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On integrability in $F$-spaces

by

Mikhail M. Popov (Kharkov)

Abstract. Some usual and unusual properties of the Riemann integral for functions $x : [a, b] \rightarrow X$ where $X$ is an $F$-space are investigated. In particular, a continuous integrable $L^p$-valued function ($0 < p < 1$) with non-differentiable integral function is constructed. For some class of quasi-Banach spaces $X$ it is proved that the set of all $X$-valued functions with zero derivative is dense in the space of all continuous functions, and for any two continuous functions $x$ and $y$ there is a sequence of differentiable functions which tends to $x$ uniformly and for which the sequence of derivatives tends to $y$ uniformly. There is also constructed a differentiable function $x$ with $x'(t_0) = x_0$ for given $t_0$ and $x_0$ and $x'(t) = 0$ for $t \neq t_0$.

Consider the classical definition of the Riemann integral in the setting of vector-valued functions $x : [a, b] \rightarrow X$ where $X$ is an $F$-space (i.e. a complete metric linear space with an invariant metric). For a partition $T = \{t_k\}_{k=0}^n (a = t_0 < t_1 < \ldots < t_n = b)$ of $[a, b]$ and a collection $A = \{\lambda_k\}_{k=1}^n (\lambda_k \in [t_{k-1}, t_k])$ define the Riemann sum

$$\mathcal{S}(T, A) = \sum_{k=1}^n x(\lambda_k) \Delta t_k, \quad \Delta t_k = t_k - t_{k-1}.$$  

A function $x$ is said to be integrable on $[a, b]$ if $\mathcal{S}(T, A)$ has a limit as $\max_k \Delta t_k \to 0$, i.e. if there exists an element $\int_a^b x(t) \, dt \in X$ such that for any $\varepsilon > 0$ there is a $\delta > 0$ such that for each partition $T$ of $[a, b]$ with $\max_k \Delta t_k < \delta$ and each $A$,

$$\left\| \mathcal{S}(T, A) - \int_a^b x(t) \, dt \right\| < \varepsilon.$$  

Some usual properties of the Riemann integral remains true: each integrable function is bounded and the integrability of $x$ on both intervals $[a, b]$
and \([b, c]\) implies its integrability on \([a, c]\) with
\[
\int_a^c x(t) \, dt = \int_a^b x(t) \, dt + \int_b^c x(t) \, dt.
\]

Other properties are not so trivial or even false. For example, an \(F\)-space \(X\) is locally convex if and only if every continuous function \(x : [a, b] \to X\) is integrable [4], [5, p. 121].

In Section 1 we prove some further properties of the Riemann integral. Section 2 is devoted to the construction of an integrable continuous function \(y : [0, 1] \to I_p\) \((0 < p < 1)\) for which the function
\[
x(t) = \int_0^t y(s) \, ds
\]
is not differentiable on the right at \(t = 0\).

The main unsolved question here is: does every continuous function \(y : [a, b] \to X\) have a primitive? (of course, we assume that \(X\) is not locally convex.) The result of Section 2 shows that the usual way of obtaining primitives fails even for integrable continuous functions. Another way of getting primitives is by passage to the limit. In Section 3 we show that this may also fail, by proving that for some class of \(F\)-spaces \(X\) and for any two continuous functions \(x, y : [a, b] \to X\) there exists a sequence of differentiable functions \(\{x_n\}_{n=1}^\infty\) such that \(x_n\) tends to \(x\) and \(x'_n\) tends to \(y\) uniformly on \([a, b]\). Finally, we show that there are differentiable functions with derivatives having points of discontinuity of the first kind.

We denote by \(\|x\|\) the \(F\)-norm of \(X\), i.e. \(\|x\| = \rho(x, 0)\) where \(\rho\) is the metric of \(X\), and \(\mathcal{L}(X)\) denotes the space of all continuous linear operators acting in \(X\).

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1. Some connections with other properties

Proposition 1.1. Let \(X\) be an \(F\)-space and \(x : [a, b] \to X\) be an integrable function on \([a, b]\). Then the set of all Riemann sums of \(x\) on \([a, b]\) is bounded in \(X\) (in particular, \(x\) is bounded).

Proof. First we show that \(x\) is bounded. Supposing the contrary, let \(s_n \downarrow 0\) and \(s_n \in [a, b]\) be numbers such that \(\|x(s_n)\| \geq \delta_0 > 0\) for each \(n\). Choose \(\delta > 0\) so that for every partition of \([a, b]\) with diameter \(< \delta\) any corresponding Riemann sum \(\mathcal{S}\) satisfies

\[
\sup_{0 < \varepsilon \leq \delta_1} \left\| \mathcal{S} - \int_a^b x(t) \, dt \right\| < \frac{\delta_0}{3}.
\]

Let \(m \geq (b - a)/\delta\). Decompose \([a, b]\) into intervals \(\{I_k\}_{k=1}^m\) of length \((b - a)/m\). Let \(k_0\) be an index such that there are infinitely many \(s_n\)'s in \(I_{k_0}\); say, \(\{s_{n_k}\}_{k=1}^\infty \subset I_{k_0}\). Choose any \(\varepsilon_n \in I_k\) \((k = 1, \ldots, m)\) and put
\[
\mathcal{S}_0 = \frac{b - a}{m} \sum_{k \neq k_0} x(\xi_k), \quad \mathcal{S}_n = \mathcal{S}_0 + \frac{b - a}{m} x(s_{n_k}).
\]

Then
\[
\sup_{0 < \varepsilon \leq \delta_1} \left\| \mathcal{S}_n - \int_a^b x(t) \, dt \right\| < \frac{\delta_0}{3}
\]
for each \(n\) and hence
\[
\left\| x(s_{n_k}) \right\| = \left\| \frac{b - a}{m} x(s_{n_k}) \right\|
\leq \left\| \mathcal{S}_0 \right\| + \left\| \frac{b - a}{m} \left( \mathcal{S}_n - \int_a^b x(t) \, dt \right) \right\| + \left\| \mathcal{S}_n - \int_a^b x(t) \, dt \right\|,
\]

Since each of the terms on the right hand side can be made \(< \delta_0/3\) for \(n\) large enough, the last inequality contradicts the assumption \(\|x(s_{n_k})\| \geq \delta_0\).

Thus, \(x\) is bounded. Let \(\varepsilon_0 > 0\). It is not hard (using the integrability of \(x\)) to choose \(\delta > 0\) such that for each collection \(\{I_k\}_{k=1}^m\) of subintervals of \([a, b]\) with disjoint interiors and with max \(\mu(I_k) < \delta\), and for any \(\eta \in I_k\),

\[
\sup_{0 < \varepsilon \leq \delta_1} \left\| \mathcal{S} \left( \sum_{k=1}^m x(\eta_k) \mu(I_k) - \sum_{k=1}^m \int_{I_k} x(t) \, dt \right) \right\| < \frac{\varepsilon_0}{2}.
\]

Using boundedness of \(x\), choose \(\varepsilon_1 \in [0, 1]\) so that

\[
\sup_{0 < \varepsilon \leq \delta_1} \left\| x(\mathcal{S} - \int_a^b x(t) \, dt) \right\| < \frac{\varepsilon_0}{2}.
\]

Let \(\mathcal{S}\) be an arbitrary Riemann sum constructed for some partition \(a = t_0 < \ldots < t_n = b\). Denote by \(n_0\) the number of intervals \([t_{k-1}, t_k]\) of length \(\geq \delta\). Clearly, \(n_0 \leq (b - a)/\delta\). Now denote by \(\mathcal{S}_0\) the part of \(\mathcal{S}\) which is obtained by summing over intervals of length \(< \delta\). Then for \(0 < \varepsilon \leq \varepsilon_1\), we have

\[
\left\| \mathcal{S} \right\| \leq \left\| \mathcal{S}_0 \right\| + \sum_{\mu(I_k) \geq \delta} \left\| x(t_k - t_{k-1}) \right\|.
\]

Then
\[
\leq \frac{\varepsilon_0}{2} + \sum_{\mu(I_k) \geq \delta} \frac{\varepsilon_0 \delta}{2(b - a)} = \frac{\varepsilon_0}{2} + n_0 \frac{\varepsilon_0 \delta}{2(b - a)} \leq \varepsilon_0.
\]
Now suppose that \( y \) is integrable on \([a, b]\) and \( t_0 \in [a, b] \). If \( X \) is locally convex then one can show that
\[
x(t) = \int_{t_0}^{t} y(s) \, ds
\]
is a differentiable function at each point of continuity of \( y \) and the main formula of Integral Calculus is valid: \( x' = y \). The situation changes when we pass to non-locally convex spaces. In general, we can only prove continuity of \( x \).

**Proposition 1.2.** Let \( y : [a, b] \to X \) be integrable on \([a, b]\) (where \( X \) is an \( F \)-space). Then the function \( x \) defined by (1) is uniformly continuous on \([a, b]\).

**Proof.** By the Cantor theorem it is enough to prove the continuity of \( x \) at any point \( t_1 \in [a, b] \). For given \( \varepsilon > 0 \) choose \( \delta > 0 \) so that for every partition \( T = \{\tau_k\}_{k=1}^m \) of \([a, b]\), \( a = \tau_0 < \ldots < \tau_m = b \), with \( \text{diam} \, T = \max_{i,j} (\tau_j - \tau_i) < \delta \) and for any points \( \xi_k \in [\tau_i-1, \tau_i] \) \((\xi = (\xi_k)_{k=1}^m)\) the corresponding Riemann sum \( \mathcal{S}(T, \xi) \) satisfies
\[
\left\| \mathcal{S}(T, \xi) - \int_{a}^{b} y(t) \, dt \right\| < \frac{\varepsilon}{4}.
\]

Suppose that \( t \in [a, b] \) and \( 0 < |t - t_1| < \delta \). Let \( T_1 = \{ t_1 = \tau_0 < \ldots < \tau_m = t \} \) be any partition of \([t_1, t]\) \((t < t_1)\) and \( \eta_i \in [\tau_{i-1}, \tau_i] \) for \( i = 1, \ldots, m \) \((\eta = (\eta_i)_{i=1}^m)\). Let us supplement the collection \( \{ \tau_i \}_{i=1}^m \) with points from \([a, b] \setminus [t_1, t]\) so that the new partition \( T_1 \) of \([a, b]\) satisfies \( \text{diam} \, T < \delta \). Denote by \( T' \) the partition of \([a, b]\) which is obtained from \( T_1 \) by removing \( t_1, \ldots, t_m \). Since \( |t - t_1| < \delta \) and \( \tau_i \in [t_1, t] \) and the ends \( \tau_0 = t_1, \tau_m = t \) are still in \( T_1 \), we have \( \text{diam} \, T' < \delta \). Choose a collection of points \( \xi \) for \( T_1 \) as follows. For the intervals \([\tau_{i-1}, \tau_i]\) retain the points \( \eta_i \) which have already been chosen and choose the left ends of the remaining intervals \( \text{(right ends if } t < t_1 \text{)} \). For the partition \( T' \) denote by \( \xi' \) the collection of the left ends of intervals from \( T' \). Then
\[
\mathcal{S}(T, \xi) - \mathcal{S}(T', \xi') = \sum_{i=1}^{m} \int_{\tau_{i-1}}^{\tau_i} y(t) \, dt - \int_{t_0}^{t} y(t) \, dt = \int_{t_0}^{t} y(t) \, dt - \int_{t_0}^{t} y(t) \, dt = 0.
\]

On the other hand, since \( \text{diam} \, T < \delta \) and \( \text{diam} \, T' < \delta \), we have
\[
\left\| \mathcal{S}(T, \xi) - \mathcal{S}(T', \xi') \right\| \leq \left\| \mathcal{S}(T, \xi) - \int_{a}^{b} y(t) \, dt \right\| + \left\| \int_{a}^{b} y(t) \, dt - \mathcal{S}(T', \xi') \right\| < \frac{\varepsilon}{2}.
\]

Choose \( \delta_1 > 0 \) so that if \( |\alpha| < \delta_1 \) then \( |\alpha y(t_1)| < \varepsilon/2 \). Putting \( \delta_2 = \min(\delta, \delta_1) \), we find that if \( |t - t_1| < \delta_2 \) then for each partition \( T_1 \) of \([t_1, t]\) \((t < t_1)\) and each \( \eta_i \in [\tau_{i-1}, \tau_i] \), where \( \tau_i \) are the points of \( T_1 \), we have
\[
\left\| \sum_{i=1}^{m} \eta_i \mathcal{S}(T_1) \right\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Since \( \sum_{i=1}^{m} \eta_i \mathcal{S}(T_1) \) is an arbitrary Riemann sum for \( y \) on \([t_1, t]\) \((t < t_1)\), we conclude that
\[
\left\| x(t) - x(t_1) \right\| = \left\| \int_{t_1}^{t} y(t) \, dt \right\| \leq \varepsilon.
\]

**Corollary 1.3.** Let \( y : [a, b] \to X \) be integrable on \([a, b]\) and \( \{ a_n \}_{n=0}^{\infty} \) be a numerical sequence satisfying \( a < a_{n+1} < a_n < a_0 = a \) for \( n \geq 1 \) and \( \lim_{n \to \infty} a_n = a \). Then the series
\[
\sum_{n=1}^{\infty} \int_{a_n}^{a_{n+1}} y(t) \, dt = \int_{a}^{b} y(t) \, dt
\]
converges in \( X \).

The following two propositions investigate the connections between convergence of improper integrals and integrability, for the needs of Section 2. We omit their proofs which are natural and straightforward.

**Proposition 1.4.** Let \( X \) be a non-locally convex \( F \)-space. There exists a continuous function \( x : [0, 1] \to X \) such that
\begin{enumerate}
  \item \( x(0) = 0 \);
  \item \( x \) is integrable on \([\varepsilon, 1]\) for every \( \varepsilon \in (0, 1) \) and the limit
    \[\lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} x(t) \, dt\]
    exists;
  \item \( x \) is not integrable on \([0, 1]\).
\end{enumerate}

**Proposition 1.5.** Let \( X \) be an \( F \)-space and \( x : [a, b] \to X \) be a bounded function. Suppose that for some sequence \( T_n \to a, T_n \in [a, b] \), the following hold:
(i) $x$ is integrable on $[T_n, b]$ for each $n$ and the limit
\[ I = \lim_{n \to \infty} \int_{T_n}^b x(t) \, dt \]
exists;

(ii) for every $\varepsilon > 0$ there is a $\delta > 0$ such that for each partition $T_n = \tau_0 < \tau_1 < \ldots < \tau_m = b$ with $\max_k \Delta\tau_k = \max_k (\tau_k - \tau_{k-1}) < \delta$ and each $\zeta_k \in [\tau_{k-1}, \tau_k]$ the Riemann sum
\[ \mathcal{S}_0 = \sum_{k=1}^m x(\zeta_k) \Delta\tau_k \]
satisfies the condition
\[ \left\| \mathcal{S}_0 - \int_{T_n}^b x(t) \, dt \right\| < \varepsilon. \]

Then $x$ is integrable on $[a, b]$ and
\[ \int_a^b x(t) \, dt = I. \]

2. An integrable continuous function having non-differentiable integral function. We show that if the space $l_p$ with $0 < p < 1$ embeds isomorphically in an $F$-space $X$ then there exists a function $y : [0, 1] \to X$ as in the title of this section. Clearly, we may simply assume that $X = l_p$. However, we do not know whether there exists a continuous integrable function for which the integral function is non-differentiable at each point or even almost everywhere.

**Theorem 2.1.** There exists a continuous Riemann integrable function $y : [0, 1] \to l_p$ ($0 < p < 1$) such that the function
\[ x(t) = \int_0^t y(s) \, ds \]
does not have a right derivative at $t = 0$.

For the proof of Theorem 2.1 we need some facts. Denote by $\{e_n\}_{n=1}^\infty$ the standard basis in $l_p$. Put
\[
c_n = 2^{-n}, \quad a_1 = 1, \quad b_1 = 1/2, \quad a_2 = a_3 = 2^{-(1-p)/2}, \quad b_2 = b_3 = c_2/2,
\]
\[
a_{(n-1)n+1} = a_{(n-1)n+2} = \ldots = a_{(n-1)n+2}/2,
\]
\[
b_{(n-1)n+1} = b_{(a-1)n+2} = \ldots = b_{(a-1)n+1}/2 = c_n/n.
\]

Obviously,
\[ \sum_{k=1}^\infty b_k = \sum_{n=2}^\infty c_n = 1. \]

Now put
\[ t_1 = 1, \quad t_n = 1 - \sum_{k=1}^{n-1} b_k = \sum_{k=n}^\infty b_k \quad (n \geq 2), \quad d_k = b_k/2 \quad (k \geq 1). \]

**Lemma 2.2.** (a) $t_{k+1} + d_{k+1} = t_k - d_k$ for all $k \geq 1$;

(b) for the function $y(t)$ defined on $[t_k - d_k, t_k + d_k]$ by
\[ y(t) = y_k(1 - |t - t_k|/d_k) \]
where $y_k = a_k^{1/p} e_k$, every Riemann sum $\mathcal{S}_k$ on this interval is estimated as
\[ \|\mathcal{S}_k\| \leq 2^p d_k^p, \]

(c) $\sum_{k=1}^\infty d_k^p < \infty$.

**Proof.** (a) Since $t_k - t_{k-1} = b_k$, we have
\[ t_{k+1} + d_{k+1} = t_{k+1} + \frac{b_{k+1}}{2} \leq t_{k+1} + \frac{b_k}{2} = t_{k+1} + \frac{t_k - t_{k+1}}{2} \]
\[ = t_k - \frac{t_k - t_{k+1}}{2} = t_k - \frac{t_k - b_k}{2} = t_k - d_k. \]

(b) Let $t_k - d_k = s_0 < s_1 < \ldots < s_n = t_k + d_k$ be any partition, and let $\xi_i \in [s_{i-1}, s_i]$, $\Delta s_i = s_i - s_{i-1}$. Then
\[ \mathcal{S}_k = \sum_{i=1}^n y(\xi_i) \Delta s_i = \sum_{i=1}^n y_k(1 - |\xi_i - t_k|/d_k) \Delta s_i \]
\[ = y_k \sum_{i=1}^n (1 - |\xi_i - t_k|/d_k) \Delta s_i. \]

Then by the definition of $y_k$,
\[ \|\mathcal{S}_k\| = a_k^{1/p} \sum_{i=1}^n (1 - |\xi_i - t_k|/d_k) \Delta s_i |^p \leq \sum_{i=1}^n \Delta s_i |^p a_k, \]
since $|1 - |\xi_i - t_k|/d_k| \leq 1$. Thus, $\|\mathcal{S}_k\| \leq a_k (2d_k)^p \leq 2^p d_k^p$. 

(c) Note that
\[ \sum_{k=1}^{n(n+1)/2} b_k^p = \sum_{i=1}^{n} \sum_{j=(i-1)/2+1}^{i(i+1)/2} b_j^p \]
\[ = \sum_{i=1}^{n} \sum_{j=(i-1)/2+1}^{i(i+1)/2} \left( \frac{c_j}{i} \right)^p = \sum_{i=1}^{n} \frac{c_i^p}{2^{2p}} \leq \sum_{i=1}^{2^{2p}} \frac{1}{2^{2p}}. \]
This implies the convergence of \( \sum_{k=1}^{\infty} b_k^p \) and hence that of \( \sum_{k=1}^{\infty} d_k^p \). The lemma is proved.

Define \( y(t) \) on \([0, 1]\) by putting
\[ y(t) = y_k(1 - |t - t_k|/d_k) \]
for \( t \in [t_k - d_k, t_k + d_k] \cap [0, 1] \), \( k = 1, 2, \ldots \), and \( y(t) = 0 \) for \( t \in [0, 1] \setminus \bigcup_{k=1}^{n_0} [t_k - d_k, t_k + d_k] \). Clearly, \( y \) is continuous on \([0, 1]\) and \( \|y(t)\| \leq 1 \) for each \( t \in [0, 1] \).

**Lemma 2.3.** Let \( \lambda \in (0, 1) \) and \( 0 = \tau_0 < \tau_1 < \ldots < \tau_s = \lambda \) be an arbitrary partition of \([0, \lambda]\): \( \xi_i \in [\tau_{i-1}, \tau_i], i = 1, \ldots, s \), and \( \Delta \tau_i = \tau_i - \tau_{i-1} \). Let \( n_0 \) be an integer such that \( \lambda \leq n_0 \). Then the Riemann sum \( S = \sum_{i=1}^{s} y(\xi_i) \Delta \tau_i \) of the function \( y \) defined above has the following estimate:
\[ \|S\| \leq (2^{p+1} + 4 + 2^p) \sum_{k=n_0}^{\infty} d_k^p. \]

**Proof.** Decompose \( S \) as
\[ S = S' + S'' + \sum_{k=n_0}^{\infty} S_k \]
where \( S_k \) is the sum of \( y(\xi_i) \Delta \tau_i \) over all \( i \) for which \( [\tau_{i-1}, \tau_i] \subset [t_k - d_k, t_k + d_k] \), \( S' \) is the sum over \( i \) for which \( y(\xi_i) = 0 \), and \( S'' \) over the remaining \( i \) (i.e., over those \( i \) for which \( [\tau_{i-1}, \tau_i] \) lies partly in \( \tilde{F} = \bigcup_{k=n_0}^{\infty} [t_k - d_k, t_k + d_k] \), and partly in \([0, \lambda] \setminus \tilde{F}\).

We now give an upper estimate of the sums \( S' \), \( S'' \) and \( S_k \) (of course, there are only finitely many non-zero elements among \( S_k \)). Clearly, \( S' = 0 \). To estimate \( \|S''\| \), denote by \( I \) the set of corresponding indices \( i \). Let \( i \in I \). Since \( y(\xi_i) \neq 0 \), there is a (unique) \( k = k(i) \geq n_0 \) such that \( \xi_i \in [t_k - d_k, t_k + d_k] \). Now put
\[ K = \{ k \geq n_0 : (\exists i \in I) \ k = k(i) \}. \]

Suppose that \( K = \{ k_1, \ldots, k_r \} \) where \( n_0 \leq k_1 < \ldots < k_r \). Then
\[ \|S''\| = \left\| \sum_{i \in I} y(\xi_i) \Delta \tau_i \right\| \leq \sum_{i \in I} y(\xi_i) \|\Delta \tau_i\|^p \]
\[ \leq \sum_{j=1}^{r} \sum_{i \in I} y(\xi_i) \|\Delta \tau_i\|^p \leq \sum_{j=1}^{r} \sum_{k(i) = k_j} y(\xi_i) \|\Delta \tau_i\|^p. \]

Note that if \( k(i) = k_j \) then
\[ \Delta \tau_i \leq t_{k_j-1} + d_{k_j} - t_{k_j} + d_{k_j} \]
since the definitions of \( k_1, \ldots, k_r \) and of \( I \) and \( K \) imply
\[ t_{k_r} - d_{k_r} < \tau_i \leq \tau_i < \tau_{k_r} - d_{k_r} \]
(\( k_0 = n_0 \) is understood). Hence the right-hand side of (2) is bounded by
\[ \sum_{j=1}^{r} \sum_{k(i) = k_j} \|y(\xi_i)\| (t_{k_j-1} - t_{k_j} + d_{k_j} - d_{k_j})^p. \]

Note that for a given \( k \in K \) there are at most two indices \( i', i'' \) in \( I \) such that \( k = k(i') = k(i'') \) since \([\tau_i - \tau_{i-1}, \tau_{i+1} - \tau_i] \) should intersect \([t_k - d_k, t_k + d_k] \) without being contained in it. Thus, (3) is estimated by
\[ \leq \sum_{j=1}^{2} \sum_{k(i) = k_j} \|y(\xi_i)\| (t_{k_j-1} - t_{k_j} + d_{k_j} - d_{k_j})^p \]
\[ \leq \sum_{j=1}^{2} \sum_{k(i) = k_j} \|y(\xi_i)\| (t_{k_j-1} - t_{k_j} + d_{k_j} - d_{k_j})^p + 2 \sum_{j=1}^{r} \|y(\xi_i)\| d_{k_j-1}^p + 2 \sum_{j=1}^{r} \|y(\xi_i)\| d_{k_j}^p. \]

To estimate \( S_k \), note that it is a Riemann sum for \( y \) on \([t_k - d_k, t_k + d_k] \)
(the missing terms of the form \( y(\xi)(\tau_i - t_k - d_k) \) and \( y(\xi)(t_k + d_k - \tau_i) \) are
considered to be zero since \( y(\xi) = y(\eta) = 0 \) for \( \xi = t_k - d_k \) and \( \eta = t_k + d_k \). Thus, by Lemma 2.2(b),

\[
\|S_k\| \leq 2^p d_k^p,
\]

and so

\[
\left\| \sum_{k=n_0}^{\infty} S_k \right\| \leq \sum_{k=n_0}^{\infty} \|S_k\| \leq 2^p \sum_{k=n_0}^{\infty} d_k^p.
\]

Combining all the above estimates we obtain the assertion of the lemma.

**Proof of Theorem 2.1.** Now we prove that \( y \) satisfies the assumptions of Proposition 1.5 with \( T_y = t_n - d_n \) and \([a, b] = [0, 1]\) (the definition of \( y \) was given before Lemma 2.3).

Since \( y \) is piecewise linear on \([T_n, 1]\), it is integrable. In order to prove the existence of the limit in (1), we calculate the integral

\[
\int_{T_k}^{T_{k+1}} y(t) \, dt = \int_{t_k-d_k}^{t_k+d_k} y_k(1 - |t - t_k|/d_k) \, dt
\]

\[
= y_k \int_{t_k-d_k}^{t_k+d_k} (1 - |t - t_k|/d_k) \, dt = d_k y_k.
\]

Hence if \( n > n_0 \) then

\[
\int_{T_n}^{T_n^0} y(t) \, dt = \sum_{k=n_0+1}^{n} \int_{T_k}^{T_{k+1}} y(t) \, dt = \sum_{k=n_0+1}^{n} d_k y_k,
\]

and also

\[
\left\| \int_{T_n}^{T_n^0} y(t) \, dt \right\| \leq \sum_{k=n_0+1}^{\infty} d_k^p \left\| y_k \right\| < \sum_{k=n_0+1}^{\infty} d_k^p.
\]

This means that \( \int_{T_n}^{T_n^0} y(t) \, dt \) is a Cauchy sequence. Define

\[
I = \lim_n \frac{1}{T_n} \int_{T_n}^{T_n^0} y(t) \, dt.
\]

To prove assumption (ii) of Proposition 1.5, fix \( \varepsilon > 0 \) and choose \( n_0 \) so that every Riemann sum for \( y \) on \([0, T_n^0]\) is less than \( \varepsilon/4 \) (this is possible by Lemma 2.3) and so that for each \( n \geq n_0 \),

\[
\left\| \int_{T_n}^{T_n^0} y(t) \, dt \right\| < \frac{\varepsilon}{4}.
\]

Now pick \( \delta > 0 \) so that every Riemann sum \( S_1 \) for \( y \) on \([T_n, 1]\), corresponding to any partition \( U \) of \([T_n, 1]\) with \( \text{diam} U < \delta \), satisfies

\[
\left\| S_1 - \frac{1}{T_n} \int_{T_n} y(t) \, dt \right\| < \frac{\varepsilon}{4}.
\]

and also that \( \delta^p < \varepsilon/4 \). Let \( n > n_0 \) and let \( S = \{T_n = \tau_0 < \tau_1 < \ldots < \tau_m = 1\} \) be a partition of \([T_n, 1]\) with \( \text{diam} S < \delta \). Put

\[
S_0 = \sum_{k=1}^{m} y(\xi_k) \Delta \tau_k
\]

where \( \Delta \tau_k = \tau_k - \tau_{k-1} \) and \( \xi_k \in [\tau_{k-1}, \tau_k] \). Choose \( j \in \{1, \ldots, m\} \) so that \( T_{n_0} \in [\tau_j-1, \tau_j] \). Note that \( S_1 = \sum_{k=1}^{m} y(\xi_k) \Delta \tau_k \) is a Riemann sum for \( y \) on \([T_{n_0}, 1]\) since we can pick \( \xi = T_{n_0} \in [T_{n_0}, 1] \) with \( y(\xi) = 0 \). Finally, \( S' = \sum_{k=1}^{m-1} y(\xi_k) \Delta \tau_k \) is a Riemann sum for \( y \) on \([0, T_{n_0}] \), since \( y(0) = y(T_{n_0}) = 0 \). Thus,

\[
\left\| S_0 - \frac{1}{T_n} \int_{T_n} y(t) \, dt \right\| \leq \left\| S' \right\| + \left\| \frac{1}{T_n} \int_{T_n} y(t) \, dt \right\|
\]

\[
< 4 \frac{\varepsilon}{4} = \varepsilon.
\]

Suppose that \( n \leq n_0 \), i.e. \( T_{n_0} \leq T_n \). It is easily seen that (since \( y \) is piecewise linear on \([T_{n_0}, T_n]\)) there is a Riemann sum \( S'' \) for \( y \) on \([T_{n_0}, T_n] \) such that

\[
S'' = \int_{T_{n_0}}^{T_n} y(t) \, dt.
\]

Hence also for \( n \leq n_0 \) we have

\[
\left\| S_0 - \frac{1}{T_n} \int_{T_n} y(t) \, dt \right\| = \left\| S_0 + S'' - \frac{1}{T_{n_0}} \int_{T_{n_0}}^{T_n} y(t) \, dt \right\| < \frac{\varepsilon}{4} < \varepsilon,
\]

since \( S_0 + S'' \) is a Riemann sum for \( y \) on \([T_{n_0}, 1]\).

Thus, we have proved that \( y \) is integrable on \([0, 1]\). Now we prove that the function

\[
x(t) = \int_{0}^{t} y(s) \, ds
\]
does not have a right derivative at 0. By (4),
\[ x(T_{n_0}) = \lim_{n \to \infty} (x(T_n) - x(T_{n+1})) = \sum_{k=n_0+1}^{\infty} d_k y_k. \]
Hence
\[ \|T^{-1}_{n_0}(x(T_{n_0}) - x(0))\| = T^{-1}_{n_0} \|x(T_{n_0})\| = T^{-1}_{n_0} \sum_{k=n_0+1}^{\infty} d_k y_k. \]
By the definition of \( y_k \),
\[ \| \sum_{k=(n-1)/2+1}^{n} d_k y_k \| = \sum_{k=(n-1)/2+1}^{\infty} d_k^p \| y_k \| \]
\[ = 2^{-p} \sum_{k=(n-1)/2+1}^{\infty} b_k^p a_k = 2^{-p} \sum_{i=1}^{\infty} \sum_{k=(i-1)/2+1}^{\infty} b_k^p a_k \]
\[ = 2^{-p} \sum_{i=1}^{\infty} \frac{k^{(i+1)/2}}{i^p} \left( \frac{1}{i} \right)^{1-p} / 2 \]
\[ = 2^{-p} \sum_{i=1}^{\infty} c_i^p \left( \frac{1}{i} \right)^{1-p} / 2 \]
\[ \geq 2^{-p} n^{(1-p)/2} \sum_{i=n}^{\infty} c_i^p \]
\[ \geq 2^{-p} n^{(1-p)/2} \left( \sum_{i=n}^{\infty} c_i^p \right)^p = \frac{n^{(1-p)/2}}{2^p} \left( \sum_{k=(n-1)/2+1}^{\infty} b_k \right)^p \]
\[ = 2^{-p} n^{(1-p)/2} \left( \frac{n^{(1-p)/2}}{2^p} \right)^p = 2^{-p} n^{(1-p)/2}(T_{(n-1)/2} - d_{(n-1)/2})^p. \]
Thus,
\[ \|T^{-1}_{(n-1)/2}(x(T_{(n-1)/2}) - x(0))\| \]
\[ \geq T_{(n-1)/2} - 2^{-p} n^{(1-p)/2} \left( \frac{n^{(1-p)/2}}{2^p} \right)^p \]
\[ \geq 2^{-p} n^{(1-p)/2} \left( 1 - \frac{1}{2^p} \right)^p \]
\[ \geq 2^{-p} n^{(1-p)/2} \left( 1 - \frac{1}{2^p} \right)^p \]
\[ = 2^{-p} n^{(1-p)/2} \left( 1 - \frac{n}{2(n-1)} \right)^p. \]
Thus, the absence of the right derivative at 0 for \( x \) is proved.

3. On the impossibility of passage to a limit under the derivaution

**Theorem 3.1.** Let \( x, y : [a, b] \to L_p \) \((0 < p < 1)\) be continuous. There exists a sequence \( x_n : [a, b] \to L_p, n \geq 1 \), of functions differentiable on \([a, b]\) such that \( x_n \) tends to \( x \) uniformly on \([a, b]\) and \( x_n' \) tends to \( y \) uniformly on \([a, b]\).

For the proof we need a few lemmas.

**Lemma 3.2.** For each \( x_0 \in L_p \setminus \{0\} \) \((0 < p < 1)\) there exists a constant \( M < \infty \) such that for each \( y_0 \in L_p \) there is a \( T \in L(L_p) \) with \( Tx_0 = y_0 \) and \( \|T\| \leq M \|y_0\| \).

**Lemma 3.3.** The set of all functions \( x : [a, b] \to L_p \) \((0 < p < 1)\) with zero derivative is dense in the space \( C([a, b], L_p) \) of all continuous functions from \([a, b]\) into \( L_p \).

**Proof.** Fix \( y \in C([a, b], L_p), \varepsilon > 0 \) and \( \alpha < \beta \). First we show that there is a constant \( M \) such that for each \( x, y \in L_p \) there exists a differentiable function \( z : [\alpha, \beta] \to L_p \) with the properties

(i) \( z(\alpha) = x, z(\beta) = y, \)
(ii) the oscillation of \( z \) on \([\alpha, \beta]\) satisfies
\[ \omega(z, [\alpha, \beta]) := \sup_{t \in [\alpha, \beta]} \|z(t) - z(s)\| \leq M\|x - y\|, \]
(iii) \( z'(t) = 0 \) for each \( t \in [\alpha, \beta] \).

Let \( u : [a, b] \to L_p \) be some non-constant differentiable function with zero derivative (such a function exists in each \( F \)-space with trivial dual \([2]\)). By continuity of \( u \), there are numbers \( \alpha_1, \beta_1 (\alpha \leq \alpha_1 < \beta_1 \leq b) \) such that
\[ 0 < \omega(u, [\alpha_1, \beta_1]) \leq 1. \]
Again by continuity of \( u \), there are \( \alpha_2, \beta_2 (\alpha \leq \alpha_2 < \beta_2 < \beta_1 \leq \beta) \) with
\[ \omega(u, [\alpha_1, \beta_1]) = \|u(\alpha_2) - u(\beta_2)\| \]
and hence
\[ 0 < \omega(u, [\alpha_2, \beta_2]) = \|u(\alpha) - u(\beta)\| \leq 1. \]

By Lemma 3.2, for \( x_0 = u(\beta_2) - u(\alpha_2) \), choose \( M < \infty \) so that for each \( y_0 \in L_p \) there is a \( T \in L(L_p) \) with \( Tu(\beta_2) - u(\alpha_2) = y_0 \) and \( \|T\| \leq M\|y_0\| \). Putting \( y_0 = y - x \), choose \( T \in L(L_p) \) with the above properties. For each \( t \in [\alpha_2, \beta_2] \) put
\[ u(t) = T(u(t) - u(\alpha_2)) + x. \]
It is not hard to see that \( v \) satisfies the following conditions:

(i') \( v(\alpha_2) = x, v(\beta_2) = y \),

(ii') \( v'(t) = 0 \) for each \( t \in [\alpha_2, \beta_2] \).

We show

(ii') \( \omega(v, [\alpha_2, \beta_2]) \leq M\|x - y\| \).

Fix any \( t, s \in [\alpha_2, \beta_2] \). Then

\[
\|v(t) - v(s)\| = \|T(u(t) - u(s))\|.
\]

By (5) we obtain \( \|u(t) - u(s)\| \leq 1 \), hence

\[
\|v(t) - v(s)\| \leq \|T\| \leq M\|y - x\|.
\]

Finally, define \( z \) as the composition of the linear bijection of \([\alpha, \beta]\) onto \([\alpha_2, \beta_2]\) and the function \( v \). Thus, (i'), (ii'), (iii') imply (i), (ii), (iii) for \( z \).

Since \( y \) is uniformly continuous on \([\alpha, \beta]\), we can decompose \([\alpha, \beta]\) into small intervals \( a = t_0 < \ldots < t_n = b \) so that for each \( k = 1, \ldots, n \),

\[
\omega(y, [t_{k-1}, t_k]) \leq \frac{\varepsilon}{M + 1}.
\]

For each \( k = 1, \ldots, n \), choose \( z_k : [t_{k-1}, t_k) \rightarrow L_p \) so that

(i) \( z_k(t_{k-1}) = y(t_{k-1}) \), \( z_k(t_k) = y(t_k) \),

(ii) \( \omega(z_k, [t_{k-1}, t_k]) \leq \|M\|y(t_k) - y(t_{k-1})\| \),

(iii) \( z_k'(t) = 0 \) for each \( t \in [t_{k-1}, t_k) \).

Then we piece together the functions \( z_k \):

\[
x(t) = z_k(t) \quad \text{if} \quad t \in [t_{k-1}, t_k), \quad k = 1, \ldots, n.
\]

Thus, \( x \) is defined on \([a, b]\) and has zero derivative. To estimate \( \|x - y\| \), fix any \( t \in [a, b] \); say, \( t \in [t_{k-1}, t_k) \). Then

\[
\|x(t) - y(t)\| = \|z_k(t) - y(t)\| \\
\leq \|z_k(t) - z_k(t_k)\| + \|y(t_k) - y(t)\| \\
\leq \omega(z_k, [t_{k-1}, t_k]) + \omega(y, [t_{k-1}, t_k]) \\
\leq M\|y(t_k) - y(t_{k-1})\| + \varepsilon/(M + 1) \\
\leq Mu(y, [t_{k-1}, t_k]) + \frac{\varepsilon}{M + 1} \leq \frac{\varepsilon}{M + 1}(M + 1) = \varepsilon.
\]

**Lemma 3.4.** Let \( y_1 \) be a continuous piecewise linear function from \([a, b]\) into an \( F \)-space \( X \). Then \( y_1 \) is integrable on \([a, b]\) and has a primitive of the form

\[
x(t) = \int_a^t y_1(s) \, ds.
\]

The proof is straightforward.

**Proof of Theorem 3.1.** Fix \( \varepsilon > 0 \). Since \( y \) is uniformly continuous on \([a, b]\), we can construct a continuous piecewise linear function \( y_1 : [a, b] \rightarrow L_p \) such that \( \|y_1(t) - y(t)\| < \varepsilon \) for each \( t \in [a, b] \). Let \( x \) be a primitive of \( y_1 \) on \([a, b]\). By Lemma 3.3, choose a differentiable function \( x_\varepsilon \) with zero derivative such that \( \|x_\varepsilon(t) - x(t) + x(t)\| < \varepsilon \) for each \( t \in [a, b] \). Finally, put \( x_\varepsilon(t) = x(t) + x(t) \) for each \( t \in [a, b] \). Thus, we obtain \( \|x_\varepsilon(t) - x(t)\| < \varepsilon \) and \( \|x_\varepsilon(t) - y(t)\| = \|y_1(t) - y(t)\| < \varepsilon \) for each \( t \in [a, b] \).

**Remark.** We can prove Theorem 3.1 in a more general case. Recall that an \( F \)-space \( X \) is called a quasi-Banach space if there exists an \( F \)-norm on \( X \) equivalent to the original one which is \( p \)-homogeneous for some \( p \in (0, 1) \) (i.e. \( \|\lambda x\| = |\lambda|^p \|x\| \)). In this case \( X \) is also called a \( p \)-Banach space. If \( X \) is a \( p \)-Banach space then the space \( L(X) \) of all continuous linear operators \( T : X \rightarrow X \) is also a \( p \)-Banach space with respect to the \( p \)-norm \( \|T\| = \sup\{|\langle Tx, x\rangle| : \|x\| \leq 1\} \). A quasi-Banach space \( X \) is called boundedly transitive \([3, p. 151]\) if there is a constant \( M < \infty \) such that if \( x, y \in X \) with \( \|x\| = \|y\| = 1 \) then there exists a \( T \in L(X) \) with \( Tx = y \) and \( \|T\| \leq M \). But we need a weaker property of \( X \). We say that a quasi-Banach space \( X \) is pointwise-boundedly transitive if for each \( x \in X \setminus \{0\} \) there exists a constant \( M < \infty \) such that for each \( y \in X \) there is a \( T \in L(X) \) with \( Tx = y \) and \( \|T\| \leq M \|y\| \). Now we are ready to formulate an exact result.

**Theorem 3.1'.** Let \( X \) be a pointwise-boundedly transitive quasi-Banach space for which there exists a non-constant function \( u : [a, b] \rightarrow X \) with zero derivative on \([a, b]\). Let \( x, y : [a, b] \rightarrow X \) be continuous. Then there exists a sequence \( x_n : [a, b] \rightarrow X, n \geq 1 \), of functions differentiable on \([a, b]\) such that \( x_n \) tends to \( x \) uniformly on \([a, b]\) and \( x'_n \) tends to \( y \) uniformly on \([a, b]\).

The proof is just the same.

**Theorem 3.5.** Let \( X \) be a quasi-Banach space satisfying the conditions of Theorem 3.1'. Then there exists a differentiable function \( x : [a, b] \rightarrow X \) with derivative having a point of discontinuity of the first kind.

**Proof.** Fix \( t_0 \in (a, b) \), \( x_0 \in X \) and construct a differentiable function \( x : [a, b] \rightarrow X \) with \( x'(t_0) = x_0 \) and \( x'(t) = 0 \) for \( t \in (a, b) \setminus \{t_0\} \). For this purpose, choose any sequence \( \delta_n \downarrow 0 \) with \( a < t_0 - \delta_1 \) and \( t_0 + \delta_1 < b \). Using Lemma 3.3 for the space \( X \) instead of \( L_p \), for \( n = 1, 2, \ldots \) construct a function

\[
x : [t_0 - \delta_n, t_0 - \delta_{n+1}] \cup [t_0 + \delta_{n+1}, t_0 + \delta_n] \rightarrow X
\]

having zero derivative such that \( x(s) = s x_0 \) for \( s = t_0 \pm \delta_n \) and \( s = t_0 \pm \delta_{n+1} \), and

\[
\left| \frac{x(t) - x(t_0)}{t - t_0} \right| < \frac{1}{n}
\]
for each $t$ with $\delta_{n+1} \leq |t - t_0| \leq \delta_n$. Finally, define $x(t_0) = t_0x_0$. We show that $x'(t_0) = x_0$. If $\delta_{n+1} \leq |\Delta t| \leq \delta_n$, then

$$\frac{\left\|x(t_0 + \Delta t) - x(t_0) - x_0\right\|}{\Delta t} \leq \left\|\frac{x(t_0 + \Delta t) - (t_0 + \Delta t)x_0}{\Delta t}\right\| < \frac{1}{n}.$$ 

PROBLEM. Let $X$ be an $F$-space with trivial dual. Does every continuous function from $[a, b]$ into $X$ have a primitive? What happens for $X = L_p$ with $0 \leq p < 1$?

Addendum (January 1994). Recently Professor N. J. Kalton sent me his short preprint “The existence of primitives for continuous functions in quasi-Banach space” which contains an affirmative answer to the Problem in the setting of quasi-Banach spaces.

References


A recurrence theorem for square-integrable martingales

by

GEROLD ALSMeyer (Kiel)

Abstract. Let $(M_n)_{n \geq 0}$ be a zero-mean martingale with canonical filtration $(\mathcal{F}_n)_{n \geq 0}$ and stochastically $L_2$-bounded increments $Y_1, Y_2, \ldots$, which means that

$$P(|Y_n| > \varepsilon | \mathcal{F}_{n-1}) \leq 1 - H(\varepsilon)$$

a.s. for all $n \geq 1$, $\varepsilon > 0$ and some square-integrable distribution $H$ on $[0, \infty)$. Let $V^2 = \sum_{n \geq 1} E(Y_n^2 | \mathcal{F}_{n-1})$. It is the main result of this paper that each such martingale is a.s. convergent on $\{V < \infty\}$ and recurrent on $\{V = \infty\}$, i.e. $P(M_n \in [-c, c])$ i.o. $V = \infty = 1$ for some $c > 0$. This generalizes a recent result by Durrett, Kesten and Lawler [4] who consider the case of only finitely many square-integrable increment distributions. As an application of our recurrence theorem, we obtain an extension of Blackwell’s renewal theorem to a fairly general class of processes with independent increments and linear drift function.

1. Introduction. Let $(S_n)_{n \geq 0}$ be a random walk with i.i.d. zero-mean, non-vanishing increments $X_1, X_2, \ldots$. Then $(S_n)_{n \geq 0}$ is recurrent with recurrence set $\mathbb{R} = \mathbb{R}$ in case of non-arithmetic increments, and $\mathbb{R} = d\mathbb{Z}$ if $X_1, X_2, \ldots$ are $d$-arithmetic for some $d > 0$. In any case

$$P(|S_n| \leq c \text{ i.o.}) = 1$$

(1.1)

for all $c > 0$. Dispensing with the stationarity assumption on $X_1, X_2, \ldots$, (1.1) need no longer be true. Durrett, Kesten and Lawler [4] give an example of a random walk $(S_n)_{n \geq 0}$ which converges a.s. to $0$, even though its increments are independent and drawn from a set of merely two zero-mean distributions. On the other hand, they also show that (1.1) holds true for sufficiently large $c$ provided that $X_1, X_2, \ldots$ are independent and drawn from a finite set of distributions with mean 0 and finite, positive variances. In fact, their result is even stated for so-called controlled random walks, that is, general martingales with square-integrable conditional increment distributions drawn from a finite set. Although their proof uses the finiteness of the latter