Topologies and bornologies determined by operator ideals, II

by

NGAI-CHING WONG (Kao-hsiung)

Abstract. Let $\mathfrak{A}$ be an operator ideal on LCS's. A continuous seminorm $p$ of a LCS $X$ is said to be $\mathfrak{A}$-continuous if $Q_p \in \mathcal{A}^{\text{cont}}(X, X_p)$, where $X_p$ is the completion of the normed space $X_p = X/p^{-1}(0)$ and $Q_p$ is the canonical map. $p$ is said to be a Groth$(\mathfrak{A})$-seminorm if there is a continuous seminorm $q$ of $X$ such that $p \leq q$ and the canonical map $Q_p : X_q \to X_p$ belongs to $\mathcal{A}(X_q, X_p)$. It is well known that when $\mathfrak{A}$ is the ideal of absolutely summing (resp. precompact, weakly compact) operators, a LCS $X$ is a nuclear (resp. Schwartz, infra-Schwartz) space if and only if every continuous seminorm $p$ of $X$ is $\mathfrak{A}$-continuous if and only if every continuous seminorm $p$ of $X$ is a Groth$(\mathfrak{A})$-seminorm. In this paper, we extend this equivalence to arbitrary operator ideals $\mathfrak{A}$ and discuss several aspects of these constructions which were initiated by A. Grothendieck and D. Randtke, respectively. A bornological version of the theory is also obtained.

1. Introduction. Let $X$ be a LCS (locally convex space) and $p$ a continuous seminorm of $X$. Denote by $X_p$ the quotient space $X/p^{-1}(0)$ equipped with the quotient seminorm (in fact, norm) $\| \cdot \|_p$. $Q_p$ denotes the canonical map from $X$ onto $X_p$, and $\tilde{Q}_p$ denotes the unique map induced by $Q_p$ from $X$ into the completion $\tilde{X}_p$ of $X_p$. If $q$ is a continuous seminorm of $X$ such that $p \leq q$ (i.e. $p(x) \leq q(x)$, $\forall x \in X$), the canonical maps $Q_{pq} : X_q \to X_p$ and $\tilde{Q}_{pq} : \tilde{X}_q \to \tilde{X}_p$ are continuous.

Let $\mathfrak{B}$ be an operator ideal on Banach spaces. Following A. Pietsch [10], we call a LCS $X$ a Groth$(\mathfrak{B})$-space if for each continuous seminorm $p$ of $X$ there is a continuous seminorm $q$ of $X$ such that $p \leq q$ and $\tilde{Q}_{pq} \in \mathcal{B}(\tilde{X}_q, \tilde{X}_p)$. This amounts to saying that the completion $\tilde{X}$ of $X$ is a topological projective limit $\lim_{\to} \tilde{Q}_{pq} \tilde{X}_q$ of Banach spaces of type $\mathfrak{B}$ (cf. [7]). A. Grothendieck's construction of nuclear spaces is a model of Groth$(\mathfrak{B})$-spaces. In fact, a LCS $X$ is a nuclear (resp. Schwartz, infra-Schwartz) space if it is a Groth$(\mathfrak{B})$-space.

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space (resp. Groth($\mathfrak{R}$)-space, Groth($\mathfrak{M}$)-space), where $\mathfrak{R}$ (resp. $\mathfrak{R}_n$, $\mathfrak{M}$) is the ideal of nuclear (resp. precompact, weakly compact) operators. It is known that a locally convex space $X$ is a Groth($\mathfrak{R}$)-space if and only if the identity operator $id_X$ of $X$ belongs to the right superextension $\mathfrak{R}_{\text{sup}}^\ast$ of $\mathfrak{R}$ to LCS's [10].

Another usual way to deal with this kind of spaces is due to D. Randtke [11]. A continuous seminorm $p$ of a LCS $X$ is said to be $\mathfrak{R}$-continuous if the canonical map $\tilde{Q}_p : X \to \tilde{X}_p$ belongs to the injective hull $\mathfrak{R}_{\text{inj}}$ of $\mathfrak{R}$. $X$ is said to be an $\mathfrak{R}$-topological space if every continuous seminorm of $X$ is $\mathfrak{R}$-continuous. For example, a LCS $X$ is nuclear (resp. Schwartz, infra-Schwartz) if $X$ is an $\mathfrak{R}$- (resp. $\mathfrak{R}_n$, $\mathfrak{M}$) topological space.

The advantage of the construction of Grothendieck is that we need only pay attention to Banach spaces operators, while the construction of Randtke appears to be simpler and easier to apply. In this paper, we shall prove that these two constructions are in fact equivalent. Motivated by those examples of classical spaces, we define the notions of ideal topologies (Groth($\mathfrak{R}$)-topologies in §3) and Grothendieck topologies (Groth($\mathfrak{R}$)-topologies in §4) associated with an operator ideal $\mathfrak{R}$. Our main result, Theorem 5.1, says that Groth($\mathfrak{R}$)-topology = $\mathfrak{R}_{\text{sup}}^\ast$-topology on LCS's. In particular, a LCS $X$ is a Groth($\mathfrak{R}$)-space if and only if $X$ is an $\mathfrak{R}$-topological space.

We also discuss dual concepts of Grothendieck spaces and $\mathfrak{R}$-topological spaces, i.e. co-Grothendieck spaces and $\mathfrak{R}$-bbornological spaces, which also attract some research interests covering co-nuclear spaces, co-Schwartz spaces, semi-Montel spaces, and semi-reflexive spaces.

Finally, we refer the readers to [4, 5, 7–10, 20] concerning Groth($\mathfrak{R}$)-spaces and co-Groth($\mathfrak{R}$)-spaces, and to [6, 9, 11, 13–17, 19, 20] concerning $\mathfrak{R}$-topological spaces and $\mathfrak{R}$-bbornological spaces for further information, and in particular to [21] for a quick review of the theory of ideal topologies and bornologies.

2. Notations and preliminaries. We shall follow the terminology of [21]. Let $X$ and $Y$ be LCS's. We denote by $\mathcal{P}(X,Y)$, $\mathcal{L}(X,Y)$, and $L^X(X,Y)$ the collection of all operators from $X$ into $Y$ which are bounded (i.e. sending a 0-neighborhood to a bounded set), continuous, and locally bounded (i.e. sending bounded sets to bounded sets), respectively. Denote by $X_0$ a vector space $X$ equipped with a locally convex (Hausdorff) topology $\mathfrak{T}$, and by $X^{\text{W}}$ a vector space $X$ equipped with a convex vector (separated) bornology $\mathfrak{N}$. $U_Y$ always denotes the closed unit ball of a normed space $N$.

A subset $B$ of a LCS $X$ is said to be a disk if $B$ is absolutely convex, i.e. $\lambda B + \beta B \subseteq B$ whenever $|\lambda| + |\beta| \leq 1$. A disk $B$ is said to be an $\sigma$-disk, or absolutely $\sigma$-convex if $\sum_n \lambda_n b_n$ converges in $X$ and the sum belongs to $B$ whenever $(\lambda_n) \in U_1$ and $b_n \in B$, $n = 1, 2, \ldots$. With each bounded disk $B$ in $X$ there is associated a normed space $X(B) = \bigcup_{\lambda \neq 0} \lambda B$ equipped with the gauge $\gamma_B$ of $B$ as its norm, where $\gamma_B(x) = \inf \{ \lambda > 0 : \frac{x}{\lambda} \in \lambda B \}$, $\forall x \in X(B)$. The canonical map $J_B$ sending $x$ in $X(B)$ to $\frac{x}{\lambda}$ in $X$ is continuous. Moreover, if $A$ is a bounded disk in $X$ such that $B \subseteq A$ then the canonical map $J_{AB}$ sending $x$ in the normed space $X(A)$ to $x$ in the normed space $X(A)$ is bounded. A bounded disk $B$ in $X$ is said to be infracomplete if $X(B)$ is complete with respect to $\gamma_B$. It is known that a bounded and closed disk in $X$ is absolutely $\sigma$-convex if and only if $B$ is infracomplete [21]. $X$ is said to be infracomplete if the von Neumann bornology $\mathfrak{N}_{\text{vn}}(X)$, i.e. the bornology of all topologically bounded subsets of $X$, has a basis consisting of infracomplete subsets of $X$, or equivalently, $\sigma$-disks of subsets of $X$. In other words, $(X, \mathfrak{N}_{\text{vn}}(X))$ is a complete convex bornological vector space (cf. [2]).

Let $X$ and $Y$ be LCS's. $Q^1$ in $L(X,Y)$ is said to be a bornological surjection if $Q^1$ is onto and induces the bornology of $Y$ (i.e. for each bounded subset $B$ of $Y$ there is a bounded subset $A$ of $X$ such that $Q^1 A = B$). Let $\mathfrak{C}$ be either the class $\mathbb{L}$ of locally convex spaces or the class $\mathbb{B}$ of Banach spaces. An operator ideal $\mathfrak{A}$ on $\mathfrak{C}$ is said to be bornologically surjective if whenever $T$ is a continuous operator from $X$ into $Y$ and $Q$ is a bornological surjection from $X_0$ onto $X$ such that $TQ \in \mathfrak{A}(X_0, Y)$, we have $T \in \mathfrak{A}(X, Y)$, $X_0, Y \in \mathfrak{C}$. The bornologically surjective hull $\mathfrak{A}_{\text{sur}}$ of $\mathfrak{A}$ is the intersection of all bornologically surjective operator ideals containing $\mathfrak{A}$. If $\mathfrak{C} = \mathbb{B}$, we have $\mathfrak{A}_{\text{sur}} = \mathfrak{A}_{\text{sup}}$, but if $\mathfrak{C} = \mathbb{L}$ then they are, in general, different (cf. [18]). We would like to mention that since a surjection is not always a bornological surjection (cf. [12, Ex. 4.9 and 4.20] or [18]), Theorem 4.10(c) in [21] should be rewritten by replacing the word "surjective" by the phrase "bornologically surjective". All other results in [21] are unaffected.

We quote two recent results for later reference.

**Proposition 2.1 ([11]).** We can associate with each LCS $Y$ a LCS $Y^\infty$ and an injection $J^\infty_Y \in L(Y, Y^\infty)$ such that the injective hull $\mathfrak{A}_{\text{inj}}$ of an operator ideal $\mathfrak{A}$ on LCS's is given by

\[ \mathfrak{A}_{\text{inj}}(X, Y) = \{ T \in L(X, Y) : J^\infty_Y T \in \mathfrak{A}(X, Y^\infty) \} \]

**Proposition 2.2 ([18]).** We can associate with each LCS $X$ a LCS $X^1$ and a bornological surjection $Q^1_X$ in $L(X^1, X)$ such that the bornologically surjective hull of an operator ideal $\mathfrak{A}$ on LCS's is given by

\[ \mathfrak{A}_{\text{sur}}(X, Y) = \{ T \in L(X, Y) : J^\infty_{Q^1_X} T \in \mathfrak{A}(X^1, Y) \} \]

3. $\mathfrak{A}$-topologies and $\mathfrak{A}$-bornologies. Let $\mathfrak{A}$ be an operator ideal on $\mathfrak{C}$, where $\mathfrak{C}$ is either the class of LCS's or the class of Banach spaces. The $\mathfrak{A}$-topology $\mathcal{T}(\mathfrak{A})(X_0)$ of an $X_0$ in $\mathfrak{C}$ is defined to be the projective topology of
In the sequel, \( \mathcal{B} \) denotes either the class of LCS's or the class of Banach spaces. The following includes a result of Jarchow [6, Proposition 3] in the context of Banach spaces.

**Theorem 3.4.** Let \( \mathfrak{A} \) be a surjective operator ideal on \( \mathcal{B} \) and \( X, Y \in \mathcal{B} \). If \( Y \) is a (topological) quotient space of \( X \) then the \( \mathfrak{A} \)-topology of \( Y \) is the quotient topology induced by the \( \mathfrak{A} \)-topology of \( X \).

**Proof.** Let \( Q \) be the quotient map from \( X \) onto \( Y \). Let \( X_{\mathcal{A}} \) (resp. \( Y_{\mathcal{A}} \)) denote the LCS \( X \) (resp. \( Y \)) equipped with the \( \mathfrak{A} \)-topology. We have \( Q \in \mathcal{L}(X_{\mathcal{A}}, Y_{\mathcal{A}}) \) [21, Theorem 3.8]. This implies that the \( \mathfrak{A} \)-topology of \( Y \) is weaker than the quotient topology induced by the \( \mathfrak{A} \)-topology of \( X \). Let \( p \) be an \( \mathfrak{A} \)-bounded seminorm of \( X \) and \( q \) the quotient seminorm of \( Y \) induced by \( p \). Let \( \tilde{Q}_p : X \to \tilde{X}_p, \tilde{Q}_q : Y \to \tilde{Y}_q \) and \( \tilde{Q}_p : X_{\mathcal{A}} \to \tilde{X}_{\mathcal{A}} \) be the canonical maps. By Proposition 3.1 (or [21, Lemma 3.3] for the Banach space version), \( \tilde{Q}_p \in \mathcal{L}(\tilde{X}_p, \tilde{X}_{\mathcal{A}}) \). Now \( \tilde{Q}_q \tilde{Q}_p = \tilde{Q}_{qp} \tilde{Q}_p \in \mathcal{L}(\tilde{X}_p, \tilde{Y}_{\mathcal{A}}) \) implies \( \tilde{Q}_q \in (\mathcal{L}(\tilde{Y}_{\mathcal{A}}) \) since \( Q \) is a surjection. However, \( (\mathcal{L}(\tilde{Y}_{\mathcal{A}}) \) is always surjective, by Proposition 2.1. As a result, \( (\mathcal{L}(\tilde{Y}_{\mathcal{A}}) \subseteq (\mathcal{L}(\tilde{Y}_{\mathcal{A}}) \). Thus \( \tilde{Q}_q \in (\mathcal{L}(\tilde{Y}_{\mathcal{A}}) \subseteq (\mathcal{L}(\tilde{Y}_{\mathcal{A}}) \) since \( \mathfrak{A} \) is surjective. This implies that \( q \) is \( \mathfrak{A} \)-continuous. Therefore, the \( \mathfrak{A} \)-topology of \( Y \) coincides with the quotient topology induced by the \( \mathfrak{A} \)-topology of \( X \). \( \square \)

**Corollary 3.5.** Let \( \mathfrak{A} \) be a surjective operator ideal on \( \mathcal{B} \). Then a quotient space of an \( \mathfrak{A} \)-topological space is again an \( \mathfrak{A} \)-topological space.

### 3.2. \( \mathfrak{A} \)-bornologies and \( \mathfrak{A} \)-bornological spaces

**Proposition 3.6.** Let \( \mathfrak{A} \) be an operator ideal on LCS's. The \( \mathfrak{A} \)-bornology coincides with the \( \mathcal{L}(\mathfrak{A}) \)-bornology on every LCS \( X \). Moreover, any bounded subset \( B \) of a LCS \( Y \) is \( \mathfrak{A} \)-bounded if and only if \( J_B \in (\mathcal{L}(\mathfrak{A}(B), Y) \). The absolute convex hull of \( B \) in \( Y \) is \( \mathfrak{A} \)-bounded in \( Y \) and one can choose a bounded disk \( A \) in a LCS \( X \) and a \( \mathfrak{A} \)-bounded in \( Y \). Conversely, if \( B \) is \( \mathfrak{A} \)-bounded in \( Y \), \( B \) is also \( \mathfrak{A} \)-bounded in \( Y \) and we can choose a bounded disk \( A \) in a LCS \( X \) and a \( \mathfrak{A} \)-bounded in \( Y \) such that \( TA = B \). So we have a \( T_{\mathfrak{A}} \) in \( (\mathcal{L}(\mathfrak{A}(X), Y) \) such that \( T_{\mathfrak{A}} = J_{B_{\mathfrak{A}}} \). Now \( T_{\mathfrak{A}} \) is \( \mathfrak{A} \)-bounded in \( Y \) and the bornological surjectivity of \( T_{\mathfrak{A}} \) implies \( J_{B_{\mathfrak{A}}} \in (\mathcal{L}(\mathfrak{A}(B), Y) \). The last assertion follows from [18, Corollary 2.6] which says that \( \mathfrak{A}(B, Y) = \mathfrak{A}(Y, B) \) for every normed space \( N \) and every LCS \( Y \) if \( \mathfrak{A} \) is surjective. \( \square \)
Recall that a LCS $Y$ is said to be $\mathfrak{A}$-bornological if its von Neumann bornology (i.e., the family of all topologically bounded subsets of $Y$) coincides with the $\mathfrak{A}$-bornology (cf. [21]).

**Corollary 3.7.** Let $\mathfrak{A}$ be an operator ideal on LCS's and $Y$ a LCS. The following are all equivalent.

1. $Y$ is $\mathfrak{A}$-bornological.
2. $L^b(X, Y) \subseteq L^b(\mathfrak{A}(X, Y), \mathfrak{A}(Y, Z))$ for every LCS $X$.
3. $L(N, Y) = \mathfrak{A}^b(N, Y)$ for every normed space $N$.

In case $Y$ is infracomplete, they are all equivalent to

(3)’ $L(E, Y) = \mathfrak{A}^b(E, Y)$ for every Banach space $E$.

If $\mathfrak{A}$ is surjective, we can replace $L^b$ by $\mathfrak{A}^b$ in all the above statements.

**Proof.** We just mention that the last assertion follows from [18, Corollary 2.6].

Example 3.8. When $\mathfrak{A}$ is the ideal $\mathfrak{M}$ of nuclear operators or the ideal $\mathfrak{P}$ of absolutely summing operators (resp. the ideal $\mathfrak{F}_p$ of precompact operators, the ideal $\mathfrak{M}$ of weakly compact operators), the corresponding $\mathfrak{A}$-bornological spaces are co-nuclear spaces (resp. semi-Montel spaces and semi-reflexive spaces). Corollary 3.7 serves as a prototype of a class of theorems concerning these spaces (see e.g. [3]).

Let $\mathfrak{C}$ be either the class of LCS’s or the class of Banach spaces.

**Theorem 3.9.** Let $\mathfrak{A}$ be an injective operator ideal on $\mathfrak{C}$ and $X, Y \in \mathfrak{C}$. If $Y$ is a (topological) subspace of $X$ then the $\mathfrak{A}$-bornology of $Y$ is the subspace bornology inherited from the $\mathfrak{A}$-bornology of $X$.

**Proof.** Similar to Theorem 3.4. Note that we have $(\mathfrak{A}^b)^{\text{inj}} = (\mathfrak{A}^b)^{\text{inj}}$ by Propositions 2.1 and 2.2 in this case.

**Corollary 3.10.** Let $\mathfrak{A}$ be an injective operator ideal on $\mathfrak{C}$. Then a subspace of an $\mathfrak{A}$-bornological space is again an $\mathfrak{A}$-bornological space.

### 4. Grothendieck Topologies and Grothendieck Bornologies

**4.1. Grothendieck Topologies and Grothendieck Bornologies**

**Definition.** Let $\mathfrak{A}$ be an operator ideal on Banach spaces. We call a continuous seminorm $p$ of a LCS $X$ a Grothendieck $\mathfrak{A}$-seminorm if there is a continuous seminorm $q$ of $X$ such that $p \leq q$ and $\overline{Q}_p \in \mathfrak{A}(\overline{X}_q, \overline{X}_p)$.

**Remark.** Two operator ideals $\mathfrak{A}$ and $\mathfrak{B}$ on Banach spaces are said to be equivalent if there are positive integers $m$ and $n$ such that $\mathfrak{A}^n \subseteq \mathfrak{B}$ and $\mathfrak{B}^m \subseteq \mathfrak{A}$. In this case, a continuous seminorm $p$ of a LCS $X$ is a Grothendieck $\mathfrak{A}$-seminorm if and only if $p$ is a Grothendieck $\mathfrak{B}$-seminorm (cf. [10] or [7]). An operator ideal $\mathfrak{A}$ is said to be quasi-injective if $\mathfrak{A}$ is equivalent to an injective operator ideal. For example, the ideal $\mathfrak{M}$ of nuclear operators is quasi-injective since it is equivalent to the injective ideal $\mathfrak{P}$ of absolutely summing operators. In fact, $\mathfrak{M} \subseteq \mathfrak{P} \subseteq \mathfrak{P}$ (cf. [20, p. 145]).

**Proposition 4.1.** Let $\mathfrak{A}$ be an operator ideal on Banach spaces and let $p, p_1, \ldots, p_n$ be Grothendieck $\mathfrak{A}$-seminorms of a LCS $X$.

(a) $\mathfrak{A}$ is a Grothendieck $\mathfrak{A}$-seminorm of $X$ for all $\lambda > 0$.
(b) If $p_0$ is a continuous seminorm of $X$ such that $p_0 \leq p$ then $p_0$ is a Grothendieck $\mathfrak{A}$-seminorm.

(c) $p_1 + \ldots + p_n$ is a Grothendieck $\mathfrak{A}(\text{inj})$-seminorm. In case $\mathfrak{A}$ is quasi-injective, $p_1 + \ldots + p_n$ is a Grothendieck $\mathfrak{A}$-seminorm.

**Proof.** (a) and (b) are trivial. For (c), let $(q_1, \ldots, q_n)$ be continuous seminorms of $X$ such that $p_i \leq q_i$ and $Q_t = \overline{Q}_{p_i} \in \mathfrak{A}(\overline{X}_q, \overline{X}_p)$, $i = 1, \ldots, n$. Let $p_0 = p_1 + \ldots + p_n$ and $Q_0 = q_1 + \ldots + q_n$. Let $J_p : \overline{X}_p \rightarrow \bigoplus_{\ell_t} \overline{X}_{p_t}$ and $J_q : \overline{X}_q \rightarrow \bigoplus_{\ell_t} \overline{X}_{q_t}$ be the canonical isometric embeddings. Let $j_k : \overline{X}_p \rightarrow \bigoplus_{\ell_t} \overline{X}_{p_t}$ and $i_k : \bigoplus_{\ell_t} \overline{X}_{p_t} \rightarrow \overline{X}_q$, $k = 1, \ldots, n$, be the canonical embeddings and projections, respectively. We want to prove that $Q_0 = \overline{Q}_0 \in \mathfrak{A}(\overline{X}_q, \overline{X}_p)$ belongs to $\mathfrak{A}(\overline{X}_q, \overline{X}_p)$. Note that $Q_j Q_0 = (j_1 Q_1 + \ldots + j_n Q_n) Q_0$. Since $Q_0 \in \mathfrak{A}(\overline{X}_q, \overline{X}_p)$, $k = 1, \ldots, n$, it follows that $Q_j Q_0 \in \mathfrak{A}(\overline{X}_q, \bigoplus_{\ell_t} \overline{X}_{p_t})$ and hence $Q_0 \in \mathfrak{A}(\overline{X}_q, \overline{X}_p)$. That is, $p_0$ is a Grothendieck $\mathfrak{A}(\text{inj})$-seminorm of $X$.

**Definition.** Let $\mathfrak{A}$ be a quasi-injective operator ideal on Banach spaces and $X$ a LCS. The Grothendieck $\mathfrak{A}$-topology of $X$ is defined to be the locally convex Hausdorff topology of $X$ determined by all Grothendieck $\mathfrak{A}$-seminorms.

Recall that a LCS $X$ is called a Grothendieck $\mathfrak{A}$-space for some operator ideal $\mathfrak{A}$ on Banach spaces if $\text{id}_X \in \mathfrak{A}(\text{inj})$, $(X, X)$ (cf. [10]). It is easy to see that for a quasi-injective operator ideal $\mathfrak{A}$ on Banach spaces, a LCS $X$ is a Grothendieck $\mathfrak{A}$-space if and only if the topology of $X$ coincides with the Grothendieck $\mathfrak{A}$-topology. In this case, the completion $\overline{X}$ of $X$ is a locally convex topological vector space (see [8]).

**4.2. Grothendieck $\mathfrak{A}$-Bornologies and co-Grothendieck $\mathfrak{A}$-Spaces**

**Definition.** Let $\mathfrak{A}$ be an operator ideal on Banach spaces. A bounded $\sigma$-disk $B$ in a LCS $X$ is said to be Grothendieck $\mathfrak{A}$-bounded in $X$ if there is a bounded $\sigma$-disk $A$ in $X$ such that $B \subseteq A$ and the canonical map $J_{AB} : \mathfrak{A}(X(B), X(A))$. Note that, in this case, both $X(A)$ and $X(B)$ are Banach spaces.

**Remark.** If $\mathfrak{A}$ and $\mathfrak{B}$ are two equivalent operator ideals on Banach spaces then a bounded $\sigma$-disk $B$ in a LCS $X$ is Grothendieck $\mathfrak{A}$-bounded if and
only if $B$ is Groth($\mathfrak{A}$)-bounded (cf. [7]). An operator ideal $\mathfrak{A}$ is said to be quasi-surjective if $\mathfrak{A}$ is equivalent to a surjective operator ideal.

**Proposition 4.2.** Let $\mathfrak{A}$ be an operator ideal on Banach spaces and let $B_1, B_2, \ldots, B_n$ be Groth($\mathfrak{A}$)-bounded $\sigma$-disks in a LCS $X$.

(a) If $B$ is Groth($\mathfrak{A}$)-bounded for all $\lambda \geq 0$. (b) If $B_0$ is a bounded subset of $X$ and $B_0 \subset B$ then the $\sigma$-disk hull $\Gamma_{\sigma}(B_0)$ of $B_0$ exists in $X$ and is Groth($\mathfrak{A}$)-bounded in $X$.

(c) $\Gamma_{\sigma}(B_1 \cup \ldots \cup B_n)$ is Groth($\mathfrak{A}$)-bounded in $X$. In case $\mathfrak{A}$ is quasi-surjective, $\Gamma_{\sigma}(B_1 \cup \ldots \cup B_n)$ is Groth($\mathfrak{A}$)-bounded in $X$.

**Proof.** Similar to Proposition 4.1. ■

**Definition.** Let $\mathfrak{A}$ be a quasi-surjective operator ideal on Banach spaces. The Groth($\mathfrak{A}$)-borelology of a LCS $X$ is defined to be the convex vector borelology of $X$ determined by all Groth($\mathfrak{A}$)-bounded $\sigma$-disks in $X$.

**Definition.** A LCS is called a co-Groth($\mathfrak{A}$)-space if all bounded $\sigma$-disks in $X$ are Groth($\mathfrak{A}$)-bounded. It is equivalent to say that id$_X \in \mathfrak{A}^{\text{map}}(X, X)$.

It is easy to see that for a quasi-surjective operator ideal $\mathfrak{A}$ on Banach spaces, an infcomplete LCS $X$ is a co-Groth($\mathfrak{A}$)-space if and only if the von Neumann borelology $\mathcal{M}_{\text{von}}(X)$ of $X$ coincides with the Groth($\mathfrak{A}$)-borelology. In this case, the complete convex borelological space $X$ is a borelological inductive limit $\lim \frac{J_{AB}X(B)}{B}$ of Banach spaces of type $\mathfrak{A}$.

5. Coincidence of ideal topologies (borelologies) and Grothendieck topologies (borelologies)

**Theorem 5.1.** Let $\mathfrak{A}$ be an operator ideal on Banach spaces. The Groth($\mathfrak{A}$)-topology coincides with the $\mathfrak{A}^{\text{map}}$-topology on every LCS $X$ and the Groth($\mathfrak{A}$)-borelology coincides with the $\mathfrak{A}^{\text{map}}$-borelology on every infcomplete LCS. In particular, we have

(a) A LCS $X$ is a Groth($\mathfrak{A}$)-space if and only if $X$ is an $\mathfrak{A}^{\text{map}}$-topological space.

(b) An infcomplete LCS $X$ is a co-Groth($\mathfrak{A}$)-space if and only if $X$ is an $\mathfrak{A}^{\text{map}}$-borelological space.

(c) The $\mathfrak{A}$-topology (resp. $\mathfrak{A}$-borelology) coincides with the Groth($\mathfrak{A}$)-topology (resp. Groth($\mathfrak{A}$)-borelology) on Banach spaces.

**Proof.** Let $p$ be an $\mathfrak{A}^{\text{map}}$-continuous norm on $X$. Then $\tilde{\mathcal{Q}}_p \in (\mathfrak{A}^{\text{map}})^{(1)}(X, \hat{X}_p) = (\mathfrak{A}^{\text{map}})^{(1)}(X, \hat{X}_p)$, by [18, Proposition 3.5]. Consequently, a factorization $\tilde{\mathcal{Q}}_p = ST$ exists, where $S \in \mathfrak{A}^{\text{map}}(E, \hat{X}_p)$ and $T \in \mathcal{L}(X, E)$ for some Banach space $E$. Define

$$q(x) = \|S\| \|Tx\|, \quad \forall x \in X.$$  

Then $q$ is a continuous seminorm of $X$ such that

$$p(x) = \|\tilde{\mathcal{Q}}_p(x)\| = \|STx\| \leq \|S\| \|Tx\| = q(x), \quad \forall x \in X.$$  

Note that $T$ induces $R$ in $\mathcal{L}(\hat{X}_p, E)$ such that $T = R\tilde{\mathcal{Q}}_p$. Hence, $\tilde{\mathcal{Q}}_{\mathcal{Q}} = SR \in \mathfrak{A}^{\text{map}}(\hat{X}_p, \hat{X}_p)$. Therefore, $\tilde{\mathcal{Q}}$ is a Groth($\mathfrak{A}$)-seminorm of $X$.

Conversely, if $p$ is a Groth($\mathfrak{A}$)-seminorm of $X$ then there exists a continuous seminorm $q$ of $X$ such that $p \leq q$ and $\tilde{\mathcal{Q}}_{\mathcal{Q}} \in \mathfrak{A}^{\text{map}}(\hat{X}_p, \hat{X}_p)$. As a result, $\tilde{\mathcal{Q}}_p = \tilde{\mathcal{Q}}_{\mathcal{Q}}\tilde{\mathcal{Q}}_\mathcal{Q} \in \mathfrak{A}^{\text{map}}(\hat{X}_p, \hat{X}_p)$ and thus $p$ is $\mathfrak{A}$-continuous.

We leave the borelological case to the reader, and comment that the infcomplete assumption is merely to give us a chance to utilize the extension condition.

**Remark.** Let $\mathfrak{A}$ be an operator ideal on Banach spaces and $\mathfrak{A}_0$ be an extension of $\mathfrak{A}$ to LCS's. It is plain that if $\mathfrak{A}_0 \subset \mathfrak{A}^{\text{map}}$ then $\mathfrak{A}_0$-topology = $\mathfrak{A}^{\text{map}}$-topology; and if $\mathfrak{A}_0 \subset \mathfrak{A}^{\text{map}}$ then $\mathfrak{A}_0$-borelology = $\mathfrak{A}^{\text{map}}$-borelology at least on infcomplete LCS's. For instance, $\mathfrak{A}$ = $\mathfrak{A}^{\text{map}}$ [20, p. 144], where $\mathfrak{A}$ is the quasi-injective ideal of nuclear operators between Banach spaces. Consequently, $\mathfrak{A}$-topology = Groth($\mathfrak{A}$)-topology on every LCS. This explains why the construcions of Grothendieck and Randtke match in the case of nuclear spaces. The discussion is similar for Schwartz and infra-Schwartz spaces and their "co-spaces".

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by
WOJCIECH WOJTYŃSKI (Warszawa)

Abstract. An algebraic scheme for Lie theory of topological groups with "large" families of one-parameter subgroups is proposed. Such groups are quotients of ER-groups, i.e. topological groups equipped additionally with the continuous exterior binary operation of multiplication by real numbers, and generated by special ("exponential") elements. It is proved that under natural conditions on the topology of an ER-group its group multiplication is described by the B-C-H formula in terms of the associated Lie algebra.

1. Introduction. The notion of a Lie group of infinite dimensions is not well founded. The differential manifold approach which is basic for the classical (finite-dimensional) theory may be successfully applied only to Banach-Lie groups ([1], [9]). This class, however, appears to be too restrictive to incorporate most of interesting infinite-dimensional examples. Difficulties in extending the manifold approach beyond the frames of Banach case are of two kinds, which correspond to the two main limitations of the differential calculus in non-Banach spaces: lack of the existence and uniqueness theorem for ordinary differential equations and lack of the inverse map theorem for smooth mappings. In the classical theory one associates with a given group $G$ its Lie algebra $\mathfrak{g}$ which is usually defined to be the Lie algebra of all left (or equivalently right) invariant vector fields on $G$. This step presents no difficulty whatsoever, but to make it meaningful $\mathfrak{g}$ has to be better connected with the group structure of $G$. Classically this is achieved by associating with each $X \in \mathfrak{g}$ its properly selected integral curve, which happens to be a one-parameter subgroup of $G$. Thus the validity of the existence and uniqueness theorem provides a one-to-one map $i$ from $\mathfrak{g}$ to $\mathscr{A}(G)$, the set of all continuous one-parameter subgroups of $G$. In the absence of this theorem, e.g. for Fréchet-Lie groups, it is not known whether such a group has a single nontrivial one-parameter subgroup (cf. [9]). On the other hand, no examples disproving bijectivity of $i : g \to \mathscr{A}(G)$ are known in this case. Concluding, the lack of methods for establishing bijectivity of $i : g \to \mathscr{A}(G)$

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