When is there a discontinuous homomorphism from $L^1(G)$?

by

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Abstract. Let $A$ be an $A^*$-algebra with enveloping $C^*$-algebra $C^*(A)$. We show that, under certain conditions, a homomorphism from $C^*(A)$ into a Banach algebra is continuous if and only if its restriction to $A$ is continuous. We apply this result to the question in the title.

Introduction. One of the central questions in automatic continuity is the following: For which Banach algebras $A$ is there a Banach algebra $B$ and a discontinuous homomorphism $\theta : A \to B$?

A fundamental result obtained independently by H. G. Dales and J. Esterle (see [Dal] for a streamlined exposition) asserts that if $X$ is a locally compact Hausdorff space and if the continuum hypothesis holds, then there is a discontinuous homomorphism from $C_0(X)$ into a Banach algebra if and only if $X$ is infinite. Surprisingly, this result cannot be proved within the confines of Zermelo-Fraenkel set theory and the axiom of choice ([D-W]). In this note, we shall not delve into these set theoretic intricacies and assume throughout that the continuum hypothesis holds.

In [A-D], E. Albrecht and Dales conjectured the following non-commutative version of the Dales–Esterle theorem:

Conjecture A. Let $A$ be a $C^*$-algebra. Then there is a discontinuous homomorphism from $A$ into a Banach algebra if and only if there is $n \in \mathbb{N}$ such that $A$ has an infinite number of inequivalent, $n$-dimensional, irreducible $*$-representations.

Albrecht and Dales were able to prove the “if” part of their conjecture and to confirm the “only if” part for so-called $AW^*$-algebras, a class of $C^*$-algebras containing all commutative $C^*$-algebras and all closed ideals of $AW^*$-algebras.

Let $G$ be a locally compact, abelian group. Then $L^1(G)$ is a Banach function algebra whose character space can be identified with $\hat{G}$, the dual group of $G$. From the general normability theory of complex algebras ([Dal, Chapter 5]), it follows that there is a discontinuous homomorphism from $L^1(G)$ if and only if $\hat{G}$ is infinite.

This fact and the Albrecht–Dales conjecture motivate the following conjecture, which was stated in [Run].

**Conjecture B.** Let $G$ be a locally compact group. Then there is a discontinuous homomorphism from $L^1(G)$ into a Banach algebra if and only if there is an infinite number of inequivalent, n-dimensional, irreducible unitary representations.

In [Run], we verified the “if” part of this conjecture for groups $G$ having an open subgroup $H \in \{ FIA \}$ such that $[G : H] < \infty$.

Let $G$ be a locally compact group. If both Conjecture A and Conjecture B are correct, then there is a discontinuous homomorphism from $L^1(G)$ if and only if there is one from $C^*(G)$. This leads to the following question: Given a discontinuous homomorphism $\theta$ from $C^*(G)$ into a Banach algebra, is $\theta L^1(G)$ also discontinuous?

In this note, we investigate this question from a more abstract point of view: Let $A$ be an $A^*$-algebra with enveloping $C^*$-algebra $C^*(A)$, and let $\theta$ be a discontinuous homomorphism from $C^*(A)$ into a Banach algebra. Is $\theta | A$ discontinuous? We prove a fairly general theorem which asserts that for many $A^*$-algebras $A$ a homomorphism $\theta$ from $C^*(A)$ into a Banach algebra is continuous if and only if $\theta | A$ is continuous. We use this theorem to confirm the “if” part of Conjecture B for certain groups $G$. In particular, we obtain a generalization of [Run, Corollary 5.2].

1. **Homomorphisms from $A^*$-algebras.** A $C^*$-norm on an $A^*$-algebra $A$ is an algebra norm $| \cdot |$ on $A$ satisfying

$$|a^* a| = |a|^2 \quad (a \in A).$$

A Banach $A^*$-algebra endowed with a $C^*$-norm is called an $A^*$-algebra. The most prominent examples of $A^*$-algebras are $C^*$- and group algebras. Every $A^*$-algebra $A$ has a largest $C^*$-norm. When writing $| \cdot |$ we always mean this largest $C^*$-norm; we denote the original Banach algebra norm on $A$ by $\| \cdot \|$. The completion of $A$ with respect to $| \cdot |$ is called the enveloping $C^*$-algebra of $A$ and denoted by $C^*(A)$. In case $A = L^1(G)$ for a locally compact group $G$, we have $C^*(A) = C^*(G)$.

Let $A$ be an $A^*$-algebra. We let $\text{Prim}^*_c(A)$ denote the collection of the kernels of the (topologically) irreducible $^*$-representations of $A$; $\text{Prim}^*_c(A)$ is naturally endowed with the Jacobson topology. In case $A$ is a $C^*$-algebra, $\text{Prim}^*_c(A)$ can be identified with $\text{Prim}(A)$, the space of primitive ideals of $A$.

Recall that $A$ is said to be

- **hermitian** if $\sigma_A(a) \subset \mathbb{R}$ for all self-adjoint $a \in A$,
- **$^*$-regular** if the map $\text{Prim}(C^*(A)) \ni P \mapsto P \cap A$ is a homeomorphism between $\text{Prim}(C^*(A))$ and $\text{Prim}^*_c(A)$,
- **locally regular** if there is a dense subset $S$ of the set of self-adjoint elements of $A$ such that for all $a \in S$ the Banach subalgebra of $A$ generated by $a$ is regular.

**Remarks.** 1. The question whether the map $\text{Prim}(C^*(A)) \ni P \mapsto P \cap A$ is a homeomorphism seems to have been investigated for the first time in [B-L-Sch-V] (in the context of group algebras).

2. Local regularity was introduced by B. A. Barnes in [Bar 1]. It implies $^*$-regularity ([Bar 1, Theorem 4.3]).

Suppose $F$ is a closed subset of $\text{Prim}^*_c(A)$. We call $F$ a set of synthesis for $A$ if $\ker(F)$ is the only closed ideal of $A$ whose hull equals $F$.

For any subset $S$ of $C^*(A)$ we write $S^*$ for the closure of $S$ in $C^*(A)$ with respect to $\| \cdot \|$. If $S \subset A$ we denote the closure of $S$ with respect to $\| \cdot \|$ by $S^*$.  

**Lemma 1.1.** Let $A$ be a hermitian, locally regular $A^*$-algebra, and let $I$ be a (not necessarily closed) ideal of $C^*(A)$ whose hull is a set of synthesis for $A$. Then $(I \cap A)^* = I \cap A$.  

**Proof.** First, note that $I \cap A$ is $\| \cdot \|$-closed. Further, it follows easily from the $^*$-regularity of $A$ that $I^\perp$ and $I^\perp \cap A$ have the same hull, say $F$. By [H-K-K, Lemma 1.2], there is an ideal $J(F)$ of $A$ whose hull equals $F$ and which is contained in every ideal of $A$ with this hull. This means that $J(F) \subset I^\perp \cap A$, and since $F$ is a set of synthesis for $A$, we have $J(F)^* = I^\perp \cap A$. From the construction of $J(F)$ in the proof of [H-K-K, Lemma 1.2], it is evident that $J(F)$ is contained in the Pedersen ideal $\mathcal{P}(I^\perp)$ of $I^\perp$. Since by [Ped, Theorem 5.6.1], $\mathcal{P}(I^\perp) \subset I^\perp \cap A$, we have

$$J(F) \subset \mathcal{P}(I^\perp) \cap A \subset I^\perp \cap A \subset I^\perp \cap A.$$ 

Taking the closures with respect to $\| \cdot \|$, we obtain

$$I^\perp \cap A = J(F)^* \subset (I \cap A)^* \subset I^\perp \cap A,$$

i.e. $(I \cap A)^* = I^\perp \cap A$ as claimed.  

For the main result of this section recall two concepts from automatic continuity.
Let $A$ and $B$ be Banach algebras, and let $\theta : A \to B$ be a homomorphism. Then
\[
S(\theta) := \{ b \in B : \text{there is a sequence } \{a_n\}_{n=1}^{\infty} \text{ in } A \\
\text{such that } a_n \to 0 \text{ and } \theta(a_n) \to b \}
\]
is called the separating space of $\theta$. It is easy to see that $S(\theta)$ is a closed ideal of the closure of $\theta(A)$, and obviously, by the closed graph theorem, $\theta$ is continuous if and only if $S(\theta) = \{0\}$. For a comprehensive account of the properties of $S(\theta)$, see the monographs [Dal] and [Sin 2].

Equally important is the continuity ideal of $\theta$, defined as
\[
I(\theta) := \{ a \in A : \theta(a)S(\theta) = S(\theta)\theta(a) = \{0\} \}.
\]
It is easy to see that $I(\theta)$ is indeed an ideal of $A$. The continuity ideal is of special importance if $A$ is a $C^*$-algebra ([Sin 1]). In particular, $I(\theta)^{-}$ has finite codimension in this case.

**Theorem 1.2.** Let $A$ be a hermitian, locally regular $A^*$-algebra with the following properties:

(I) Every closed ideal of $A$ with finite codimension has a bounded left approximate identity.

(II) Every finite subset $F$ of $\text{Prim}(A)$ such that each $P \in F$ has finite codimension is a set of synthesis for $A$.

Suppose $\theta$ is a homomorphism from $C^*(A)$ into a Banach algebra. Then the following are equivalent:

(i) $\theta$ is continuous.

(ii) $\theta | A$ is continuous.

(iii) $I(\theta) \cap A$ is closed (with respect to $\| \cdot \|$).

**Proof.** The implications from (i) to (ii) and from (ii) to (iii) are trivial. Suppose $I := I(\theta) \cap A$ is $\| \cdot \|$-closed. Obviously, $I(\theta)^{-}$ is a left Banach $I$-module. From Lemma 1.1, we conclude that $I$ has finite codimension in $A$. Hence, by assumption, $I$ has a bounded left approximate identity, say $\{c_n\}$. Note that
\[
I^-% = (I(\theta) \cap A)^- = (I(\theta)^{-} \cap A)^- \quad \text{by Lemma 1.1} \]
\[
= I(\theta)^{-} \quad \text{by [Bar 1, Theorem 4.2].}
\]
As a consequence, $\{c_n\}$ is also a bounded approximate identity for the left Banach $I$-module $I(\theta)^{-}$. Let $c_0(I(\theta)^{-})$ denote the left Banach $I$-module of all sequences $\{x_n\}_{n=1}^{\infty}$ in $I(\theta)^{-}$ such that $x_n \to 0$. It is easy to see that $\{c_n\}$ is a bounded approximate identity for $c_0(I(\theta)^{-})$ as well. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence belonging to $c_0(I(\theta)^{-})$. The module version of Cohen’s factorization theorem ([B-D, Theorem 11.10]) then asserts that there is $a \in I$ and a sequence $\{y_n\}_{n=1}^{\infty}$ in $I(\theta)^{-}$ such that
\[
x_n = ay_n \quad (n \in \mathbb{N}) \quad \text{and} \quad |y_n| \to 0.
\]
Since $I \subset I(\theta)$, this means that
\[
\theta(x_n) = \theta(ay_n) \to 0,
\]
i.e. $\theta(I(\theta)^{-})$ is continuous. Since $I(\theta)^{-}$ has finite codimension in $C^*(A)$, this implies the continuity of $\theta$.

**Remarks.** 1. Assumption (I) of Theorem 1.2 is automatically satisfied when $A$ is amenable ([Hel, Proposition VII.2.31]). If $G$ is a locally compact group, then $L^1(G)$ satisfies (I) if and only if $G$ is amenable.

2. We do not know if in Theorem 1.2 the demand that $A$ be hermitian and locally regular cannot be relaxed. Weakening this hypothesis would require a substitute for [H-K-K, Lemma 1.2] in the proof of Lemma 1.1.

**2. Applications to $L^1(G)$.** We now wish to apply Theorem 1.2 to the special case of a group algebra.

Recall the definitions of some classes of locally compact groups (compare [Pal]):

[Moore]: Moore groups. Groups all of whose irreducible, unitary representations are finite-dimensional.

[MAP]: Maximally almost periodic groups. Groups $G$ such that the finite-dimensional, irreducible unitary representations of $G$ separate its points.

[PG]: Groups with polynomial growth. Groups $G$ such that for each compact neighborhood $K$ of 1 there is $k \in \mathbb{N}$ such that
\[
|K^n| = O(n^k) \quad (n \in \mathbb{N})
\]
\[
|K^n| \text{ denoting Haar measure of } K^n.
\]

[Her]: Hermitian groups. Groups $G$ such that $L^1(G)$ is hermitian.

Our first result on homomorphisms from group algebras is a rather straightforward application of Theorem 1.2.

**Theorem 2.1.** Let $G \in [PG] \cap [Her]$, and let $\theta$ be a homomorphism from $C^*(G)$ into a Banach algebra. Then $\theta$ is continuous if and only if $\theta L^1(G)$ is continuous. In particular, if there is $n \in \mathbb{N}$ such that $G$ has an infinite number of inequivalent, $n$-dimensional, irreducible unitary representations, then there is a discontinuous homomorphism from $L^1(G)$ into a Banach algebra.
Proof. Obviously, $L^1(G)$ is hermitian. Also, by [Bar, Theorem 4.1], $L^1(G)$ is locally regular. Further, $L^1(G)$ is amenable by [B-L-Sch-V], and finally, by [Bar 2, Theorem 12], every finite subset $F$ of Prim$_*(L^1(G))$ such that each $P \in F$ has finite codimension is a set of synthesis for $L^1(G)$. All in all, $L^1(G)$ satisfies the assumptions of Theorem 1.2. This proves the first part of the present theorem.

Now, suppose there is $n \in \mathbb{N}$ such that $G$ has an infinite number of inequivalent, $n$-dimensional, irreducible unitary representations. Then $C^*(G)$ has an infinite number of inequivalent, $n$-dimensional, irreducible *-representations. By [A-D, Theorem 2.5], there is a discontinuous homomorphism from $C^*(G)$ into a Banach algebra; its restriction to $L^1(G)$ is discontinuous by the first part of the theorem.

Remarks. 1. The second part of Theorem 2.1 clearly subsumes [Run, Corollary 5.2].

2. For abelian $G$, the first part of Theorem 2.1 yields in particular that if there is a discontinuous homomorphism from $C_0(G)$ into a Banach algebra, then there is one from $L^1(G)$. This rather special case was already proved in [Lau], several years before the question whether there are discontinuous homomorphisms from commutative $C^*$-algebras was settled.

Concluding, we wish to confirm the "if" part of Conjecture B for another class of locally compact groups.

If $G$ is a locally compact group, we use $G_1$ to denote the component of $G$ containing 1. It is easy to see that $G_1$ is a closed, normal subgroup of $G$. If $G/G_1$ is compact, we call $G$ almost connected.

Theorem 2.2. Let $G$ be an almost connected group, and suppose there is $n \in \mathbb{N}$ such that $G$ has an infinite number of inequivalent, $n$-dimensional, irreducible unitary representations. Then there is a discontinuous homomorphism from $L^1(G)$ into a Banach algebra.

Proof. Let $N$ denote the intersection of the kernels of all $n$-dimensional, irreducible unitary representations of $G$. Clearly, $N$ is a closed, normal subgroup. It is easy to see that the class of almost connected groups is stable under taking quotients, i.e., $G/N$ is almost connected as well. Moreover, $G/N \in [MAP]$ by the definition of $N$. By [G-M, Theorem 2.18], this means that $G/N \in [Moore]$. By Theorem 2.1 (or alternatively by [Run, Corollary 5.2]) there is a discontinuous homomorphism $\theta$ from $L^1(G/N)$ into a Banach algebra. The canonical map $\pi : L^1(G) \to L^1(G/N)$ being open, we conclude that $\theta \circ \pi$ is discontinuous.

Remarks. 1. Let $[X]$ be any class of locally compact groups such that (a) $[X]$ is stable under the formation of quotients, and (b) $[X] \cap [MAP] \subset [PG] \cap [Her]$. Then, if $G \in [X]$, and if there is $n \in \mathbb{N}$ such that $G$ has an infinite number of inequivalent, $n$-dimensional, irreducible unitary representations, the same argument as in the proof of Theorem 2.2 shows that there is a discontinuous homomorphism from $L^1(G)$.

2. Both Theorem 2.1 and Theorem 2.2 are certainly not optimal: Let $G$ be a locally compact group having an infinite number of inequivalent, 1-dimensional, irreducible unitary representations, i.e., characters. Then $N$ defined as in the proof of Theorem 2.2 is the closed commutator subgroup of $G$. Consequently, $G/N$ is abelian with $G/N$ infinite. Therefore, there is a discontinuous homomorphism from $L^1(G/N)$ and hence one from $L^1(G)$. For $r \in \mathbb{N}$, let $F_r$ denote the free group of $r$ generators. The preceding argument shows that there is a discontinuous homomorphism from $\ell^1(F_r)$ although $F_r$ is neither almost connected, nor, for $r \geq 2$, does it belong to $[PG] \cap [Her]$.

References


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