The following theorem includes [5, p. 64, Theorem 9] whose assumption is given by \( \|S^{1/p}_{\infty}f\|_{p,1} \) in our notation.

**Theorem 5.** If \( f \in B^{1/p}_{p,1} \) (0 < \( p \leq 2 \)), then \( \sum_{k=0}^{\infty} |(f, w_k)| < \infty \).

**Proof.** From Theorem A(i) and the definition of \( B^{x}_{p,q} \), we have \( B^{1/p}_{p,1} \subset B^{x}_{1,2} \) for 0 < \( p \leq 2 \). Thus, we obtain the theorem from (3.12) and Theorem 1.

**References**


[8] E. A. Storozenko, V. G. Krotov and P. Oswald, **Direct and converse theorems of Jackson type in \( L^p \) spaces**, 0 < \( p < 1 \), Math. USSR-St. 27 (1975), 355–374.


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we have
\[\int \Phi(Tf) \, dx = \int \phi(\lambda) \int_{[x : Tf(x) > \lambda]} \phi(\lambda') \, d\lambda' \, dx \leq C \int \frac{\phi(\lambda)}{\lambda} \int_{[|f| > \lambda/K]} |f(x)| \, dx \, d\lambda \]
\[= C \int |f(x)| \int_0^\infty \phi(\lambda) \frac{d\lambda}{\lambda} \, dx \leq \int \Phi(|f(x)|) \, dx \]
provided \(\phi\) satisfies a Dini condition, that is,
\[\int_0^\infty \frac{\phi(s)}{s} \, ds \leq C \phi(x).\]
\(\phi\) Dini is equivalent to a \(\Delta_2\) condition on the N-function complementary to \(\Phi\). So an operator like the Hardy–Littlewood maximal operator \(M\) satisfies an \(L_\Phi\) integral inequality provided this complementary \(\Delta_2\) condition holds. The converse is also true [14].

To avoid \(\Delta_2\), one needs to improve this argument, or avoid interpolation. In [3], we characterized weighted \(L_\Phi\) integral inequalities for generalized Hardy operators, obtaining both weak- and strong-type inequalities with no \(\Delta_2\) assumptions. With hopes high, we turned to the Hardy–Littlewood maximal operator.

The weighted theory for the maximal operator,
\[Mf(x) = \sup \left\{ \frac{1}{|I|} \int_I |f(y)| \, dy : I \text{ is a cube in } \mathbb{R}^n \text{ containing } x \right\}\]
is quite beautiful. A weight \(w\) is a function on \(\mathbb{R}^n\) which is positive and finite almost everywhere. \(w\) belongs to the Muckenhoupt \(A_p\) class if
\[\left( \frac{1}{|I|} \int_I w(x) \, dx \right) \left( \frac{1}{|I|} \int_I w(x)^{-1/(p-1)} \, dx \right)^{p-1} \leq C\]
for all cubes \(I\). The maximal operator \(M\) is a bounded operator from \(L^p(\mathbb{R}^n, w(x) \, dx)\) to \(L^p(\mathbb{R}^n, w(x) \, dx)\) if and only if \(w \in A_p, 1 < p < \infty\). See [10].

In [7], Kerman and Torchinsky extended this to Orlicz spaces. They examined the inequality
\[\int \Phi(Mf(x))w(x) \, dx \leq \int \Phi(C|f(x)|)w(x) \, dx.\]
Assuming both \(\Phi\) and its complement were in \(\Delta_2\), they showed that (1) holds if and only if \(w \in A_p\), where \(1/p\) is the Boyd upper index of \(\Phi\).

Since then, researchers have been chipping away at the \(\Delta_2\) assumptions. Bagby [1], Gogatishvili and Pick [6], Pick [11], and Quinsheng [12] have all tackled weak-type problems for the maximal operator. Pick and Quinsheng do strong-type as well. Pick has made the most progress. He showed that (1), even with two weights, forces a local \(\Delta_2\) condition on the complement to \(\Phi\). This suggests that the Dini condition used for interpolating is actually indispensable. That turns out to be true.

We say an N-function \(\Phi \in \Delta_2^\kappa\) if its complement (see Section 2) belongs to \(\Delta_2\). Our main result is

**Theorem 1.** Let \(\Phi\) be an N-function and let \(w\) be a weight on \(\mathbb{R}^n\). Then the following are equivalent:

(a) For all \(f,\)
\[\int \Phi(Mf(x))w(x) \, dx \leq \int \Phi(C|f(x)|)w(x) \, dx.\]

(b) \(\Phi \in \Delta_2^\kappa\) and the weak-type boundedness
\[\Phi(\lambda)w([x : Mf(x) > \lambda]) \leq \int \Phi(|f(x)|)w(x) \, dx\]
holds for all \(f\).

(c) \(\Phi \in \Delta_2^\kappa\) and \(w\) satisfies the condition
\[\int_I \Psi \left( \frac{\Phi(\lambda)w(I)}{C\lambda^\lambda w(x)} \right) w(x) \, dx \leq \Phi(\lambda)w(I) < \infty\]
for all cubes \(I\), where \(\Psi\) is the complementary Young function to \(\Phi\).

This paper is organized into three further sections. In the next section, we describe the Orlicz space theory that we will use. Section 3 gives weak-type results. These are quite general, with two Young functions and four weights. Some interesting consequences are described in the special one \(\Phi\), one weight setting. In the last section, we present the strong-type theory and prove Theorem 1.

2. Orlicz spaces. The standard theory of Orlicz spaces can be found in Zygmund [15], Krasnosel'skiĭ and Rutitskiĭ [9], or Rao and Ren [13], and weighted theory is developed in the recent book by Kokilashvili and Krbec [8]. An N-function \(\Phi : \mathbb{R}^+ \to \mathbb{R}^+\) is a nonnegative, convex function satisfying
\[\lim_{x \to 0} \frac{\Phi(x)}{x} = 0, \quad \lim_{x \to \infty} \frac{\Phi(x)}{x} = \infty.\]
\(\Phi\) has a derivative \(\phi\) which is nondecreasing and nonnegative, \(\phi(0) = 0\), and \(\phi(\infty) = \infty\), so that
\[\Phi(x) = \int_0^x \phi(t) \, dt,\]
and we can and will take \(\phi\) to be right-continuous.
Set
\[ \psi(x) = \inf \{ y : \phi(y) \geq x \} \]
and
\[ \Psi(x) = \int_0^x \psi(y) \, dy. \]
This is also an N-function, and is known as the complementary Young function to \( \Phi \). We have Young's inequality
\[ ab \leq \Phi(a) + \Phi(b) \quad \text{for all } a, b > 0, \]
and \( \Psi \) satisfies
\[ \Psi(x) = \sup \{ xy - \Phi(y) \}. \]

If \( (X, \mu) \) is a \( \sigma \)-finite measure space, then the Orlicz space \( L_\Phi = L_\Phi(X, d\mu) \) is the Banach space on which we take the Luxemburg norm
\[ \| f \|_\Phi = \inf \left\{ \lambda > 0 : \int_X \frac{f(x)}{\lambda} \, d\mu(x) \leq 1 \right\}. \]
The original norm used by Orlicz is
\[ \| f \|_\Phi = \sup \left\{ \int f g \, d\mu : \int \Psi(|g|) \, d\mu \leq 1 \right\}. \]
These two norms are equivalent, and one has the Hölder inequality
\[ \int f g \, d\mu \leq C \| f \|_\Phi \| g \|_\Psi. \]

The following easy lemma is taken from [3].

**Lemma 2.** Let \( \Phi \) be an N-function with complementary function \( \Psi \). Let \( x \) and \( y > 0 \). Then
\[ \Phi(x) \leq x \phi(x) \leq \Phi(2x), \]
\[ \Phi(x) + \Phi(y) \leq \Phi(x + y) \]
and
\[ \Phi \left( \frac{\psi(x)}{x} \right) \leq \Psi(x). \]

We will also need the inequality
\[ x \leq \Phi^{-1}(x) \Psi^{-1}(x) \leq 2x. \]
The left inequality follows immediately from (5), and the other is just Young's inequality.

We say \( \Phi \in \Delta_2 \) if \( \Phi(2x) \leq C \Phi(x) \) for all \( x > 0 \), and \( \Phi \in \text{local } \Delta_2 \) if this holds for all \( x \geq x_0 \). \( \Phi \in \Delta_2^0 \) if the complement \( \Psi \) is in \( \Delta_2 \).

\( C \) will always denote a universal constant, and may change in subsequent appearances.

**Proposition 3** (Bari and Stechkin [2]). Let \( \Phi \) be an N-function with derivative \( \phi \). Then the following are equivalent:
\[(a) \ \phi \ \text{is Dini, i.e.,} \]
\[ \int_0^x \frac{\phi(s)}{s} \, ds \leq C \phi(x) \]
for all \( x > 0 \).
\[(b) \ \text{There exists a } \delta > 0 \text{ such that } \phi(\delta x) \leq \frac{1}{2} \phi(x) \text{ for all } x > 0. \]
\[(c) \ \Phi \in \Delta_2^0. \]

**Proof.** \( (a) \Rightarrow (b) \). Let \( \Psi \) be the complement and let \( 0 < \delta < 1 \). We have
\[ \int_0^x \frac{\phi(y)}{y} \, dy = \int_0^1 \int_0^y \phi(s) \, dy \, ds = \int_0^x \phi(s) \log \frac{x}{s} \, ds \]
\[ \geq \int_0^x \phi(s) \log \frac{x}{s} \geq \left( \log \frac{1}{\delta} \right) \phi(\delta x). \]
Choose \( \delta \) so that \( \log \frac{1}{\delta} = 2C \). Then the Dini condition gives
\[ 2\phi(\delta x) \leq \phi(x). \]

\( (b) \Rightarrow (c) \). Fix \( y \) and put \( x > \psi(y)/\delta \). Since \( \psi(y) = \inf \{ t : \phi(t) \geq y \} \) and \( \phi \) is nondecreasing, we must have \( y \leq \phi(\delta x) \). Thus \( 2y \leq \phi(x) \), or \( \psi(2y) \leq \phi(\phi(x)) \leq x \).

Since this holds for every \( x > \psi(y)/\delta \), we in fact have
\[ \psi(2y) \leq \frac{1}{\delta} \psi(y) \]
and \( \Delta_2 \) follows on integrating \( \psi \).

\( (c) \Rightarrow (a) \). Finally, if \( \Psi \in \Delta_2 \), then
\[ \psi(2^n x) \leq C^n \psi(x), \quad \text{for } n = 0, 1, 2, \ldots, \]
and we can take \( C > 1 \), obviously. So
\[ \int_0^x \frac{\phi(y)}{y} \, dy = \sum_{n=0}^{\infty} \int_{x/2^n}^{x/2^{n+1}} \frac{\phi(y)}{y} \, dy \leq \log C \sum_{n=0}^{\infty} \phi \left( \frac{x}{C^n} \right). \]
But
\[ \psi \left( 2^n \phi \left( \frac{x}{C^n} \right) \right) \leq C^n \psi \left[ \phi \left( \frac{x}{C^n} \right) \right] \leq x \]
and by the definition of \( \psi \),
\[ 2^n \phi \left( \frac{x}{C^n} \right) \leq \phi(x). \]
Hence,
\[ \int_0^\infty \phi(y) \, dy \leq 2(\log C) \phi(x). \]

We would like to thank our referee for pointing out the reference [2].

There is an important connection between norm inequalities and integral inequalities. This next proposition is from [3].

**Proposition 4.** Suppose \( T \) is a linear operator acting from a \( \sigma \)-finite measure space \((X, d\mu)\) to a \( \sigma \)-finite measure space \((Y, dv)\). Let \( \Phi \) be an N-function and let \( L_{\Phi, e\mu}(X) \) be the Orlicz space with the norm
\[ ||f||_{\Phi, e\mu} = \inf \left\{ \lambda > 0 : \int_X \Phi \left( \frac{|f(x)|}{\lambda} \right) e \, d\mu(x) \leq 1 \right\} \]
and let \( L_{\Phi, e\nu}(Y) \) be defined similarly. Then
\[ \int Y \Phi(|Tf(y)|) \, d\nu(y) \leq \int X \Phi(C|f(x)|) \, d\mu(x) \]
if and only if
\[ ||Tf||_{\Phi, e\nu} \leq C ||f||_{\Phi, e\mu} \]
for all \( \varepsilon > 0 \), with \( C \) independent of \( \varepsilon \).

3. **Weak-type integral inequalities.** The complete characterization of weak-type weighted \( L_p \) inequalities for monotone operators on \( \mathbb{R}^n \) was given in Theorem 3.1 of [3]. The argument used there can be applied to other operators. For example, it is easy to adapt it to obtain

**Theorem 5.** Let \( 0 < \alpha < n \), and let \( M_\alpha f(x) \) be the fractional maximal operator on \( \mathbb{R}^n \),
\[ M_\alpha f(x) = \sup \left\{ |f|^\alpha/n-1 \int_I |f(y)| \, dy : I \text{ is a cube containing } x \right\}. \]

Let \( t, u, v, \) and \( w \) be weights on \( \mathbb{R}^n \), \( \Phi_1 \) and \( \Phi_2 \) be N-functions with complements \( \Psi_1 \) and \( \Psi_2 \) respectively. Assume further that \( \Phi_2 \ominus \Phi_1^{-1} \) is convex. Then weak-type boundedness,
\[ \Phi_2^{-1} \left[ \int_{M_\alpha f > \lambda} \Phi_2(\lambda w(u(x)) tu(x)) \, dx \right] \leq \Phi_1^{-1} \left[ \int_{\mathbb{R}^n} \Phi_1(C|f(x)| u(x)) v(x) \, dx \right], \]
holds if and only if
\[ \int \Psi_1 \left[ \frac{\gamma(\lambda, I)}{C \lambda u(y) v(y)} |f|^\alpha/n-1 \right] v(y) \, dy \leq \gamma(\lambda, I) < \infty \]
holds for each cube \( I \), where
\[ \gamma(\lambda, I) = \Phi_1 \circ \Phi_2^{-1} \left[ \int_I \Phi_2(\lambda w(u(y)) tu(y)) \, dy \right]. \]

A couple of remarks about this theorem may be in order. In the Lebesgue setting, and in an expression like \( \int (|f(x)| u(x))^{1/p} v(x) \, dx \), the weights \( u \) and \( v \) can obviously be combined. This is not true for general \( \Phi \), and that necessitates confronting such unpleasant four-weight inequalities. The convexity assumption in the theorem corresponds to the Riesz triangle \( p \leq q \) in the Lebesgue setting.

**Proof of Theorem 5.** The necessity is essentially the argument of [3], with \( \varepsilon \) chosen so that
\[ \int E \Phi_1 \left( \frac{\varepsilon}{u w} \right) \frac{v(x)}{\varepsilon} \, dx = 2C \lambda |T|^\alpha/n \], \( E \subset I \),
and
\[ f(x) = \frac{1}{C} \Phi_2 \left( \frac{\varepsilon}{u w} \right) \frac{v(x)}{\varepsilon} \chi_E(x). \]

For the sufficiency, let \( \Gamma \subset \{ x : M_\alpha f(x) > \lambda \} \) be a compact set. For each \( x \in \Gamma \), there exists an open cube \( I \) with \( |I|^\alpha/n-1 \int_I |f| > \lambda \), and \( \Gamma \) is covered by finitely many of these cubes. Let \( I_k \) be the largest, and \( I_{k+1} \) the largest remaining cube disjoint from \( I_1 \cup \ldots \cup I_k \). Let \( J_k \) be the cube concentric with \( I_k \) but triple its side length. We have
\[ \lambda < |J_k|^\alpha/n-1 \int_{I_k} |f|, \quad \Gamma \subset \bigcup_{I_k} \]
and the \( I_k \)'s are disjoint. Hence,
\[ 2\gamma(\lambda, \Gamma_k) \leq \int_{I_k} |f(x)| \frac{|J_k|^\alpha/n-1}{\lambda} \frac{\gamma(\lambda, J_k)}{C \lambda u(x) v(x)} |J_k|^\alpha/n-1 v(x) \, dx \]
\[ = \int I_k \left( 2 \cdot 3^{n-\alpha} C |f(x)| u(x) \cdot \frac{\gamma(\lambda, J_k)}{C \lambda u(x) v(x)} |J_k|^\alpha/n-1 v(x) \right) \, dx \]
\[ \leq \sum_{I_k} \Phi_1(2 \cdot 3^{n-\alpha} C |f(x)| u(x)) v(x) \, dx \]
\[ + \sum_{I_k} \Psi_1 \left( \frac{\gamma(\lambda, J_k)}{C \lambda u(x) v(x)} |J_k|^\alpha/n-1 \right) v(x) \, dx \]
\[ \leq \sum_{I_k} \Phi_1(2 \cdot 3^{n-\alpha} C |f(x)| u(x)) v(x) \, dx + \gamma(\lambda, J_k), \]
so
\[ \gamma(\lambda, J_k) \leq \int_{I_k} \Phi_1(2 \cdot 3^n - \alpha \cdot C |f(x)| u(x)) v(x) \, dx \]
and thus
\[ \int_{J_k} \Phi_2(\lambda w(y)) \mu(y) \, dy \leq \Phi_2 \circ \Phi_1^{-1} \left[ \int_{I_k} \Phi_1(2 \cdot 3^n - \alpha \cdot C |f(x)| u(x)) v(x) \, dx \right]. \]
Summing over \( k \) gives
\[ \int_{I} \Phi_2(\lambda w(y)) \mu(y) \, dy \leq \sum_{k} \Phi_2 \circ \Phi_1^{-1} \left[ \int_{I_k} \Phi_1(2 \cdot 3^n - \alpha \cdot C |f(x)| u(x)) v(x) \, dx \right]. \]
But (4) applies to \( \Phi_2 \circ \Phi_1^{-1} \), so this last sum is bounded by
\[ \Phi_2 \circ \Phi_1^{-1} \left( \sum_{k} \int_{I_k} \Phi_1(2 \cdot 3^n - \alpha \cdot C |f(x)| u(x)) v(x) \, dx \right) \leq \Phi_2 \circ \Phi_1^{-1} \left[ \int_{\mathbb{R}^n} \Phi_1(2 \cdot 3^n - \alpha \cdot C |f(x)| u(x)) v(x) \, dx \right], \]
and that proves the theorem.

This last result has some history. For the Hardy–Littlewood maximal operator \((\alpha = 0)\), Bagby did a one-weight version of this, when \( \Phi_1 = \Phi_2 \). We say a weight \( w \in W_\Phi \) if
\[ \Phi(\lambda) w([x : M f(x) > \lambda]) \leq \int \Phi(C |f(x)|) w(x) \, dx. \]
Bagby, in [1], characterized the weights for which a slightly modified form of this inequality holds. This is the first weighted Orlicz space paper that completely escaped the \( \Delta_2 \) conditions, and we are quite indebted to this work. Pick extended this to a two-weight setting [11], assuming a doubling condition, which he and Gogatishvili eliminated in [6]. For a further generalization, see [5].

In the Lebesgue setting, for \( 1 < p < \infty \), it is well known that \( M \) is of weak-type \((p, p)\) with respect to \( w \) if and only if it is of strong-type \((p, p)\) with respect to \( w \), and this is equivalent to \( w \in A_p \). For Orlicz spaces \( L_\Phi \), with \( \Phi \) and its complement \( \Psi \) satisfying \( \Delta_2 \), the Kerman–Torchinsky \( A_\Phi \) condition and the condition (9) are equivalent. What happens away from \( \Delta_2 \)?

Recall that a weight \( w \in A_p, 1 < p < \infty \), provided
\[ \left( \frac{1}{|I|} \int_I w(x) \, dx \right) \left( \frac{1}{|I|} \int_I w(x)^{-1/(p-1)} \, dx \right)^{p-1} \leq C \]
for all cubes \( I \). The limiting condition as \( p \rightarrow 1^+ \),
\[ \frac{1}{|I|} \int_I w(x) \, dx \leq C \text{ess inf } w(x), \]
is denoted by \( A_1 \). The \( A_p \) classes are nested, that is, \( A_p \subset A_q \) if \( p < q \), and the union of these classes is customarily called \( A_\infty \). The \( L_\Phi \) integral inequality for the maximal operator proven in [7] relied heavily on the "reverse Hölder" property of \( A_\infty \) weights. It would be nice if \( A_\infty \) were necessary beyond \( \Delta_2 \). Nice, and also true:

**Theorem 6.** We have
\[ \bigcap_{\Phi \text{ N-functions}} W_\Phi = A_1, \quad \bigcup_{\Phi \text{ N-functions}} W_\Phi = A_\infty. \]

**Proof.** Let \( w(I) \) denote \( \int_I w(x) \, dx \). By Theorem 5, if \( \Phi \) is an \( N \)-function with complement \( \Psi \), then \( w \in W_\Phi \) if and only if, for each \( \lambda > 0 \) and cube \( I \),
\[ \int_I \psi \left( \frac{\Phi(\lambda) w(I)}{C \lambda |I| w(x)} \right) w(x) \, dx \leq \Phi(\lambda) w(I) < \infty. \]
So assume that \( w \in A_1 \), and take \( \psi(x) = \int_0^x \psi(y) \, dy \), where \( \psi(x) = \text{inf} \{ y : \psi(y) \geq x \} \). By (8),
\[ \int_I \psi \left( \frac{\Phi(\lambda) w(I)}{C \lambda |I| w(x)} \right) w(x) \, dx \leq \Phi(\lambda) w(I) \int_I \psi \left( \frac{\Phi(\lambda) w(I)}{C \lambda |I| w(x)} \right) \, dx. \]
Now \( w \in A_1 \) means
\[ \frac{w(I)}{|I|} \leq C \text{ ess inf } w(x). \]
So, with this \( C \),
\[ \frac{w(I)}{C |I| w(x)} \leq 1 \quad \text{a.e. on } I. \]
Thus
\[ \psi \left( \frac{\Phi(\lambda) w(I)}{C \lambda |I| w(x)} \right) \leq \psi(\Phi(\lambda)) \leq \lambda \]
and (10) follows, so long as \( C \) is taken to be \( \geq 1 \).

Conversely, if \( w \not\in A_1 \), it will suffice to construct an \( N \)-function \( \Psi \) for which (10) fails, for the \( \lambda \) with \( \Phi(\lambda) = 1 \). In other words, we will construct such a \( \Psi \) for which
\[ \int_I \psi \left( \frac{w(I)}{C |I| w(x)} \right) w(x) \, dx \leq w(I) \]
must fail for every choice of $C$. Now $w \not\in A_1$ means there are cubes $I_k$ on which

$$\frac{1}{|I_k|} \int_{I_k} w \geq 2k^2 \text{ess inf } w.$$ 

In particular, if

$$E_k = \left\{ x \in I_k : w(x) < \frac{1}{k^2 |I_k|} \int_{I_k} w \right\}$$

then

$$t_k = \frac{|E_k|}{|I_k|} > 0.$$ 

Set

$$a(k) = \max_{1 \leq j \leq k} \frac{j}{f_j}.$$ 

This is a nondecreasing sequence which tends to infinity. So we can choose a subsequence $\{k_j\}$ on which $a(k_j)$ is strictly increasing. Define $\psi(t)$ to be continuous, strictly increasing, $\psi(0) = 0$, and $\psi(k_j) = a(k_j)$, and let

$$\psi(x) = \int_0^x \psi(t) \, dt.$$ 

Were (11) to hold for $\Psi$, then, by (3), we must have

$$\frac{1}{|I|} \int_{I} \psi\left( \frac{w(I)}{2C|I|w} \right) \leq 2C.$$ 

But

$$\frac{1}{|I_k|} \int_{I_k} \psi\left( \frac{w(I_k)}{k_j |I_k|w} \right) \geq \frac{1}{|I_k|} \int_{E_{k_j}} \psi\left( \frac{w(I)}{k_j |I_k|w} \right) \geq \psi(k_j) \frac{|E_{k_j}|}{|I_k|} = a(k_j)t_{k_j} \geq k_j$$

so that (12) fails.

For the union, clearly $A_\infty \subset \bigcup W_\Phi$. Conversely, if $w \in W_\Phi$, we will show that $w$ must satisfy the fundamental inequality of Coifman and C. Fefferman [4], that there exists a $\beta > 0$ such that, for each cube $I$,

$$\left[ \left\{ x : w(x) > \frac{\beta \cdot w(I)}{|I|} \right\} \right] \geq \frac{1}{2} |I|.$$ 

(13) implies a reverse Hölder inequality, and so $A_\infty$.

Let

$$E_{t,I} = \left\{ x \in I : \frac{w(I)}{|I|w(x)} \geq t \right\} \quad \text{and} \quad F_{t,I} = I \sim E_{t,I}.$$ 

By (12), with $\Psi$ the complement and $\psi = \psi'$,

$$2C \geq \frac{1}{|I|} \int_{E_{t,I}} \psi\left( \frac{w(I)}{2C/I|w} \right) \geq \psi\left( \frac{t}{2C} \right) \frac{|E_{t,I}|}{|I|}$$

and so

$$\frac{|E_{t,I}|}{|I|} \leq \frac{2C}{\psi(t/(2C))} \to 0 \quad \text{as } t \to \infty.$$ 

So we can choose $t$ so large that $|E_{t,I}|/|I| \leq 1/2$. Obviously, then, $|F_{t,I}| \geq \frac{1}{2} |I|$, and so (13) holds with $\beta = 1/t$.

4. Strong-type integral inequalities. Pick has shown that a weighted $L_\infty$ integral inequality for the maximal operator $M$ forces the complement $\Psi$ to be in local $A_2$ [11]. A look at his proof shows that this results not from the integral inequality, but from the weaker norm inequality. Since integral inequalities are actually equivalent to a uniform family of norm inequalities, it is not hard to modify Pick's proof to obtain $A_2$.

**Theorem 7.** Let $\Phi$ be an $N$-function, and let $t$, $u$, $v$, and $w$ be weights on $\mathbb{R}^n$. Then, in order for

$$\int \Phi(Mf) \, t(x) \, dx \leq \int \Phi(Cu(x)|f(x)|v(x)) \, dx$$

to hold for all $f$, we must have $\Phi \in A_2$.

**Proof.** Since these are weights, there exists a constant $K > 0$ such that the set

$$E = \left\{ x \in \mathbb{R}^n : K^{-1} \leq t(x), u(x), v(x), w(x) \leq K \right\}$$

has positive measure. Let $x$ be a point of density of $E$. Then there exists an $r_0 > 0$ such that

$$B(r, x) \cap E \geq \frac{1}{2} B(r, x)$$

for all $0 < r \leq r_0$, where $B(r, x) = \{ y : |x - y| < r \}$. Let

$$B_m = B(2^{-m}/r_0, x), \quad m = 0, 1, 2, \ldots,$$

and let $f_m = \chi_{B_m}$. Pick's construction showed that

$$\int_{(B_m \cap B_m) \sim B_m} \frac{dy}{|x - y|^n} \geq Cm$$

and

$$Mf_m(y) \geq C |E \cap B_m||x - y|^{-n}$$

when $y \not\in B_m$. From (14) and Proposition 4, we have

$$\|Mf_m\|_{\psi, \infty} \leq C \|u f_m\|_{\Phi, \psi}$$
for all \( \varepsilon > 0 \) and \( m \), with \( C \), of course, independent of \( \varepsilon \) and \( m \). We claim that
\[
\|u_{f_m}\|_{\psi, \varepsilon} \leq \frac{K}{\psi^{-1}[1/(\varepsilon v(E \cap B_m))]}
\]

(18)

For, if this number is \( \lambda \), then
\[
\int \psi\left(\frac{u_{f_m}}{\lambda}\right) \varepsilon v = \int_{E \cap B_m} \psi\left(\frac{K}{\lambda}\right) \varepsilon v(E \cap B_m) = 1,
\]

proving the claim.

Let \( \Psi \) be the complement of \( \Phi \). Using (6), (18) shows that
\[
\|u_{f_m}\|_{\phi, \varepsilon} \leq K \varepsilon v(E \cap B_m) \Psi^{-1}\left[\frac{1}{\varepsilon v(E \cap B_m)}\right]
\]
\[
\leq K^2 \varepsilon |E \cap B_m| \Psi^{-1}\left[\frac{K}{|E \cap B_m|}\right]
\]
\[
\leq K^2 \varepsilon |E \cap B_m| \Psi^{-1}\left[\frac{2K}{|B_m|}\right]
\]
\[
= K^2 \varepsilon |E \cap B_m| \Psi^{-1}\left[\frac{C^2 m^2 K}{\varepsilon |B_0|}\right]
\]

Let
\[
g = \Psi^{-1}\left[\frac{1}{v(B_0 \cap E)}\right] v_{B_0 \cap E}.
\]

Then \( \|g\|_{\phi, \varepsilon} \leq 1 \), and using Hölder's inequality, we have
\[
\|w M f_m\|_{\phi, \varepsilon} \geq C \int w(M f_m) \, dt
\]
\[
\geq C K^{-2} \varepsilon \Psi^{-1}\left[\frac{1}{v(B_0 \cap E)}\right] |E \cap B_m| \int_{B_0 \cap E} \frac{dy}{|x - y|^n}
\]
by (16). Hence, by (15),
\[
\|w M f_m\|_{\phi, \varepsilon} \geq C \varepsilon \Psi^{-1}\left[\frac{1}{v(B_0 \cap E)}\right] |E \cap B_m|
\]
\[
\geq C \varepsilon K^{-2} \Psi^{-1}\left[\frac{1}{v(B_0 \cap E)}\right]
\]

Now, fix \( y > 0 \). Choose \( \varepsilon \) so that
\[
\frac{1}{\varepsilon K |B_0|} = \psi(y).
\]

We have shown that
\[
\|w M f_m\|_{\phi, \varepsilon} \geq C \varepsilon K^{-2} \varepsilon |E \cap B_m| y
\]
and
\[
\|u_{f_m}\|_{\psi, \varepsilon} \leq K^2 \varepsilon |E \cap B_m| \Psi^{-1}(C^2 m^2 K^2 \psi(y)).
\]

By (17), we have
\[
m_y \leq C K^4 \Psi^{-1}(C^2 m^2 K^2 \psi(y)).
\]

Now fix \( m \), chosen so that \( m \geq 2CK^4 \). For that \( m \), which does not depend on \( y \), we have
\[
2y \leq \Psi^{-1}(C^2 m^2 K^2 \psi(y))
\]
or
\[
\Psi(2y) \leq C^2 m^2 K^2 \psi(y),
\]
proving the theorem.

We now have most of the ingredients for Theorem 1. To finish off, we will need a series of lemmas, modeled on the Kerman–Torčhinsky proof. Henceforth, let \( I(w) \) denote the average of \( w \) over the cube \( I \),
\[
I(w) = \frac{1}{|I|} \int_I w(x) \, dx.
\]

**Lemma 8.** If \( w \in W_\Phi \), then there exists a constant \( C \) such that
\[
I(w) \frac{1}{C} I\left(\psi\left(\frac{w}{\varepsilon}\right)\right) \leq C \varepsilon
\]
holds for every cube \( I \) and every \( \varepsilon > 0 \).

**Proof.** Standard arguments using (3) show that (2), and hence \( w \in W_\Phi \), is equivalent to
\[
\frac{1}{|I|} \int_I \psi\left[\frac{\Phi(\lambda I(w))}{\lambda C \psi(w)}\right] \, dx \leq C \lambda
\]
for each cube \( I \) and \( \lambda > 0 \).

Given \( \varepsilon > 0 \), since \( \Phi(\lambda)/\lambda \) has full range, we can choose \( \lambda \) so that \( (\Phi(\lambda)/(C \lambda)) I(w) = \varepsilon \). Then (20) says that \( I(\psi(\varepsilon/w)) \leq C \lambda \) or
\[
\phi\left[\frac{1}{2C} I\left(\psi\left(\frac{\varepsilon}{w}\right)\right)\right] \leq \phi\left(\frac{\lambda}{2}\right) \leq 2 \frac{\Phi(\lambda)}{\lambda} = 2 \frac{\Phi(\lambda)}{I(w)},
\]
which is (19) with \( 2C \) replacing \( C \).

**Lemma 9.** Let \( w \in W_\Phi \) with \( \Phi \in \Delta_2 \). Then \( v = \psi(w)^r \) satisfies a reverse Hölder inequality
\[
I(v^r)^{1/r} \leq C I(v)
\]
for all cubes \( I \), with \( C > 0 \) and \( r > 1 \) independent of \( \varepsilon \).
Proof. Set $E_\alpha = \{ x \in I : v(x) \leq \alpha I(v) \}$. We must show that there exists an $\alpha > 0$, independent of $\varepsilon$, with $|E_\alpha| \leq \frac{3}{2} |I|$. By (19),

$$\frac{C}{\phi((1/C)I(v))} \geq \frac{1}{|I|} \int_{E_\alpha} \frac{w(x)}{\varepsilon} \, dx.$$ 

On $E_\alpha$, $\psi(\varepsilon/w) \leq \alpha I(v)$ and so

$$\Psi\left(\frac{\varepsilon}{w}\right) \leq \frac{\varepsilon}{w} \alpha I(v).$$

From (8), we get

$$\frac{\varepsilon}{w} \Phi^{-1}\left(\frac{\varepsilon}{w} \alpha I(v)\right) \leq \frac{2\varepsilon}{w} \alpha I(v).$$

Thus

$$\frac{\varepsilon}{w} \alpha I(v) \leq \Phi(2\alpha I(v)) \leq 2\alpha I(v) \Phi(2\alpha I(v)),$$

or

$$\frac{\varepsilon}{w} \leq 2\phi(2\alpha I(v)).$$

So

$$\frac{2C}{\phi((1/C)I(v))} \geq \frac{|E_\alpha|}{|I|} \frac{1}{\phi(2\alpha I(v))}.$$ 

By iterating Proposition 3(b), we can find a $t > 0$ sufficiently small that $\phi(tx) \leq \phi(x)/(4C)$. Choose $\alpha = t/(2C)$. Then

$$\frac{|E_\alpha|}{|I|} \leq 2C \frac{\phi((t/C)I(v))}{\phi((1/C)I(v))} \leq \frac{1}{2}.$$

**Lemma 10.** Let $w \in B_\delta$ and $\Phi \in \Delta^\delta_2$. Then there exists an $r > 1$ such that $w \in W_{\Phi^r}$, where $\Phi^r$ is the $N$-function with derivative

$$\Phi^r(x) = \phi(x^{1/r}).$$

Proof. Let $\phi^r(x) = \phi(x^{1/r})$ and $\Phi^r(x) = x\psi(x)^r$. Choose $r > 1$ so that (21) holds. We claim that there exists a constant $C$ such that, for every $\lambda > 0$,

$$\frac{1}{|I|} \int_I \psi\left(\frac{\Phi^r(\lambda)}{C\lambda w} I(w)\right)^r \leq C\lambda.$$

For this, let $C/2$ be the constant in (20) and let $K$ be the reverse Hölder constant in (21). Then, with $\varepsilon = (\Phi^r(\lambda)/(C\lambda))I(w)$, we have

$$\frac{1}{|I|} \int_I \psi\left(\frac{\Phi^r(\lambda)}{C\lambda w} I(w)\right)^r \leq K^r \frac{1}{|I|} \int_I \psi\left(\frac{\Phi^r(\lambda)}{C\lambda w} I(w)\right)^r \leq K^r \frac{1}{|I|} \int_I \psi\left(\frac{\Phi^r(\lambda)}{C\lambda w} I(w)\right)^r \leq (KC)^r \lambda$$

by (20), proving the claim with $C$ replaced by $(KC)^r$.

Hence,

$$\int I \psi\left(\frac{\Phi^r(\lambda)}{C\lambda w} I(w)\right) w \leq (KC)^r \lambda,$$

for all cubes $I$. This is (2), and would give $w \in W_{\Phi^r}$ were we lucky enough to know that $\Psi^r$ is the complement of $\Phi^r$. That, however, need not be the case. Still, the proof of the sufficiency side of Theorem 5 uses only Young’s inequality, so it would suffice to show that $\Phi_{\varepsilon}$ and $\Psi_{\varepsilon}$ satisfy Young’s inequality. In fact, the proof would carry through verbatim if we just had

$$xy \leq \Phi^r(2x^r) + \Phi_{\varepsilon}(y).$$

We show this. Let $r'$ be the conjugate exponent to $r$, $1/r + 1/r' = 1$. Using Young’s inequality for $\Phi(x)$ and for $x^{r/r'}$, we get

$$xy = x^{1/r'} x^{1/r} y \leq x^{1/r'} (\Phi^r(x^{1/r}) + \Phi_{\varepsilon}(y)) \leq x \Phi^r(x^{1/r}) + x^{1/r'} \psi(y) \leq \Phi_{\varepsilon}(2x) + y \left[ \frac{x}{r'} + \psi(y)^r \right]$$

and so

$$xy \leq r\Phi_{\varepsilon}(2x) + \Phi_{\varepsilon}(y)$$

and (22) follows from the convexity of $\Phi_{\varepsilon}$. This completes the proof of the lemma.

Set

$$k(s) = \sup_{t > 0} \frac{\Phi^r(st)}{\Phi^r(t)}$$

and

$$\alpha = \lim_{r \to 0^+} \frac{\log k(s)}{\log s}.$$

$\alpha$ is the Orlicz–Mazimanda lower index for $\Phi$.

Notice, for $\Phi \in \Delta^\delta_2$, that by Proposition 3(b), with that $\delta$,

$$\Phi\left(\frac{\delta^n}{2}\right) \leq \left(\frac{\delta}{2}\right)^n \Phi(t)$$
and so, if \( \delta^{n+1/2} < s \leq \delta^n/2 \),

\[
k(s) \leq \left( \frac{\delta}{2} \right)^n,
\]

but then

\[
\frac{\log k(s)}{\log s} \geq \frac{n \log(2/\delta)}{\log 2 + (n + 1) \log(1/\delta)}
\]

and so, \( \Phi \in \Delta^0_2 \) forces \( \alpha > 1 \).

**Lemma 11.** Let \( \alpha \) be given by (23) with \( \Phi \in \Delta^0_2 \), and let \( q < \alpha \). Then there exists a constant \( C \) depending on \( q \) for which

\[
\Phi(st) \leq Cs^q \Phi(t)
\]

for all \( t > 0 \) and \( 0 < s < 1 \).

**Proof.** There exists an \( s_0 \) such that

\[
\frac{\log k(s)}{\log s} \geq q
\]

whenever \( 0 < s \leq s_0 \). Thus \( k(s) \leq s^q \) for such \( s \), and the lemma holds with

\[
C = s_0^{-q}.
\]

The heart of the argument is contained in our last lemma:

**Lemma 12.** Let \( w \in W_\Phi \) and \( \Phi \in \Delta^0_2 \). Then there exists a \( p < \alpha \), with \( \alpha \) given by (23), such that \( w \in A_p^\Phi \).

**Proof.** As in [7], it will suffice to prove that \( w \) is of weak-restricted type \((p,p)\) for some \( p < \alpha \), and that requires showing

\[
\frac{|E|}{|I|} \leq C \frac{w(E)^{1/p}}{w(I)}
\]

for all cubes \( I \) and measurable sets \( E \subset I \). Clearly (24) will hold if it holds whenever \( w(E)/w(I) \leq \varepsilon_0 \). Fix \( E \) and put

\[
\varepsilon = \frac{w(E)}{w(I)}.
\]

Let \( \varepsilon > 1 \). Then there exists an \( s_0 \) such that

\[
\frac{\log k(s)}{\log s} \geq \varepsilon \alpha
\]

for all \( 0 < s \leq s_0 \). Hence \( 1/k(s) \leq s^{-\varepsilon \alpha} \), or

\[
\Phi^{\varepsilon \alpha}(st) \leq \sup_t \Phi^\alpha(t).
\]

and we can find a \( t \) with \( s^{\varepsilon \alpha} \leq 2\Phi(st)/\Phi(t) \). Using (3), we have

\[
\phi^{\varepsilon \alpha - 1} \left( \frac{t}{2} \right) \leq 4 \Phi(st).
\]

Now

\[
\phi(st) = \phi^\alpha((st)^\alpha) \leq \Phi^\varepsilon(st^\varepsilon) \left( \frac{t}{2} \right)^\varepsilon
\]

and so

\[
\phi^{\varepsilon \alpha - 1 + r} \Phi^\varepsilon \left( \frac{t}{2} \right)^r \leq 2^{2-r} \Phi^\varepsilon \left( 2^{1+r} \frac{t}{2} \right)^r.
\]

Thus, there exists a \( y \) with

\[
s^{\varepsilon \alpha - 1 + r} \Phi^\varepsilon(y) \leq 2^{2-r} \Phi^\varepsilon \left[ 2^{1+r} \frac{t}{2} \right]^r.
\]

Choose \( s \) so that \( s^{\varepsilon \alpha - 1 + r} = \varepsilon \) and set

\[
p = \frac{\varepsilon \alpha - 1 + r}{r}.
\]

Notice that \( p < \alpha \) if and only if \( \varepsilon \alpha - 1 < r \alpha - 1 \). Since this holds when \( \varepsilon = 1 \), we can choose \( \varepsilon = \varepsilon(r) \) so that \( p < \alpha \). Since \( s^{\varepsilon} = e^{1/p} \), we have

\[
e \Phi^\varepsilon(y) \leq \Phi^\varepsilon \left( 2^{1+r} e^{1/p} \right).
\]

Let \( \lambda = 2^{1+r} e^{1/p} y \) and let \( \Phi^\varepsilon \) be the complement of \( \Phi^\varepsilon \). Then

\[
\frac{\text{Vol}(E) \Phi^\varepsilon(\lambda)}{\text{Vol}(I)}
\]

\[
= \int \frac{\text{Vol}(E) \Phi^\varepsilon(\lambda)}{\text{Vol}(I)} \cdot \psi_E(z) w(z) dx
\]

\[
\leq \Phi^\varepsilon(y) \text{Vol}(E) + \int \frac{\text{Vol}(E) \Phi^\varepsilon(\lambda)}{\text{Vol}(I)} w \text{ by Young's inequality}
\]

\[
\leq \Phi^\varepsilon(y) \text{Vol}(E) + \Phi^\varepsilon(\lambda) \text{Vol}(I),
\]

since \( w \in W_{\Phi^\varepsilon} \). Dividing by \( \text{Vol}(I) \) gives

\[
\frac{|E|}{|I|} \Phi^\varepsilon(\lambda) / \text{Vol}(I) \leq \varepsilon \Phi^\varepsilon(y) + \Phi^\varepsilon(\lambda) \leq (1 + 2^{2-r}) \Phi^\varepsilon(\lambda)
\]

by (25), and so

\[
\frac{|E|}{|I|} \leq C 2^{1+r} (1 + 2^{2-r}) e^{1/p},
\]

proving the lemma.
Now we prove Theorem 1. All that is left to show is the implication (c) $\Rightarrow$ (a). Let $\alpha$ be the index given by (23). Fix some $p < \alpha$ for which $w \in A_p$, as in the last lemma, and let $p < q < \alpha$. Since $w \in A_p$,

$$w([x : Mf(x) > \lambda]) \leq \lambda^{-p} \int_{[x : 2|f(x)| > \lambda]} |f(x)|^p w(x) \, dx,$$

for all $\lambda > 0$. Hence,

$$\int \Phi(Mf) w = \int_0^\infty \Phi(\lambda) w([x : Mf(x) > \lambda]) \, d\lambda \leq C \int_0^\infty \Phi(\lambda) \lambda^{-p} \int_{[x : 2|f(x)| > \lambda]} |f(x)|^p w(x) \, dx \, d\lambda = C \int_0^{2|f(x)|} |f(x)|^p w(x) \int_0^{2|f(x)|} \Phi(\lambda) \lambda^{-p} \, d\lambda \, dx \leq C \int_0^{2|f(x)|} |f(x)|^p w(x) \int_0^{2|f(x)|} \Phi(2\lambda) \lambda^{-p-1} \, d\lambda \, dx.$$

Now, by Lemma 11,

$$\Phi(2\lambda) = \Phi\left(\frac{\lambda}{2|f(x)|} |4f(x)| \right) \leq C \left(\frac{\lambda}{2|f(x)|} \right)^q \Phi(4|f(x)|),$$

and so

$$\int \Phi(Mf) w \leq C \int |f(x)|^{p-q} \Phi(4|f(x)|) w(x) \int_0^{2|f(x)|} \lambda^{q-p-1} \, d\lambda \, dx = C \int \Phi(4|f(x)|) w(x) \, dx \leq \int \Phi(4C|f(x)|) w(x) \, dx.$$

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