Isomorphism of some anisotropic Besov and sequence spaces

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Abstract. An isomorphism between some anisotropic Besov and sequence spaces is established, and the continuity of a Stieltjes-type integral operator, acting on some of these spaces, is proved.

1. Introduction. This paper gives a description of some anisotropic Besov spaces $B^q_{p,q}(I^d), \alpha = (\alpha_1, \ldots, \alpha_d)$. It is proved (Theorem A.1) that these spaces are isomorphic to some sequence spaces $u_{p,q}^\beta$, and the isomorphism is given by the coefficients of a function in the tensor product Franklin system of sufficiently high order. The one-dimensional version of Theorem A.2 was proved in [8]. In several dimensions the case of isotropic Besov spaces $B^q_{p,q}(I^d), s \in \mathbb{R}$, was treated in [4] and [5]. It was proved in these papers that the isotropic Besov space $B^q_{p,q}(I^d)$ is isomorphic to some sequence space, and the isomorphism is given by the coefficients of a function in a specially constructed spline basis. The functions forming such a basis have the property that they are concentrated on small cubes, while the tensor products of Franklin functions, which are used in this paper, are concentrated on parallelepipeds.

The second part of Theorem A says that for some $\alpha$ we can obtain another isomorphism of $B^q_{p,q}(I^d)$ and a sequence space by taking the coefficients of a function in the tensor product Schauder system (normalized in $L^2$). The one-dimensional version of Theorem A.2 was proved in [6].

Theorem B says that the Stieltjes-type integral operator $I(F,G)$,

$$I : B^q_{p,1}(I^d) \times B^q_{p,\infty}(I^d) \to B^q_{p,\infty}(I^d),$$

with $1/p < \alpha_i < 1 - 1/p, \beta_i = 1 - \alpha_i$, is bounded as a bilinear operator. Its one-dimensional version was also proved in [6].

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The author thinks that the characterization of anisotropic Besov-type spaces of the type considered in this paper can be useful in investigation of fractional Wiener fields with multidimensional time parameter.

2. Preliminaries and notation. For $I = [0, 1]$ we will denote by $W_0^m(I^d)$ the space $L^p(I^d)$ of all functions integrable with $p$th power for $1 \leq p < \infty$, and the space $C(I^d)$ of continuous functions on $I^d$ for $p = \infty$.

Let $e_i = (\delta_{1,i}, \ldots, \delta_{d,i}) \in \mathbb{R}^d$ for $i = 1, \ldots, d$ be the unit vectors, and $D = \{1, \ldots, d\}$. For $f : I^d \to \mathbb{R}$, $n \in \mathbb{N}$, $t \in \mathbb{R}^d$ and $i \in D$ we define

$$I^{(n)}(t_i, t, i) = \{x \in I^d : x = nt_i + t, i \in D\},$$

$$\Delta_{n,i}^3 f(x) = \begin{cases} \sum_{j=0}^{n-1} (-1)^{n+1} \binom{n}{j} f(x + jte_i) & \text{for } x \in I^d(t_i, t, i), \\ 0 & \text{for } x \notin I^d(t_i, t, i). \end{cases}$$

Let $A = \{t_1, \ldots, t_d\} \subset I^d$, $\tilde{t} = (t_1, \ldots, t_d) \in \mathbb{R}^d$ and $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$; then we set

$$\Delta_{n,i}^3 f = \Delta_{n,1}^3 \circ \ldots \circ \Delta_{n,i}^3 f.$$

The modulus of smoothness of order $n$ in directions $A$ in the $L^p$-norm is defined as follows:

$$\omega_{n,p}(f, \tilde{t}) = \sup_{|h_1| \leq 1} \sup_{|h_d| \leq 1} \|\Delta_{n,i}^3 f\|_p$$

for $\tilde{t} \in I^d$, $0 < t_j \leq 1/n_j$.

For $A = \emptyset$ we put

$$\omega_{n,p}(f, \tilde{t}) = \|f\|_p.$$

For $h = (h_1, \ldots, h_d) \in \mathbb{R}^d$ and $A = \{t_1, \ldots, t_d\}$ we will write $h(A) = (h_1, \ldots, h_d) \in \mathbb{R}^d$, where $h_i = h_i$ for $i \in A$ and $h_i = 0$ for $i \notin A$. For $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$ and $\tilde{t} = (t_1, \ldots, t_d) \in \mathbb{R}^d$ the following abbreviations will be used:

$$t^n = \prod_{i=1}^d t_i^{n_i}, \quad D^n f = \frac{\partial^{n_1}}{\partial x_{n_1}} \ldots \frac{\partial^{n_d}}{\partial x_{n_d}} f.$$

We will also need some spaces of spline functions with dyadic knots. Let us introduce the notation

$$s_0 = 0, \quad s_1 = 1, \quad s_n = \frac{2n - 1}{2^{d+1}}$$

for $n > 1$, $n = 2^\nu + \mu$, $\mu \geq 0$, $1 \leq \nu \leq 2^d$.

Then for each $m \in \mathbb{N} \cup \{0\}$ and $n \geq -m$ we define

$$S_{n}^{(m)}(I) = \begin{cases} \text{the space of polynomials degree } n + m, \text{ restricted to } I, \\ \text{the space of spline functions on } I \text{ of degree } m+1 \text{ and maximal smoothness, with knots } s_0, \ldots, s_n. \end{cases}$$

Now the system of Franklin functions of order $m+2$ is defined as follows: $f^{(m)}_{-m} = 1$, and for $n > -m$, $f^{(m)}_n \in S^{(m)}_{n}(I)$ and is orthogonal (in $L^2(I)$) to $S^{(m)}_{n-1}(I)$, $\|f^{(m)}_n\|_2 = 1$.

We will also need the Haar and Schauder systems on $I$:

- the Haar system: $h_1 = 1$, and for $n > 1$, $n = 2^\nu + \mu$, $\mu \geq 0$, $1 \leq \nu \leq 2^d$,

$$h_n(t) = \begin{cases} 2^{\nu/2} & \text{for } t \in \left[\frac{2^{\nu-1}}{2^d+1}, \frac{2^\nu}{2^d+1}\right], \\ -2^{\nu/2} & \text{for } t \in \left[\frac{2^\nu}{2^d+1}, \frac{2^{\nu+1}}{2^d+1}\right] \text{ if } \nu < 2^d, \\ 0 & \text{otherwise in } I, \end{cases}$$

- the Schauder system:

$$\phi_0(t) = 1, \quad \phi_n(t) = \int_0^t h_n(u) \, du, \quad n \geq 1,$$

- the normalized Schauder system $\phi_n^* = \phi_n / \|\phi_n\|_2$.

For $n = (n_1, \ldots, n_d)$ we introduce the tensor product Franklin, Haar and Schauder systems on $I^d$:

$$f^{(m)} = f^{(m)}_{n_1} \otimes \ldots \otimes f^{(m)}_{n_d}$$

for $n_i \geq -m$,

$$h^{(m)} = h^{(m)}_{n_1} \otimes \ldots \otimes h^{(m)}_{n_d}$$

for $n_i \geq 1$,

$$\phi^{(m)} = \phi^{(m)}_{n_1} \otimes \ldots \otimes \phi^{(m)}_{n_d}$$

for $n_i \geq 0$,

$$\phi^{*}_{n_i} = \phi^{*}_{n_1} \otimes \ldots \otimes \phi^{*}_{n_d}$$

for $n_i \geq 0$.

It is well known that the tensor product Franklin system of order $m+2$, \{\(f^{(m)}_{n_i} : n_i \geq -m\), properly ordered (in the so-called rectangular order, described for instance in [3]) is a Schauder basis in $W_0^m(I^d)$ (cf. [3]). Similarly, \{\(h^{(m)}_{n_i} : n_i \geq 1\) \} (in rectangular order) is a Schauder basis in $L^p(I^d)$ for $1 \leq p < \infty$, and \{\(\phi^{(m)}_{n_i} : n_i \geq 0\) \} (also in rectangular order) is a Schauder basis in $C(I^d)$.

Remark. Throughout this paper we sum over a $d$-dimensional set of parameters $(N^d_1$ or $N^d_2$), we always mean that this set is arranged in rectangular order.
3. Function and sequence spaces. Let \( n = (n_1, \ldots, n_d) \in \mathbb{N}^d \), \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d \), \( 0 < \alpha_i < n_i \) for \( i = 1, \ldots, d \), \( 1 \leq p \leq \infty \), \( 1 \leq q \leq \infty \), \( \gamma = (1, \ldots, 1) \in \mathbb{N}^d \). For \( f \in W_p^\gamma(I^d) \) and \( 1 \leq q < \infty \) we define

\[
\|f\|_{p,q} = \left( \int_0^{1/n_1} \cdots \int_0^{1/n_d} \left( \sum_{k \in \mathbb{Z}^d} \omega_{\gamma, \beta, \alpha}(f, \xi) \|\xi\|^{-\alpha} \right)^q \|\xi\|^{-\gamma} d\xi \right)^{1/q},
\]

and for \( q = \infty \) we put for \( f \in W_p^\gamma(I^d) \),

\[
\|f\|_{p,\infty} = \sup_{0 < \xi_1 \leq 1/n_1} \cdots \sup_{0 < \xi_d \leq 1/n_d} \|\xi\|^{-\alpha} \omega_{\gamma, \beta, \alpha}(f, \xi).
\]

For \( 1 \leq p, q \leq \infty \) we consider the Besov-type function spaces

\[
B_{p,q}^\gamma(I^d) = \{ f \in W_p^\gamma(I^d) : \| f \|_{p,q} < \infty \}.
\]

Remark. Let \( n, m \in \mathbb{N}^d \) be such that \( \alpha_i < \alpha_i \) and \( \alpha_i < \beta_i \) for \( i = 1, \ldots, d \), and let \( B_{p,q}^\gamma(I^d)_n \) and \( B_{p,q}^\gamma(I^d)_m \) denote the spaces defined as above, corresponding to \( n \) and \( m \) respectively. Then it follows from the Marchaud-type inequalities for \( L_p \)-valued functions (cf. [4], Proposition 2.1) that \( B_{p,q}^\gamma(I^d)_n = B_{p,q}^\gamma(I^d)_m \) (the sets are equal and the norms are equivalent).

Now we define the sequence spaces. For a given integer \( m \) define \( N_m = \{-m - 2, -m - 1, \ldots, m, \ldots\} \), and for \( j \in \mathbb{N}_m \),

\[
\tilde{N}_j = \begin{cases} \{ j + 2 \} & \text{for } j = -m - 2, \ldots, -1, \\ \{ 2^j + k : k = 1, \ldots, 2^j \} & \text{for } j \geq 0. 
\end{cases}
\]

For \( j = (j_1, \ldots, j_d) \in N_m^d \) define

\[
\tilde{N}_j = \tilde{N}_{j_1} \times \cdots \times \tilde{N}_{j_d}.
\]

Observe that \( N_{m-2}^d = \bigcup_{j \in \mathbb{N}_m^d} \tilde{N}_j \).

For a real number \( \alpha > 0 \) and \( 1 \leq p \leq \infty \) set

\[
c(j, \alpha, p) = \begin{cases} 1/2j(1/2 - 1/p + \alpha) & \text{for } j \leq 0, \\ 1/2j(1/2 - 1/p + \alpha) & \text{for } j > 0. 
\end{cases}
\]

For \( j \in \mathbb{N}_m^d \) and \( \alpha = (\alpha_1, \ldots, \alpha_d) \) put

\[
c(j, \alpha, p) = c(j_1, \alpha_1, p) \cdots c(j_d, \alpha_d, p).
\]

Now for a given sequence of real numbers \( \alpha = (\alpha_k)_{k \in N_{m-2}^d} \) let

\[
\|g\|_{p,\alpha} = \left( \sum_{j \in \mathbb{N}_m^d} \left( \sum_{k \in \tilde{N}_j} |g_k|^p \right)^{1/p} \right)^{1/q}
\]

(with the sums of \( p \)-th or \( q \)-th powers replaced by suprema over the same set of indices if \( p = \infty \) or \( q = \infty \)), and

\[
b_{p,\alpha} = \{ (a_k)_{k \in N_{m-2}^d} : \|a\|_{p,\alpha} < \infty \}.
\]

4. Results

**Theorem A.1.** Let \( m \in \mathbb{N} \cup \{0\} \), \( \alpha = (\alpha_1, \ldots, \alpha_d) \), \( 0 < \alpha_i < m + 1 \) for \( i = 1, \ldots, d \), \( 1 \leq p, q \leq \infty \). Then the spaces \( B_{p,q}^{\alpha}(I^d) \) and \( b_{p,\alpha} \) are isomorphic, and the isomorphism is given by the coefficients of a function in the basis \( \{ f_{k}^{(m)} : k \in N_{m-2}^d \} \) of tensor products of Franklin functions of order \( m + 2 \).

**Remark.** As \( f_k^{(m)} \in B_{p,q}^{\alpha}(I^d) \) for \( 0 < \alpha_i < m + 1, k \in N_{m-2}^d \), it follows from Theorem A.1 that these functions form a Schauder basis in \( B_{p,q}^{\alpha}(I^d) \) for \( 1 \leq q < \infty \), and in some separable subspace of \( B_{p,\infty}^{\alpha}(I^d) \). Analogously, it follows from Theorem A.2 that if \( 1/p < \alpha_i < 1 \), then \( \{ \phi_k : k \in N_{m-2}^d \} \) form a Schauder basis in \( B_{p,q}^{\alpha}(I^d) \) for \( 1 \leq q \leq \infty \), and in their linear span in case \( q = \infty \).

**Theorem B.** Let \( 1 \leq p \leq \infty \), \( 1/p + 1/p' = 1 \), \( \alpha = (\alpha_1, \ldots, \alpha_d) \), \( 1/p < \alpha_i < 1/p' \), \( \beta_i = 1 - \alpha_i \). For \( F \in B_{p,q}^{\alpha}(I^d) \) and \( G \in B_{p,\infty}^{\beta}(I^d) \),

\[
F = \sum_{j \in \mathbb{N}_m^d} \sum_{k \in \tilde{N}_j} F_k h_k, \quad G = \sum_{j \in \mathbb{N}_m^d} \sum_{k \in \tilde{N}_j} G_k \phi_k,
\]

and \( g = (g_1, \ldots, g_d) \in I^d \) define

\[
I(F, G)(g) = \sum_{j \in \mathbb{N}_m^d} \sum_{k \in \tilde{N}_j} \sum_{l \in \mathbb{Z}^d} \int_0^1 \cdots \int_0^1 h_k(u_i) h_{l_i}(u_i) \, du_i.
\]

There exists a constant \( C = C(g, p) \) such that for all \( F \in B_{p,q}^{\alpha}(I^d) \) and \( G \in B_{p,\infty}^{\beta}(I^d) \),

\[
\|I(F, G)\|_{p,\alpha} \leq C \|F\|_{p,\alpha} \|G\|_{p,\infty}.
\]

**Remark.** As \( D^2 \phi_k = h_k \) for \( n \in \mathbb{N}^d \), we can write

\[
I(F, G)(g) = \int_0^1 \cdots \int_0^1 F \, dG.
\]
This operator can be useful in investigation of multidimensional Stratonovich integrals.

5. Properties of the moduli of smoothness. The following properties of $\omega_{n,p,A}(f,\ell)$ will be needed:

5.1. For each $B \subset A$, $f \in W_p^n(B)(I^d)$ and $\ell = (\ell_1, \ldots, \ell_d) \in \mathbb{R}^d$, $0 < \ell_k < 1/n_i$,

$$\omega_{n,p,A}(f,\ell) \leq \ell_\delta(B) \omega_{n,p,A,B}(D \ell(B),f,\ell).$$

5.2. For each $f \in W_p^n(I^d)$, $\ell = (\ell_1, \ldots, \ell_d) \in \mathbb{N}^d$, $\ell = (\ell_1, \ldots, \ell_d)$, $0 < \ell_k \leq 1/n_i$, $\ell_\delta(\ell_1, \ldots, \ell_d) \in \mathcal{D}$, and $A = \{\ell_1, \ldots, \ell_d\} \subset \mathcal{D}$,

$$\omega_{n,p,A}(f,\ell) \leq E(\ell) \omega_{n,p,A}(f,\ell).$$

The following extension lemma will be useful.

**Lemma 5.3.** Let $Q = [a_1, a_1+i_1] \times \cdots \times [a_d, a_d+i_d]$ and $S$ be two compact parallelepipeds in $\mathbb{R}^d$ with $Q \subset S$. Then there exists an extension operator $T : W_p^n(Q) \to W_p^n(S)$ such that

$$\|Tf\|_p(Q) \leq C \|f\|_p(Q), \quad \omega_{n,p,A}(Tf,\ell)(S) \leq C \omega_{n,p,A}(f,\ell)(Q)$$

for all $f \in W_p^n(Q)$, $A = \{\ell_1, \ldots, \ell_d\} \subset \mathcal{D}$ and $\ell \in \mathbb{R}^d$ with $0 < \ell_k \leq i_1/n_i$.

**Proof.** It is enough to prove the lemma for $Q, S$ of the form

$$Q = [-a, 0] \times Q_0, \quad S = [-a, a] \times Q_0$$

(where $Q_0$ is a compact parallelepiped in $\mathbb{R}^{d-1}$). Then for Whitney’s extension

$$Tf(x_1, x') = \begin{cases} f(x_1, x') & \text{for } -a \leq x_1 \leq 0, \\ \sum_{j=0}^{n_1} a_j f(-2^{-j}x_1, x') & \text{for } 0 < x_1 \leq a, \end{cases}$$

where $\sum_{j=0}^{n_1} a_j (-1/2)^j = 1$ for $k = 0, \ldots, n_1$, there is a constant $C$ such that for all $f \in W_p^n(Q)$, $A = \{1\}$,

$$(2) \quad \|Tf\|_p(S) \leq C \|f\|_p(Q),$$

$$(3) \quad \omega_{n,p,A}(Tf,\ell)(S) \leq C \omega_{n,p,A}(f,\ell)(Q)$$

(cf. Proposition 2.9 of [4]). Observe that for $A' \subset \mathcal{D}$ with $1 \not\in A'$ we have

$$\Delta_{n,A'}^{\ell} \circ T(f) = T \circ \Delta_{n,A'}^{\ell}(f),$$

so from (2) we get

$$\omega_{n,p,A}(Tf,\ell)(S) \leq C \omega_{n,p,A}(f,\ell)(Q).$$

If $A = \{1\} \cup A'$ then

$$\Delta_{n,A'}^{\ell} \circ T(f) = \Delta_{n,1}^{\ell_1} \circ T \circ \Delta_{n,A'}^{\ell}(f),$$

and it follows from (3) that

$$\sup_{|\ell_1| \leq 1/n_1} \|\Delta_{n,1}^{\ell_1} \circ T \circ \Delta_{n,A'}^{\ell}(f)\|_p(S) \leq C \sup_{|\ell_1| \leq 1/n_1} \|\Delta_{n,1}^{\ell_1} \circ \Delta_{n,A'}^{\ell}(f)\|_p(Q),$$

so the lemma follows from the definition of $\omega_{n,p,A}(f,\ell)$. \hfill \Box

In the sequel we will need the equivalence between the modulus of smoothness $\omega_{n,p,A}(f,\ell)$ and the $K$-functional defined by the formula

$$K_{n,A,p}(f,\ell) = \inf \left\{ \|f - \sum_{\emptyset \not= B \subset A} g_B\|_p + \|\sum_{\emptyset \not= B \subset A} \ell_\delta(B) \omega_{n,p,A,B}(D \ell(B),g_B,\ell)\|_p : g_B \in W_p^n(B)(I^d), \emptyset \not= B \subset A \right\}$$

for $f \in W_p^n(I^d)$ and $\ell = (\ell_1, \ldots, \ell_d) \in \mathbb{R}^d$ with $0 < \ell_k < 1/n_i$.

**Lemma 5.4.** Let $1 \leq p \leq \infty$, $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$ and $A = \{i_1, \ldots, i_k\} \subset \mathcal{D}$ be given. There exist constants $C_k = C_k(n, p, A)$, $k = 1, 2, \ldots$ such that

$$C_1 \omega_{n,p,A}(f,\ell) \leq K_{n,A,p}(f,\ell) \leq C_2 \omega_{n,p,A}(f,\ell)$$

for every $f \in W_p^n(I^d)$ and $\ell = (\ell_1, \ldots, \ell_d) \in \mathbb{R}^d$ with $0 < \ell_k \leq 1/n_i$.

**Proof.** The left inequality is a consequence of 5.1.

Now let $f$ be the extension of $f$ to the parallelepiped $S = [0, n_1^2 + 1] \times \cdots \times [0, n_d^2 + 1]$, given by Lemma 5.3; then for $\ell = (\ell_1, \ldots, \ell_d)$ with $0 < \ell_k \leq 1/n_i$,

$$\omega_{n,p,A}(f,\ell)(S) \leq C \omega_{n,p,A}(f,\ell)(Q).$$

For $B \subset A$, $B = \{i_1, \ldots, i_m\}$, define a function $g_B \in W_p^n(B)(I^d)$ by the Stieltjes means

$$g_B(x) = \sum_{k_{i_1}=1}^{n_{i_1}} \cdots \sum_{k_{i_m}=1}^{n_{i_m}} (-1)^{k_{i_1} + \cdots + k_{i_m}} (k_{i_1}) \cdots (k_{i_m})$$

$$\times \int \cdots \int \int f(x + \sum_{i=1}^{n_{i_1}} k_{i_1} t_{i_1} (s_{i_1}(t_{i_1}) + \cdots + s_{i_m}(t_{i_m}) \ell_{i_m})) \, ds_{i_1} \cdots ds_{i_m}. $$

Then

$$\|f - \sum_{\emptyset \not= B \subset A} g_B\|_p \leq C_1 \omega_{n,p,A}(f,\ell)(S)$$

and it follows from 5.2 that for some constant $C_2 > 0$,

$$K_{n,A,p}(f,\ell) \leq C_2 \omega_{n,p,A}(f,\ell)(S),$$

which together with Lemma 5.3 completes the proof. \hfill \Box
6. Proof of Theorem A. Let us start with the following definitions.
For $g \in W_p^0(I)$, $m \in \mathbb{N} \cup \{0\}$ and $\mu \in \mathbb{N}_m$ let

$$P_{\mu}^{(m)} g = \sum_{j=-m}^{\mu} \sum_{k \in \mathbb{N}_j} (g, f_{k}^{(m)}) f_{k}^{(m)}.$$ 

The following one-dimensional results will be needed (cf. [2], [3]).

**Lemma 6.1.** Let $m \in \mathbb{N} \cup \{0\}$. There exists a constant $C = C(m)$ such that for all $1 \leq p \leq \infty$ and $\mu \in \mathbb{N}_m$,

$$\|P_{\mu}^{(m)}\|_p \leq C.$$ 

**Lemma 6.2.** Let $m \in \mathbb{N} \cup \{0\}$ and $1 \leq p \leq \infty$. Then there exists a constant $C > 0$ such that for all $\mu \geq 0$ and $f \in W_{p}^{m+1}(I)$,

$$\|f - P_{\mu}^{(m)} f\|_p \leq C \frac{1}{2^{m+1}} \|D^{m+1} f\|_p.$$ 

For $f \in W_p^0(I^d)$ and $i \in D$ let

$$P_{\mu, i}^{(m)} f(x_1, \ldots, x_d) = \sum_{j=-m}^{\mu} \sum_{k \in \mathbb{N}_j} \int_{I} f(x_1, \ldots, x_{i-1}, u_{i}, x_{i+1}, \ldots, x_d) f_{k}^{(m)}(u_i) \, du_j f_{k}^{(m)}(x_i)$$

and for $A = \{i_1, \ldots, i_k\}$ and $\mu = (\mu_1, \ldots, \mu_d)$,

$$P_{\mu, A}^{(m)} f = \text{Id} - (\text{Id} - P_{\mu_1, i_1}^{(m)}) \circ \ldots \circ (\text{Id} - P_{\mu_k, i_k}^{(m)}).$$

Define

$$N^*_m(\mu, A) = \{k = (k_1, \ldots, k_d) \in \mathbb{N}^d_{m-2} : \exists \xi \in A \quad \exists 1 \leq s, t_k \in \mathbb{N}_\xi\},$$

$$Q^{(m)}_{\mu, A, p} = \text{span}_{W_p^0(I^d)} \{f_k^{(m)} : k \in N^*_m(\mu, A)\}.$$ 

For $f \in W_p^0(I^d)$ let us introduce

$$\mathcal{E}_{\mu, A, p}^{(m)}(f) = \inf_{g \in Q^{(m)}_{\mu, A, p}} \|f - g\|_p.$$ 

Observe that $P_{\mu, A}^{(m)}$ is a projection of $W_p^0(I^d)$ onto $Q^{(m)}_{\mu, A, p}$. As a consequence of Lemma 6.1 we obtain

**Lemma 6.3.** Let $m \in \mathbb{N} \cup \{0\}$. There exists a constant $C = C(m, d)$ such that for all $1 \leq p \leq \infty$, $\mu = (\mu_1, \ldots, \mu_d) \in \mathbb{N}^d_m$ and $\emptyset \neq A \subset D$,

$$\|P_{\mu, A}^{(m)}\|_p \leq C.$$ 

For $\mu \in (\mathbb{N} \cup \{0\})^d$ let $\mathbf{t}_\mu = (1/2^\mu_1, \ldots, 1/2^\mu_d).$

**Lemma 6.4.** Let $m \in \mathbb{N} \cup \{0\}$, $1 \leq p \leq \infty$, $\mu = (\mu_1, \ldots, \mu_d) \in \mathbb{N}^d_m$, $\lambda \in \mathbb{N}$, $2^{\lambda-1} \leq m + 1 < 2^\lambda$. There exists a constant $C = C(m, d, p, \mu) > 0$ such that for each $\mu' = (\mu_1, \ldots, \mu_d)$ with $\mu_i \geq \lambda$, $A \subset D$ and $f \in W_p^0(I^d)$,

$$\|f - P_{\mu, A}^{(m)} f\|_p \leq C \omega_{\mu, p, A}(f, \mathbf{t}_\mu).$$

**Proof.** The proof is by induction on $\#(A \setminus \emptyset)$ (i.e. the cardinality of $A$). For $\#(A \setminus \emptyset) = 1$ this lemma is a consequence of Proposition 7.15 of [5] and Lemma 6.3.

Now let $\#(A \setminus \emptyset) > 1$. Let $\{i_1, \ldots, i_k\} \subset D$ and $g \in C^\infty(I^d)$ and $f \neq g$. From Lemma 6.2 we obtain

$$\|g - P_{\mu, A}^{(m)} g\|_p = \|(\text{Id} - P_{\mu_1, i_1}^{(m)}) \circ \ldots \circ (\text{Id} - P_{\mu_k, i_k}^{(m)}) g\|_p \leq C \frac{1}{2^{m+1}} \|D^{m+1} g - P_{\mu, A}^{(m)} D^{m+1} g\|_p.$$ 

As $\#(A \setminus \emptyset) < \#(A \setminus \emptyset)$, it follows from the induction hypothesis that for any $g \in C^\infty(I^d)$,

$$\|g - P_{\mu, A}^{(m)} g\|_p \leq C \frac{1}{2^{m+1}} \omega_{\mu, p, A, B}(D^{m+1} g, \mathbf{t}_\mu).$$ 

Now let $f \in W_p^0(I^d)$ and $g \in C^\infty(I^d)$ for each $f \neq g$. Using the last inequality and Lemma 6.3 we obtain

$$\|f - P_{\mu, A}^{(m)} f\|_p \leq \|f - \sum_{\emptyset \neq B \subset A} g_B\|_p + \sum_{\emptyset \neq B \subset A} \|g_B - P_{\mu, A}^{(m)} g_B\|_p + \|P_{\mu, A}^{(m)} \left( \sum_{\emptyset \neq B \subset A} g_B - f \right)\|_p \leq C_2 \frac{1}{2^{m+1}} \omega_{\mu, p, A, B}(D^{m+1} g, \mathbf{t}_\mu).$$

As this inequality holds for every choice of $g \in C^\infty(I^d)$, we obtain

$$\|f - P_{\mu, A}^{(m)} f\|_p \leq C_2 \omega_{\mu, p, A, B}(f, \mathbf{t}_\mu).$$

This, together with Lemma 6.4, completes the proof. $\blacksquare$

Now we are ready to prove one of the inequalities needed in Theorem A.1. For $f \in W_p^0(I^d)$ and $i \in D$ we introduce the notation

$$F_{p, i}^{(m)}(f) = \left\| \sum_{k \in \mathbb{N}_i} (f, f_k^{(m)}) f_k^{(m)} \right\|_p \quad F_{p, i}^{(m)}(f) = \left( \sum_{k \in \mathbb{N}_i} |(f, f_k^{(m)})|^p \right)^{1/p}.$$ 

It follows from the properties of one-dimensional Franklin functions (cf. [2]) that
\[ F_{\xi, P}^{(m)}(f) \sim \prod_{i=1}^{d} 2^{j_i(1/2-1/p)}|f_{\xi, P}^{(m)}(f)|. \]

For \( \mu = (\mu_1, \ldots, \mu_d) \), define \( D_\mu = \{ i \in D : \mu_i \geq -m - 2 \} \). Observe that
\[
\left| \sum_{\xi \in N_{\mathbf{b}}} (f_{\xi, B}^{(m)}, f_{\xi, P}^{(m)}) \right| = \left| \sum_{\mu=\mu_1}^{J_1} \cdots \sum_{\mu_d=\mu_d}^{J_d} (-1)^{\sum_{i=1}^{d} (j_i - \mu_i)} P_{\mu, B}^{(m)} (f) \right|.
\]

Let \( \lambda \in \mathbb{N} \) be chosen as in Lemma 6.4 and \( A_{\xi} = \{ i : j_i \geq 2^{\lambda} \} = \{ i_1, \ldots, i_k \} \), \#A_{\xi} = k. Then from the definition of \( P_{\mu, B}^{(m)} \), Lemmas 6.3, 6.4 and (5.2) we obtain, for \( \mathbf{b} = (m + 1, \ldots, m + 1) \in \mathbb{N}^d \),
\[
F_{\xi, P}^{(m)}(f) = \left\| \sum_{\mu=\mu_1}^{J_1} \cdots \sum_{\mu_d=\mu_d}^{J_d} (-1)^{\sum_{i=1}^{d} (j_i - \mu_i)} P_{\mu, B}^{(m)} (f) \right\|_p.
\]

Now the reverse inequality will be proved. Let \( \mu = (\mu_1, \ldots, \mu_d), \mu_i \geq \lambda, A \subset \mathbb{D} \) and \( \mathbf{b} = (b_1, \ldots, b_d) \in \mathbb{R}^d \), then \( |b_i| \leq 1/2^m \). It follows from the properties of spline functions of one variable (cf. [2], Lemma 9.2) and Fubini’s theorem that there exists \( C > 0 \) such that for all \( f \in B_{\mathbf{b}, P}^{(m)}(\mathbb{D}) \),
\[
\left( \sum_{\xi \in N_{\mathbf{b}}} \left( c(\xi, \alpha, p) \left( \sum_{\xi \in N_{\mathbf{b}}} |f_{\xi, B}^{(m)}| \right)^p \right)^{1/p} \right)^{1/q} \leq C \| f \|_{B_{\mathbf{b}}^{(m)}}.
\]

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\[
\left\| \Delta_{\mathbf{b}, A} (\sum_{\xi \in N_{\mathbf{b}}} a_{\xi, B}^{(m)}) \right\|_p \leq C \left\| \Delta_{\mathbf{b}, A} (\sum_{\xi \in N_{\mathbf{b}}} a_{\xi, B}^{(m)}) \right\|_p.
\]

Defining \( W(\mu, B) = \{ \xi \in N_{\mathbf{b}}^m : j_i \leq \mu_i \} \), for \( i \in B, j_i > \mu_i \) for \( i \notin B \) for}

\[ B \subset A, \] we obtain, for \( f \in W_{\mathbf{b}}^{(m)}(\mathbb{D}) \),
\[
\| \Delta_{\mathbf{b}, A} f \|_p \leq \sum_{\xi \in N_{\mathbf{b}}} \| \Delta_{\mathbf{b}, A} (\sum_{\xi \in N_{\mathbf{b}}} (f_{\xi, B}^{(m)}, f_{\xi, P}^{(m)})) \|_p
\]
\[
\leq C \sum_{B \subset A, \xi \in W(\mathbf{b}, B)} \| \Delta_{\mathbf{b}, B} (\sum_{\xi \in N_{\mathbf{b}}} (f_{\xi, B}^{(m)}, f_{\xi, P}^{(m)})) \|_p
\]
\[
\leq C_1 \sum_{i \in A} \prod_{i \in A} 2^{(m+1) \min(\mu_i, j_i)} F_{\xi, P}^{(m)}(f),
\]
which gives
\[
\omega_{\mathbf{b}, p, A}(f, \mathbf{b}) \leq C_1 \sum_{i \in A} \prod_{i \in A} 2^{(m+1) \min(\mu_i, j_i)} F_{\xi, P}^{(m)}(f).
\]

Using this inequality and (4) we obtain
\[
\omega_{\mathbf{b}, p, A}(f, \mathbf{b}) \leq C_1 \sum_{i \in A} \prod_{i \in A} 2^{(m+1) \min(\mu_i, j_i)} F_{\xi, P}^{(m)}(f).
\]

Now the reverse inequality will be proved. Let \( \mu = (\mu_1, \ldots, \mu_d), \mu_i \geq \lambda, A \subset \mathbb{D} \) and \( \mathbf{b} = (b_1, \ldots, b_d) \in \mathbb{R}^d \), then \( |b_i| \leq 1/2^m \). It follows from the properties of spline functions of one variable (cf. [2], Lemma 9.2) and Fubini’s theorem that there exists \( C > 0 \) such that for all \( f \in B_{\mathbf{b}, P}^{(m)}(\mathbb{D}) \),
\[
\left( \sum_{\xi \in N_{\mathbf{b}}} \left( c(\xi, \alpha, p) \left( \sum_{\xi \in N_{\mathbf{b}}} |f_{\xi, B}^{(m)}| \right)^p \right)^{1/p} \right)^{1/q} \leq C \| f \|_{B_{\mathbf{b}}^{(m)}}.
\]

Now the reverse inequality will be proved. Let \( \mu = (\mu_1, \ldots, \mu_d), \mu_i \geq \lambda, A \subset \mathbb{D} \) and \( \mathbf{b} = (b_1, \ldots, b_d) \in \mathbb{R}^d \), then \( |b_i| \leq 1/2^m \). It follows from the properties of spline functions of one variable (cf. [2], Lemma 9.2) and Fubini’s theorem that there exists \( C > 0 \) such that for all \( \xi \in N_{\mathbf{b}}^m \) and \( \mathbf{b} \in \mathbb{N}^d \),
\[
\left\| \Delta_{\mathbf{b}, A} (\sum_{\xi \in N_{\mathbf{b}}} a_{\xi, B}^{(m)}) \right\|_p \leq C \left\| \Delta_{\mathbf{b}, A} (\sum_{\xi \in N_{\mathbf{b}}} a_{\xi, B}^{(m)}) \right\|_p.
\]

Defining \( W(\mu, B) = \{ \xi \in N_{\mathbf{b}}^m : j_i \leq \mu_i \} \), for \( i \in B, j_i > \mu_i \) for \( i \notin B \) for
\begin{align*}
&\sum_{\mu_1=\lambda}^{\infty} \cdots \sum_{\mu_k=\lambda}^{\infty} \left( \prod_{i \in A} 2^{\alpha_i \mu_i} \omega_{B_p,A}(f, [\xi_i]) \right)^q \\
&\leq C \prod_{i \in A} \sum_{\mu_1=\lambda}^{\infty} \cdots \sum_{\mu_k=\lambda}^{\infty} \sum_{j \in \mathbb{N}_d} \left( c(j, \lambda; \bar{p}) \tau_{j, p}^{(m)}(f) \right)^q \\
&\times \prod_{i \in A} 2^{[m+1] \min(\mu, j_i) - \alpha_i j_i - \mu_i (m+1-\alpha_i)} \prod_{i \in A} 2^{-\alpha_j j_i} \\
&\leq C \sum_{j \in \mathbb{N}_d} \left( c(j, \lambda; \bar{p}) \tau_{j, p}^{(m)}(f) \right)^q \\
&\times \prod_{\mu_1=\lambda}^{\infty} \cdots \sum_{\mu_k=\lambda}^{\infty} \prod_{i \in A} 2^{[m+1] \min(\mu, j_i) - \alpha_i j_i - \mu_i (m+1-\alpha_i)}.
\end{align*}

There exists a constant $C_0 > 0$ such that for all $j \in \mathbb{N}_d$,
\begin{align*}
\sum_{\mu_1=\lambda}^{\infty} \cdots \sum_{\mu_k=\lambda}^{\infty} \prod_{i \in A} 2^{[m+1] \min(\mu, j_i) - \alpha_i j_i - \mu_i (m+1-\alpha_i)} \leq C_0,
\end{align*}
so
\begin{align*}
\sum_{\mu_1=\lambda}^{\infty} \cdots \sum_{\mu_k=\lambda}^{\infty} \left( \prod_{i \in A} 2^{\alpha_i \mu_i} \omega_{B_p,A}(f, [\xi_i]) \right)^q \leq C \sum_{j \in \mathbb{N}_d} \left( c(j, \lambda; \bar{p}) \tau_{j, p}^{(m)}(f) \right)^q,
\end{align*}
which (together with 5.2) implies
\begin{align*}
\left( \int_0^1 \cdots \int_0^1 \left( \omega_{B_p,A}(f, \xi) \right)^q \frac{1}{\xi^{2(A)}(A)} \frac{1}{\xi^{2(A)}(A)} \right)^{1/q} \\
&\leq C_0 \left( \sum_{j \in \mathbb{N}_d} \left( c(j, \lambda; \bar{p}) \tau_{j, p}^{(m)}(f) \right)^q \right)^{1/q},
\end{align*}
and the proof of Theorem A.1 is complete.

Now Theorem A.2 will be proved. Its proof is based on the main idea of the proof of Theorem III.6 of [6]. For convenience we set $f_k = f_k^{(0)}$. Observe first that if $1/p < \alpha_i < 1$ for all $i = 1, \ldots, d$, $f \in B^2_{p, q}(I^d)$ and $f = \sum_{j \in \mathbb{N}_d} \sum_{k \in \mathbb{N}_d} a_k f_k$, then
\begin{align*}
\left\| \sum_{k \in \mathbb{N}_d} a_k f_k \right\|_p \leq C \prod_{i=1}^d 2^{j_i/2} \sup_{k \in \mathbb{N}_d} |a_k|
\end{align*}
and this implies $f \in C(I^d)$.

Now let $f \in B^2_{p, q}(I^d)$,
\begin{align*}
f = \sum_{j \in \mathbb{N}_d} a_j f_j = \sum_{k \in \mathbb{N}_d} b_k \phi_k^*,
\end{align*}
(recall that $\phi_k^*$ denotes a tensor product Schauder function, normalized in $L^2(I^d)$). To prove the second part of Theorem A it is enough to show the existence of constants $M_1, M_2 > 0$ such that for each $f \in B^2_{p, q}(I^d)$,
\begin{align}
M_1 \left\| a_j \right\|_p^{(0)} \leq \left\| a_j \right\|_p^{(0)} \leq M_2 \left\| a_j \right\|_p^{(0)},
\end{align}
where $g = (a_j)_{j \in \mathbb{N}_d}, b = (b_k)_{k \in \mathbb{N}_d}$. Set
\begin{align*}
\tau_{j, p}^{(m)}(g) = \left( \sum_{j \in \mathbb{N}_d} |a_j|^p \right)^{1/p}, \quad \tau_{j, p}^{(m)}(b) = \left( \sum_{k \in \mathbb{N}_d} |b_k|^p \right)^{1/p}.
\end{align*}
First we will show the existence of $M_1$.

Observe that (cf. (4))
\begin{align*}
\tau_{j, p}^{(m)}(g) \sim \prod_{i=1}^d 2^{j_i(1/2-1/p)} \left\| \sum_{k \in \mathbb{N}_d} a_k f_k \right\|_p.
\end{align*}

It follows from Lemma 6.3 that $\left\| f - P_n^{(0)} f \right\|_p \sim E_n f^{(0)} (f)$, so we have
\begin{align*}
\left\| \sum_{k \in \mathbb{N}_d} a_k f_k \right\|_p &= \left\| \sum_{\mu_1=1}^{j_1+1} \cdots \sum_{\mu_d=1}^{j_d+1} (-1)^{\sum_{i=1}^d (j_i+1-\mu_i)} \omega_{B_p,A}(f, [\xi_i]) \right\|_p \\
&\leq C_1 \sum_{\mu_1=1}^{j_1+1} \cdots \sum_{\mu_d=1}^{j_d+1} E_n^{(0)} f^{(0)}(f) \leq C_2 E_n^{(0)} f^{(0)}(f) \\
&\leq C_3 \left\| f - \sum_{k \in \mathbb{N}_d} b_k \phi_k^* \right\|_p.
\end{align*}
\[
\begin{align*}
C_3 \sum_{t_1 = j_1}^{\infty} \ldots \sum_{t_d = j_d}^{\infty} \sum_{y \in N_d^\xi} b_y \phi_{y_t}^* (y)_p \\
C_3 \sum_{t_1 = j_1}^{\infty} \ldots \sum_{t_d = j_d}^{\infty} \sum_{y_{t,j} \in \mathcal{G}_{t,j}} \left( \int \phi_{y_t}^* (y)_p \right) (b)
\end{align*}
\]

(because for each pair \( \xi_1 \neq \xi_2, \eta_1, \eta_2 \in \tilde{N}_\xi \), the supports of the functions \( \phi_{\xi_1}^* \) and \( \phi_{\xi_2}^* \) are disjoint), so

\[
\prod_{i=1}^{d} 2^{i(1/2-1/p)} \phi_{\xi_i}^* (a)_p \leq C_3 \sum_{t_1 = j_1}^{\infty} \ldots \sum_{t_d = j_d}^{\infty} \sum_{y_{t,j} \in \mathcal{G}_{t,j}} \left( \int \phi_{y_t}^* (y)_p \right) (b).
\]

As \( \sum_{t_1 = j_1}^{\infty} \ldots \sum_{t_d = j_d}^{\infty} \prod_{i=1}^{d} 2^{-\alpha_i \xi_i} \sim \prod_{i=1}^{d} 2^{-\alpha_i \xi_i} \), from Jensen's inequality we get

\[
(c_j, \omega_p, \tau_{\xi,j}^*)_p (a)_q \leq C_3 \left( \prod_{i=1}^{d} 2^{\alpha_i \xi_i} \right)^q \left( \sum_{t_1 = j_1}^{\infty} \ldots \sum_{t_d = j_d}^{\infty} \sum_{y_{t,j} \in \mathcal{G}_{t,j}} \left( \int \phi_{y_t}^* (y)_p \right) (b) \right)^q \prod_{i=1}^{d} 2^{-\alpha_i \xi_i} \leq C_6 \prod_{i=1}^{d} 2^{\alpha_i \xi_i} \sum_{t_1 = j_1}^{\infty} \sum_{t_d = j_d}^{\infty} \sum_{y_{t,j} \in \mathcal{G}_{t,j}} \left( \int \phi_{y_t}^* (y)_p \right) (b). \]

This implies

\[
\sum_{y \in N_d^\xi} (c \xi, \omega_p, \tau_{\xi,j}^*)_p (a)_q \leq C_6 \sum_{y \in N_d^\xi} (c \xi, \omega_p, \tau_{\xi,j}^*)_p (b)_q \prod_{i=1}^{d} 2^{-\alpha_i \xi_i} \sum_{j_1 = j_1}^{\infty} \sum_{j_d = j_d}^{\infty} \prod_{i=1}^{d} 2^\alpha \xi_i \leq C_7 \sum_{y \in N_d^\xi} (c \xi, \omega_p, \tau_{\xi,j}^*)_p (b)_q,
\]

which gives the left inequality in (5).

To prove the right inequality in (5) we will need the formula for the coefficients of a function \( f \in C(I^d) \) in the basis \( \phi_{y_t}^* \):

\[
\begin{align*}
b_{y_t} (f) &= \Delta_{h_{1,1}} \ldots \Delta_{h_{d,d}} f, \\
&= \Delta_{h_{1,1}} \ldots \Delta_{h_{d,d}} f,
\end{align*}
\]

where

\[
\begin{align*}
\Delta_{h_{1,1}} f &= \begin{cases} f(x_{1,1}, \ldots, x_{1,i-1}, 0, x_{1,i+1}, \ldots, x_d) & \text{for } k = 0, \\
\frac{1}{h_{1,1}} \left( f(x_{1,1}, \ldots, x_{1,i-1}, x_{1,i}, x_{1,i+1}, \ldots, x_d) - f(x_{1,1}, \ldots, x_{1,i-1}, 0, x_{1,i+1}, \ldots, x_d) \right) & \text{for } k = 1, \\
\frac{1}{h_{1,1}} \left( f(x_{1,1}, \ldots, x_{1,i-1}, 1, x_{1,i}, x_{1,i+1}, \ldots, x_d) - f(x_{1,1}, \ldots, x_{1,i-1}, x_{1,i}, x_{1,i+1}, \ldots, x_d) \right) & \text{for } 2^j < k \leq 2^j + 1, k = 2^j + v.
\end{cases}
\end{align*}
\]

This formula follows from the formulae for functions of one variable and the fact that linear combinations of tensor products of continuous functions of one variable are dense in \( C(I^d) \).

The exponential estimates for \( b_k \) will also be needed: there exist \( C > 0 \) and \( 0 < \nu < 1 \) such that for all \( j \in N_d^\xi, \xi \in \tilde{N}_\xi, \) and \( t \in I^d \),

\[
|b_k (f)\|_t \leq C \prod_{i=1}^{d} 2^{i/2} \gamma^{|i^{-1} \xi_i - k_i + 2^{i+1}|^i}.
\]

(This is a straightforward consequence of exponential estimates for Franklin functions of one variable, cf. [2], [1].)

It follows from (6), (7) and the definition of \( f \)-s that for \( \xi \in \tilde{N}_\xi, \xi \in \tilde{N}_\xi, \)

\[
b_{y_t} (f) = 0 \text{ if } j_i > \xi_i \text{ for some } 1 \leq i \leq d, \\
b_{y_t} (f) \leq C \prod_{i=1}^{d} 2^{i(\xi_i - j_i)/2} \gamma^{|i^{-1} \xi_i - (k_i - 2^{i+1}) - j_i + 2^{i+1}|} \text{ if } j_i \leq \xi_i \text{ for all } 1 \leq i \leq d,
\]

so for \( f = \sum_{\xi \in N_d^\xi} \sum_{\eta \in \tilde{N}_\xi} a_{\xi, \eta} \phi_{\xi, \eta} \) and \( \xi \in \tilde{N}_\xi \) we get

\[
|b_{y_t} (f)| \leq C \sum_{\xi_1 = j_1}^{\infty} \ldots \sum_{\xi_d = j_d}^{\infty} \prod_{i=1}^{d} 2^{(\xi_i - j_i)/2} \sum_{\eta \in N_d^\xi} |a_{\eta}| \gamma^{|i^{-1} \xi_i - (k_i - 2^{i+1}) - j_i + 2^{i+1}|}.
\]

Defining

\[
x (\xi, \eta) = \sum_{\eta \in N_d^\xi} |a_{\eta}| \gamma^{|i^{-1} \xi_i - (k_i - 2^{i+1}) - j_i + 2^{i+1}|},
\]

we have

\[
x (\xi, \eta) \leq C \sum_{\xi_1 = j_1}^{\infty} \ldots \sum_{\xi_d = j_d}^{\infty} \prod_{i=1}^{d} 2^{(\xi_i - j_i)/2} \left( \sum_{\eta \in N_d^\xi} |a_{\eta}| \right)^{2/p}.
\]

But for \( \Theta (\xi, \eta) = \prod_{i=1}^{d} \theta_i (j_i) \) for \( j_i \in Z^d \),

\[
A(x, y) = \begin{cases} |a_{\eta}| & \text{if } \eta \in \tilde{N}_\xi, \\
0 & \text{if } \eta \notin \tilde{N}_\xi.
\end{cases}
\]
and \( n(\xi, k) = (2^{\alpha_i-j_i} (k_1 - 2^j_1) + 2^{\alpha_i}, \ldots, 2^{\alpha_i-j_d} (k_d - 2^j_d) + 2^{\alpha_i}) \) we have

\[
z(\xi, k) = (A(\xi, \cdot) * \Theta)(n(\xi, k)),
\]

so

\[
\left( \sum_{k \in \mathbb{N}_j} |z(\xi, k)|^p \right)^{1/p} \leq \left( \sum_{n \in \mathbb{N}_d} |(A(\xi, \cdot) * \Theta)(n)|^p \right)^{1/p} \leq \|\Theta\|_1 \|A(\xi, \cdot)\|_p = C_4 \tau^{(m)}_{i,p}(\alpha).
\]

This implies

\[
\prod_{i=1}^d 2^{j_i/2} \tau^{(m)}_{i,p}(\alpha) \leq C_5 \sum_{\xi = (\xi_1, \ldots, \xi_d)} \prod_{i=1}^d 2^{j_i/2} \tau^{(m)}_{i,p}(\alpha)
\]

and

\[
c(\xi, \alpha, p) \tau^{(m)}_{i,p}(\alpha) \leq C_6 \prod_{i=1}^d 2^{j_i/2} \tau^{(m)}_{i,p}(\alpha).
\]

As \( 1/p < \alpha_i \) for all \( 1 \leq i \leq d \), it follows that

\[
\sum_{\xi = (\xi_1, \ldots, \xi_d)} \prod_{i=1}^d 2^{j_i/2} \tau^{(m)}_{i,p}(\alpha) \sim \prod_{i=1}^d 2^{j_i/2} \tau^{(m)}_{i,p}(\alpha),
\]

and from Jensen’s inequality we obtain

\[
(c(\xi, \alpha, p) \tau^{(m)}_{i,p}(\alpha))^q \leq C_7 \prod_{i=1}^d 2^{j_i/2} \tau^{(m)}_{i,p}(\alpha)^q \sum_{\xi = (\xi_1, \ldots, \xi_d)} \prod_{i=1}^d 2^{j_i/2} \tau^{(m)}_{i,p}(\alpha)^q,
\]

and

\[
\sum_{\xi \in \mathbb{N}_d} (c(\xi, \alpha, p) \tau^{(m)}_{i,p}(\alpha))^q \leq C_8 \sum_{\xi \in \mathbb{N}_d} (c(\xi, \alpha, p) \tau^{(m)}_{i,p}(\alpha))^q \prod_{i=1}^d 2^{j_i/2} \tau^{(m)}_{i,p}(\alpha)^q,
\]

which gives the right inequality in (5) and completes the proof of Theorem A.2.

7. Proof of Theorem B. Recall that for \( \alpha = (\alpha_1, \ldots, \alpha_d) \) with \( 1/p < \alpha_i < 1/p' \), \( \beta_i = 1 - \alpha_i \), \( \beta = (\beta_1, \ldots, \beta_d) \), \( F \in B^B_{p,1}(I^d) \), \( G \in B^B_{p,\infty}(I^d) \),

\[
F = \sum_{i \in \mathbb{N}_d} \sum_{k \in \mathbb{N}_j} F_k \varphi_k, \quad G = \sum_{\xi \in \mathbb{N}_d} \sum_{k \in \mathbb{N}_j} G_\xi \varphi_\xi,
\]

and \( g = (g_1, \ldots, g_d) \in I^d \) we have defined

\[
I(F, G)(\alpha) = \sum_{\xi \in \mathbb{N}_d} \sum_{k \in \mathbb{N}_j} \sum_{\xi \in \mathbb{N}_d} \sum_{k \in \mathbb{N}_j} F_k G_\xi \prod_{i=1}^d \int_0^{s_i} h_{\xi_i}(u_i) h_{\eta_i}(u_i) \, du_i.
\]

We are to prove that there exists a constant \( C = C(\alpha, p) \) such that for all \( F \in B^B_{p,1}(I^d) \) and \( G \in B^B_{p,\infty}(I^d) \),

\[
\|I(F, G)(\alpha)\|_p \leq C \|F\|_{B^B_{p,1}(I^d)} \|G\|_{B^B_{p,\infty}(I^d)}.
\]

Recall that by \( s_k, k \in \mathbb{N} \setminus \{0\} \), we denote the dyadic points in \( I \), described by (1).

There are the following formulae for inner products of one-dimensional Haar functions:

- for \( k \in \mathbb{N}_j \) and \( \eta \in \mathbb{N}_\xi \) with \( \xi < j \):

\[
\int_0^s h_k(u) h_\eta(u) \, du = h_\eta(s_k) \varphi_k(s),
\]

- for \( k, \eta \in \mathbb{N}_j \), \( j > -1 \):

\[
\int_0^s h_k(u) h_\eta(u) \, du = \delta_{k,\eta} \left( \sum_{s_k \in \mathbb{N}_\xi} h_{\eta}(s_k) \varphi_k(s) \right),
\]

- for \( k = \eta = 1 \):

\[
\int_0^s h_k(u) h_\eta(u) \, du = \varphi_1(s).
\]

Define

\[
J(F, G)(\alpha) = \sum_{\xi \in \mathbb{N}_d} \sum_{k \in \mathbb{N}_j} \sum_{\xi \in \mathbb{N}_d} \sum_{k \in \mathbb{N}_j} |F_k| |G_\xi| \prod_{i=1}^d |h_{\xi_i}(s_k)| |h_{\eta_i}(s_k)|
\]

\[
\times \prod_{i : j_i = \xi_i} \delta_{k,\eta_i} \left( \sum_{s_k \in \mathbb{N}_\xi} |h_{\eta_i}(s_k)| |\varphi_k(s)| \right)
\]

\[
\times \prod_{i : j_i > \xi_i} |h_{\eta_i}(s_k)| |\varphi_k(s)|.
\]

As for each \( g \in I^d \) this is a series with non-negative components, it can be
rearranged as
\[ J(F, G)(\xi) = \sum_{j \in N_1} \sum_{k \in \tilde{N}_1} J_k \phi_k(\xi). \]

Observe that for \( F = (F_k) \) we have
\[ \|J(F, G)(\xi)\|_{p,1} \leq C_1 \|F\|_{p,1}^\alpha, \]
where \( C_1 > 0 \) does not depend on \( F \in B_{p,1}^\alpha(I^d) \) (the proof of this statement is a simple calculation and is omitted here). Also, set \( G^* = (G_k^*), G_k^* = \|\phi_k\|_2 G_k \) and \( J^* = (J_k^*) \). It will be shown that
\[ \|J^*(\xi)\|_{p,\infty} \leq C_2 \|J(F, G)(\xi)\|_{p,1}, \]
where \( C_2 > 0 \) does not depend on \( F \in B_{p,1}^\alpha(I^d) \) and \( G \in B_{p,\infty}^\alpha(I^d) \). Then it will follow from Theorem A.2 that \( J(F, G) \in B_{p,\infty}^{\alpha}(I^d) \subset C(I^d) \). Moreover, we will show that the series defining \( I(F, G)(\xi) \) is absolutely convergent for each \( \xi \in I^d \), and therefore it can be rearranged as
\[ I(F, G)(\xi) = \sum_{j \in N_1} \sum_{k \in \tilde{N}_1} I_k \phi_k(\xi), \]
with \( |I_k| \leq J_k^* \), so for \( J^* = (J_k^*) \) we will have \( I^* \in \mathcal{B}_{p,\infty} \), and
\[ \|I^*(\xi)\|_{p,\infty} \leq \|J^*(\xi)\|_{p,1}, \]
and Theorem B will follow from (8) and Theorem A.2.

It remains to prove inequality (9).

For \( A, B, C \subset D \) with \( A \cap B = A \cap C = B \cap C = \emptyset \) and \( A \cup B \cup C = D \), let
\[ N(A, B, C) = \{(j, \xi) \in N_1^d \times N_1^d : j_i < \xi_i \text{ for } i \in A, j_i = \xi_i \text{ for } i \in B \text{ and } j_i > \xi_i \text{ for } i \in C\}, \]
and
\[ J_{A, B, C}(\xi) = \sum_{(j, \xi) \in N(A, B, C)} \sum_{k \in \tilde{N}_1} |F_k||G_k| \prod_{i \in A} |h_{k_i}(\xi_{k_i})| \phi_{k_i}(\xi_{k_i}) \prod_{i \in B} \delta_{k_i, m_i} \left( \sum_{\xi_i = -1}^1 \sum_{\xi_i \in \tilde{N}_1} |h_{k_i}(\xi_{k_i})| \phi_{k_i}(\xi_{k_i}) \right) \prod_{i \in C} |h_{k_i}(\xi_{k_i})| \phi_{k_i}(\xi_{k_i}). \]
Setting \( S(\xi) = \{\xi = (\xi_1, \ldots, \xi_d) : \xi_i < j_i \text{ for } i \in A \cup C, \xi_i > j_i \text{ for } i \in B\}, \)
and for \( k \in \tilde{N}_1 \) and \( \eta \in \tilde{N}_1 \),
\[ a(k, \eta) = \begin{cases} \eta_i & \text{for } i \in A \cup B, \\ k_i & \text{for } i \in C, \end{cases} \]
\[ b(k, \eta) = \begin{cases} \eta_i & \text{for } i \in A, \\ k_i & \text{for } i \in C, \text{ and } \eta_i & \text{for } i \in B \cup C, \end{cases} \]
and denoting by \( \chi_n \) the characteristic function of \( \{t \in I : h_n(t) \neq 0\} \) we get
\[ J_{A, B, C}(\xi) = \sum_{i \in \tilde{N}_1} \sum_{k \in \tilde{N}_1} W_k \phi_k(\xi), \]
where for \( k \in \tilde{N}_1 \),
\[ W_k = \sum_{\xi \in S(\xi)} \prod_{i \in \tilde{N}_1} 2^{\xi_i/2} \prod_{i \in \tilde{N}_1} 2^{k_i/2} \times \sum_{\eta \in \tilde{N}_1} |F_{a(k, \eta)}| |G_{b(k, \eta)}| \prod_{i \in A \cup C} \chi_{n_{k_i}}(\eta_{k_i}) \prod_{i \in B} \chi_{n_{k_i}}(\eta_{k_i}). \]
As for given \( k \in \tilde{N}_1 \) and \( \xi \in S(\xi) \) we have
\[ \#\{\eta \in \tilde{N}_1 : \prod_{i \in A \cup C} \chi_{n_{k_i}}(\eta_{k_i}) \prod_{i \in B} \chi_{n_{k_i}}(\eta_{k_i}) \neq 0\} = \prod_{i \in \tilde{N}_1} 2^{1 - j_i}, \]
we obtain
\[ \sum_{\eta \in \tilde{N}_1} |F_{a(k, \eta)}| |G_{b(k, \eta)}| \prod_{i \in A \cup C} \chi_{n_{k_i}}(\eta_{k_i}) \prod_{i \in B} \chi_{n_{k_i}}(\eta_{k_i}) \leq \prod_{i \in \tilde{N}_1} 2^{(\xi_i - j_i)/(1/p' - 1/p)} \left( \sum_{\eta \in \tilde{N}_1} |F_{a(k, \eta)}| |G_{b(k, \eta)}| \prod_{i \in A \cup C} \chi_{n_{k_i}}(\eta_{k_i}) \prod_{i \in B} \chi_{n_{k_i}}(\eta_{k_i}) \right)^{1/p} \times \left( \sum_{\eta \in \tilde{N}_1} |G_{b(k, \eta)}| \prod_{i \in A \cup C} \chi_{n_{k_i}}(\eta_{k_i}) \prod_{i \in B} \chi_{n_{k_i}}(\eta_{k_i}) \right)^{1/p} = \prod_{i \in \tilde{N}_1} 2^{(\xi_i - j_i)/(1/p' - 1/p)} \left( \sum_{i \in A \cup C} \sum_{\eta \in \tilde{N}_1} |F_{a(k, \eta)}| \prod_{i \in A \cup C} \chi_{n_{k_i}}(\eta_{k_i}) \prod_{i \in B} \chi_{n_{k_i}}(\eta_{k_i}) \right)^{1/p} \times \left( \sum_{i \in A \cup C} \sum_{\eta \in \tilde{N}_1} |G_{b(k, \eta)}| \prod_{i \in A \cup C} \chi_{n_{k_i}}(\eta_{k_i}) \prod_{i \in B} \chi_{n_{k_i}}(\eta_{k_i}) \right)^{1/p} \leq \prod_{i \in \tilde{N}_1} 2^{(\xi_i - j_i)/(1/p' - 1/p)} \left( \sum_{i \in A \cup C} \sum_{\eta \in \tilde{N}_1} |F_{a(k, \eta)}| \prod_{i \in A \cup C} \chi_{n_{k_i}}(\eta_{k_i}) \prod_{i \in B} \chi_{n_{k_i}}(\eta_{k_i}) \right)^{1/p} \times \left( \sum_{i \in A \cup C} \sum_{\eta \in \tilde{N}_1} |G_{b(k, \eta)}| \prod_{i \in A \cup C} \chi_{n_{k_i}}(\eta_{k_i}) \prod_{i \in B} \chi_{n_{k_i}}(\eta_{k_i}) \right)^{1/p}. \]
Using this inequality we obtain

\[
\left( \sum_{k \in \mathbb{Z}} |W_k|^p \right)^{1/p} \leq \sum_{\xi \in \mathbb{Z}^d} \prod_{i \in \mathbb{A} \cup \mathbb{C}} 2^{\xi_i/2} \prod_{i \in \mathbb{B}} 2^{\xi_i(1/2 + 1/p - 1/p') + \xi_i(1/p' - 1/p)}
\times \left( \sum_{k \in \mathbb{Z}^d} |F_k|^p \right)^{1/p} \left( \sum_{\xi \in \mathbb{N} \cup \mathbb{Q}} |G_{\xi}|^p \right)^{1/p}.
\]

Writing

\[
F_k = c(\xi, \mathcal{F}, p) \left( \sum_{k \in \mathbb{N}^d} |G_{k}|^p \right)^{1/p}, \quad G_{\xi} = c(\xi, \mathcal{A}, p) \left( \sum_{\xi \in \mathbb{N}^d} |G_{\xi}|^p \right)^{1/p},
\]

\[
W_k^* = c(\xi, \mathcal{A}, p) \left( \sum_{k \in \mathbb{N}^d} |W_k|^p \right)^{1/p},
\]

where \( W_k^* = \|\phi_k\|_2 W_k \) and \( W^*(A, B, C) = (W_k^*) \), we obtain from the last inequality

\[
W_k^* \leq M \prod_{i \in \mathbb{B}} 2^{\xi_i(\alpha_i - 1/p') - \xi_i(1 + 1/p - \alpha_i)} \prod_{i \in \mathbb{C}} 2^{\xi_i(\alpha_i - 1/p') - \xi_i(1 + 1/p - \alpha_i)}
\times \sum_{\xi \in \mathbb{N}^d} \mathcal{F}_{\xi} \sum_{\xi \in \mathbb{Q}} |\mathcal{G}_{\xi}|^p \prod_{i \in \mathbb{A}} 2^{\xi_i(\alpha_i - 1/p') - \xi_i(1 + 1/p - \alpha_i)}
\]

As \( \alpha_i < 1/p' \) for all \( i = 1, \ldots, d \), this implies

\[
W_k^* \leq M \|G^*\|_{p, \mathbb{C}} \prod_{i \in \mathbb{B}} 2^{\xi_i(\alpha_i - 1)} \sum_{\xi \in \mathbb{N}^d} \mathcal{F}_{\xi} \prod_{i \in \mathbb{C}} 2^{\xi_i(1/p' - \alpha_i)}
\]

and

\[
\|W^*(A, B, C)\|_{p, \mathbb{C}} \leq M \|E\|_{p, \mathbb{A}} \|G_*\|_{p, \mathbb{C}}.
\]

As \( J(F, G) = \sum_{A, B, C} J_{A, B, C} \), we get

\[
\|J^*\|_{p, \mathbb{C}} \leq M \|E\|_{p, \mathbb{A}} \|G_*\|_{p, \mathbb{C}}.
\]

This completes the proof of Theorem B. \( \blacksquare \)

References

[3] Z. Ciesielski and J. Domsta, Construction of an orthonormal basis in \( C^\infty(I^d) \) and \( W^*(I^d) \), ibid. 41 (1972), 211–224.