( \varepsilon > 0 \) there exists a \( \sigma > 0 \) such that for every \( Y \in \mathcal{K}(E) \) and every \( \theta \in \Delta \),

\begin{enumerate}
\item[(i)] \( M_Y(b_b(0)) \subset B_{E}(x_2, \varepsilon) \), \( M_Y(b_b(1)) \subset B_{E}(x_1, \varepsilon) \),
\item[(ii)] \( M_Y(b_b(t)) \subset B_{E}(x_1, \varepsilon) \cup B_{E}(x_2, \varepsilon) \) for every \( t \in [0,1] \).
\end{enumerate}

\textbf{Proof.} For (i) it suffices to prove the first inclusion (the proof of the second being analogous). Suppose that, on the contrary, there exist an \( \varepsilon > 0 \), a sequence \( \{ Y_n \} \subset \mathcal{K}(E) \) converging to \( Z \), and a sequence \( \{ \theta_n \} \subset \Delta \) such that

\[ M_{Y_n}(a(t_\theta)(n)) \nsubseteq B_{E}(x_2, \varepsilon), \quad n \in \mathbb{N}. \]

Passing to a subsequence, we assume that \( \{ \theta_n \} \) converges to a \( \theta \in \Delta \). Let \( \{ y_n \} \subset E \) be a sequence such that

\[ y_n \in M_{Y_n}(a(t_\theta)(n)) \setminus B_{E}(x_2, \varepsilon), \quad n \in \mathbb{N}. \]

Thus \( y_n \in Y_n \), and \( \|y_n - a(t_\theta)(n)\| = e_2(a(t_\theta)(n)) \), \( n \in \mathbb{N} \). Since \( \{ \theta_n \} \) converges to \( \theta \) and \( \{ Y_n \} \) converges to \( Z \), there exists a sequence \( \{ \sigma_n \} \), \( \sigma_n > 0 \), converging to zero such that

\[ \|y_n - a(t_\theta)\| \geq e_2(a(t_\theta)) - \sigma_n, \quad n \in \mathbb{N}. \]

As \( y_n \in Y_n \), and \( \{ Y_n \} \) converges to \( Z \), there exists a sequence \( \{ \varepsilon_n \} \subset Z \) satisfying

\[ \lim_{n \to \infty} \|y_n - y_n\| = 0. \]

Clearly,

\[ \|y_n - a(t_\theta)\| \geq e_2(a(t_\theta)) - \sigma_n - \|y_n - y_n\|, \quad n \in \mathbb{N}. \]

Hence \( \{ \varepsilon_n \} \) is a maximizing sequence of \( \max(a(t_\theta), Z) \), and so, by Lemma 2.3(ii), there is a subsequence, say \( \{ \varepsilon_n \} \), which converges to a point \( \varepsilon \in [\varepsilon_1, \varepsilon_2] \). Since \( z \in M_Z(a(t_\theta)) \) and, by Lemma 2.3(i), \( M_Z(a(t_\theta)) = x_2 \), we have \( z = x_2 \). Consequently, there exists an \( n_0 \in \mathbb{N} \) such that \( x_n \in B_{E}(x_2, \varepsilon/2) \) for \( n \geq n_0 \). Thus, by (2.8), there exists an \( n_1 \geq n_0 \) such that \( y_n \in B_{E}(x_2, \varepsilon) \) for \( n \geq n_1 \), contrary to (2.7). We can conclude that, given \( \varepsilon > 0 \), there exists a \( \sigma_0 > 0 \) such that for every \( Y \in \mathcal{K}(E) \), \( \theta \in \Delta \), (i) is satisfied. It remains to prove (ii). Suppose that it is not true. Then there exist an \( \varepsilon > 0 \), a sequence \( \{ Y_n \} \subset \mathcal{K}(E) \) converging to \( Z \), and two sequences \( \{ \theta_n \} \subset \Delta \) and \( \{ \varepsilon_n \} \subset [0,1] \) such that

\[ M_{Y_n}(b_{\theta_n}(\varepsilon_n)) \nsubseteq B_{E}(x_1, \varepsilon) \cup B_{E}(x_2, \varepsilon), \quad n \in \mathbb{N}. \]

Passing to subsequences, we can assume that \( \{ \theta_n \} \) converges to \( \theta \in \Delta \), and that \( \{ \varepsilon_n \} \) converges to \( t \in [0,1] \). Now let \( \{ y_n \} \subset E \) be a sequence such that

\[ y_n \in M_{Y_n}(b_{\theta_n}(\varepsilon_n)) \setminus (B_{E}(x_1, \varepsilon) \cup B_{E}(x_2, \varepsilon)), \quad n \in \mathbb{N}. \]

As in the proof of (i), one can construct a sequence \( \{ z_n \} \subset Z \) which satisfies (2.8) and is maximizing for \( \max(b_b(t), Z) \). Then, by Lemma 2.3(ii), a subsequence, say \( \{ z_n \} \), converges to a point \( z \in [x_1, x_2] \). This and (2.8) imply that there exists an \( n_0 \in \mathbb{N} \) such that \( y_n \in B_{E}(x_1, \varepsilon) \cup B_{E}(x_2, \varepsilon) \) for \( n \geq n_0 \), contrary to (2.9). Hence, given \( \varepsilon > 0 \), there exists a \( 0 < \sigma < \sigma_0 \) such that for every \( Y \in \mathcal{K}(E) \), \( \theta \in \Delta \), (ii) as well as (i) are satisfied. This completes the proof.

\textbf{3. Main result}

\textbf{Theorem 3.1.} Let \( E \) be a strictly convex separable Banach space of dimension at least 2. Then

\[ \mathcal{K}^0 = \{ X \in \mathcal{K}(E) \mid A(X) \text{ is everywhere uncountable in } E \} \]

is a residual dense subset of \( \mathcal{K}(E) \).
Proof. We follow some ideas from Klee [10] and Zamfirescu [15]. For $a \in E$ and $s > 0$, set

$$N_{a,s} = \{ X \in K(E) | A(M_X) \cap B_2(a,s) \text{ is empty or at most countable} \}.$$

Claim. $N_{a,s}$ is nowhere dense in $K(E)$.

For this it suffices to show that, given $X \in K(E)$ and $0 < \varrho < s$, both arbitrary, there exist $Z \in K(E)$ and $\sigma > 0$ such that

$$B_{K(E)}(Z, \sigma) \subset B_{K(E)}(X, \varrho) \cap (K(E) \setminus N_{a,s}).$$  \hfill (3.1)

Case 1. Suppose $X \neq \{a\}$. Take $x_0 \in X$ such that $\|x_0 - a\| = r$, where $r = \varepsilon(x,a)$, and set

$$x_1 = a + \left(1 + \frac{\varrho}{4r}\right)(x_0 - a).$$

We have $\|x_1 - a\| = R$, where $R = r + \varrho/4$. Next take $x_2 \in E$ such that

$$\|x_2 - a\| = \|x_1 - a\|, \quad \|x_2 - x_1\| = \varrho/4.$$

Define $Z = \overline{B}(x_1, x_2) \cup \overline{B}(x_2, x_3)$. Clearly $Z \in K(E)$. By construction, $\|x_1 - x_0\| = \varrho/4$ and $\|x_2 - x_1\| = \|x_2 - x_0\| = \varrho/2$, thus $h(Z, X) \leq \varrho/2$.

Set $\Delta = \{d/8, d/4\}$, where $d = (R - r)/R$. Now define $a_i(\theta) = a + \theta(x_i - a), i = 1, 2$, and $b_0(\theta) = (1 - t)a(\theta) + ta_0(\theta), t \in [0, 1]$.

By Lemma 2.2, there exists an $\varepsilon > 0$ with

$$B_2(x_1, \varepsilon) \cap B_2(x_2, \varepsilon) = \emptyset$$

such that for every $\theta \in \Delta$, and every $C_1 \subset B_2(x_1, \varepsilon), C_2 \subset B_2(x_2, \varepsilon)$ with $C_1, C_2 \neq \emptyset$, there exists a $t_0 \in [0, 1]$ (depending on $C_1$ and $C_2$) such that

$$e_{C_1}(b_0(t_0)) = e_{C_2}(b_0(t_0)).$$  \hfill (3.3)

By Lemma 2.4, given $\varepsilon/2$, there exists a $\sigma$ with

$$0 < \sigma < \min\{\varepsilon/2, \varrho/2\}$$

such that for every $Y \in B_{K(E)}(Z, \sigma)$ and every $\theta \in \Delta$ we have

$$M_Y(b_0(t)) \subset B_2(x_1, \varepsilon/2) \cup B_2(x_2, \varepsilon/2), \quad t \in [0, 1].$$  \hfill (3.4)

Now, let $Y \in B_{K(E)}(Z, \sigma)$ be arbitrary. Set $C_1 = Y \cap B_2(x_1, \varepsilon/2), C_2 = Y \cap B_2(x_2, \varepsilon/2)$ and observe that $C_1$ and $C_2$ are compact, and also nonempty since $x_i \in Z, i = 1, 2$, and $\sigma < \varepsilon/2$. Let $t_0 \in [0, 1]$ be such that (3.3) is satisfied, with $C_1$ and $C_2$ defined above.

We claim that

$$M_Y(b_0(t_0)) \cap \overline{B}(x_1, \varepsilon/2) \neq \emptyset, \quad i = 1, 2.$$  \hfill (3.5)

Indeed, let $y_i \in C_i, i = 1, 2$, be such that

$$\|y_i - b_0(t_0)\| = e_{C_i}(b_0(t_0)), \quad i = 1, 2.$$

Clearly, $e_{C_i}(b_0(t_0)) \leq e_Y(b_0(t_0)), i = 1, 2$. Suppose that for $i = 1$ or $i = 2$ the strict inequality holds. Then, by (3.3),

$$e_{C_i}(b_0(t_0)) < e_Y(b_0(t_0)),$$  \hfill (3.6)

Now let $y \in Y$ be such that $\|y - b_0(t_0)\| = e_Y(b_0(t_0)), y \in M_Y(b_0(t_0))$.

From (3.4) it follows that for $i \in \{1, 2\}$, say $i = 1$, we have $y \in B_2(x_1, \varepsilon/2)$. Hence $y \in C_1$, and so $e_{C_1}(b_0(t_0)) \geq \|y - b_0(t_0)\|$, which gives $e_{C_1}(b_0(t_0)) \geq e_Y(b_0(t_0))$, contrary to (3.6). Hence,

$$e_{C_i}(b_0(t_0)) = e_Y(b_0(t_0)),$$  \hfill (3.7)

Since $C_1 \subset Y, i = 1, 2$, it follows that

$$M_{C_1}(b_0(t_0)) \subset M_Y(b_0(t_0)),$$  \hfill (3.8)

Moreover,

$$M_{C_1}(b_0(t_0)) \subset \overline{B}_2(x_1, \varepsilon/2), \quad i = 1, 2.$$  \hfill (3.9)

Combining (3.7) and (3.8) gives (3.3).

From (3.2) and (3.5) it follows that $b_0(t_0) \in A(M_Y)$. Furthermore, $b_0(t_0) \in B_2(a, \varepsilon)$, for

$$\|b_0(t_0) - a\| \leq \theta R \leq \frac{d}{4} R \leq \frac{\varepsilon}{16} < \varepsilon.$$

Hence $b_0(t_0) \in A(M_Y) \cap B_2(a, \varepsilon)$. As the set of such points $b_0(t_0)$ with $\theta \in \Delta$ is uncountable, we see that $Y \in K(E) \setminus N_{a,s}$. Since, in addition, $Y \in B_{K(E)}(Z, \sigma)$ is arbitrary, we have

$$B_{K(E)}(Z, \sigma) \subset K(E) \setminus N_{a,s}.$$  \hfill (3.10)

On the other hand, each $Y \in B_{K(E)}(Z, \sigma)$ satisfies $h(Y, X) \leq h(Y, Z) + h(Z, X) < \sigma + \varrho/2 \leq \varrho$ for, by construction, $\sigma \leq \varrho/2$ and $h(Z, X) \leq \varrho/2$.

Hence,

$$B_{K(E)}(Z, \sigma) \subset B_{K(E)}(Z, \sigma).$$

Combining this with (3.9) gives (3.1), and thus the claim that $N_{a,s}$ is nowhere dense in $K(E)$ is proved, in Case 1.

Case 2. Suppose $X = \{a\}$. Take an $x_0 \in E$ with $\|x_0 - a\| = \varrho/4$, and fix $x_1, x_2 \in E$ as in Case 1. Set $Z = \overline{B}(x_0, x_1, x_2)$. Clearly $Z \in K(E)$, and $h(Z, X) = \varrho/2$. From this point the proof is as in Case 1 and so it is omitted.

Now we are ready to prove that the set $K^*$ is residual in $K(E)$. To this end, let $D \subset E$ be a countable set everywhere dense in $E$, and let $Q^+$ be the set of all strictly positive rationals. Define

$$K^* = \bigcap_{a \in D} (K(E) \setminus N_{a,s}).$$
Clearly, $K^*$ is residual in $K(\mathbb{E})$. Furthermore, $K^* \subseteq K^0$. Indeed, let $X \in K^*$, $x \in \mathbb{E}$ and $r > 0$. Take $a \in A$ and $s \in \mathbb{Q}^+$ so that $B_E(a,s) \subset B_E(x,r)$. Since $X \notin N_a$, the set $A(M_X) \cap B_E(a,s)$ is nonempty and uncountable. This shows that $A(M_X)$ is everywhere uncountable in $\mathbb{E}$, and so $X \in K^0$. Hence $K^* \subseteq K^0$, and $K^0$ is residual in $K(\mathbb{E})$, for $K^*$ is so. As $K(\mathbb{E})$ is complete, $K^0$ is dense in $K(\mathbb{E})$. This completes the proof.

Remark 3.1. Let $\mathbb{E} = \mathbb{R}^n$ be endowed with the Euclidean norm. From Theorem 3.1 and the Mazur property it follows that most $X \in K(\mathbb{R}^n)$, in the Baire category sense, can be represented as the intersection of a family of closed balls containing $X$, having on their boundary at least two points of $X$.

Remark 3.2. If $X$ is a nonempty closed convex bounded subset of $\mathbb{E}$, beside the ambiguous locus of uniqueness $A^u(M_X)$ given by $A^u(M_X) = A(M_X)$, one can consider the ambiguous locus of $x \in \mathbb{E}$ such that $\max \{a, X\} = \emptyset$. The ambiguous locus of well posedness $A^w(M_X) = \{a \in \mathbb{E} \mid \max \{a, X\} = \emptyset\}$ is not well posed. We recall that a maximization problem $\max \{a, X\}$ is said to be well posed if it has one and only one solution, say $z$, and every maximizing sequence converges to $z$. Clearly, $A^u(M_X) \cup A^u(M_X) \subseteq A^w(M_X)$. However, while the local cardinality of the set $A^w(M_X)$ can be studied, under appropriate hypotheses, by adapting the preceding approach, the investigation of the sets $A^u(M_X)$ and $A^w(M_X)$ seems to require a different approach.

Whenever $X \in K(\mathbb{E})$, we have $A^u(M_X) = \emptyset$ and $A^w(M_X) = A^u(M_X) = A(M_X)$, where the latter set is the ambiguous locus considered in Theorem 3.1.

Finally, we observe that the main result of this paper, proved for the farthest distance mapping from sets $X \in K(\mathbb{E})$, has no analog for the nearest distance mapping since, in this case, the corresponding ambiguous locus is empty for each $X \in K(\mathbb{E})$. A comprehensive treatment of nearest distance problems from closed sets can be found in Borwein and Fitzpatrick [3].

References


