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References


On the characterization of Hardy–Besov spaces on the dyadic group and its applications

by

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Dedicated to Professor C. Watari
on the occasion of his sixtieth birthday

Abstract. C. Watari [12] obtained a simple characterization of Lipschitz classes $\text{Lip}^p_{\alpha}(W)$ ($1 \leq p \leq \infty$, $\alpha > 0$) on the dyadic group using the $L^p$-modulus of continuity and the best approximation by Walsh polynomials. Onoe and Weyi [4] characterized homogeneous Besov spaces $B^p_{\alpha,q}$ on locally compact Vilenkin groups, but there are still some gaps to be filled up. Our purpose is to give the characterization of Besov spaces $B^p_{\alpha,q}$ by oscillations, atoms and others on the dyadic groups. As applications, we show a strong capacity inequality of the type of the Marčya inequality, a weak type estimate for maximal Cesàro means and a sufficient condition of absolute convergence of Walsh–Fourier series.

0. Introduction and notation. The dyadic group, $2^a$, is viewed classically as the set of all sequences of 0's and 1's with addition (mod 2) defined pointwise, and is supplied with the usual product topology. Our results are stated in the situation that $2^a$ is the additive subgroup of the ring of integers in the 2-series field $K$ of formal Laurent series in one variable over $GF(2)$ (see [9]). Such a field $K$ is a particular instance of a local field; that is, a locally compact, totally disconnected, non-discrete, complete field. The results of this paper have extensions to any local field.

We need to set some basic notation. It is taken from [9] where the fundamentals are detailed. For the additive subgroup $K^+$ of the 2-series field $K$, we may choose a Haar measure $dx$. Let $d(\alpha x) = |\alpha| dx$ and call $|\alpha|$ the valuation of $\alpha$. Let $|0| = 0$. The mapping $x \rightarrow |x|$ has the following properties: $|x| = 0 \Leftrightarrow x = 0$, $|xy| = |x| \cdot |y|$, $|x + y| \leq \max(|x|, |y|)$.

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(If \(|x| \neq |y|\), then \(|x + y| = \max(|x|, |y|)\).) Let \(P^0 = \{x \in K : |x| \leq 1\}\) and \(P^1 = \{x \in K : |x| < 1\}\). \(K\) is totally disconnected, hence the value is discrete valued. Thus there is an element of \(P^1\) of maximum value. Let \(p\) be a fixed element of maximum value. Then an element \(x \in K\) is represented as

\[
x = \sum_{k=-\infty}^{\infty} a_k p^k, \quad a_k \in GF(2),
\]

which can contain a finite number of terms with negative powers of \(p\). The addition and multiplication of two power series are defined in a natural fashion. The ring of integers \(P^0 = \{x = \sum_{k=-\infty}^{\infty} a_k p^k\}\) coincides with the dyadic group \(2^\mathbb{Z}\) as an additive group.

For \(E\) a measurable subset of \(K\), let \(|E| = \int_K \Phi_E(x) \, dx\), where \(\Phi_E\) is the characteristic function of \(E\) and \(dx\) is Haar measure normalized so \(|P^0| = 1\). Then \(|P^1| = |\Phi| = 2^{-1}\). Let \(P^k = \{x \in K : |x| \leq 2^{-k}\}\) and \(\Phi_k\) be its characteristic function.

For \(x = x_0 + \sum_{k=-\infty}^{\infty} a_k p^k\), \(a_k \in GF(2)\), \(x_0 \in P^0\), set

\[
w(x) = \begin{cases} -1, & k = -1, \\ 1, & k < -1, \\ 0, & k = 0. \end{cases}
\]

Then \(w\) is a function on \(K\) that is trivial on \(P^0\), but is non-trivial on \(P^{-1}\). For \(x, y \in K\), let \(w(x, y) = w(y \cdot x)\). \(w\) is constant on cosets of \(P^0\) and if \(y \in P^k\) then \(w_y\) is constant on cosets of \(P^{-k}\).

We assume that all functions are complex-valued and measurable.

If \(f \in L^1(K)\) the Fourier transform of \(f\) is the function \(\hat{f}\) defined by

\[
\hat{f}(y) = \int_K f(u) w_y(u) \, du.
\]

Then we have \(\Phi_k = 2^{-k} \Phi_{-k}\).

The space of test functions, \(S(K)\), is the space of finite linear combinations of functions of the form \(h \ast \Phi_k\), \(h \in K, k \in \mathbb{Z}\). Then \(\phi \in S(K)\) if and only if there are integers \(k, l\) such that \(\phi\) is constant on cosets of \(P^k\) and supported in \(P^l\) (see [9, p. 36, Theorem (3.2)]). The space of distributions, \(S'(K)\), is the space of continuous linear functionals on \(S(K)\).

Let \((u(n))_{n=0}^{\infty}\) be a complete list of distinct coset representatives of \(P^0\) in \(K^+\). Define \(u(0) = 0\), \(u(1) = p^{-1}\) and for \(n = b_0 + b_1 \cdot 2 + b_2 \cdot 2^2 + \cdots + b_s 2^s\) \((b_i = 0 \text{ or } 1, b_1 = 1)\), \(u(n) = u(b_0) + p^{-1} u(b_1) + \cdots + p^{-s} u(b_s)\). Then \((w(u(n))_{n=0}^{\infty}\) is a complete set of characters on \(P^0\). This is the Walsh–Paley system (see [9, p. 85, Proposition (6.4)]).

The Dirichlet kernels are the functions \(D_n(x) = \sum_{k=0}^{n-1} u_k(x), n \geq 1, D_0(x) \equiv 0\). If \(f \in L^1(P^0)\) the Walsh–Fourier coefficients \((c_k)_{k=0}^{\infty} = (\hat{f}(u(k)))_{k=0}^{\infty}\) are given by \(c_k = \int_{P^0} f(x) u_k(x) \, dx\). The Walsh–Fourier series is given by \(f(x) \sim \sum_{k=0}^{\infty} c_k u_k(x)\). The \(n\)th partial sum of the Walsh–Fourier series of \(f\) is denoted by \(S_n f(x)\) and is defined as \(S_n f(x) = \sum_{k=0}^{n-1} c_k u_k(x)\).

If \(f \in L^1(P^0)\) \((n \geq 0)\) then \(S_{2n} f(x) = 2^n \int_{P^0} f(t) \, dt\), as follows from the fact that \(D_{2n} = 2^n \Phi_{-k}\).

\[S = S(P^0)\] is the collection of test functions on \(P^0\). We have \(\phi \in S\) if and only if \(\phi\) is constant on cosets of \(P^n\) for some \(n \geq 0\). If \(\phi \in S\) then \(\phi\) is a "polynomial", that is, \(\phi = \sum_{k=0}^{n-1} \Phi_k u_k(x)\) for some \(n \geq 0\). Let \(S(j)\) be a subset of \(S\) spanned by \(\Phi_k u_k (0 \leq i \leq j)\). \(S^j = S(P^j)\) is the space of distributions on \(P^j\). If \(j \in S'\) then \(f \in S(P^0)\) if \(f = \sum_{k=0}^{\infty} (\hat{u}_k f) u_k(x)\) for some \(n \geq 0\). Let \(S(f)\) be the subset of \(S(P^0)\) on \(u_k(x)\). Then, that is, \(f\) is a "normal Walsh–Fourier series". The Fejér kernels, \(K_n^0(x)\) and \(K_n^0(x) (n \geq 0)\), are the functions

\[
K_n^0(x) = \frac{1}{A_n} \sum_{k=0}^{n-1} A_k u_k(x), \quad n \geq 1.
\]

\[
K_0^0 \equiv 0, \quad A_n^0 = (\beta + 1) (\beta + n) \cdots n! \simeq C(\beta) n^\beta,
\]

and

\[
K_n^0(x) = \frac{1}{A_n} \sum_{i=0}^{s} b_i A_i K_j^0, \quad n = \sum_{i=0}^{s} b_i 2^i, \quad b_i = 0, b_s = 1, b_i = 1 \text{ or } 0.
\]

Let \(\sigma_n^0 f(x) = f \ast K_n^0(x)\) be the Cesàro means of order \(\beta\) of the partial sums of the Walsh–Fourier series of \(f\) whenever \(f \in S(P^0)\).

Let \(\{\Delta_j\}_{j=0}^{\infty}\) be a family of functions on \(P^0\) satisfying \(\Delta_j(x) = 2^j \Phi_j(x) - 2^{j-1} \Phi_j - 1(x)\) for \(j \geq 1\) and \(\Delta_0 = \Phi_0\). Since \(D_0(x) = 2^j \Phi_j(x), \Delta_j(x) = D_{2j}(x) - D_{2j-1}(x)\) for \(j \geq 1\).

The Besov space \(B^p_q (P^0) = B^p_q (0 < p, q \leq \infty \text{ and } -\infty < \alpha < \infty)\) is the collection of all \(f \in S(P^0)\) such that

\[
\|f\|_{B^p_q} = \left( \sum_{j=0}^{\infty} 2^{\alpha q j} \|\Delta_j \ast f\|_p^q \right)^{1/q}
\]

is finite (modification if \(q = \infty\)).

\(C\) denotes a constant, not always the same one.

1. Characterization by means of difference, oscillations and approximations. To give a characterization of \(B^p_q\) by means of difference and oscillations, let

\[
\|S^p_q f\|_{p,q} = \left( \sum_{j=0}^{\infty} 2^{\alpha q j} \int_{P^p} \int_{P^q} \left| f(x + u) - f(x) \right|^p \, du \, dx \right)^{1/p}
\]

and
\[ \|D^\alpha f\|_{p,q} = \left( \sum_{j=0}^\infty 2^{j\alpha q} \|S_{\alpha j}((f - S_{\alpha j}f)(\cdot\gamma))\|_{p,q}^q \right)^{1/q}. \]

Furthermore, to give a characterization by approximation, let
\[ \|E^\alpha f\|_{p,q} = \left( \sum_{j=0}^\infty 2^{j\alpha q} E_p(2^j, f)^q \right)^{1/q}, \]
where
\[ E_p(2^j, f) = \inf \{ \|f - g\|_p : g \in S(j) \}, \quad j = 0, 1, \ldots. \]

For \( q = \infty \) or \( r = \infty \) we have the usual modifications.

The following theorem generalizes and improves [4, Theorem 5(a)] in the inhomogeneous Besov case in the dyadic group setting. The corresponding theorem and its proof for the \( \mathbb{R}^n \) case are somewhat complicated (see [11, p. 101, Theorem; p. 105, Theorem; and p. 81, Theorem]).

**Theorem 1.** Let \( 0 < p, q \leq \infty \) and \( r \geq 1 \). If \( \alpha > \max(1/p - 1, 0) \), then
\[ \|f\|_{b_{p,q}^\alpha} \approx \|S_{\alpha j}f\|_{p,q} + \|f\|_p \approx \|D^\alpha f\|_{p,q} + \|f\|_p \approx \|E^\alpha f\|_{p,q} + \|f\|_p. \]

We shall need the following theorems (cf. [11, p. 22, Theorem, and p. 129, Theorem]).

**Theorem A.** (i) (Nikolskii’s inequality) Let \( 0 < p \leq \infty \). If \( \phi \in S(j) \), then \( \|\phi\|_q \leq 2^{(1/p - 1/\delta)}\|\phi\|_p \).

(ii) (Embedding Theorem) Let \( 0 < p \leq 1 \) and \( \alpha > 1/p - 1 \). Then \( B_{p,q}^\alpha \subset L^1 \cap L^p \). The inclusion maps are continuous.

**Proof.** (i) As \( \phi \in S(j) \), we may write \( \phi(x) = 2^j \int_{\mathbb{R}^n} \phi(y)\Phi_j(x - y) dy \). If \( 0 < p \leq 1 \), then \( \|\phi\|_q \leq 2^j \sup_{x \in \mathbb{R}^n} \|\phi(y)\|_q \int_{\mathbb{R}^n} \Phi_j(x - y) dy \). Taking the supremum with respect to \( x \), we have \( \|\phi\|_1 \leq 2^j/p\|\phi\|_p \). Similarly, if \( p \leq \infty \), then \( \|\phi\|_p \leq \|\phi\|_{1/p} \|\phi\|_p \). If \( \alpha > 1/p - 1 \), then \( \|\phi\|_p \leq 2^{j\alpha q}/p\|\phi\|_p \leq 2^{j\alpha q}/p\|\phi\|_p \).

(ii) By (i), \( \|D_j \ast f\|_1 \leq 2^{j(1/p - 1/\delta)}\|D_j \ast f\|_p \). Hence we have \( B_{p,q}^\alpha \subset B_{1,p}^{(1/p - 1/\delta)} \). On the other hand, \( B_{1,p}^{(1/p - 1/\delta)} \subset B_{11}^0 \) is well known (see [11, p. 47, Proposition 2(ii)]). Therefore \( \|f\|_1 \leq \|f\|_{b_{p,q}^\alpha} \approx \|f\|_{b_{p,q}^\alpha} \).

We shall use the type of reverse H"older inequality due to DeVore and Sharpley [1, p. 26, Theorem 4.3] in the dyadic group setting. This plays a vital role in the proof of Theorem 1. Let \( 0 < \gamma < s \leq 1 \leq r > 0 \). For \( f \in S'(P^0) \) and \( Q = x + P^j \) \((j \geq 0)\) set
\[ f_{\alpha j}(x) = f_j(x) = \sup_{n \geq j} 2^{n\beta} \{ S_n((f - S_{\alpha j}f)(\cdot\gamma))/(x)\}^{1/\gamma}. \]

**Theorem B.** If \( f \in S' \), then there exists a constant \( C \) such that
\[ \|f - S_{\alpha j}f\|_{L^r(Q)} \leq C \|f_{\alpha j}(x)\|_{L^r(Q)}. \]

**Proof.** First, we show
\[ [(f - S_{\alpha j}f)(\cdot\gamma)](t) \leq C \int_t^{2^{-j}} f_j^\gamma(u)\nu^{\alpha - 1} du + C t^{\alpha} f_j^\gamma(t), \]

\( 0 < t < 2^{-j-1} \), where \( \Phi_Q \) and \( \gamma \) denote the characteristic function of \( Q \) and the decreasing rearrangement of \( f \) respectively. Let \( E = \{ u \in Q : f_j(u) > f_j^\gamma(t) \} \), so that \( |E| \leq t \). Let \( i \) be the integer with \( 2^{-i+1} \leq t < 2^{-i+1} \) \((i \geq 1)\). Since \( S_{\alpha j}f \) and \( S_{\alpha j+1}f \) are constant on cosets of \( p^{k+1} \), we have, on \( x + p^{k+1} \),
\[ |S_{\alpha j}f(u) - S_{\alpha j+1}f(u)|^2 \leq |S_{\alpha j}f - S_{\alpha j+1}f|^2(u) \leq 2S_{\alpha j+1}(|S_{\alpha j}f - f|^\gamma(u)) + 2S_{\alpha j+1}(|S_{\alpha j+1}f - f|^\gamma(u)) \leq 2S_{\alpha j+1}(|S_{\alpha j}f - f|^\gamma(u)) + 2S_{\alpha j+1}(|S_{\alpha j+1}f - f|^\gamma(u)) \leq 2^{-1+\delta \nu} \inf_{u \in E(p^{k+1})} (f_j(u)) + 2^{-1+\delta \nu} \inf_{u \in E(p^{k+1})} (f_j(u)) \leq C 2^{-1+\delta \nu} \inf_{u \in E(p^{k+1})} (f_j(u)). \]

Hence, using the monotone property of \( f_j^\gamma \) with respect to \( \gamma \) and the inequality \( \inf_{u \in E(p^{k+1})} (f_j(u)) \leq f_j^\gamma(2^{-[1+k+1]}) \) for \( x \in Q \setminus E \), we have,
\[ |S_{\alpha j}f(x) - S_{\alpha j+1}f(x)| \leq \sum_{k=j}^{\infty} \|S_{\alpha k}f - S_{\alpha k+1}f\|_{L^\infty(Q(p^{k+1}))} \leq C \sum_{k=j}^{\infty} 2^{-1+\delta \nu} \inf_{u \in E(p^{k+1})} (f_j(u)) \leq C \sum_{k=j}^{\infty} \int_{Q(p^{k+1})} f_j^\gamma(v)\nu^{\alpha - 1} dv \leq C \sum_{k=j}^{\infty} \int_{Q(p^{k+1})} f_j^\gamma(v)\nu^{\alpha - 1} dv \leq C \int_{Q(p^{k+1})} f_j^\gamma(v)\nu^{\alpha - 1} dv. \]

On the other hand, since \( S_{\alpha j}f \) is \( f \) a.e. as \( j \to \infty \), for \( x \in Q \setminus E \),
\[ |S_{\alpha j+1}^\gamma f(x) - f(x)| \leq \sum_{k=j+1}^{\infty} |S_{\alpha j}f(x) - S_{\alpha j+1}f(x)| \leq C \sum_{k=j+1}^{\infty} 2^{-1+\delta \nu} f_j(x) \leq C 2^{-1} 2^{2^{-1+\delta \nu} f_j^\gamma(t) \leq C t^{\alpha} f_j^\gamma(t). \]

Therefore, we have
\[ |S_{\alpha j}f(x) - f(x)| \leq C \int_t^{2^{-1+\delta \nu}} f_j^\gamma(u)\nu^{\alpha - 1} du + C t^{\alpha} f_j^\gamma(t) \quad \text{for} \quad x \in Q \setminus E. \]
Since $|E| \leq t$, by the property of decreasing rearrangement we have the desired inequality.

Next, taking the $L^r$-norm ($r \geq 1$) over $[0, 2^{-(j+1)}]$ and using Hardy’s inequality, we have
\[
\left( \int_0 ^{2^{-(j+1)}} \left| \int_0 ^t f_j(x)^{\beta-1} \, dx \right|^r \, dt \right)^{1/r} \leq C \left( \int_0 ^{2^{-(j+1)}} \int_0 ^t f_j(v)^{\beta-1} \, dv \, dt \right)^{1/r} + C \left( \int_0 ^{2^{-(j+1)}} |f_j(t)|^r \, dt \right)^{1/r} \leq C \left( \int_0 ^{2^{-(j+1)}} |f_j(t)|^r \, dt \right)^{1/r} \leq C \|f_j\|_{L^r(Q)} = C \|f\|_{L^r(Q)} \leq C \|f\|_{L^r(Q)}.
\]

Then
\[
\|S_\alpha f\|_{p,q} \leq \left( \sum_{j=0} ^{\infty} 2^{aq_j} \left[ \int_0 ^{2^j} \left| \int_0 ^t (f - S_\alpha f)^{\gamma} \, dx \right|^q \, dt \right]^{p/q} \right)^{1/p}. \]

Hence, by the fact that $\mathcal{E} \subset \ell^1 (0 < p \leq 1)$, Minkowski’s inequality and then H"older’s inequality, we obtain
\[
\left( \sum_{j=0} ^{\infty} 2^{aq_j} \|S_\alpha f - f\|_{p,q} \right)^{p/q} \leq C \left( \sum_{j=0} ^{\infty} 2^{aq_j} \left( \sum_{k=0} ^{\infty} 2^{qk} \|S_\alpha^{k+1} f - S_\alpha^{k+1} f\|_{p}\right)^{q/p} \right)^{1/q}\]
\[
\leq C \left( \sum_{k=0} ^{\infty} 2^{aq_k} \left( \sum_{k=0} ^{\infty} 2^{qk} \|S_\alpha^{k+1} f - S_\alpha^{k+1} f\|_{p}\right)^{q/p} \right)^{1/q} \leq C \left( \sum_{k=0} ^{\infty} 2^{aq_k} \left( \sum_{j=0} ^{\infty} 2^{qk} \|S_\alpha^{j+1} f - S_\alpha^{j+1} f\|_{p}\right)^{q/p} \right)^{1/q} \leq C \left( \sum_{k=0} ^{\infty} 2^{aq_k} \left( \sum_{j=0} ^{\infty} 2^{qk} \|S_\alpha^{j+1} f - S_\alpha^{j+1} f\|_{p}\right)^{q/p} \right)^{1/q} \leq C \left( \sum_{k=0} ^{\infty} 2^{aq_k} \left( \sum_{j=0} ^{\infty} 2^{qk} \|S_\alpha^{j+1} f - S_\alpha^{j+1} f\|_{p}\right)^{q/p} \right)^{1/q} \leq C \left( \sum_{k=0} ^{\infty} 2^{aq_k} \left( \sum_{j=0} ^{\infty} 2^{qk} \|S_\alpha^{j+1} f - S_\alpha^{j+1} f\|_{p}\right)^{q/p} \right)^{1/q}
\]
which proves the desired inequality.
\[ \|D^\alpha f\|_{p-q}^q = \sum_{j=0}^{\infty} 2^{ajq} \left\{ \int_{\mathbb{R}^d_{2^j}} \|f - S_{2^j} f\|_{p-2^j}^p dx \right\}^{q/p} \]
\[ \leq C \sum_{j=0}^{\infty} 2^{ajq} \left\{ \int_{\mathbb{R}^d_{2^j}} \|f\|_{L^p(2^j \mathbb{R}^d)}^p dx \right\}^{q/p} \]
\[ = C \sum_{j=0}^{\infty} 2^{ajq} \left\{ \int_{\mathbb{R}^d_{2^j}} \left( \sup_{n \geq j} 2^n \beta S(n, \gamma)(t)^s \right) dt \right\}^{p/s} \]

where \( S(n, \gamma)(t) = \{ S(n, \gamma)(t) \}_{n \geq j} \). Replacing \( \sup_{n \geq j} \) with \( \sum_{n=j}^{\infty} \) and using the fact that \( \ell^s \subset \ell^1 \) \((0 < s < 1)\), we have
\[ 2^{aj} \int_{\mathbb{R}^d_{2^j}} \left( \sup_{n \geq j} 2^n \beta S(n, \gamma)(t)^s \right) dt \]
\[ = \int_{\mathbb{R}^d_{2^j}} \left( \sum_{n=j}^{\infty} 2^n \beta S(n, \gamma)(t)^s \right) dt \]
\[ \leq \int_{\mathbb{R}^d_{2^j}} \left( \sum_{n=j}^{\infty} 2^n \beta S(n, \gamma)(t)^s \right) dt \]
\[ \leq \int_{\mathbb{R}^d_{2^j}} \left( \sum_{n=j}^{\infty} 2^n \beta S(n, \gamma)(t)^s \right) dt ^{p/s} \]

Then, by Minkowski's inequality twice,
\[ \|D^\alpha f\|_{p-q}^q \leq C \sum_{j=0}^{\infty} 2^{aj} \left\{ \int_{\mathbb{R}^d} \left( \sum_{n=j}^{\infty} 2^n \beta S(n, \gamma)(t)^s \right) dt \right\}^{p/s} \]
\[ \leq C \sum_{j=0}^{\infty} 2^{aj} \left[ \int_{\mathbb{R}^d} \left( \sum_{n=j}^{\infty} 2^n \beta S(n, \gamma)(t)^s \right) ^{p/s} \right] \]
\[ = C \sum_{j=0}^{\infty} 2^{aj} \left[ \int_{\mathbb{R}^d} \left( \sum_{n=j}^{\infty} 2^n \beta S(n, \gamma)(t)^s \right)^{p/s} \right] \]
\[ \leq C \sum_{j=0}^{\infty} 2^{aj} \left[ \int_{\mathbb{R}^d} \left( \sum_{n=j}^{\infty} 2^n \beta S(n, \gamma)(t)^s \right)^{p/s} \right] \]

Hence, by the mean convergence theorem for partial sums \([13, \text{Theorem } 2, \text{ or } 5, \text{ p. 103, Corollary } 6]\) we obtain finally
\[ \|D^\alpha f\|_{p-q}^q \leq C \left( \sum_{m=0}^{\infty} \left[ \int_{\mathbb{R}^d} \left( \sum_{k=m}^{\infty} 2^n \beta S(n, \gamma)(t)^s \right)^{p/s} \right]^{q/p} \right) \]
\[ \leq C \left( \sum_{m=0}^{\infty} \left[ \int_{\mathbb{R}^d} \left( \sum_{k=m}^{\infty} 2^n \beta S(n, \gamma)(t)^s \right)^{p/s} \right]^{q/p} \right) \]

In order to get the second inequality, we write
\[ \|D^\alpha f\|_{p-q}^q = \sum_{j=0}^{\infty} 2^{aj} \left\{ \int_{\mathbb{R}^d} \left( \sum_{n=j}^{\infty} 2^n \beta S(n, \gamma)(t)^s \right) ^{p/s} \right\} \]
\[ \leq \sum_{j=0}^{\infty} 2^{aj} \left\{ \int_{\mathbb{R}^d} \left( \sum_{n=j}^{\infty} 2^n \beta S(n, \gamma)(t)^s \right) ^{p/s} \right\} \]

By the mean convergence theorem, the fact that \( \ell^s \subset \ell^1 \) \((p < 1)\) and Hölder's inequality, for \( 0 < \epsilon < \alpha \), \( \|D^\alpha f\|_{p-q}^q \) is majorized by
\[ \sum_{j=0}^{\infty} 2^{aj} \left\{ \int_{\mathbb{R}^d} \left( \sum_{n=j}^{\infty} \Delta_{k+1} f \right) ^{p/s} \right\} \]
\[ \leq \sum_{j=0}^{\infty} 2^{aj} \left\{ \int_{\mathbb{R}^d} \left( \sum_{n=j}^{\infty} \Delta_{k+1} f \right) ^{p/s} \right\} \]

Therefore we have
\[ \|D^\alpha f\|_{p-q}^q \leq C \|f\|_{B^\alpha_{p,s}} \]

The third step is to prove (1.3).
This proof is based on [11, p. 81, Theorem]. Since there exists a step function of best approximation for any $L^p$ function, we can choose a sequence $g_j \in S(\mathcal{g})$ such that $\|f - g_j\|_p \leq 2E_p(2^j, f), j = 0, 1, \ldots$. Hence we have $\|\Delta_j \ast f\|_p \leq \sum_{k=0}^{\infty} \|\Delta_j \ast (g_k - g_{k-1})\|_p^2$. On the other hand, by Theorem $\Lambda(i)$ or [11, p. 26, Theorem], we see that
\[
\|\Delta_j \ast (g_{k-1} - g_k)\|_p^2 \leq C P^{-k} \|\Delta_j \|_p^2 \|g_{k-1} - g_k\|_p^2 = C 2^{(k-j)(1-p)\|\Delta_j \|_p} \|g_{k-1} - g_k\|_p^2.
\]
Therefore, by Minkowski's inequality, we have
\[
\|f\|_{L^p(x)} = \sum_{j=0}^{\infty} 2^{aj} \|\Delta_j \ast f\|_p^p \leq \sum_{j=0}^{\infty} 2^{aj} \left( \sum_{k=j}^{\infty} 2^{(k-j)(1-p)\|\Delta_j \|_p} \|g_{k-1} - g_k\|_p^p \right)^{q/p} \leq \sum_{j=0}^{\infty} 2^{aj} \left( \sum_{k=j}^{\infty} 2^{(k-j)(1-p)\|\Delta_j \|_p} \|g_{k-1} - g_k\|_p^p \right)^{q/p} \leq \sum_{j=0}^{\infty} 2^{aj(a-\alpha)\|\Delta_j \|_p} \left( \sum_{k=j}^{\infty} 2^{aj\|\Delta_j \|_p} \|g_{k-1} - g_k\|_p^p \right)^{q/p} \leq \sum_{j=0}^{\infty} 2^{aj\|\Delta_j \|_p} \|g_{j-1} - g_j\|_p^p + C 2^{aj\|\Delta_j \|_p} \|f\|_p^p \leq \sum_{j=0}^{\infty} 2^{aj\|\Delta_j \|_p} \|g_{j-1} - g_j\|_p^p + C 2^{aj\|\Delta_j \|_p} \|f\|_p^p.
\]
Taking the infimum over all such sequences $\{g_j\}$ gives
\[
\|f\|_{L^p(x)} \leq C(\|E^\alpha f\|_{L^p(x)} + \|f\|_p).
\]
The last step is to prove the inequality (1.4).

By the definition of best approximation, we have
\[
E_p(2^j, f) \leq \|f - S_{2^j} f\|_p = \left\{ \int_{p^0} \left( \int f(x) - f(x + h) dh \right)^p dx \right\}^{1/p} \leq \sum_{h=0}^{\infty} \left\{ \int_{p^0} \left( \int f(x) - f(x + h) dh \right)^p dx \right\}^{1/p}.
\]
Then we have the desired inequality:
\[
\|E^\alpha f\|_{L^p} = \sum_{j=0}^{\infty} 2^{aj} E_p(2^j, f) \leq \|E^\alpha f\|_{L^p}.
\]
These steps together with Theorem $\Lambda(ii)$ complete the proof of Theorem 1.

To obtain a variation of Theorem 1 for the case $r = q = \infty$, we let $\|f\|_B = \sup_{j \geq 0} 2^{aj} \|\Delta_j \ast f\|_p$, $\|f\|_D = \sup_{j \geq 0} 2^{aj} \|f - S_{2^j} f\|_p$, $\|f\|_E = \sup_{j \geq 0} 2^{aj} E_p(2^j, f)$.

The following corollary generalizes the result of [12, Theorem], [4, Theorem 1] and [10, Theorem 3].

**Corollary.** If $0 < p \leq \infty$ and $\alpha > \max(1/p - 1, 0)$, then
\[
\|f\|_B \approx \|f\|_S + \|f\|_D \approx \|f\|_D + \|f\|_E \approx \|f\|_E + \|f\|_D.
\]

**Proof.** The proof is carried out for the case $0 < p < 1$, the remaining cases are similar. The outline of the proof is to show $\|f\|_S \leq C \|f\|_D, \|f\|_D \leq C \|f\|_S, \|f\|_E \leq C \|f\|_D$, and then $\|f\|_E \leq C \|f\|_S$.

Since $S_{2^j} f(x) = S_{2^j} f(x + h) \text{ for } \|h\| = 2^{-j}$, we have
\[
\|f\|_S \leq \sup_{j \geq 0} 2^{aj} \left\| \int_{p^0} \Delta_k \ast f(x) dx \right\|_p = \sum_{h=0}^{\infty} \left\| \int_{p^0} \Delta_k \ast f(x) dx \right\|_p = \sum_{h=0}^{\infty} \left\| \int_{p^0} \Delta_k \ast f(x) dx \right\|_p = \sum_{h=0}^{\infty} \left\| \int_{p^0} \Delta_k \ast f(x) dx \right\|_p = \sum_{h=0}^{\infty} \left\| \int_{p^0} \Delta_k \ast f(x) dx \right\|_p = \sum_{h=0}^{\infty} \left\| \int_{p^0} \Delta_k \ast f(x) dx \right\|_p.
\]

Thus, we get $\|f\|_S \leq C \|f\|_D$.

Next, since $S_{2^j} f(x) = \sum_{k=j+1}^{\infty} \Delta_k \ast f(x)$ for a.e. $x$,
\[
\|f\|_D \leq \sum_{j \geq 0} \left\| \sum_{k=j+1}^{\infty} \Delta_k \ast f(x) \right\|_p \leq \sum_{j \geq 0} 2^{aj} \left( \sum_{k=j+1}^{\infty} \left\| \Delta_k \ast f(x) \right\|_p \right)^{1/p} = C \sum_{j \geq 0} 2^{aj} \left( \sum_{k=j+1}^{\infty} \left\| \Delta_k \ast f(x) \right\|_p \right)^{1/p}.
\]
We shall use Hölder's inequality and then a simple variation of [9, p. 179, Lemma (2.1)]:

**Suppose $\alpha > 0$ and $\{a_k\}$ is a sequence of non-negative numbers such that $\sup_{k} 2^k a_k < \infty$. Then $\sup_{k} 2^k \sum_{j=k+1}^{\infty} a_j \leq C \sup_{k} 2^k a_k$.**

For $0 < \varepsilon < \alpha$, we have, by Hölder's inequality,
\[
\sup_{j} 2^{aj} \left\| \sum_{k=j+1}^{\infty} \Delta_k \ast f(x) \right\|_p \leq C \sup_{j} 2^{aj} \left\| \sum_{k=j+1}^{\infty} \Delta_k \ast f(x) \right\|_p \leq C \sup_{j} 2^{aj} \left\| \sum_{k=j+1}^{\infty} \Delta_k \ast f(x) \right\|_p.
\]
From the third step of the proof of Theorem 1, we get easily $\|f\|_B \leq C(\|f\|_E + \|f\|_D)$.

To prove $\|f\|_E \leq C \|f\|_S$, we rewrite [8, p. 359, Lemma 2.1] as follows:
Let $f \in S(n)$. Then for any integer $k \geq 0$, there exists $g \in S(k)$ such that $\|f - g\|_p \leq C 2^k \int_{P^k} \|f(\cdot + h) - f(\cdot)\|^p_p \, dh$.

If $\|f - f_n\|_p \to 0$ as $n \to \infty$, then $E_p(2^k, f_n) \to E_p(2^k, f)$ as $n \to \infty$. Therefore,

$$E_p(2^k, f) \leq C 2^k \int_{P^k} \|f(\cdot + h) - f(\cdot)\|^p_p \, dh$$

for $f \in L^p$. Hence, we have

$$\sup_{k \geq 0} 2^{ak} E_p(2^k, f) \leq C \sup_{k \geq 0} \left( 2^k \int_{P^k} \|f(\cdot + h) - f(\cdot)\|^p_p \, dh \right)^{1/p}$$

$$\leq C \sup_{k \geq 0} \left( 2^k \int_{P^k} |h|^{-\alpha_p} |f(\cdot + h) - f(\cdot)|^p_p \, dh \right)^{1/p}$$

$$\leq C \sup_h |h|^{-\alpha} \|f(\cdot + h) - f(\cdot)\|_p = C \|f\|_s.$$

Thus the corollary is proved. □

2. Characterization by atoms. We shall show that each $f \in B^\alpha_{pq}(P^0)$ can be decomposed into a sum of atoms.

We define an $(\alpha, p)$-atom $a(x)$ ($-\infty < \alpha < \infty, 0 < p \leq \infty$) to be a function satisfying, for some point $x_0 \in P^0$ and non-negative integer $k$,

$$(2.1) \quad \text{supp } a \subset x_0 + P^k,$$

$$(2.2) \quad |a(x)| \leq C |P^k|^{|\alpha|} |x|^\alpha,$$

$$(2.3) \quad |a(x - y) - a(x)| \leq C |P^k|^{\alpha - 1/p - 1} |y|^{\frac{\alpha}{p-1}} \quad \text{if } |y| < 2^{-k},$$

and

$$(2.4) \quad \int a(x) \, dx = 0,$$

where $l > \alpha$. The constant function $a(x) \equiv 1$ on $P^0$ is also considered to be an atom. We write $a_Q$ for an atom satisfying (2.1)–(2.4) for a given $Q = x_0 + P^k$.

We call $m(x)$ an $(\alpha, p)$-molecule if there exist a non-negative integer $k$ and a point $x_0 \in P^0$ such that

$$(2.5) \quad |m(x)| \leq C |P^k|^{\alpha - 1/p} (1 \vee 2^k |x - x_0|)^{-2\alpha},$$

$$(2.6) \quad |m(x - y) - m(x)| \leq C |P^k|^{\alpha - 1/p} |y|^{\alpha} (1 \vee 2^k |x - x_0|)^{-2\alpha},$$

and

$$(2.7) \quad \int m(x) \, dx = 0,$$

where $1 \vee f(x) = \max(1, f(x))$, $l > \alpha$ and $M = \max[1/p - \alpha, 1]$. We also write $m_Q$ for an $(\alpha, p)$-molecule satisfying (2.5)–(2.7) for a given coset of $P^k$ in $P^0$, $Q = x_0 + P^k$.

It is easy to check that a non-constant $(\alpha, p)$-atom is an $(\alpha, p)$-molecule.

The following decomposition theorem for $B^\alpha_{pq}(P^0)$ is based on comparable $\mathbb{R}$-results [2, Theorem 2.6 and Theorem 3.1] and improves [4, Theorem 6] in the dyadic group setting.

**Theorem 2.** Let $-\infty < \alpha < \infty, 0 < p, q \leq \infty$.

(a) Each $f \in B^\alpha_{pq}$ can be decomposed as follows:

$$f = \sum_{j=0}^\infty \sum_{|Q|=2^{-j}} \lambda_Q a_Q,$$

where the $a_Q$'s are $(\alpha, p)$-atoms. The numbers $\lambda_Q$ satisfy

$$\left\{ \sum_{j=0}^\infty \left( \sum_{|Q|=2^{-j}} |\lambda_Q|^q \right)^{q/p} \right\}^{1/q} = \|f\|_{B^\alpha_{pq}},$$

(b) Suppose $f = \sum_{j=0}^\infty \sum_{|Q|=2^{-j}} \lambda_Q m_Q$, where the $m_Q$'s are $(\alpha, p)$-molecules. Then

$$\|f\|_{B^\alpha_{pq}} \leq C \left\{ \sum_{k=0}^\infty \left( \sum_{|Q|=2^{-k}} |\lambda_Q|^q \right)^{q/p} \right\}^{1/q}.$$

**Proof.** (a) For each $f \in S'(P^0)$, we have

$$f = \sum_{j=0}^\infty f * \Delta_j * \Delta_j = \sum_{j=0}^\infty \int (f * \Delta_j)(t) \Delta_j(x - t) \, dt$$

$$= \sum_{j=0}^\infty \sum_{p \geq 2^{-j}} \int (f * \Delta_j)(t) \Delta_j(x - t) \, dt,$$

where $\{Q\}$ are cosets of $P^j$. Since $f * \Delta_j$ and $\Delta_j$ are constant functions on each coset $Q$ of $P^j$,

$$f = \sum_{j=0}^\infty \sum_{|Q|=2^{-j}} |Q| (f * \Delta_j)(y) \Delta_j(x - y).$$

For $y \in Q$, define $\lambda_Q = |Q|^{-\alpha + 1} |f * \Delta_j(y)|$ and $a_Q = |Q|^{-\alpha + 1} \Delta_j(x - y)$. For $|Q| = 2^{-j} (1 \leq j)$, we have $\sup_{a_Q} = y + P^{j-1}$, $|a_Q| \leq |Q|^{-\alpha + 1/p + 1} \times 2^{j-1} = C |P^{j-1}|^{\alpha - 1/p}$ and $\int a_Q(x) \, dx = |Q|^{-\alpha + 1/p} \int \Delta_j(x - t) \, dx = 0$. If $|Q| \leq 2^{-j}$, then $|a_Q(x - t) - a_Q(x)| = 0$. For $j = 0$, $a_Q(x) = |Q|^{-\alpha + 1}$ is a
constant function on $P^0$. Thus, $a_Q$ is an $(\alpha, p)$-atom. Moreover, we have
\[
\left\{ \sum_{j=0}^{\infty} \left( \sum_{|Q|=2^{-j}, \gamma \in Q} |\lambda_Q|^p \right)^{q/p} \right\}^{1/q} = \left\{ \sum_{j=0}^{\infty} \left( \sum_{|Q|=2^{-j}, \gamma \in Q} |Q|^{-\alpha p} (f \ast \Delta_j)(\gamma) \right)^p \right\}^{1/q}
\]
\[
= \left\{ \sum_{j=0}^{\infty} \left( \sum_{|Q|=2^{-j}} |Q|^{-\alpha p} \int_{Q} |f \ast \Delta_j(t)|^p dt \right)^{q/p} \right\}^{1/q} = \left\| f \right\|_{B_p^2}.
\]
(b) To get the norm estimate, we write
\[
\left\| \Delta_j \ast m_Q \right\|^p_p = \left\| \Delta_j \ast \left( \sum_{k=0}^{\infty} \sum_{|Q|=2^{-k}} \lambda_Q m_Q \right) \right\|^p_p \leq \left( \sum_{k=0}^{\infty} \sum_{k=j+1}^{\infty} \sum_{|Q|=2^{-k}} |\lambda_Q|^p \right) \left\| \Delta_j \ast m_Q \right\|^p_p.
\]
We shall use the following pointwise inequalities:
(2.8) $|\Delta_j \ast m_Q(x)| \leq C 2^{(j-k)+k(1/p-\alpha)} (1 \vee 2^k|x|)^{-M}$ if $k \leq j - 1$,
(2.9) $|\Delta_j \ast m_Q(x)| \leq C 2^{j-k}(1 \vee 2^k|x|)^{-M}$ if $j - 1 \leq k$.

If we can show (2.8) and (2.9), then since $\theta^p \subset \ell^1$ ($0 < p < 1$), we have
\[
\left\| f \right\|_{B_p^2}^q = \sum_{j=0}^{\infty} 2^{aqj} \left\| \Delta_j \ast f \right\|^q_{\ell^q_p} \leq C \sum_{j=0}^{\infty} 2^{aqj} \left\{ \sum_{k=0}^{j} \sum_{|Q|=2^{-k}} |\lambda_Q|^p 2^{-(j-k)p+(1-p)k(1-\alpha)} \int (1 \vee 2^k|x|)^{-Mp} dx \right\}^{q/p}
\]
\[
+ \sum_{k=j+1}^{\infty} \sum_{|Q|=2^{-k}} |\lambda_Q|^p 2^{(j-k)p+k(1-\alpha)} \int (1 \vee 2^k|x|)^{-Mp} dx \right\}^{q/p} = C \sum_{j=0}^{\infty} 2^{aqj} \left\{ \sum_{k=0}^{j} \sum_{|Q|=2^{-k}} |\lambda_Q|^p \right\}^{q/p} + \sum_{k=j+1}^{\infty} \sum_{|Q|=2^{-k}} |\lambda_Q|^p \right\}^{q/p}.
\]
Applying Young's inequality, we have
\[
\left\| f \right\|_{B_p^2}^p \leq C \sum_{j=0}^{\infty} \left\{ \sum_{|Q|=2^{-j}} |\lambda_Q|^p \right\}^{q/p} \left\{ \sum_{j=0}^{\infty} 2^{-j\alpha p} (1 \vee 2^k|x|)^{-M} \int |\Delta_j(t)| dt \right\}^{q/p} \leq C \sum_{j=0}^{\infty} \left\{ \sum_{|Q|=2^{-j}} |\lambda_Q|^p \right\}^{q/p},
\]
since $l - \alpha > 0$ and $Mp - 1 + p \alpha > 0$.

We shall now work towards a proof of (2.8) and (2.9). Consider (2.8) first. By translation, we may assume $x_0 = 0$ in (2.6). Using the fact that
\[
|\Delta_j(t)| dt = \int (m_Q(x - t) - m_Q(x)) \Delta_j(t) dt
\]
\[
\leq C |x|^k (a/p - 1) (1 \vee 2^k|x|)^{-M} \int |\Delta_j(t)| dt \leq C 2^{(j-k)+k(1/p-\alpha)} (1 \vee 2^k|x|)^{-M} \int |\Delta_j(t)| dt
\]
\[
= C 2^{(j-k)+k(1/p-\alpha)} (1 \vee 2^k|x|)^{-M}.
\]
The proof of (2.9) is similar. By (2.7) and (2.5), we have
\[
|\Delta_j \ast m_Q(x)| = \int |m_Q(x - t) - m_Q(x)| \Delta_j(t) dt \leq C \int |m_Q(x - t)| \Delta_j(t) dt \leq C \left( \int_{|t| \leq 2^{-j}} \int_{|t| = 2^{-j+1}} \int_{|t| \geq 2^{-j+2}} \right) 2^{k(1/p-\alpha)} (1 \vee 2^k|x - t|)^{-M} |\Delta_j(t) - \Delta_j(x)| dt
\]
\[
= I + II + III.
\]
If $|x| \leq 2^{-j}$, then $\Delta_j(x) = \Delta_j(t) = 2^{j-1}$ in $I$, $|x - t| = 2^{-j+1}$ and $\Delta_j(x) = -\Delta_j(t) = 2^{-j-1}$ in $II$, and $|x - t| \geq 2^{-j+2}$ and $\Delta_j(x) = 2^{j-1}$ and $\Delta_j(t) = 0$ in $III$. Hence, $I = 0$,
\[
II = C 2^{k(1/p-\alpha)} \int_{|t| = 2^{-j+1}} (1 \vee 2^k|x - t|)^{-M} - 2^{j-1} - 2^{-j-1} dt \leq C 2^{k(1/p-\alpha)} + (j-k)M.
\]
and
\[
III = C 2^{k(1/p-\alpha)} \int_{|t| \geq 2^{-j+2}} (1 \vee 2^k|x - t|)^{-M} - 2^{j-1} - 2^{-j-1} dt
\]
We may assume $Q$'s are mutually disjoint. By (3.2), we have
\[ 
\int_0^\infty \text{Cap}_{\alpha,p} \left( \{ x \in P^0 : |f(x)| > \lambda \} \right) \lambda^{p-1} \, d\lambda 
= \sum_{n=-\infty}^{\infty} \int_0^{2^{n+1}} \text{Cap}_{\alpha,p} \left( \{ x \in P^0 : |f(x)| > \lambda \} \right) \lambda^{p-1} \, d\lambda 
\leq \sum_{n=-\infty}^{\infty} \sum_{j=0}^{2^n} \sum_{Q \in 2^{-j}} |\lambda_Q| |a_Q(x)| > 2^n \right) 
\leq \sum_{j=0}^{\infty} \sum_{Q \in 2^{-j}} \sum_{n=-\infty}^{\infty} 2^n \text{Cap}_{\alpha,p} \left( \{ x \in P^0 : |\lambda_Q| |a_Q(x)| \geq 2^n \} \right).
\]

If $|Q| = 2^{-j}$, then $n \leq \log_2 |\lambda_Q| - j(\alpha - 1/p) + \log_2 C$ follows from the atom condition (2.2), and
\[ 
\text{Cap}_{\alpha,p} \left( \{ x \in P^0 : |\lambda_Q| |a_Q(x)| \geq 2^n \} \right) \leq \text{Cap}_{\alpha,p}(Q) \leq ||\Phi_Q||_{B^{\alpha,p}}. 
\]
By translation, we may assume $Q = P$. Now an easy calculation shows that
\[ 
||\Phi_Q||_{B^{\alpha,p}} = \sum_{m=0}^{2^\alpha} \sum_{j} 2^m \text{Cap}_{\alpha,p} \left( \{ x \in P^0 : |f(x)| > \lambda \} \right) \lambda^{p-1} \, d\lambda 
\]
\[ 
\sum_{m=0}^{2^\alpha} \sum_{j} 2^m \text{Cap}_{\alpha,p} \left( \{ x \in P^0 : |f(x)| > \lambda \} \right) \lambda^{p-1} \, d\lambda 
\leq \sum_{j=0}^{\infty} \sum_{Q \in 2^{-j}} |\lambda_Q|^{-j(\alpha - 1/p) + \log_2 C} \leq 2^{(\alpha - 1/p) \gamma_j/2},
\]

We have $\lambda_Q$ and $a_Q$ such that
\[ 
|f|_B^{\alpha,p} = \sum_{j=0}^{\infty} \sum_{Q \in 2^{-j}} |\lambda_Q|^{1/p} |a_Q| \leq C |f|_B^{\alpha,p}.
\]

3. Applications. As the first application of Theorem 2 we shall show the strong capacity inequality of the type of the Maz'ya inequality ([3, p. 54, Theorem 1]).

For $0 < p \leq \infty$ and a compact subset $A$ of $P^0$, we set
\[ 
\text{Cap}_{\alpha,p}(A) = \inf \{ ||f||_{B^{\alpha,p}} : f \geq 1 \text{ on } A, \ f \in B^{\alpha,p}(P^0) \}.
\]

Basic properties of $\text{Cap}_{\alpha,p}$ will be useful:
\[ 
\text{Cap}_{\alpha,p}(A_1) \leq \text{Cap}_{\alpha,p}(A_2), \quad A_1 \subset A_2, \tag{3.1}
\]
\[ 
\text{Cap}_{\alpha,p}(A) = \sum_{i=0}^{\infty} \text{Cap}_{\alpha,p}(A_i), \tag{3.2}
\]
\[ 
\text{Cap}_{\alpha,p}(\emptyset) = 0, \tag{3.3}
\]
\[ 
\text{Cap}_{\alpha,p}(\{ x \in P^0 : |f(x)| > \lambda \} \leq \lambda^{-\alpha} ||f||_{B^{\alpha,p}}. \tag{3.4}
\]

THEOREM 3. If $0 < p \leq \infty$ and $\max(1/p - 1, 0) < \alpha < \infty$, then
\[ 
\int_0^{\infty} \text{Cap}_{\alpha,p} \left( \{ x \in P^0 : |f(x)| > \lambda \} \right) \lambda^{p-1} \, d\lambda \leq C ||f||_{B^{\alpha,p}}.
\]

Proof. Let $f \in B^{\alpha,p}$, By Theorem 2 we have $\lambda_Q$ and $a_Q$ such that
\[ 
|f|_B^{\alpha,p} = \sum_{j=0}^{\infty} \sum_{Q \in 2^{-j}} |\lambda_Q|^{1/p} |a_Q| \leq C ||f||_{B^{\alpha,p}}.
\]

As the second application of Theorem 2 we shall show a weak type estimate for maximal Cesàro means.

We list some properties of $K^0_n(x)$ and $K^0_n(x)$ for $n = \sum_{i=0}^t b_i 2^i$, $b_s = 1$, $b_t = 0$ or 1 (see [15], [5, p. 46]):
\[ A_n^\beta |K_n^\beta(x) = \sum_{i=0}^{2^i-1} b_i w_{2^i, \ldots, 2^{i+1}-1}(x) - \sum_{k=1}^{2^{i-2}} k K_n^\beta(x) A_{n-2^i - \ldots - 2^{i+1}-1} + \sum_{k=1}^{2^{i-2}} k K_n^\beta(x) A_{n-2^i - \ldots - 2^{i+1}-1} + D_{n-1}(x) A_{n-2^i - \ldots - 2^{i+1}-1} \] 

\[ |nK_n^\beta(x)| \leq 3nK_n^\beta(x) = 3 \sum_{i=0}^{i} b_i 2^i K_n^\beta(x), \] 

\[ K_n^\beta(x) = (2^i-1 + 1/2) \Phi_i(x) + \sum_{r=0}^{i-1} 2^{r-1} \Phi_r(x - r^+). \] 

Let \( a(x) \) be an \((\alpha, p)\)-atom supported by \( x_0 + P^u \). Then 

\[ (|a| \ast \Phi_i)(x) \leq C \Phi_i(x - x_0) 2^{-u(\alpha-1/p)-u\vee i}, \] 

where \( i \wedge u = \min[i, u] \) and \( i \vee u = \max[i, u] \). 

In fact, by (2.2), 

\[ (|a| \ast \Phi_i)(x) = \int_{(x + P^i) \cap (x_0 + P^u)} |a(t)| dt \]

\[ = \Phi_i \wedge u(x - x_0) \int_{(x + P^i) \cap (x_0 + P^u)} |a(t)| dt \]

\[ \leq C \Phi_i \wedge u(x - x_0) \|(x + P^i) \cap (x_0 + P^u)\|^\alpha/p \]

\[ = C \Phi_i \wedge u(x - x_0) 2^{-u(\alpha-1/p)-u\vee i}. \] 

From (3.7) and (3.8) it will follow that 

\[ \sigma_2 |a(x)| \leq C 2^{-u(\alpha-1/p+1)} \left\{ (2^{i-1} + 1)^2 \Phi_i(x - x_0) \right. \]

\[ + \sum_{r=0}^{i-1} 2^{r-1} \Phi_r(x - x_0 - r^+) \left. \right\} \] 

if \( l \leq u \) and 

\[ \sigma_2 |a(x)| \leq C 2^{-u(\alpha-1/p)} \left\{ 2 \Phi_u(x - x_0) \right. \]

\[ + \sum_{r=0}^{i-1} 2^{r-1} \Phi_r(x - x_0 - r^+) \left. \right\} \] 

if \( l > u \). 

We shall use the following obvious estimate. 

If \( 0 \leq i \leq j \) and \( 0 \leq k \), then 

\[ \sum_{l=i}^{j} 2^{k |\Phi_l(x)| \leq C \Phi_i(x) \left| x \right|^\alpha. \] 

**Theorem 4.** Suppose \( 1/2 \leq p < 1, \) \( 0 < q \leq \infty, \) \( \alpha > 0, \) and \( \beta = 1/p - 1. \) If \( f \in B_p^\alpha \) then for all \( \lambda > 0, \)

\[ \{ x \in P^0 : \sup_n |\sigma_n^\alpha f(x) > \lambda \} \leq (C \|f\|_{B_p^\alpha})^p. \] 

We shall prove below the following lemma.

**Lemma 1.** Suppose \( 1/2 \leq p < 1 \) and \( \beta = 1/p - 1. \) If \( a_Q(x) \) is a non-constant \((\alpha, p)\)-atom supported in \( Q \) then

\[ \{ x \in P^0 : \sup_n |\sigma_n^\alpha a_Q(x) > \lambda > 0 \} \leq (C |Q|^{\alpha/p})^p. \] 

If \( a(x) \) is constant on \( P^0 \) then

\[ \{ x \in P^0 : \sup_n |\sigma_n^\alpha a_Q(x) > \lambda > 0 \} \leq (C \lambda)^p. \] 

**Theorem 4** follows directly from Lemma 2 whose proof is similar to [7, p. 85, Lemma (1.8)].

**Lemma 2.** Suppose \( 0 < p < 1 \) and \( \{a_Q\} \) is a sequence of \((\alpha, p)\)-atoms such that for each \( Q \) and each \( \lambda > 0, \)

\[ \{ x \in P^0 : |a_Q(x) > \lambda > 0 \} \leq (C |Q|^{\alpha/p})^p. \] 

If \( \{\lambda_Q\} \) is a sequence such that \( \{\sum (|Q|^{\alpha/p})^{1/2} \leq \|f\|_{B_p^\alpha} \), then

\[ \left\{ x \in P^0 : \sum \lambda_Q a_Q(x) > \lambda \right\} \leq (C \|f\|_{B_p^\alpha})^p. \] 

**Proof of Lemma 1.** This proof is based on [16]. There is no loss in generality if we assume \( a(x) \) to be supported in \( x_0 + P^u \). We have \( a(k) = 0 \) for \( 0 < k < 2^u \) and \( S_n a(x) = \sigma_n^\alpha a(x) = 0 \) for \( 0 \leq n < 2^u \). Then we assume \( n > 2^u \). The proof is carried out for the number \( n = \sum_{i=0}^{[\log n]} b_i 2^i, b_{[\log n]} = 1, \) \( b_i = 0 \) for \( 1 < i < [\log n] \) and \( 1/2 < p < 1 \). We have, from (3.5),

\[ |\sigma_n^\alpha a(x)| \leq C \sum_{i=0}^{[\log n]} \left\{ \sum_{j=1}^{2^i-1} (2^j + j)^{\beta-2} |\sigma_j a(x)| + 2^{i(\beta-1)} |(2^i - 1)| \sigma_{2^i-1} a(x) + 2^i |S_{2^i} a(x)| \right\}. \]
\[
\begin{align*}
&= \frac{C}{n^\beta} \sum_{i=u+1}^{[\log n]} \sum_{j=1}^{2^n} 2^{i(\beta-2)} j |\sigma_j a(x)| + \frac{C}{n^\beta} \sum_{i=u+1}^{[\log n]} \sum_{j=2^{u+1}}^{2^i-1} j^{\beta-2} |\sigma_j a(x)|
+ \frac{C}{n^\beta} \sum_{i=u+1}^{[\log n]} 2^{i(\beta-1)} (2^i - 1) |\sigma_{2^i-1} a(x)| + \frac{C}{n^\beta} \sum_{i=u+1}^{[\log n]} 2^{i\beta} |\sigma_{2^i} a(x)|

= I + II + III + IV.
\end{align*}
\]

Now, \( I = 0 \) since \( \sigma_j a(x) = 0 \) for \( 1 \leq j \leq 2^u \). We estimate \( II + III \) by making use of (3.6), (3.9), (3.10) and (3.11):

\[
II + III \leq C \sup_{j>2^u} \sigma_j \|a(x)\| \frac{1}{n^\beta} \left\{ \sum_{i=u+1}^{[\log n]} \sum_{j=2^u+1}^{2^i-1} j^{\beta-1} + \sum_{i=u+1}^{[\log n]} 2^{i\beta} \right\}
\]

\[
\leq C \sup_{j>2^u} \sigma_j \|a(x)\|
\]

\[
\leq C \sup_{j>2^u} \left\{ \frac{1}{j} \sum_{i=0}^{u} 2^i \sigma_{2^i} \|a(x)\| + \frac{1}{j} \sum_{i=u+1}^{[\log j]} 2^i \sigma_{2^i} \|a(x)\| \right\}
\]

\[
\leq C 2^{-u(\alpha-1/p+2)} \sum_{i=0}^{u} \left\{ 2^{2i} \Phi_i(x) + C \sum_{r=0}^{i-1} 2^{r\beta} \Phi_i(x - \rho^r) \right\}
+ C \sup_{i>u} \sigma_{2^i} \|a(x)\|
\]

\[
\leq C 2^{-u(\alpha-1/p)} \left\{ \sum_{i=0}^{u} 2^{i\beta} \Phi_i(x) + \sum_{r=0}^{u-1} 2^{\beta} \Phi_i(x - \rho^r) \right\} + C \sup_{i>u} \sigma_{2^i} \|a(x)\|
\]

\[
\leq C 2^{-u(\alpha-1/p)} \left\{ \left| x \right|^{-1/p} \Phi_0(x) + \sum_{r=0}^{u-1} 2^{\beta} \frac{\Phi_{r+1}(x - \rho^r)}{|x - \rho^r|} \right\}
+ C 2^{-u(\alpha-1/p)} \left[ \Phi_u(x) + \sup_{i>u} \sum_{r=0}^{i-1} 2^{r(1-p)} \Phi_u(x - \rho^r) \right].
\]

Finally, we estimate \( IV \) by (3.8):

\[
IV \leq \frac{C}{n^\beta} \sum_{i=u+1}^{[\log n]} 2^{i(\beta+1)} |\Phi_i \|a(x)\|)
\]

\[
\leq \frac{C}{n^\beta} 2^{-u(\alpha-1/p)} \Phi_u(x) \sum_{i=u+1}^{[\log n]} 2^{i\beta} \leq C 2^{-u(\alpha-1/p)} \Phi_u(x).
\]

Collecting these estimates, we have

\[
\left\{ x : \sigma_n^\alpha a(x) > \lambda \right\}
\]

\[
\leq \left\{ x : C 2^{-u(\alpha-1/p)} \left| x \right|^{-1/p} \Phi_0(x) + C 2^{-u(\alpha-1/p)} \sum_{r=0}^{u-1} 2^{\beta} \frac{\Phi_{r+1}(x - \rho^r)}{|x - \rho^r|} \right\}
+ \left\{ x : C 2^{-u(\alpha-1/p)} \Phi_u(x) + C \sup_{i>u} \sum_{r=0}^{i-1} 2^{r(1-p)} \Phi_u(x - \rho^r) \right\}
\]

\[
\leq \left\{ x : \frac{C 2^{-u(\alpha-1/p)}}{\lambda} > \left| x \right|^{1/p} \right\}
+ C \sum_{r=0}^{u-1} \left\{ x : \frac{C 2^{-u(\alpha-1/p)}}{\lambda} 2^{\beta} \Phi_{r+1}(x - \rho^r) > |x - \rho^r| \right\}
+ \left\{ x : \frac{C 2^{-u(\alpha-1/p)}}{\lambda} \Phi_u(x) > 2^{-u/p} \right\}
+ \left\{ x : \frac{C 2^{-u(\alpha-1/p)}}{\lambda} \sup_{i>u} \sum_{r=0}^{i-1} 2^{r(1-p)} \Phi_u(x - \rho^r) > 2^{-u/p} \right\}
\]

\[
\leq \left( \frac{C 2^{-u(\alpha-1/p)}}{\lambda} \right)^p + \left( \frac{C 2^{-u(\alpha-1/p)}}{\lambda} \sum_{r=0}^{u-1} 2^{\beta} + \sum_{r=u+1}^{u-1} 2^{r(1-p)} \right)^p
+ \left( \frac{C 2^{-u(\alpha-1/p)}}{\lambda} \right)^p \left( \frac{C 2^{-u(\alpha-1/p)}}{\lambda} \right)^{\sum_{i>u} \sum_{r=0}^{i-1} 2^{(r-i)p}}.
\]

When \( p = 1/2 \), \( I = II = 0 \) and we can estimate \( III \) and \( IV \) in the same way. Thus we proved Lemma 2. □

Remark. We can see by the same technique of the proof of the theorem that Theorem 4 is valid for \( 0 < p < 1/2 \). However, we shall need a more precise formula for \( K_n^\alpha \) and an inductive proof for a range of \( p \).

As the third application of Theorem 1 we shall show a sufficient condition for the absolute convergence of the Walsh-Fourier series. The well known Bernstein-Stechkin criterion says

\[
\sum_{j=0}^{\infty} 2^{j/2} E_2(2^j, f) < \infty \text{ implies } \sum_{k=0}^{\infty} \left| \left( f, w_k \right) \right| < \infty,
\]

where \( (f, w_k) \) denotes the action of \( f \in S'(P^0) \) on \( w_k \in S(P^0) \) (see [14]).
The following theorem includes [5, p. 64, Theorem 9] whose assumption is given by $\|S_{1/p}f\|_{p,1}$ in our notation.

**Theorem 5.** If $f \in B_{p,q}^{1/p}$ ($0 < p \leq 2$), then $\sum_{k=0}^{\infty} |(f, w_k)| < \infty$.

**Proof.** From Theorem A(i) and the definition of $B_{p,q}^{1/p}$ we have $B_{p,q}^{1/p} \subset B_{1/2}^{1/2}$ for $0 < p \leq 2$. Thus, we obtain the theorem from (3.12) and Theorem 1.

References


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Weighted Orlicz space integral inequalities for the Hardy–Littlewood maximal operator

by

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Abstract. Necessary and sufficient conditions are given for the Hardy–Littlewood maximal operator to be bounded on a weighted Orlicz space when the complementary Young function satisfies $\Delta_2$. Such a growth condition is shown to be necessary for any weighted integral inequality to occur. Weak-type conditions are also investigated.

1. Introduction. For an N-function $\Phi$, the Orlicz space $L_\Phi(X, d\mu)$ is the Banach space normed by

$$\|f\|_\Phi = \inf \left\{ \lambda > 0 : \int_X \Phi\left( \frac{|f(x)|}{\lambda} \right) d\mu(x) \leq 1 \right\}.$$

The usual Lebesgue spaces $L^p(X, d\mu)$, $1 < p \leq \infty$, arise from the N-function $\Phi(x) = x^p/p$. These Lebesgue spaces satisfy a $\Delta_2$ condition; that is,

$$\Phi(2\lambda) \leq C \Phi(\lambda).$$

Most papers trying to describe a weighted operator theory in an Orlicz space setting have made $\Delta_2$ assumptions. It is easy to see where these arise. Marcinkiewicz interpolation is one of our most cherished tools. Suppose we start with a sublinear operator $T$. Then $T$ is of type $(\infty, \infty)$ and of weak-type $(1, 1)$ if and only if, for each $\lambda > 0$,

$$\int \frac{|f(x)|}{\lambda} d\mu(x) \leq \frac{C}{\lambda} \int \Phi\left( \frac{|f(x)|}{\lambda} \right) dx$$

where $K = 2\|T\|_\infty$. Writing $\Phi$ as

$$\Phi(x) = \int_0^x \phi(t) dt,$$

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