Concerning entire functions in $B_0$-algebras

by

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Abstract. We construct a non-m-convex non-commutative $B_0$-algebra on which all entire functions operate. Our example is also a $Q$-algebra and a radical algebra. It follows that some results true in the commutative case fail in general.

A $B_0$-algebra (an algebra of type $B_0$) is a topological algebra whose underlying topological vector space is a completely metrizable locally convex space. The topology of a $B_0$-algebra $A$ can be given by means of a sequence $(\| \cdot \|_i)$ of seminorms such that

$$|x|_1 \leq |x|_2 \leq \ldots \quad \text{for all } x \in A$$

and

$$|xy|_i < C_i|x|_{i+1}|y|_{i+1} \quad \text{for all } x, y \in A, \ i = 1, 2, \ldots,$$

where $C_i$ are positive constants (one can easily have $C_i = 1$ for all $i$, but here it is more convenient to have inequalities of the form (2)). A $B_0$-algebra $A$ is said to be multiplicatively-convex ($m$-convex for short) if the seminorms

(1) $$|x|_1 \leq |x|_2 \leq \ldots \quad \text{for all } x \in A$$

and

(2) $$|xy|_i < C_i|x|_{i+1}|y|_{i+1} \quad \text{for all } x, y \in A, \ i = 1, 2, \ldots,$$

where $C_i$ are positive constants (one can easily have $C_i = 1$ for all $i$, but here it is more convenient to have inequalities of the form (2)). A $B_0$-algebra $A$ is said to be multiplicatively-convex ($m$-convex for short) if the seminorms

(3) $$|xy|_i \leq |x|_i|y|_i \quad \text{for all } x, y \in A, \ i = 1, 2, \ldots$$

Note that (1) implies that if $\| \cdot \|$ is a continuous seminorm on a $B_0$-algebra $A$, then there is an index $m$ and a positive constant $C$ such that

(4) $$\|x\| \leq C|x|_m \quad \text{for all } x \in A.$$

An element $x$ of an algebra $A$ is said to be quasi-invertible if there is an element $y$ in $A$, called a quasi-inverse of $x$, such that $xy = yx = 0$, where $xy = xy + x \cdot y$. This is equivalent to $(x + c) \cdot (y + c) = (y + c) \cdot (x + c) = c$, if $A$ has a unit element $e$, or to this relation in the unitization $A_1$ of $A$, if there is no unit in $A$. That means that the quasi-inverse of an element $x$ is uniquely determined by $x$. ```
A topological algebra $A$ is said to be a $Q$-algebra if the set of all its quasi-invertible elements is open. If $A$ has a unit, then $A$ is a $Q$-algebra if and only if the set of all invertible elements of $A$ is open. Clearly the unitization of a $Q$-algebra without unit is again a $Q$-algebra.

One can prove that the complexification of a real $Q$-algebra is a complex $Q$-algebra. Also, one can easily see that if for an element $x$ of a topological algebra $A$ the series $\sum_{i=1}^{\infty} (-1)^i x^i$ is convergent, then $x$ is quasi-invertible in $A$ with quasi-inverse $\sum_{i=0}^{\infty} (-1)^i x^i$.

Let $\varphi(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n$ be an entire function of a complex variable $\zeta$. We say that $\varphi$ operates on a complex topological algebra $A$ if the series $\sum_{n=1}^{\infty} a_n x^n$ converges for every $x$ in $A$. If $A$ has a unit element $e$, we can start the summation from $0$, setting $x^0 = e$ for each $x$ in $A$. The same definition can be given for a real algebra $A$, provided all coefficients $a_i$ are real numbers.

If $A$ is a real or complex $m$-convex $B_0$-algebra then all entire functions (with real coefficients in case of a real algebra) operate on $A$. This follows immediately from the formula (3) and the estimate

$$\sum_{n} |a_n x^n|_1 \leq \sum_{n} |a_n| |x|^n, \quad x \in A, \quad i = 1, 2, \ldots$$

The main result in [2] gives a partial converse:

**Theorem A.** If $A$ is a commutative complex $B_0$-algebra, then $A$ is $m$-convex if and only if all entire functions operate on $A$.

The same proof works for real algebras, provided we only consider functions with real coefficients.

It is a long-standing question ([2], Problem 3, see also [5], Problem 13.15, [6], Problem 16.8, and [8], Problem 17) whether Theorem A is also true for non-commutative algebras. In this paper we give a counterexample showing that the condition of commutativity cannot be dropped.

In [2] it was also shown that for every entire function $\varphi$ there is a commutative non-$m$-convex algebra $A_\varphi$ such that $\varphi$ operates on $A_\varphi$. Thus we cannot substantially relax the condition that all entire functions operate on the algebra in question.

Turpin [3] constructed a commutative completely metrizable locally pseudoconvex algebra $A$ with exponent $p$, $0 < p < 1$, on which all entire functions operate but which is not $m$-convex. (The definitions are similar to those for $B_0$-algebras. The only difference is that the seminorms satisfying (1), (2), or (3) are not homogeneous, but $p$-homogeneous with exponent $p$, i.e., $|\lambda x| = |\lambda|^p |x|$ for each scalar $\lambda$ and element $x$.) Thus the condition of local convexity cannot be relaxed either.

Later the author [7] showed that there is a complete, commutative non-$m$-convex locally convex algebra on which all entire functions operate. Thus we cannot relax the condition of metrizability. All that means that Theorem A gives the strongest possible result.

Using Theorem A, the author obtained in ([5], Theorem 13.17) the following result:

**Theorem B.** Let $A$ be a commutative complex $B_0$-algebra with unit which is a $Q$-algebra. Then $A$ is multiplicatively-convex.

The same proof gives the result for an algebra without unit, and since the complexification of a $Q$-algebra is again a $Q$-algebra, the result is also true for real algebras. It was an open question (see [8], Problem 26) whether Theorem B is true in the non-commutative case. Our example here also provides a negative answer to that question.

Turpin [3] extended Theorem B to the non-metrizable case:

**Theorem C.** Let $A$ be a commutative complex complete locally convex algebra with unit which is a $Q$-algebra. Then $A$ is a multiplicatively-convex algebra provided the operation of taking inverse $x \rightarrow x^{-1}$ is continuous in $A$.

Similarly to Theorem B, this result can be extended to algebras without unit (provided the operation of taking quasi-inverse is continuous) and to real algebras. Our example shows that the problem of extending Theorem C to the non-commutative case ([8], Problem 27) has a negative answer.

Using Theorem B, the author proved ([5], Theorem 13.18)

**Theorem D.** If a commutative complex $B_0$-algebra $A$ has a closed radical $\text{rad} A$, then this radical is an $m$-convex algebra.

Here again our example shows that the above result fails to be true if $A$ is non-commutative.

For more information on the classes of topological algebras mentioned above the reader is referred to [1] and [4]–[6].

When presenting the above-mentioned example of a pseudoconvex algebra, Turpin used the following lemma given in [2] (see Lemmas 2.1 and 2.2).

**Lemma E.** For any continuous function $v(t) > 0$, $0 \leq t < \infty$, such that $\lim_{t \to \infty} v(t)/t = \infty$, there exists a continuous function $u(t) > 0$, $0 \leq t < \infty$, such that $\lim_{t \to \infty} u(t)/t = \infty$ and

$$u(t_1 + \ldots + t_n) \leq \delta [u(t_1) + \ldots + u(t_n)] + v(n), \quad 0 \leq t_1 < \infty.$$

Our construction will also be based upon this lemma. Following Turpin we choose $v$ so that $u(n) = n (\log n)^{1/2}$ for $n \geq 2$. Thus

$$u(n) = n \log n$$

and for the corresponding function $u$ we have

$$v(n) = r_n \log n$$

with $\lim_{n \to \infty} r_n = 0$.
Denote by $A$ the completion of $A_0$ in the topology given by the seminorms (11). The algebra $A$ consists of elements of the form (10) with infinite summation, such that all seminorms (11) are finite. Clearly these seminorms satisfy (1) and (2), so that $A$ is a $B_0$-algebra.

Let $x$ be an arbitrary element of $A$ and let $m$ be a natural number. We have

$$x^m = \sum_{k_0=0}^{\infty} \sum_{n_1=1}^{\infty} \sum_{n_2}^{\infty} \mu(k, n) t_{n_1} \cdots t_{n_2}$$

where

$$\mu(k, n) = \sum_{k_1 + \cdots + k_m = k} \xi(n, k_1) \xi(n + k_1 + 1, k_2 - 1) \cdots \xi(n + k_1 + \cdots + k_{m-1} + 1, k_m - 1).$$

Note that among the $m!$ products of elements of the form

$$\xi(n + k_1 + \cdots + k_j + 1, k_{j+1} - 1) t_{n+k_1+\cdots+k_{j}+1} \cdots t_{n+k_1+\cdots+k_{j+1}}$$

only one product is different from zero. Thus we have

$$|x^m|_j = \sum_{n, k} |\mu(k, n)| \exp(8^j u(k + 1))$$

$$\leq \sum_{n, k} \sum_{k_1 + \cdots + k_m = k} |\xi(n, k_1)| \xi(n + k_1 + 1, k_2 - 1) \cdots \xi(n + k_1 + \cdots + k_{m-1} + 1, k_m - 1)|$$

$$\times \exp(8^j [8^j 8u(k_1 + 1) + 8u(k_2) + \cdots + 8u(k_m) + u(m)])$$

$$= \exp(8^j v(m)) \sum_{n, k} \sum_{k_1 + \cdots + k_m = k} |\xi(n, k_1)| \exp(8^j u(k_1 + 1)) \cdots \xi(n + k_1 + \cdots + k_{m-1} + 1, k_m - 1)| \exp(8^j u(k_m)).$$

On the other hand, we have

$$|x|_{m+1} \geq m! \sum_{n, k} \sum_{k_1 + \cdots + k_m = k} |\xi(n, k_1)| \exp(8^j u(k_1 + 1)) \cdots \xi(n + k_1 + \cdots + k_{m-1} + 1, k_m - 1)| \exp(8^j u(k_m)).$$

Now (12) and (13) imply

$$|x^m|_j \leq \frac{\exp(8^j v(m))}{m!} |x|_{m+1}^j$$

for all $x$ in $A$ and all natural $j$. 

(14)
Put $a_{m,j} = \exp(8^j v(m))/m!$. Then (5) implies $a_{m,j} = m^j a_{m,j} / m!$. For large $m$ we have $8^j r_m \leq 1/2$, and so

$$a_{m,j} \leq \left( \frac{m^j}{m!} \right)^{1/2} \frac{1}{(m^j)!^{1/2}}.$$ 

But $\lim_{m}(m^j/m!)^{1/m} = e$, thus $a_{m,j} \leq C_m/(m!)^{1/2}$, for large $m$, where $C$ is a positive constant depending only upon $j$. Since $\lim_{m}(CM)^{m} / (m!)^{1/2} = 0$ for each fixed $M$, it follows that the right hand side of (14) tends to zero as $m \to \infty$ for each fixed $j = 2, \ldots$. This means that $\lim_{m} x_m = 0$ for each $x$ in $A$.

Note that $\lim_{m} a_{m,j} = 0$ implies $a_{m,j} \leq C_j$ for all $m$, where $C_j$ is a positive constant. Thus (14) implies

$$|x_m| \leq C_j |x|_{j+1}^{m+1}$$

for all $x$ in $A$ and all positive integers $m$ and $j$.

It remains to be shown that $A$ is a non-$m$-convex algebra. Suppose to the contrary that $A$ is $m$-convex. Then there is a sequence $(\| \cdot \|_i)$ of seminorms on $A$ satisfying (1) and (3) and giving the same topology as the sequence $(\| \cdot \|_i)$. Thus, by (4), there is a constant $c_1$ and a seminorm $\| \cdot \|_i$ such that $c_1 |x|_i \leq \|x\|_i$ for all $x$ in $A$. Similarly, there is an index $k$ and a positive $c_2$ such that together with the previous inequality we have

$$c_1 |x|_i \leq \|x\|_i \leq c_2 |x|_k, \quad x \in A.$$ 

By (3), this implies that for any sequence $(x_i)$ in $A$ such that $c_2 |x_i|_k \leq 1/2$, we have

$$\lim_n |x_1 \ldots x_n|_i = 0.$$ 

Put $x_i = e_i$, and choose a positive $\varepsilon$ so that $c_2 |e_i|_k = c_2 \varepsilon \exp(8^k u(1)) \leq 1/2$. Then $\lim_n e_i |t_1 \ldots t_n|_i = 0$. But $e_i |t_1 \ldots t_n|_i = e_i \exp(8^k u(n))$, and the right hand term, in view of (6), tends to infinity for each positive $\varepsilon$. The conclusion follows.

As corollaries we obtain the following results.

**Theorem 2.** There exists a non-$m$-convex algebra $A$ on which all entire functions operate.

**Proof.** Let $A$ be the algebra constructed in Theorem 1. Let $\varphi(c_i) = \sum_n a_n c_n^i$ be an entire function. Then (15) implies

$$\sum_n a_n x^n |_i \leq C_j \sum_n a_n \|x\|_{j+1}^m < \infty$$

for all $x$ in $A$. Thus all entire functions operate on the algebra $A$, which is non-$m$-convex.

If we wish to have in Theorem 2 an algebra with unit element, we just take the unitization $A_1$ of $A$. It is a non-$m$-convex algebra, and every commutative subalgebra of $A_1$ is $m$-convex, being the unitization of an $m$-convex algebra (see the Corollary below). Thus all entire functions operate on $A_1$.

**Corollary.** There exists a non-$m$-convex $B_0$-algebra with all commutative subalgebras $m$-convex.

**Theorem 3.** There exists a non-$m$-convex $B_0$-algebra which is a $Q$-algebra.

**Proof.** Let $A$ be the algebra of Theorem 1 and let $x$ be an element in $A$. Then $\lim_n |2^n x^n|_i = 0$ for every $j$, and so there is a positive constant $C_j$ such that $|2^n x^n|_i \leq C_j$ for all $n$. This implies

$$\sum_{n=1}^{\infty} |2^n x^n|_i \leq \sum_{n=1}^{\infty} \frac{C_j}{2^n} < \infty$$

and the element $x$ has quasi-inverse $\sum_{n=0}^{\infty} (-1)^n x^n$. Thus $A$ is a $Q$-algebra.

The radical of a non-commutative algebra with unit is the intersection of all its maximal left ideals (equal to the intersection of all its maximal right ideals). Let $A_1$ be the unitization of the algebra $A$ of Theorem 1, with unit $e$. Every element of $A_1$ of the form $\lambda e + x$, where $x \in A$ and $\lambda$ is a non-zero scalar, is invertible with inverse $\sum_{n=0}^{\infty} (\lambda e + x)^n$. Thus every non-invertible element of $A_1$ is in $A$, and so $A$ is the only (two-sided, or one-sided) maximal ideal of $A$ coinciding with its radical. Thus we have

**Theorem 4.** There exists a non-$m$-convex $B_0$-algebra with closed radical which is not $m$-convex.

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