Linear operations in Saks spaces (I)

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This paper deals with metric spaces composed of elements of the unit sphere of a linear normed space, the metric of which is defined (see 1.3) by means of another norm, not necessarily homogeneous. The spaces of this kind may be considered as pseudolinear in a certain sense, and some investigations of Banach spaces can be adapted to the spaces of this kind 1).

1.1. Let $X$ be a linear space. A functional $\|x\|$ defined in $X$ will be called a $B$-norm if it satisfies the following conditions:

(a) $\|x\|=0$ if and only if $x=0$,

(b) $\|x+y\| \leq \|x\| + \|y\|$, 

(c) $\|\lambda x\| = |\lambda| \|x\|$, $\lambda$ being any real number. 

Each functional $\|x\|$ satisfying the above conditions (a), (b), and the following one:

(c') if the sequence $\{\lambda_n\}$ of real numbers tends to 0 and $\|x_n-x\| \to 0$, then $\|\lambda_n x_n - \lambda x\| \to 0$ 

will be said to be a $F$-norm.

Any functional $\|x\|$ satisfying the conditions (b) and (c), or (b) and (c'), will be termed a $B$- or $F$-pseudonorm respectively.

A Banach space or a Fréchet space is a linear space $X$ provided with a $B$- or $F$-norm (i.e. Banach norm or Fréchet norm) respectively and such that the distance

$$d(x,y) = \|x-y\|$$

makes $X$ a complete metric space.

1) The results of this paper were presented September 26th 1948 at the VI Polish Mathematical Congress in Warsaw. The second part of the present paper (to appear) will deal with investigation of sequences of operations and with applications of the results of part I.
If this distance does not define a complete space, \( X \) will be said to be an *incomplete Banach space* or *incomplete Fréchet space* respectively \(^1\).

1.1. We adopt the following symbols to denote some concrete Banach and Fréchet spaces (addition of elements, multiplication by real numbers, and definition of norm having in each case the usually accepted meaning):

\[ L^a \] — the space of the sequences \( \{ a_\alpha \} \) for which

\[
\sum_{\alpha \in A} |a_\alpha|^a < \infty \quad (a > 0),
\]

\[ m \] — the space of the bounded sequences,

\[ m_0 \] — the space of the sequences converging to 0,

\[ L^a \] — the space of the functions \( x = x(t) \) for which

\[
\int_0^1 |x(t)|^a dt < \infty \quad (a > 0),
\]

\[ M \] — the space of the functions which are measurable and equivalent to bounded functions in a closed interval \( \langle a, b \rangle \),

\[ S \] — the space of the functions measurable in \( \langle a, b \rangle \) or, more generally, in a \( n \)-dimensional interval,

\[ C \] — the space of the functions continuous in \( \langle a, b \rangle \),

\[ C^a \] — the space of the functions equivalent to continuous functions in \( \langle a, b \rangle \),

\[ V \] — the space of the functions of bounded variation in \( \langle a, b \rangle \),

\[ V^a \] — the space of the functions equivalent to functions of bounded variation in \( \langle a, b \rangle \),

\[ V_C \] — the subspace of the space \( V \), composed of the continuous functions.

The spaces \( L^p \) and \( L^\infty \) are for \( 0 < a < 1 \) Fréchet spaces, and are not equivalent to any Banach space; the same property has also the space \( S \).

\[ 1 \] S. Banach in his monograph *Théorie des opérations linéaires*, Monografie Matematyczne, Warsaw 1932, calls these spaces spaces of type \( B \) and of type \( F \) respectively. Since in the recent literature the spaces of type \( B \) are generally called Banach spaces, I adopt for the spaces of type \( F \) the term Fréchet spaces, for M. Fréchet was the first to call attention to this class of abstract spaces.

We write \( x_\alpha(t) \to x(t) \) to denote that the sequence \( \{ x_\alpha(t) \} \) converges asymptotically to \( x(t) \) in the interval considered.

We denote by \( \text{sup} \{ x(t) \} \) the least number \( k \) for which the set of the elements \( t \) satisfying the inequality \( x(t) < k \) is a null-set.

1.2. Let \( \| \cdot \| \) be a \( B \)- or \( F \)-norm in a linear space \( X \).

The sequence \( \{ x_\alpha \} \) of elements of \( X \) will be called *bounded with respect to the norm* \( \| \cdot \| \) if \( \alpha_0 \to 0 \) implies \( \alpha_0 x_\alpha \to 0 \) for each sequence \( \{ \alpha_\alpha \} \) of real numbers \(^1\).

If any sequence composed of elements of a set \( X_0 \subseteq X \) is bounded with respect to the norm \( \| \cdot \| \), the set \( X_0 \) itself will be said to be *bounded with respect to the norm* \( \| \cdot \| \).

If there is no risk of mistake, we shall omit the words "with respect to the norm" in the above defined term.

A necessary and sufficient condition for a sequence \( \{ x_\alpha \} \) of elements of \( X \) (for a set \( X_0 \subseteq X \)) to be bounded, is given any \( \epsilon > 0 \), the existence of a \( \delta_0 > 0 \) such that \( \| \alpha_0 \| < \delta_0 \) implies \( \| \alpha_0 x_\alpha \| < \epsilon \) for \( n = 1, 2, \ldots \) (\( \| \alpha_0 \| < \epsilon \) for every \( x \in X_0 \)).

1.21. If a sequence \( \{ x_\alpha \} \) of elements of \( X \) is bounded with respect to the norm \( \| \cdot \| \), then \( \text{sup} \| x_\alpha \| < \infty \).

The boundedness of the sequence implies the existence of a \( \delta > 0 \) such that \( \| \alpha x_\alpha \| < 1 \) for \( n = 1, 2, \ldots \) and \( |\alpha_0| < \delta \); putting \( k = E(1/\delta) \) and \( \vartheta = 1 - k \delta \) we get \( \| x_\alpha \| < k \| \delta x_\alpha \| + \| \alpha x_\alpha \| k + 1 \).

The condition \( \text{sup} \| x_\alpha \| < \infty \) is in the case of a general \( F \)-norm only necessary for the boundedness of the sequence. In the case, however, of \( \| \cdot \| \) being a \( B \)-norm, it is also sufficient; the set \( X_0 \) is bounded with respect to a \( B \)-norm \( \| \cdot \| \) if and only if there exists a \( K > 0 \) such that \( \| x \| < K \).

1.22. The following property of \( F \)-norms seems to be useful in the sequel:

Given \( \varphi > 0 \) and \( \theta_0 > 0 \), there exists a \( \theta_1 \) such that \( 0 < \theta_1 < \varphi \), and that \( \| x \| < \varphi \) and \( |\alpha_0| < \theta_0 \) imply \( \| \alpha x \| < \varphi \).

\[ 1 \] This concept of boundedness is due to S. Banach; see S. Mazur and W. Orlicz, *Über Folgen linearer Operationen*, Studia Mathematica 4 (1933), p. 192 - 197.
1.5. Let $X$ be a Banach space or an incomplete Banach space (fundamental space) with the norm $\| \cdot \|$, and let $\| \cdot \|$ be another norm defined in $X$. In the set $R$ of elements $x \in X$ satisfying the inequality $\|x\| < 1$ we define the distance between the elements $x, y \in R$ by the formula
\[
d(x, y) = \|x - y\|.
\]
If this metric space is complete, it will be termed a Saks space and denoted by $X_s$.

It should be pointed out that it is not supposed that the space $X$ with the distance defined as equal to $\|x - y\|$ or to $\|x - y\|^\alpha$ is a complete space (this is really the case in the example (VII) which will be considered in 1.4, p. 246).

The following notion of limit defined in the whole of the space $X$ is quite naturally associated with every Saks space: a sequence $\{x_n\}$ of elements of $X$ is said to be (l)-convergent to $x_0$ if there exists a $K$ such that $\|x_n\| < K$ for $n = 1, 2, \ldots$, and $\|x_n - x_0\|^\alpha \to 0$ (the space $X_s$ being complete, this implies $\|x_0\| < K$).

We shall write $x_n \to x_0$ to denote that the sequence $\{x_n\}$ is (l)-convergent to $x_0$; if $x_n \in X_s$, $x_n \to x_0$ means that the sequence $\{x_n\}$ tends to $x_0$ in the sense of the metric (1).

1.51. Denote by $K(x_0, \varrho)$ the open sphere with the centre $x_0$ and radius $\varrho$ in the space $X_s$.

We shall consider Saks spaces satisfying some of the following three conditions:

$(E_1)$ Given any $x_0 \in X_s$ and $\varrho > 0$, there exists a $\delta > 0$ such that every element $x \in X_s$ for which $d(x, 0) < \delta$ can be written in the form $x = x_0 - \varrho x$, with $x_0, x \in K(x_0, \varrho)$.

$(E_2)$ If $x_n \in X_s$, $x_n \to 0$, $\sum_n \alpha_n = 0$ and $\alpha_n > 0$, then there exists an increasing sequence $\{k_n\}$ of indices and a sequence $\{x_n\}$ such that
\[
d(x_n, x_{k_n}) < \alpha_n \quad \text{for} \quad n = 1, 2, \ldots,
\]
\[
\sum_{n=1}^{\infty} \alpha_n \sum_{k_n} x_{k_n} \in X_s
\]
and
\[
\sum_{n=1}^{\infty} \alpha_n x_{k_n} \to x_0 \quad \text{as} \quad i \to \infty.
\]

$(E_3)$ Let $\varepsilon(x)$ be a positive real-valued function such that $\lim_{x \to 0} \varepsilon(x) = 0$ and given any $\varepsilon > 0$ there exists a sequence $\{x_n\}$ of elements satisfying the conditions (ii), (iii) and $d(x_n, x_{k_n}) < \varepsilon(\|x_n\|^\alpha)$ for $n = 1, 2, \ldots$.

If in a Banach space (with the norm $\| \cdot \|$) we put $\|x\|^\alpha = \|x\|$, the corresponding space $X_s$ obviously satisfies all the conditions $(E_1)$, $(E_2)$ and $(E_3)$. The point of the matter is that in the most important applications we choose in $X$ as $\| \cdot \|$ another norm, which permits us to get a pseudo-banachian space $X_s$ having such properties as separability and compactness — contrarily to the unit sphere of the Banach space $X$.

1.32. We shall say that the condition $(E_1)$ is satisfied at the point $x_0 \in X_s$ if there exists a $\delta > 0$ such that
\[
d(x_0, 0) = \|x_0\|^\alpha < \delta
\]
denote the point $(E_3)$ be satisfied at any point of a set $X_s$. Then the space $X_s$ satisfies the conditions $(E_1)$, $(E_2)$ and $(E_3)$.

We first prove that the condition $(E_1)$ is satisfied. Given a sphere $K(x_0, \varrho)$, there exists a $x_0 \in K(x_0, \varrho) \cap X_s$; then there exists, by hypothesis, a $\delta > 0$ such that $\|x\|^\alpha < \delta$ implies $x = x_0 + y \in X_s$, and we may suppose that $\delta = \varepsilon(\delta/\varrho)$. Since
\[
y = x_0 - x,
\]
and
\[
d(x_0, x_0) < \delta \quad \text{and} \quad d(x_0, x_0) < \varepsilon(\|x_0\|^\alpha) < \delta,
\]
we see that the space $X_s$ satisfies the condition $(E_1)$.

1 The idea of such class of spaces and of using the condition $(E_1)$ has been suggested to me by the paper of S. Saks, On some functional transformations of the American Mathematical Society 35 (1929), 2, 549-556, and was published first by A. Alexiewicz in his paper On sequences of operations (l), this volume, p. 1-50. The notion of $(l)$-convergence appears in full generality first in the paper of A. Alexiewicz, On sequences of operations (l), ibidem, p. 209-256, and in his paper A generalised convergence in linear spaces, to appear in Comptes Rendus de la Société des Sciences et des Lettres de Wrocław. Some examples of linear spaces with $(l)$-convergence had been noted by G. Fichtenthal in his paper Sur les fonctionnelles continues au sens généralisé, Recueil Mathématique du Moscou 4 (1958), p. 199-214.
We now prove that the condition (Σ₂) is satisfied. Let the sequence \( \{x_n\} \) be \( (\beta) \)-convergent to 0, and let \( \epsilon_n > 0 \) tend to 0. We can easily construct a sequence \( \{k_i\}_{i=1,2,...} \) of indices, a sequence \( \{\lambda_{i}\}_{i=1,2,...} \) of positive numbers and a sequence \( \{y_i\} \) of points of \( X_0 \) such that for \( i = 1, 2, ... \):

\[ 1^\circ \quad \epsilon_i + \|x_{k_i}\| < \delta_{i-1}/2, \]
\[ 2^\circ \quad \delta_i < \delta_{i-1}/2, \]
\[ 3^\circ \quad \|y_i - x_{k_i}\| < \epsilon_i, \]
\[ 4^\circ \quad y_i \in X_0 \text{ and } \|y_i\| < \delta_i \text{ imply } y_i + y \in X_0. \]

We shall prove that for arbitrary indices \( i \) and \( l \), and for every sequence \( \{\lambda_i\} \) composed of \( 0 \)'s and \( 1 \)'s

\[ y_i + \lambda_{i+1} y_{i+1} + \cdots + \lambda_{i+l} y_{i+l} \in X_0. \]

Since \( 1^\circ \) and \( 3^\circ \) imply \( \|y_{i+1}\| < \delta_i/2 \), by \( 4^\circ \) we get \( 2 \) for \( l = 1 \) and for \( i = 1, 2, ... \). Suppose \( 2 \) holds for \( l = l_0 - 1 \) and \( i = 1, 2, ... \). Then

\[ \lambda_{i+1} y_{i+1} + \lambda_{i+2} y_{i+2} + \cdots + \lambda_{i+l_0} y_{i+l_0} \in X_0, \]

and by \( 1^\circ \), \( 2^\circ \) and \( 3^\circ \)

\[ \|\lambda_{i+1} y_{i+1} + \cdots + \lambda_{i+l_0} y_{i+l_0}\| \leq \sum_{j=1}^{l_0} \|y_{i+j}\| < \frac{1}{2} \sum_{j=1}^{l_0} \delta_{i+j} < \delta_i; \]

together with \( 4^\circ \) this implies \( 2 \) for \( l = l_0 \) and \( i = 1, 2, ... \). Putting \( \delta_i = y_i \), we obviously get the condition 1.31(\( ii \)); \( 3 \) implies immediately the condition 1.31(\( iii \)), and, the space \( X_0 \) being complete, the inequality

\[ \sum_{i=1}^{\infty} \|y_i\|^2 < \frac{1}{2} \sum_{i=1}^{\infty} \delta_i < \delta_0, \]

implies the condition 1.31(\( iii \)).

Finally, we prove the condition (Σ₂'). Let \( \epsilon(x) \) be a positive real-valued function such that \( \lim_{x \to 0} \epsilon(x) = 0 \); \( \{x_n\}_{n=1,2,...} \) being a sequence \( (\beta) \)-convergent to 0, it is easy to see that there exist sequences \( \{k_i\} \), \( \{\lambda_i\} \) and \( \{y_i\} \) satisfying the conditions \( 2^\circ \) and \( 4^\circ \), and such that for \( i = 1, 2, ... \):

\[ 1^\circ \quad \epsilon_i + \|x_{k_i}\| < \delta_{i-1}/2, \]
\[ 3^\circ \quad \|y_i - x_{k_i}\| < \epsilon_i \text{ and } \|y_i\| < \delta_i, \]
\[ 4^\circ \quad y_i \in X_0. \]

Arguing quite similarly as above we prove that \( 2 \) is satisfied for every \( i \) and \( l \), and putting \( \delta_i = y_i \), we get a sequence satisfying the conditions of (Σ₂').

1.4. We now give some examples of Saks spaces. In each case we indicate the fundamental Banach space \( X \), the corresponding norm \( \| \| \) and the second norm \( \| \|^* \). In all these examples, except \( (IV) \) and \( (VII) \), the defined Saks spaces are separable; in these cases we indicate a subset \( X_0 \) of \( X \) which is complete and dense. We omit the proofs of completeness and separability which are easy. The proofs of the conditions (Σ₂) and (Σ₂') are given in 1.52.

Examples of Saks spaces satisfying the conditions (Σ₂) (Σ₂') and (Σ₂').

1. \( X \) is the space \( m \) of bounded sequences \( x = (a_n) \) of real numbers, and

\[ \|x\| = \sup_{n \geq 0} |a_n|, \quad \|x\|^* = \sum_{n=0}^{\infty} \frac{1}{2^n} |a_n|; \]

\( X_0 \) is the set of sequences of rationals absolutely less than 1, and almost all of which are equal to 0.

2. \( X \) is the space \( m_p \) of the sequences \( x = (a_n) \) satisfying the inequality \( |a_n| < k p_n \), the sequence \( p = (p_n) \) being a fixed sequence of positive numbers, and

\[ \|x\| = \sup_{n \geq 0} |a_n|/p_n, \quad \|x\|^* = \sum_{n=1}^{\infty} \frac{1}{2^n} |a_n|; \]

\( X_0 \) is the set of sequences of rationals with the \( n \)th term absolutely less than \( p_n \) and almost all terms equal to 0.

3. \( X \) is the space \( m_{\infty} \) of bounded sequences \( x = (a_n) \) of real numbers, summable to 0 by a linear sequence-transformation \( T \) corresponding to the matrix \( (a_{nm}) \) satisfying the Toeplitz conditions.
Linear operations in Saks spaces (I).

\( (6') \quad \lim_{h \to 0} \int_0^\infty |K(t+h,t) - K(t,t)| \, dt = 0 \quad \text{for} \quad r \geq a; \)

\( \|x\|_\infty = \sup_{\langle x, y \rangle} |x|, \quad \|x\|_* = \max_{\langle x, y \rangle} |T(x,y)| + \sum_{n=1}^\infty \frac{1}{2^n} \|x\|_n, \text{ where } T(x) = \sum_{n=1}^\infty a_n x_n \; \text{and} \; X_0 \text{ is the set of functions } x(t) \text{ for which there exists a rational} \)

\( t_0 \text{ such that} \quad 1^o x(t) \leq 1 \quad \text{for all} \; t, \)

\( 2^o x(t) \text{ coincides with a polynomial with rational coefficients for} \quad 0 \leq t < t_0, \)

\( 3^o x(t) = 0 \quad \text{for} \quad t \geq t_0, \)

\( (IV) [IV'] \quad X \text{ is the space } M_{r} \text{ of measurable functions } x = x(t) \text{ equivalent to bounded measurable functions [to continuous functions] in } \langle 0, \infty \rangle \text{ and such that there exists the limit} (4) \text{ with } K(t,t) \text{ satisfying the conditions} \; (6)-(6'); \)

\( \|x\|_\infty = \sup_{\langle x, y \rangle} |x|, \quad \|x\|_* = \max_{\langle x, y \rangle} |T(x,y)| + \sum_{n=1}^\infty \frac{1}{2^n} \|x\|_n, \text{ where } \|x\|_n = \sup_{\langle x, y \rangle} |x(y)|; \; X_0 \text{ is in the class of the space } M_{r} \text{ for the same set as in the space} \; (IV), \text{ but with the restriction that the functions } x(t) \text{ are continuous.} \)

\( \text{The space } M_{r} \text{ is non-separable.} \)

\( (V) \quad X \text{ is the space } D \text{ of functions } x(t) \text{ vanishing for } t = a \text{ and satisfying in } \langle a, b \rangle \text{ the Lipschitz condition; } \)

\( \|x\| = \sup_{t, t' \in \langle a, b \rangle} \frac{|x(t') - x(t)|}{t - t'} \; \text{ and} \; \|x\|_* = \int_0^b |x'(t)| \, dt; \)

\( X_0 \text{ is the set of polynomials with rational coefficients, vanishing for } t = a, \text{ the derivatives of which are absolutely less than 1.} \)

\( (VI) \quad X \text{ is the space } C^0 \text{ of functions } x = x(t) \text{ bounded in } \langle a, b \rangle, \text{ continuous except at one fixed point } t, \text{ and vanishing at } t = r; \)

\( \|x\|_\infty = \sup_{\langle a, b \rangle} |x(t)|, \quad \|x\|_* = \sum_{n=1}^\infty \frac{1}{2^n} \|x\|_n. \)

\( \text{To define } \|x\|_n \text{ we choose two sequences } \{c_i\} \text{ and } \{d_i\} \text{ such that} \)

\( a < c_1 < c_1 < \ldots < c_i < c_j = r, \quad t \to r, \quad r < \ldots < t_{n+1} < t_n < c_i < b, \quad c_n \to r; \)

\( T \text{ the kernel } K(t,t) \text{ satisfying the conditions:} \)

\( (6) \quad \int_0^\infty |K(t,t)| \, dt < K \quad \text{for} \quad t \geq a, \)

\( (6') \quad \lim_{t \to a^+} |K(t,t)| \, dt = 0 \quad \text{for} \quad b \geq 0. \)
if $\tau$ is an end-point of $\langle a, b \rangle$, we pick out only one of these sequences. We denote by $A_\tau$ the union of the intervals $\langle a, t \rangle$ and $\langle t, b \rangle$, or one of these intervals in the case when $\tau$ is an end-point of $\langle a, b \rangle$, and define the pseudonorm $\|x\|_a$ by the formula

$$\|x\|_a = \max_{\tau \in A_\tau} |x(\tau)|.$$

$X_0$ is the set of polynomials of the form $m(\tau) - m(\tau_0)$, where $m(\tau)$ is any polynomial with rational coefficients, such that $|m(\tau) - m(\tau_0)| < 1$.

Remark. In (VI) we can allow the interval $\langle a, b \rangle$ to be infinite, and $\tau = \pm \infty$; in this case we omit the condition $x(\tau) = 0$.

(VII) $X$ is the space $\mathcal{V}$ of bounded functions vanishing at $t = \tau$ and being of bounded variation in all the intervals $\langle a, \tau \rangle$ and $\langle \tau, b \rangle$, where the point $\tau$ is fixed between $a$ and $b$, and $\epsilon$ runs down the set of positive numbers;

$$x = \sup_{t \in (a, b)} |x(t)|, \quad \|x\|_\epsilon = \sum_{\tau \in A_\tau} \frac{1}{2^n} \|x\|_a.$$

The pseudonorms $\|x\|_\epsilon$ are defined as follows: we choose the sequences $\{t^*_n\}$ and $\{t^*_n\}$ as in (VI), and put

$$\|x\|_\epsilon = \sup_{\tau \in A_\tau} |x(\tau)| + \|x(\tau)\|_\epsilon.$$

The space (VII) is non-separable.

1.51. We now prove two lemmas, which we need in 1.52.

(A) Suppose that

(i) $\lim_{n \to \infty} f_n = 0$ for $i = 1, 2, \ldots$,

(ii) $\|f_n\|_\epsilon < K$ for $i, n = 1, 2, \ldots$,

(iii) $\lim_{n \to \infty} f_n = f$ for $n = 1, 2, \ldots$,

(iv) $\lim_{n \to \infty} f_n = 0$.

Then, $\epsilon$ being any positive number and $\delta$ being any integer, there exist $a_0, a_0 + 1, \ldots, a_\epsilon + \delta$ such that

$$\sum_{i=0}^{\epsilon+\delta} a_i = 1,$$

and

$$\sup_{n \in \mathbb{N}} \sum_{i=0}^{\epsilon+\delta} a_i f_i - f < \epsilon.$$

To prove this lemma note first that the hypotheses mean that the sequences $f_0^\infty = \{f_n^\infty = f_0^\infty\}$ converge weakly to $\tau = \{f_\tau\}$ in the space $m_\tau$, and then apply a general theorem of Mazur.

(B) Let the functions $f_i(t)$ be continuous in $\langle a, \infty \rangle$, and suppose that

(i) $\lim_{\tau \to \infty} f_i(\tau) = 0$

for $i = 1, 2, \ldots$ and $\tau \in \langle a, \infty \rangle$,

(ii) $|f_i(\tau)| < K$

for $i = 1, 2, \ldots$ and $\tau \in \langle a, \infty \rangle$,

(iii) the sequence $\{f_i(\tau)\}$ converges to $f(\tau)$ uniformly in every finite interval,

(iv) $\lim_{\tau \to \infty} f_i(\tau) = 0$.

Then, given any $\epsilon > 0$ and an integer $\delta$, there exist $a_0, a_0 + 1, \ldots, a_\epsilon + \delta$, for which the condition (7) and the condition

$$\max_{\tau \in \mathbb{N}} \sum_{i=0}^{\epsilon+\delta} a_i f_i(t) - f(t) < \epsilon$$

are satisfied.

To prove this lemma put $f_0^\infty = \max_{\tau \in \mathbb{N}} \sum_{i=0}^{\epsilon+\delta} a_i f_i(t)$ for $i, n = 1, 2, \ldots$, and $f_0^\infty = 0$. We easily observe that the hypotheses of lemma (A) are satisfied; hence there exist $a_0, a_0 + 1, \ldots, a_\epsilon + \delta$ satisfying the conditions (7) and (8), i.e. such that

$$\sup_{\tau \in \mathbb{N}} \sum_{i=0}^{\epsilon+\delta} a_i f_i(t) - f(t) < \sup_{\tau \in \mathbb{N}} \sum_{i=0}^{\epsilon+\delta} a_i \max_{\tau \in \mathbb{N}} |f_i(\tau) - f(\tau)| < \epsilon,$$

and this gives (8).
The space (I). Choose \( N \) and \( \delta \) so that
\[
\sum_{n=1}^{N} \frac{1}{2^n} < \delta. \tag{9}
\]
and let the sequences \( x = \{a_n\} \) and \( y = \{b_n\} \) have the same meaning as in the proof relative to the space (I). Supposing that \( K > 1 \) we get \( \|yo\| \leq 1 \). Since \( b_i = 0 \) for \( i > s + r \), and since \( i < s + r \) implies
\[
|b_i| < \frac{1}{4K}, \quad \text{and} \quad |a_i| < \frac{1}{4K},
\]
we get \( \|z\| \leq 1 \) and
\[
d(m, xo) = \|xo - y\| = \sup_{i=1}^{s} |T_a(m) - T_a(x)| + \frac{1}{4K} |T_a - T_a(x)| + \frac{1}{4K} \langle x, y \rangle < \frac{1}{4} + \frac{1}{4} + \frac{1}{4} < \delta.
\]
Therefore, \( \delta = \frac{\delta}{10(b-\alpha)} \), \( y \in X_a \), and \( d(y, 0) = |y| < \delta \).

The formulae (3) and (7) imply that the sequences \( \{t_n\} \) satisfy the hypotheses of Lemma 1.51(A); hence for each integer \( s \) there exist \( a_n \) satisfying the condition (7) and such that
\[
\sup_{[a_n]} \left| \sum_{n=s}^{r} T_a(x_n) - T_a(x_n) \right| < \frac{1}{4}.
\]
and let the sequences \( x = \{a_n\} \) and \( y = \{a_n\} \) have the same meaning as in the proof relative to the space (I). Supposing that \( K > 1 \) we get \( \|yo\| \leq 1 \). Since \( b_i = 0 \) for \( i > s + r \), and since \( i < s + r \) implies
\[
|b_i| < \frac{1}{4K}, \quad \text{and} \quad |a_i| < \frac{1}{4K},
\]
we get \( \|z\| \leq 1 \) and
\[
d(m, xo) = \|xo - y\| = \sup_{i=1}^{s} |T_a(m) - T_a(x)| + \frac{1}{4K} |T_a - T_a(x)| + \frac{1}{4K} \langle x, y \rangle < \frac{1}{4} + \frac{1}{4} + \frac{1}{4} < \delta.
\]
Therefore, \( \delta = \frac{\delta}{10(b-\alpha)} \), \( y \in X_a \), and \( d(y, 0) = |y| < \delta \).

The space (III). We first prove the condition (E). We can suppose that \( \phi < 4(b-\alpha) \). Let
\[
\delta = \frac{\delta}{10(b-\alpha)}, \quad y \in X_a, \quad \text{and} \quad d(y, 0) = |y| < \delta.
\]
There exists a set \( A \subset \langle a, b \rangle \) such that \( |A| > (b-\alpha) - \frac{\phi}{4} \) and \( t \in A \) implies \( |y(t)| < \frac{\phi}{4(b-\alpha)} \).

Put
\[
m(t) = \begin{cases} \left[ 1 - \frac{\phi}{4(b-\alpha)} \right] x(t) & \text{for } t \in A, \\ 0 & \text{for } t \notin \langle a, b \rangle - A, \end{cases}
\]
\[
x(t) = m(t) + y(t) \quad \text{for } t \in \langle a, b \rangle.
\]
Obviously \( \|m\| \leq 1 \), and since \( |y(t)| < \frac{\phi}{4(b-\alpha)} \) for \( t \in A \), we get \( |x(t)| \leq 1 \) for \( t \in A \); similarly \( t \in \langle a, b \rangle - A \) implies \( |x(t)| \leq 1 \); hence \( \|z\| \leq 1 \). Finally, writing \( y = z - m \), we get
\[ d(n, x_0) = \|n - x_0\| = \frac{\theta}{4(b-a)} \int_a^b |x_0(t)| \, dt + \frac{\theta}{4(b-a)} \int_a^b |x_0(t)| \, dt < \frac{\theta}{4(b-a)} + \frac{\theta}{4(b-a)} = \theta. \]

\[ d(x, x_0) = \|x - x_0\| < \|n - x_0\| + \|y\| < \frac{\theta}{2} + \delta < \theta. \]

We now prove the condition (Σ₂). Let \( x_1 = x; x_i \downarrow 0, \ e_i \to 0, \) and \( \epsilon_i > 0. \) Since

\[ \int_a^b |x_i(t)| \, dt \to 0, \]

\( k_i \) being chosen freely so that \( \epsilon_i \leq 2(b-a), \) we can easily construct an increasing sequence of indices \( \{k_i\} \) such that

\[ x_{k_i} \leq x_{i+1} - \frac{\epsilon_i}{2}, \quad \text{for } i = 1, 2, \ldots, \]

Denote by \( A_i \) the set of points \( t \in (a, b) \) satisfying the inequality \( |x_i(t)| < \frac{\epsilon_{i+1}}{4(b-a)}. \) Then, write

\[ B_i = A_{i+1} + A_{i+2} + \ldots, \quad C_i = \langle a, b \rangle - A_i. \]

Since \( 2^i \) implies \( |A_i| < \frac{1}{4 \epsilon_{i+1}}, \) we get by \( 1^o \)

\[ |B_i| = \frac{1}{2} \epsilon_i + \frac{1}{2} \epsilon_{i+1} + \ldots < \frac{1}{2} \epsilon_i. \]

The function \( \tilde{x}_i(t) = \begin{cases} x_i(t), & \text{for } t \in C_i - B_i, \\ 1 - \frac{\epsilon_i}{2(b-a)}, & \text{for } t \in A_i - B_i, \\ 0, & \text{for } t \in B_i \end{cases} \)

satisfies the inequality

\[ d(\tilde{x}_i, x_0) = \int_a^b |\tilde{x}_i(t)| \, dt + \frac{\epsilon_{i+1}}{4(b-a)} \int_a^b |x_i(t)| \, dt < \frac{\epsilon_i}{2} + \frac{\epsilon_{i+1}}{2} = \epsilon_i. \]

Suppose \( t \in A_i - (A_{i+1} + A_{i+2} + \ldots); \) then \( t \in B_i_1 \) and \( t \in C_{i+1}, C_{i+2}, \ldots; \) hence

\[ \sum_{i=1}^n |x_i(t)| \leq \left(1 - \frac{\epsilon_i}{2(b-a)}\right) + \frac{\epsilon_n}{4(b-a)} + \frac{\epsilon_{n+1}}{4(b-a)} + \ldots < 1. \]

If \( t \text{ non-} e_{1} = e_{2} = \ldots, \) then \( t \notin C_{1} \cdot C_{2}, \ldots; \) hence

\[ \sum_{i=1}^n \lambda_i \epsilon_i < \frac{\epsilon_n}{4(b-a)} + \frac{\epsilon_{n+1}}{4(b-a)} + \ldots < 1. \]

If \( t \) belongs to an infinity of sets \( A_i, \) i.e. if \( t \notin \lim_{i \to 0} A_i, \) we have \( \lim_{i \to 0} A_i = 0, \) and we see that \( \lambda_i \) being 0 or 1 the element

\[ \sum_{i=1}^n \lambda_i x_i \]

belongs to \( X; \) moreover, \( \sum_{i=1}^n \lambda_i x_i \to x \), for

\[ \|\sum_{i=1}^n \lambda_i \tilde{x}_i\| < \frac{1}{16(b-a)} \sum_{i=1}^n \epsilon_i. \]

The space (III'). The proofs are similar to those concerning the space (III).

The space (IV). In order to prove the condition (Σ₂), we first prove the following lemma:

(*) Given any \( b > 0 \) and \( \epsilon > 0, \) there exists an \( \eta = \eta(b, \epsilon) > 0 \) such that

\[ \int_0^b |x(t)| \, dt < \eta \quad \text{and} \quad \sup_{t \in (0, b)} |x(t)| < 1 \]

imply

\[ \max_{\langle a, b \rangle} \int_0^b K(t, x(t)) \, dt < \epsilon. \]

By (6) there exists a \( \Theta \) such that

\[ \tau > \Theta \implies \int_0^b K(t, x(t)) \, dt < \frac{\epsilon}{2}. \]

By (6') the functions \( \int_0^b K(t, x(t)) \, dt \) are equicontinuous in the interval \( a < \tau < \Theta \) as \( x(t) \) varies over the sphere \( |x(t)| \leq 1; \) hence

\[ \int_0^b K(t, x(t)) \, dt = 0 \quad \text{in} \quad \langle a, \Theta \rangle \]

as \( \int_0^b |x(t)| \, dt \to 0, \) and \( |x(t)| \leq 1 \) in \( \langle 0, b \rangle. \)

Thus the lemma (*) is proved.
Now, let $x_0 = x_0(t)$, and
$$x^{(t)}_0 = x^{(t)}_0(t) = \begin{cases} x_0(t) & \text{for } 0 \leq t \leq 1, \\ 0 & \text{for } t > 1. \end{cases}$$

The conditions (6) imply that the functions $T(x^{(t)}_0, r)$ satisfy the hypotheses of Lemma 1.51(B); hence for every positive integer $s$ there exist $z_s$ satisfying the condition (7) and the condition
$$\max_{s, m} \left| \sum_{k=1}^m T(x^{(t)}_0, r) - T(x_0, r) \right| < \frac{\eta}{8K} \quad \text{with} \quad \frac{\eta}{8K} < 1. \tag{11}$$

Fix $s > 3$ so that $s = N$ implies the inequality (9); then choose $\eta < \min \left\{ \frac{\eta}{2}, \frac{\eta}{16}, \frac{\eta}{16}, \frac{\eta}{8K} \right\}$, and put
$$0 < \delta < \frac{\eta}{2^{s+1} 8K},$$
and put
$$0 < \delta < \frac{1}{2^{s+1} 8K}.$$ 

If $d(y, 0) = \|y\| < \delta$, then
$$0 < \sum_{s=1}^m \int |y(t)| \, dt < \delta; \quad \text{hence}$$
$$\int |y(t)| \, dt < \frac{\eta}{8K}. \tag{12}$$

The set $B$ of those $t$ for which $t \in \langle 0, s + r \rangle$ and $|y(t)| > \frac{\eta}{8K}$

satisfies by (12) the inequality $|B| < \delta$. Let
$$m_0(t) = (1 - \frac{\eta}{8K}) \sum_{s=1}^m \alpha s^{(t)}_0(t) \quad \text{for } t \in \langle 0, \infty \rangle,$$
$$z_0(t) = \begin{cases} y(t) & \text{for } t \in \langle 0, s + r \rangle, \\ 0 & \text{for } t \in \langle s + r, \infty \rangle, \end{cases}$$
$$v_0(t) = \begin{cases} m_0(t) & \text{for } t \in \langle 0, s + r \rangle, \\ 0 & \text{for } t \in \langle s + r, \infty \rangle. \end{cases}$$

By (6), $r \to \infty$ implies $T(m_0, r) \to 0$, $T(z_0, r) \to 0$ and $T(v_0, r) \to 0$.

It is clear that $\int |z_0(t)| \, dt < \eta$; hence by lemma (*)
$$\max_{s, m} \left| \sum_{s=1}^m T(z_0, r) \right| < \frac{\eta}{8K}.$$ 

and since
$$\sum_{s=0}^m \frac{1}{2^s} \int |z_0(t)| \, dt < \eta < \frac{\eta}{16},$$
we get $\|z_0\| < \frac{\eta}{16}$. Similarly $\|v_0\| < \frac{\eta}{16}$.

Put $m = (m_n - v_0) - z_0$ and $z = (m_n - v_0) + (y - z_0)$. The formula
$$z(t) = \begin{cases} 0 & \text{for } t \in B, \\ \left| \sum_{s=1}^m \alpha s^{(t)}_0(t) - T(z_0, r) \right| < \left( 1 - \frac{\eta}{8K} \right) + \frac{\eta}{8K} = 1 & \text{for } t \in \langle 0, \infty \rangle - B, \end{cases}$$
implies $z \in X_s$. Similarly $v \in X_s$. Obviously $y = z - m$. Now
$$d(m_0, x_0) = \|m_0 - x_0\| =$$
$$= \max_{s, m} \left| T(m_0, x_0) - \left( \sum_{s=1}^m \alpha s^{(t)}_0(t) - T(z_0, r) \right) \right| <$$
$$< \max_{s, m} \left| \sum_{s=1}^m T(x^{(t)}_0, r) - T(z_0, r) \right| + \frac{\eta}{8K} \max_{s, m} \left| \sum_{s=1}^m \alpha s^{(t)}_0(t) - T(z_0, r) \right| +$$
$$+ \frac{\eta}{8K} \sum_{s=1}^m \int |x_0(t) - x_0(t)| \, dt <$$
$$< \frac{\eta}{8} + \frac{\eta}{8} + \frac{\eta}{8} + 2\phi < \frac{5\eta}{8}.$$ 

$$d(m_0, x_0) < d(m_0, x_0) + \|y_0\| + \|y_0\| < \frac{5\eta}{8} + \frac{2\eta}{8} < \eta.$$ 

$$d(z, x_0) < d(m_0, x_0) + \|y_0\| + \|y_0\| < \frac{5\eta}{8} + \frac{2\eta}{8} + \eta < \eta.$$ 

We now prove the condition (2). Let $x_1 = x_1(t) \to 0$, $s_1 \to 0$, and $e_1 > 0$. By lemma 1.51(B) there exist functions $x_i^{(t)} = \sum_{s=1}^m \alpha s^{(t)}_0(t)$, where
$$x_i^{(t)}(t) = \begin{cases} x(t) & \text{for } 0 \leq t \leq k, \\ 0 & \text{for } t > k, \end{cases}$$
with $s_1$, $r_i$, $e_i$ satisfying the conditions of this lemma and such that $d(x_i^{(t)}(t), x_i(t)) < \eta_i/2$ for $i = 1, 2, \ldots$. Put $r_1 = s_1 + r_i$ and let $\eta_i$ denote...
the number $\eta(b, \delta)$ from lemma (8) corresponding to $b = \tau_1$ and $\delta = \delta_4$; moreover, let $\eta < \eta/4$ and $\delta = \eta/2 \tau_1$. Choose an increasing sequence of indices $|k_i|$ so as to have

$$1^{st} \; \delta_{k_i} = \frac{1}{2} \delta_{k_{i-1}},$$

$$2^{nd} \; \int_{0}^{\delta_{k_i}} |x_{k_i}^*(t)| dt < \frac{\delta_{k_i}}{4} \quad \text{for } i = 1, 2, \ldots,$$

$$3^{rd} \; \sum_{i=1}^{n} \|x_{k_i}\|^* < \infty.$$

Denoting by $A_i$ the set of these $t \in (0, \tau_{k_{i-1}})$ for which $|x_{k_i}^*(t)| > \delta_{k_{i-1}}/2$, put $B_i = A_i + A_{i+1} + \ldots$. We have $|A_i| < \delta_{k_{i-1}}/2$ and $|B_i| < \delta_{k_i}$. Write

$$\delta_{k_i}(t) = \begin{cases} (1 - \delta_i) x_{k_i}^*(t) & \text{for } t \in (0, \tau_{k_i}) - B_i \\ 0 & \text{for } t \in B_i \cup (0, \tau_{k_i}) \text{ and for } t \in (\tau_{k_i}, \infty). \end{cases}$$

The following inequality holds:

$$\int_{0}^{\tau_{k_i}} |\delta_{k_i}(t) - x_{k_i}^*(t)| dt = \int_{0}^{\tau_{k_i}} |\delta_{k_i}(t)| dt + \int_{0}^{\tau_{k_i}} |x_{k_i}^*(t)| dt < \delta_{k_i},$$

where $F_i = (0, \tau_{k_i}) - B_i$ and $G_i = (0, \tau_{k_i}) - B_i$. Hence

$$\sum_{i=1}^{n} \frac{1}{2^n} \int_{0}^{\tau_{k_i}} |\delta_{k_i}(t) - x_{k_i}^*(t)| dt < \delta_{k_1},$$

and by definition of $\eta_{k_i}$,

$$\max_{x \in F_i} |T(x, \tau_i)| \leq \|x_{k_i}^*\| < \frac{\delta_{k_i}}{4} \tau_{k_i},$$

i.e. $d(\delta_{k_i}, x_{k_i}^*) < \delta_{k_i}/4$, hence $d(\delta_{k_i}, x_{k_i}^*) < \delta_{k_i}/4$.

Given any $t$, choose $m$ so as to get $t \in (\tau_{k_{m-1}}, \tau_k)$; arguing similarly as in the proof of the condition (2), for the space (III), we can prove that

$$\sum_{i=1}^{m} |\delta_{k_i}(t)| = \sum_{i=1}^{m} |\delta_{k_i}(t)| < 1,$$

except a set of $t$ of measure 0. Since we have also

$$\|\sum_{i=1}^{m} \delta_{k_i} \| \leq \sum_{i=1}^{m} \|\delta_{k_i}\| < \sum_{i=1}^{m} \|x_{k_i}\| < \frac{\eta}{4} + \sum_{i=1}^{m} \|x_{k_i}\| \rightarrow 0$$

as $m, n \rightarrow \infty$, hence $\sum_{i=1}^{m} \delta_{k_i} \rightarrow x_{k_1} = \sum_{i=1}^{n} \delta_{k_i}$, where $\lambda_i$ are 0's or 1's arbitrarily chosen.

The proof of the condition (2) is similar.

The space (IV) (IV'). Let $x^0$ denote the same functions as in the case of the space (IV). Fix $s$ so that $s = N$ implies the inequality (9). There exist $a_i$ satisfying (7) and (11). Arguing similarly as above (p. 252-253) we can easily prove that the function $m = m_0(t)$ defined by formula (15) satisfies the inequality

$$d(m, x_s) < \frac{s}{8} \varphi.$$

Choose $\delta > 0$ such that

$$d(0, y) = \|y\|^* < \delta \quad \text{implies sup } \|y(t)\| < \frac{\delta}{8K},$$

and put $x = x(t) = m(t) + y(t)$. It follows that $x \in X_s$, since

$$|x(t)| \leq \left\{ \begin{array}{ll} |m(t)| + |y(t)| & \text{for } 0 \leq t \leq s + r, \\ |y(t)| & \text{for } t > s + r. \end{array} \right.$$

In the case of the space (IV'), note that if $x(t)$ is continuous in a finite interval and vanishes elsewhere, we have by (6) $T(x, s) \rightarrow 0$ as $r \rightarrow \infty$. Thus we can determine $s$ and $a_i$ satisfying the inequality (11). Put

$$z^{t+i}(t) = \left\{ \begin{array}{ll} \frac{x^{t+i}(t)}{s+i-t} & \text{for } s+i \leq t < \tau_i, \\ 0 & \text{for } t \in (0, \infty) - (s+i, \tau_i), \end{array} \right.$$}

where $i = 0, 1, \ldots, r$, and $\tau_i$ is choosen so that

$$\max_{t \in (s+i, \tau_i)} |T(z^{t+i}(t))| < \frac{\delta}{4(r+i)}.$$

This is possible by lemma (8). Put

$$m(t) = m_0(t) + \left(1 - \frac{\delta}{8K}\right) \sum_{i=1}^{n} a_i z^{t+i}(t).$$

It is easy to see that $m(t) \in (IV')$ and $d(m, x_{s}) < \varphi$. Write $p = E(|x| + 1)$, and choose $\delta > 0$ such that $d(y, 0) = \|y\|^* < \delta$ implies $\sup \|y(t)\| < \delta/8K$. Then $\|m + y\| < 1$, i.e. $m + y \in (IV')$. 
The space \((V, \cdot)
\). The formula
\[
x(t) = \int_a^t y(s) \, ds
to \quad \alpha \leq \beta
\]
establishes a one-one mapping between the space \(D\) and the space \(M\) of measurable functions equivalent to bounded functions. Since
\[
\sup_{t \leq \beta} \left| x(t) - x(t') \right| = \sup_{t \leq \beta} |y(t)|, \quad \int_a^b |x'(t)| \, dt = \int_a^b |y(t)| \, dt,
\]
this mapping is isometrical. The space \(M\) satisfies the conditions \((\Sigma_1), (\Sigma_3)\) and \((\Sigma_8)\); hence the space \(D\) satisfies these conditions too, for they are invariant under isometrical mappings.

The space \((V, \cdot)
\). Let \(a < \tau < b\). Let \(N\) be a positive integer satisfying the inequality \((9),\) hence such that \(1/2^{N-1} < \theta/2\). Define \(\delta\) by formula \((10)\) and write
\[
m(t) = \begin{cases} 
(1-\theta/4) x_0(t) & \text{for } a \leq t \leq t_{N-1} \\
\theta/4 & \text{for } t_{N-1} < t \leq b, \\
0 & \text{for } t_{N-1} < t \leq t_{N-1}, 
\end{cases}
\]
where
\[
\delta = \frac{m(t_{N-1})}{t_{N-1}} \left( t - t_{N-1} \right) + m(t_{N-1}) \quad \text{for } t_{N-1} < t \leq t_N
\]
(14)
\[
\lambda(t) = \frac{m(t_{N-1})}{t_{N-1}} \left( t - t_{N-1} \right) + m(t_{N-1}) \quad \text{for } t_{N-1} < t \leq t_N.
\]
If \(d(0, y) = \|y\| < \delta\), then \(\|y\| < \delta/4\) for \(n = 1, 2, \ldots, N\). The function \(m(t)\) belongs to \(C^N\), and \(x(t) = \lambda(t) + y(t) \cdot C^N\), because
\[
|m(t)| \leq |y(t)| < \frac{1}{4} + \frac{\theta}{4} \quad \text{for } a \leq t \leq t_N \quad \text{and } \tau_N < t < b,
\]
Finally
\[
d(m, x) = \|m - x\| < \delta
\]
\[
< \frac{\theta}{4} \sum_{i=1}^{N-1} \|x_i\| + \frac{\theta}{4} (\|m\| + \|x_N\|) + \frac{\theta}{4} (\|m\| + \|x_N\|) <
\]
\[
< \frac{\theta}{4} + \frac{\theta}{4} + \frac{\theta}{8} = \theta.
\]
Under these conditions the sets of the field $\mathcal{E}$ will be called $\mu$-measurable. If $\mu(E) = 0$, the set $E$ will be said to be a null set.

Two sets $E_1$ and $E_2$ of $\mathcal{E}$ which differ by null sets will be said to be $\mu$-equivalent. Two sets $E_1$ and $E_2$ of $\mathcal{E}$, the common part of which is a null set, will be said to be $\mu$-disjoint; in this case we shall write $E_1 \cap E_2 = 0$.

Consider the set $X(\mathcal{E})$ the elements of which are the classes of $\mu$-equivalent sets of $\mathcal{E}$.

We shall prove that $X(\mathcal{E})$ is a complete metric space if the distance of elements $A$ and $B$ of $X(\mathcal{E})$ is defined by the formula

$$d(A, B) = \|\mu(A - B) + \mu(B - A)\|.$$  \hfill (15)

1.64. For the proof, we need the following lemma:

Suppose that $A_n \in \mathcal{E}$ for $n = 1, 2, \ldots$, and

$$\sum_{n=1}^\infty d(A_n, A_{n+1}) < \infty.$$  \hfill (16)

Then the sets $\lim_{n \to \infty} A_n$ and $\lim_{n \to \infty} A_n$ are $\mu$-measurable.

Suppose first the sequence $\{A_n\}$ to be monotone. Then

$$\lim_{n \to \infty} A_n = \lim_{n \to \infty} A_n,$$

and

$$\lim_{n \to \infty} A_n = A_1 - \{(A_1 - A_2) + (A_2 - A_3) + \ldots\}$$

or

$$\lim_{n \to \infty} A_n = A_1 + \{(A_1 - A_2) + (A_2 - A_3) + \ldots\}.$$

The sets on the right-hand side of the above formulae being disjoint, the set $\lim_{n \to \infty} A_n$ is $\mu$-measurable by (15), (16) and 1.6(c).

In the general case we have

$$(A_1 + A_2 + \ldots + A_n) - (A_1 + A_2 + \ldots + A_n) \subset A_{n+1} - A_n;$$

thus, by what has been just proved, the set $A_1 + A_2 + \ldots$ and hence the sets $A_n + A_{n+1} + \ldots$, are $\mu$-measurable. Since

$$(A_1 + A_2 + \ldots) - (A_1 + A_2 + \ldots) \subset A_n - A_{n+1},$$

we see that the set $\lim_{n \to \infty} A_n = (A_1 + A_2 + \ldots) + (A_3 + A_4 + \ldots)$ is $\mu$-measurable. We prove similarly that $\lim_{n \to \infty} A_n$ is $\mu$-measurable.

1.62. To prove that the function (15) satisfies the triangle-inequality, note that $1.6(\alpha)$ and $1.6(\beta)$ imply the inequality

$$\|\mu(E_1 + E_2)\| < \|\mu(E_1)\| + \|\mu(E_2)\|$$

for any $E_1, E_2 \in \mathcal{E}$. It is sufficient to make use of the inclusion

$$(A - C) + (C - A) \subset (A - B) + (B - A) + (B - C) + (C - B),$$

To prove now that the space $X(\mathcal{E})$ is complete, it will be convenient to apply the characteristic functions of sets belonging to $\mathcal{E}$. If $h(t)$ and $g(t)$ are the characteristic functions of the sets $A$ and $B$ respectively, then $|h(t) - g(t)|$ is the characteristic function of their symmetric difference, i.e. of the set $(A - B) + (B - A)$.

Suppose that $E_n \in X(\mathcal{E})$ and $d(E_n, E_m) \to 0$ as $m, n \to \infty$. Let $\{k_n\}$ be an increasing sequence of indices such that

$$d(E_{k_n}, E_{k_{n+1}}) < 1/2^n \quad \text{for } p, q \gg k_n;$$

put $S_n = E_{k_n}$, $S_n = (S_{k_n} - S_{k_{n+1}}) + (S_{k_{n+1}} - S_k)$, and denote by $h_n(t)$ the characteristic function of the set $S_n$. By lemma 1.61 the sets $\{k_n\}$ and $S_n = \lim_{n \to \infty} S_n$ are $\mu$-measurable. Denote by $h(t)$ the characteristic function of the set $S$; then $h_n(t) \to h(t)$ as $n \to \infty$.

$$m > n \text{ implies } |h_n(t) - h(t)| < \frac{1}{2^n} \sum_{i=n}^m |h_i(t) - h_{i+1}(t)|,$$

we get

$$|h_n(t) - h(t)| \leq \frac{1}{2^n} \sum_{i=n}^m |h_i(t) - h_{i+1}(t)|,$$

hence

$$|S - S_0| + |S_0 - S| \subset A_n + A_{n+1} + \ldots,$$

and since the sets on the right-hand side of the formula

$$A_n + A_{n+1} + \ldots + A_k + (A_k + A_{k+1} + \ldots),$$

are $\mu$-disjoint, the set $A_n + A_{n+1} + \ldots$ is measurable by (17) and 1.6(c), and we get

$$\|\mu(A_n + A_{n+1} + \ldots)\| \leq \|\mu(A_n)\| + \sum_{n=1}^\infty \|\mu(A_{n+1} + A_n)\| \leq \frac{1}{2^n}.$$

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Hence
\[ d(S_n, S) \leq \frac{1}{2^{n+1}}. \]

Let now \( \varepsilon > 0 \) be arbitrary, and choose \( n \) so that \( 1/2^{n+1} < \varepsilon \); then \( d(S_n, S) < \varepsilon/2 \), and by (17) we have \( d(S_n, E_\alpha) < \varepsilon/2 \) for \( n \geq k_\alpha \); it follows \( d(E_\alpha, S) < \varepsilon \).

1.6.5. Denote by \( X(T) \) the space of the real-valued functions \( x = x(t) \) defined in \( T \) and bounded outside null sets; with the addition of elements and the multiplication by real numbers defined as usual, and with the norm \( ||x|| \) of the element \( x = x(t) \) as the largest lower bound of the numbers \( k \) for which \( |x(t)| < k \) holds outside a null set, \( X(T) \) is a Banach space.

Denote by \( x_\alpha = x_\alpha(t) \) the characteristic function of the set \( A \in \mathcal{E} \), and by \( X_\alpha(\mathcal{E}) \) the class of the functions \( x_\alpha \) as \( A \) runs over \( \mathcal{E} \). If the distance of two elements \( x_\alpha \), \( x_\beta \in X_\alpha(\mathcal{E}) \) is defined by the formula
\[ d(x_\alpha, x_\beta) = d(A, B), \]
we easily see by 1.6.2 that \( X_\alpha(\mathcal{E}) \) is a complete metric space.

Obviously \( X_\alpha(\mathcal{E}) \) is the subset of the space \( X(T) \).

If \( d(x_\alpha, x_\beta) \to 0 \), we shall write \( x_\alpha \to x_\beta \), analogously as in Saks spaces.

In the space \( X_\alpha(\mathcal{E}) \) we introduce the conditions (2.22), (2.21) and (2.23) by rewriting the conditions (2.22), (2.21) and (2.23) with \( X_\alpha(\mathcal{E}) \) instead of \( X_\alpha \).

We shall prove that the space \( X_\alpha(\mathcal{E}) \) satisfies the conditions (2.22), (2.21) and (2.23). Hence the space \( X_\alpha(\mathcal{E}) \) may be considered as a generalized Saks space. The difference between the Saks spaces and the space \( X_\alpha(\mathcal{E}) \) lies in the reach of the distance definition: in the former space the formula (1) defining the distance has a meaning in the whole of the fundamental Banach space \( X_\alpha \), and in the latter space the distance is defined in \( X_\alpha(\mathcal{E}) \) only.

To prove that the space \( X_\alpha(\mathcal{E}) \) satisfies the condition (2.22), let \( 0 \) denote the characteristic function of the null set, and let \( x_\alpha = x_\alpha \) be the centre of the sphere \( K(x_\alpha, 0) \). If \( d(x_\alpha, 0) < \delta = \varepsilon \), \( B = A_\alpha + (A_\alpha - A) \), and \( C = A_\alpha - A \), then we have \( x_\alpha - x_\alpha = x_\alpha \),
\[ d(x_\alpha, x_\alpha) = \| \mu(A_\alpha) \| < \delta = \varepsilon \]
and \( d(x_\alpha, x_\alpha) = \| \mu(A_\alpha A) \| < \delta = \varepsilon \).

To prove that the space \( X_\mu(\mathcal{E}) \) satisfies the condition (2.22), let \( d(x_\mu, 0) = \| \mu(E_\mu) \| < \varepsilon \), and \( \varepsilon_n \to +0 \). Pick out an increasing sequence \( \{k_n\} \) of indices such that
\[ A_n = E_{k_n} \]
implies \( \| \mu(A_{k_n+1}) \| < \varepsilon_{k_n} \),
and put
\[ A_0 = A_1, \quad A_2 = A_1 - A_1, \quad A_3 = A_2 - A_2, \ldots \]

The sets \( A_n A_{n+1} \) and \( A_n A_{n+1} \) are disjoint for any \( n \neq n' \), and
\[ \sum \| \mu(A_n A_{n+1}) \| < \infty. \]

By 1.6(c) the sets \( B_i = A_i - \bigcap_{n} A_{n+1} \) are \( \mu \)-measurable, and since the sets \( B_i \) are disjoint and \( \sum \| \mu(B_i) \| < \infty \), we get
\[ d(x_{B_i}, x_{B_i}) < \varepsilon_{k_i} \quad \text{for} \quad i = 1, 2, \ldots, \]
\[ x_{B_i} = l_1 x_{B_1} + l_2 x_{B_2} + \ldots + l_{n_k} x_{B_{n_k}} \]
and \( l_i x_{B_1} + l_i x_{B_2} + \ldots + l_i x_{B_{n_i}} \to x_{B_i} \), \( l_i \) being 0's or 1's.

The proof of (2.23) is analogous.

1.7. Now some examples of separable \( X_\alpha(\mathcal{E}) \) spaces will be given. Since the spaces \( X_\alpha(\mathcal{E}) \) arise from the spaces \( X(\mathcal{E}) \) by passing to characteristic functions, we indicate in each case the set \( T \), the class \( \mathcal{E} \) and the measure \( \mu(\mathcal{E}) \). We give also examples of denumerable and dense sets in \( X(\mathcal{E}) \), and we discuss further the separability of these spaces.

Examples of separable \( X_\alpha(\mathcal{E}) \)-spaces.

(1) \( T \) is a finite interval \( Q \) in the space \( \mathcal{E}^n \) (the \( n \)-dimensional euclidean space).

\( \mathcal{E} \) is the class of Lebesgue-measurable sets in \( Q \).

\( \mu(\mathcal{E}) = |\mathcal{E}| \) is the Lebesgue-measure of \( E \).

\( \mathcal{E}_0 \) is the class of sets which are sums of a finite number of intervals with rational vertices.

(2) \( T \) is the interval \( (-\infty, +\infty) \).

\( \mathcal{E} \) is the class of Lebesgue-measurable sets \( E \) of relative measure 0, i.e. such that \( \lim_{r \to 0} |E \cap [-r, r]|/2r = 0 \).
\( \mu(E) \) is the function \(|\langle -\tau, \tau \rangle \cdot E|/2\tau \) in an interval \( 0 < a < \tau \), the values of \( \mu(E) \) being considered as elements of the Banach space composed of the continuous functions \( x = x(t) \) in \( \langle a, \infty \rangle \) converging to 0 as \( t \to \infty \), with the norm \( \|x\| = \sup_{\langle s, \infty \rangle} |x(t)| \).

\( \mathcal{E}_a \) is the class of sets which are sums of a finite number of intervals with rational end-points.

(\( \text{III}^* \)) \( T \) is the interval \( [0, \infty) \).
\( \mathcal{E} \) is the class of Lebesgue-measurable sets \( E \) in \( (0, \infty) \) such that

\[
\lim_{\tau \to 0} \frac{1}{\tau} \int_{-\tau}^{\tau} K(t) dt = 0,
\]

where \( K(t) \) is a function satisfying the conditions (6)-(6") and positive for \( t < \tau \) and \( 0 < t < \infty \).

\( \mu(E) = \int_{-\tau}^{\tau} K(t) dt \); this function considered for \( a \leq \tau \) is an element of the Banach space defined in (\( \text{IV}^* \)).

\( \mathcal{E} \) is the same set as in (\( \text{II}^* \)).

(\( \text{IV}^* \)) \( T \) is the set of positive integers.
\( \mathcal{E} \) is the class of all subsets of \( T \).

\( \mu(E) = \sum_{n=0}^{\infty} \frac{1}{2^n} \) where \( E \) is the set \( \{n_1, n_2, \ldots\} \); thus the values of \( \mu(E) \) are real numbers.

\( \mathcal{E}_a \) is the class of finite sets composed of positive integers.

(\( \text{V}^* \)) \( T \) is the set of positive integers.
\( \mathcal{E} \) is the class of all sets \( E = \{n_1, n_2, \ldots\} \) of positive integers for which

\[
\lim_{n \to \infty} \sum_{i=1}^{n} a_i \lambda_n = 0,
\]

where \( a_i \) are non-negative, satisfy the conditions (3) and (3'),

\[
\lambda_n = \begin{cases} 1 & \text{for } n = n_i, \\ 0 & \text{elsewhere} \end{cases}
\]

\( \mu(E) \) is the sequence \( \{t_i\} \), where \( t_i = \sum \sum a_i \lambda_n \); the values of \( \mu(E) \) belong to the space \( M_{\mathcal{E}} \).

\( \mathcal{E}_a \) is the same set as in (\( \text{IV}^* \)).

(\( \text{VI}^* \)) \( T \) is the set of positive integers.
\( \mathcal{E} \) is the class of all sets \( E = \{n_1, n_2, \ldots\} \) of positive integers for which

\[
\lim_{n \to \infty} \sum_{i=1}^{\infty} a_i \lambda_n = 0,
\]

where the \( a_i \) are continuous functions, non-negative for \( t \in \langle t_i, t_i \rangle \) with \( t_i < \infty \), and satisfying the conditions:

(i) \( \sum_{n=1}^{\infty} a_i \lambda_n = 0 \) for \( t \in \langle t_i, t_i \rangle \),
(ii) \( \lim_{t \to \infty} \sum_{n=1}^{\infty} a_i \lambda_n = 0 \) for \( n = 1, 2, \ldots \),
(iii) the series \( \sum_{n=1}^{\infty} a_i \lambda_n \) converges uniformly in every interval \( \langle t_i, t_i \rangle \subset \langle t_i, t_i \rangle \).

\( \mu(E) = \sum_{n=1}^{\infty} a_i \lambda_n \), this function (continuous in \( \langle t_i, t_i \rangle \)) being considered as an element of the space \( C \).

\( \mathcal{E}_a \) is the same set as in (\( \text{IV}^* \)).

1.71. We presently prove that the sets \( \mathcal{E}_a \) defined above are dense in \( X_{\mathcal{E}} \). The cases of the spaces (\( \text{II}^* \)), (\( \text{III}^* \)) and (\( \text{IV}^* \)) require only some explanations.

In the case of the space \( (\text{III}^*) \) set for \( n = 1, 2, \ldots \) and

\[
E_n = \mathcal{E}_{\langle 0, n \rangle},
\]

\( \varphi(t) = \int_{-\tau}^{\tau} K(t) dt \), \( \psi(t) = \int_{-\tau}^{\tau} K(t) dt \).

Since the functions \( \varphi_n(t) \) and \( \psi(t) \) are continuous and

\[\varphi_n(t) \leq \varphi_{n+1}(t) \leq \varphi(t), \quad \varphi_n(t) \to \varphi(t),\]

the sequence \( \{\varphi_n(t)\} \) converges to \( \varphi(t) \) uniformly in every finite interval. Moreover, since

\[
|\varphi_n(t) - \varphi(t)| < \varepsilon \quad \text{for } t > \tau, \varepsilon > 0,
\]

we get

\[
\max_{\langle s, \infty \rangle} |\varphi_n(t) - \varphi(t)| \to 0 \text{ as } n \to \infty.
\]

Choose \( m \) so that

\[
d(E_n, E) = \max_{\langle s, \infty \rangle} |\varphi_n(t) - \varphi(t)| < \varepsilon/2,
\]

and let \( A \) be a subset of \( \langle 0, m \rangle \) composed of a finite number of intervals with rational end-points such that

\[
d(E_n, A) = \max_{\langle s, \infty \rangle} |\varphi_n(t) - \varphi(t)| < \varepsilon/2.
\]
where $M = (E_m - A) + (d - E_m)$. This is possible by Lemma (*), p. 251. Hence $d(E, A) \leq d(E_m, A) + d(E, E_m) < \varepsilon$, and this shows that the set $\mathcal{E}_9$ is dense in the set $\mathcal{E}_9$.

The space $(\Pi^o)$ is a particular case of the space $(\Pi^o)$. Let the set $E = \{a_1, a_2, \ldots\}$ belong to it, and let

$$
\lambda_0 = \begin{cases} 
1 & \text{for } n = n_0, \\
1 & \text{for } n = 1, 2, \ldots, m, \\
0 & \text{for } n > m.
\end{cases}
$$

Given any $\varepsilon > 0$, there exists a $t' \in (0, t)$ such that

$$
t' \geq t \implies \sum_{i=1}^{m} a_i(t) \lambda_0 < \varepsilon.
$$

The series $\sum_{i=1}^{m} a_i(t)$ converges uniformly in $\langle t_0, t' \rangle$; hence we have $\sum_{i=1}^{m} a_i(t) (\lambda_0 - \lambda_0^m) < \varepsilon$ for $t \in \langle t_0, t' \rangle$ and $m$ sufficiently large; this implies that

$$
\max_{t \in \langle t_0, t' \rangle} \sum_{i=1}^{m} a_i(t) (\lambda_0 - \lambda_0^m) \to 0 \quad \text{as } m \to \infty.
$$

Thus, denoting by $E_n$ the set $(a_1, a_2, \ldots, a_n, 0, 0, \ldots)$, we get $d(E, E_n) \to 0$, i.e. $\mathcal{E}_9$ is dense in $\mathcal{E}_9$.

2.1. An operation $U(x)$ from a Saks space $X_0$ to a Banach or Fréchet space $Y$ will be said to be additive, if for arbitrary $x_1, x_2 \in X_0$ and arbitrary rational $\lambda_1, \lambda_2$

$$
\lambda_1 x_1 + \lambda_2 x_2 \in X_0 \quad \text{implies} \quad U(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 U(x_1) + \lambda_2 U(x_2).
$$

The operation $U(x)$ will be termed $(X_0, Y)$-continuous, if $x_n \to x$ implies $U(x_n) \to U(x)$

according to the convergence defined in $Y$ by the norm.

Any additive and $(X_0, Y)$-continuous operation will be said to be $(X_0, Y)$-linear.

If $U(x)$ is a functional, and when no ambiguity arises, we shall simply say continuous operation or linear operation instead of $(X_0, Y)$-continuous operation or $(X_0, Y)$-linear operation respectively.

If $U(x)$ is an operation defined in a Banach or Fréchet space, a corresponding terminology will be used.

2.2. We now prove two lemmas:

(A) Let $U_n(x)$ be additive operations from $X_0$ to a Fréchet space. Each of the following conditions is sufficient for the sequence $U_n(x)$ to be equicontinuous at any point $x \in X_0$:

(a) the space $X_0$ satisfies the condition $\mathcal{E}_9$, and there exists an element $x_0$ at which the operations $U_n(x)$ are equicontinuous,

(b) the space $X_0$ is arbitrary, and there exists a $x_0$ such that $\|x_0\| < 1$ and that the operations $U_n(x)$ are equicontinuous at $x_0$.

Suppose the condition (a) is satisfied. Given any $\varepsilon > 0$ there exists a $\delta(e) > 0$ such that

$$
x \in K(x_0, \varepsilon(e)) \implies \|U_n(x) - U_n(x_0)\| < \varepsilon/2 \quad \text{for } n = 1, 2, \ldots.
$$

By (a) there exists a $\delta(e) > 0$ such that any element $y$, for which $d(y, 0) < \delta(e)$, is of the form $y = x_1 - x_2$ with $x_1, x_2 \in K(x_0, \varepsilon(e))$. The operation $U_n(x)$ being additive, we have

$$
\|U_n(y)\| < \|U_n(x_1) - U_n(x_2)\| < \varepsilon.
$$

Let $\delta$ be an arbitrary element of $X_0$; by 1.22 there exists a $\delta(e) > 0$ such that

$$
\|x - \delta\| < \delta(e) \quad \text{implies} \quad \left| \frac{x - \delta}{\delta} \right| < \delta(e).
$$

Since $(x - \delta)/2 \in X_0$, we get for $n = 1, 2, \ldots$

$$
\left| \frac{1}{2} U_n(x) - \frac{1}{2} U_n(\delta) \right| = \left| U_n \left( \frac{x - \delta}{2} \right) \right| < \varepsilon, \quad \|U_n(x) - U_n(\delta)\| < 2\varepsilon.
$$
Suppose now the condition \( (a^n) \) to be satisfied. Let \( X_x \) be arbitrary, \( \eta \) any positive, and \( \lambda \) a rational number satisfying the inequality \( \|x\| < \lambda < 1 \). Then there exists an \( \varepsilon > 0 \) such that

\[
\|x\| < \varepsilon \quad \text{and} \quad \varepsilon \in \mathcal{Y} \quad \text{implies} \quad \left\| \frac{1}{1 - \lambda} x \right\| < \eta.
\]

If \( y \in X_x \), then \( x + (1 - \lambda) y \in X_x \) and \( d(y, 0) < \lambda (\varepsilon) \) imply

\[
x + (1 - \lambda) y \in K(x, \varepsilon (\delta)) = K_x \varepsilon (\delta).
\]

Here we can take for \( K(x, \varepsilon (\delta)) \) the same sphere as in the proof of the first part of the lemma; then for \( n = 1, 2, \ldots \),

\[
U_n \left( x + (1 - \lambda) y - U_n (x, \varepsilon) \right) = \left\| (1 - \lambda) U_n (y) \right\| < \varepsilon, \quad \| U_n (y) \| < \eta.
\]

Hence the operations are equicontinuous at 0. We complete the proof as above.

In connection with lemma (A) note that the set \( X^* \) of those elements, for which \( \|x\| < 1 \), is a \( F_x \) in \( X_x \). Remark that

\[
x \xrightarrow{\delta} x \quad \text{implies} \quad \lim_{n \to \infty} \|x_n\| \geq \|x\|.
\]

In fact, suppose \( \|x\| > 0 \), and choose \( \varepsilon > 0 \) so as \( 0 < \varepsilon < \|x\| \) and \( \|x_n\| \leq \|x\| - \varepsilon < \eta \); since \( \|x_0\| < 1 \), we get \( x_0 \varepsilon \to y \) and \( y \in X_x \); on the other hand we have obviously \( y = x_0 / \|x_0\| \), and this leads to a contradiction: \( \|x_0\| < \eta \). The sets \( X_x \) of the elements \( x \) satisfying the inequality \( \|x\| < 1 \), being closed in \( X_x \), we see that \( X^* \) is a \( F_x \).

Applying this result it can be easily proved that in the spaces (1)-(VII) defined in 1.4 the set \( X^* \) is of the first category of Baire.

(B) Let \( (a_n) \) be a matrix of real numbers such that

\[
\sum_{i=1}^{\infty} |a_i| < \infty \quad \text{for} \quad i = 1, 2, \ldots \quad \text{and} \quad \lim_{n \to \infty} a_n = a_n = a \quad \text{for} \quad n = 1, 2, \ldots
\]

If for any sequence \( \{x_i\} \) composed of 0's and 1's there exists the limit \( \lim_{n \to \infty} \sum_{i=1}^{\infty} \lambda_i a_n \), then \( \lim_{n \to \infty} \sum_{i=1}^{\infty} \lambda_i a_n = 0 \).

It is sufficient to consider the case \( a_n = 0 \). Suppose that \( \lim_{n \to \infty} \lambda_n = \sup \{\lambda_n\} > 0 \); then we can define successively two increasing sequences of indices, \( \{i_k\} \) and \( \{n_k\} \), such that

\[1^\circ \quad \sum_{i=k}^{\infty} |a_{i_k}| < \frac{r}{8}, \quad 2^\circ \quad |a_{n_k}| > \frac{3}{4} r, \quad 3^\circ \quad \sum_{i=k}^{\infty} |a_{n_k+k}| < \frac{r}{8}.
\]

Put

\[
\lambda_n = \begin{cases} \frac{1}{2} (1 - (-1)^{n+1}) & \text{for} \quad n = n_k \\ 0 & \text{for} \quad n \neq n_k; \end{cases}
\]

\( k \) being even, we have

\[
\sum_{a=1}^{\infty} \lambda a_n > \sum_{a=1}^{n_k} \lambda a_n + \sum_{a=n_k+1}^{\infty} \lambda a_n > \frac{r}{2} \quad \text{if} \quad \lambda a_n > 0,
\]

\[
\sum_{a=1}^{\infty} \lambda a_n < \sum_{a=1}^{n_k} \lambda a_n + \sum_{a=n_k+1}^{\infty} \lambda a_n < -\frac{r}{2} \quad \text{if} \quad \lambda a_n < 0,
\]

and \( k \) being odd, we have

\[
\left| \sum_{a=1}^{n_k} \lambda a_n \right| \leq \frac{r}{8} \quad \text{and} \quad \left| \frac{r}{8} \right| = \frac{r}{4}.
\]

The sequence \( \sum_{a=1}^{\infty} \lambda a_n = b_i \) is divergent, and this is contradictory.

2.3. Denote by \( Y \) the space conjugate to a Banach space \( Y \). The subset \( Y^* \) of \( Y \) is called fundamental, if there exist two positive constants \( c \) and \( C \) such that

\[
(1) \quad \sup_{e^*_{Y^*}} |e^*(y)| > c |y| \quad \text{for any} \quad y \in Y,
\]

\[
(2) \quad \|e^*_{Y^*}\| < C \quad \text{for each} \quad e^*_{Y^*}.
\]

Theorem 1. If the space \( X \) satisfies the condition \((\mathcal{C}_1)\) or the condition \((\mathcal{C}_2)\), then the following conditions are jointly sufficient for an additive operation to be \((X, Y, Y)\):

\( (a) \) the space \( Y \) is a separable Banach space,

\( (b) \) there exists a fundamental set \( Y^* \) such that for any \( e^*_{Y^*} \) the functional \( e^*(U(x)) \) is continuous in \( X \).

Proof. Suppose first the space \( X \) satisfies the condition \((\mathcal{C}_1)\).

By \((a)_1\), \((a)_2\) and a general theorem \(7\) the operation \( U(x) \) is of Baire's first class, hence continuous in a residual set. It is sufficient to apply the lemma 2.2.(A).

Suppose now the space $X$ satisfies the condition $(\Sigma_0)$.

By the condition $(b)_1$, the convergence $x_n \xrightarrow{n} x_0$ implies that

\[
\lim_{n\to\infty} \left\| U(x_n) \right\| \geq \frac{c}{2} \left\| U(x_0) \right\| = k \left\| U(x_0) \right\|.
\]

By lemma 2.2(A) it is sufficient to prove that the operation $U(x)$ is continuous at 0. Suppose it is not the case; then there exists an $\varepsilon_0 > 0$ and a $x_0$ such that $x_n \xrightarrow{n} 0$ and $\left\| U(x_n) \right\| \geq \varepsilon_0$.

By $(\Sigma_0)$ and (19) there exists a sequence $\{k_n\}$ of indices and a sequence $\{x_{k_n}\}$ of elements such that the conditions (ii) and (iii) of $(\Sigma_0)$ are satisfied, and

\[
\left\| U(x_{k_n}) \right\| \geq \frac{k_{\varepsilon_0}}{2} \quad \text{for } n = 1, 2, \ldots
\]

Let $n Y_{k_n} \triangleq x_{k_n}$ be the element the existence of which is assured by the condition (iii) of $(\Sigma_0)$, we have by $(b)_2$

\[
\sum_{n=1}^{\infty} \lambda_{\eta}(U(x_{k_n})) = \eta(U(x_{k_n})).
\]

The set $Y$ being fundamental, there exists a sequence $\{\eta_i\}$ of elements of $Y_{k_n}$ such that

\[
\left| \eta_i(U(x_{k_n})) \right| \geq \frac{c}{2} \left| U(x_{k_n}) \right| \geq \frac{k_{\varepsilon_0}}{4} \quad \text{for } i = 1, 2, \ldots,
\]

and by the separability of the space $Y$ we can suppose that there exists a subsequence $(l)$ such that $\lim_{n \to \infty} \eta_i(y_i)$ exists for every $y_i \in Y$. Write $a_n = \eta_i(U(x_{k_n}))$; the matrix $(a_n)$ satisfies by (20) the hypotheses of lemma 2.2(B); hence $\left| \eta_i(U(x_{k_n})) \right| \to 0$, which is impossible, since $\left| \eta_i(U(x_{k_n})) \right| \geq \frac{k_{\varepsilon_0}}{4}$.

Theorem 1'. If the space $X$ is separable and satisfies the condition $(\Sigma_0)$ or the condition $(\Sigma_0')$, the following conditions are jointly sufficient for an additive operation $U(x)$ to be $(X, Y)$-linear:

$(a)$ $Y$ is a Banach space,

$(b)$ for any linear functional $\eta(y)$ the functional $\eta(U(x))$ is linear in $X$.

Proof. Let the sequence $\{x_n\}$ be dense in $X$; $x$ being any element of $X$, choose $x_n \xrightarrow{n} x$. Since by $(b')$ $\eta(U(x_n)) \to \eta(U(x))$ for every linear functional $\eta$, there exists by a well-known theorem a sequence of linear combinations of the elements $U(x_n)$ converging to $U(x)$ in $Y$. This implies that the range of operation $U(x)$ belongs to a separable subspace $Y \subset Y$, and it suffices to apply Theorem 1.

Theorem 2. If the space $X$ satisfies the condition $(\Sigma_0)$ or the condition $(\Sigma_0')$, and $U(x)$ is an additive operation in $X$, then the following conditions are jointly sufficient for $U(x)$ to transform any set $X_0$ bounded with respect to the norm $\left\| x \right\|$ into a bounded set: $\{x_n\}$ $X$ is a Fréchet space,

$(b_1)$ $x_n \xrightarrow{n} x_0$ implies

\[
\lim_{n \to \infty} \left\| U(x_n) \right\| \geq \left\| U(x_0) \right\|.
\]

Proof. Suppose first the space $X$ satisfies the condition $(\Sigma_0)$; $\varepsilon$ being any positive number, denote by $X_0$ the set of the elements of $X$ such that $\left| \varepsilon \right| \leq 1/n$ implies $\left| \varepsilon U(x) \right| \leq \varepsilon$. From the additivity of $U(x)$ and from $(b_1)$ it follows that for every rational $\theta$

\[
x_n \xrightarrow{n} x \quad \text{and } \left| \theta \right| \leq 1 \implies \lim_{n \to \infty} \left| \theta U(x_n) \right| \geq \left| \theta U(x) \right|.
\]

Thus the sets $X_0$ are closed; since $X_0 = \sum X_0$, one of the sets, say $X_0$, contains the sphere $K(x_0, \delta)$. By condition $(\Sigma_0)$ there exists a $\delta > 0$ such that $\left\| y \right\| < \delta$, and $y \in X_0$ implies the existence of $x_n, x_0 \in K(x_n, \delta)$ for which $y = x_n - x_0$. Hence

\[
\left| \theta U(y) \right| \leq \left| \theta U(x_n) \right| + \left| \theta U(x_0) \right| \leq \delta \quad \text{for } \left| \theta \right| \leq 1/p.
\]

There exists a rational number $\theta_0$ such that $0 < \theta_0 < 1$ and $\left| \theta_0 x \right| < \varepsilon$ for every $x \in X_0$. The inequality $\left| \varepsilon \right| \leq \theta_0/p$ implies for every $x \in X_0$

\[
\left| \theta U(x) \right| = \left| \frac{\theta}{\theta_0} U(\theta_0 x) \right| \leq \varepsilon.
\]

Thus the operation $U(x)$ maps the set $X_0$ into a bounded set.

Suppose now the space $X$ satisfies the condition $(\Sigma_0')$. We first prove that $x_n \xrightarrow{n} 0$ implies the boundedness of the sequence $\{U(x_n)\}$. 
Suppose it is not the case for a sequence \( \{x_n\} \) 1-converging to 0; then there exists an \( \varepsilon_0 > 0 \) and rational numbers \( \theta_n \to 0 \) such that \( |\theta_n| < 1 \) and \( \lim_{n \to \infty} \| \theta_n U(x_n) \| > \varepsilon_0 \).

By \((\Sigma_2), (b_3)\) and 1.22 there exists a sequence \( \{k_n\} \) of indices and a sequence \( \{x_{k_n}\} \) of elements such that the conditions (i)–(iii) of \((\Sigma_2)\) are satisfied, and that

\[
1^* \| \theta_n U(x_{k_n}) \| > \varepsilon_0 / 2 \quad \text{for} \ n = 1, 2, \ldots .
\]

2\(^*\) the series \( \sum_{n=1}^{\infty} \| \theta_n x_{k_n} \| \) converges uniformly in the set of sequences \( \{\tau_n\} \) with \( \|\tau_n\| < 1 \).

The condition (ii) of \((\Sigma_2)\) implies that, given any functional \( \xi(x) \) of norm 1 linear in \( X \),

\[
\lambda_n = 0 \text{ or } 1 \quad \Rightarrow \quad \sum_{n=1}^{\infty} \lambda_n |\xi(x_{k_n})| < 1 ;
\]

hence for any sequence \( a = \{a_n\} \) of numbers absolutely less than 1 we have

\[
\sum_{n=1}^{\infty} a_n \lambda_n \| x_{k_n} \| < 1 .
\]

Thus, by the completeness of \( X \), and by \( 2^* \) there exists a \( x^* \in X^* \) such that

\[
\sum_{n=1}^{\infty} a_n x_{k_n} \to x^* .
\]

We now define in the space \( l^* \) an operation \( F(a) \) writing

\[
F(a) = U(x_a) \quad \text{for} \ a = \{a_n\} .
\]

It is an additive operation, and by \( 2^* \) and \( (b_3) \) we easily infer that, if \( a \to a \) in the space \( l^* \), then \( \lim \| F(a_n) \| = \| F(a) \| \). This space satisfying the condition \((\Sigma_4)\) and being bounded with respect to the norm \( \| \cdot \| \), the range of the operation \( F(a) = U(x_a) \) is bounded. Hence the sequence \( U(x_a) \) is bounded. But this contradicts \( 1^* \).

Let now \( x^* \in X^* \), \( \theta_n \to 0 \), and let \( \theta_n' \) be rational numbers such that \( \theta_n' \to 0 \) and \( 1 - |\theta_n'| \theta_n' = 0 \). Since \( \theta_n x_{k_n} \to 0 \), we get

\[
\theta_n U(x_{k_n}) = \frac{\theta_n}{\theta_n'} U(\theta_n' x_{k_n}) \to 0 ,
\]

which completes the proof.

In connection with Theorem 2 we prove the following one:

If the operation \( U(x) \) is additive, and maps any bounded set into a bounded set, then

\[
x \in X \quad \text{and} \quad \theta x \in X \quad \Rightarrow \quad U(\theta x) = \theta U(x) .
\]

It is sufficient to prove that \( \theta_n x \to 0 \) implies \( U(\theta_n x) \to 0 \).

Let \( \theta_n' \) be rational numbers such that \( |\theta_n'| \theta_n' \to 0 \) and \( \theta_n' \to 0 \).

The sequence \( \{\theta_n' x\} \) being bounded, we get \( U(\theta_n' x) = \theta_n' U(\theta_n' x) \to 0 \).

Simple examples show that there exist operations additive in Saks spaces \( X \) satisfying the conditions \((\Sigma_4)\) and \((\Sigma_5)\), mapping the space \( X \) into a bounded set, but discontinuous in \( X \).

2.4.1. We will say that a Fréchet space \( Y \) satisfies the condition \( (Z) \) if boundedness of all the sums \( \sum_{n=1}^{\infty} \lambda_n y_n \) with \( \lambda_n = 0 \) or 1 implies the convergence in \( Y \) of the series \( \sum_{n=1}^{\infty} y_n \).

Theorem 3. If the space \( X \) satisfies the condition \((\Sigma_4)\), then the conditions \((a_3), (b_3)\) of Theorem 2 and the following condition:

\((c_3)\) the space \( Y \) satisfies the condition \((Z)\)

are jointly sufficient for an operation \( U(x) \) additive in \( X \), to be \((X, Y)-linear\).

Proof. By Lemma 2.2(A) \((a)\) it is sufficient to prove that \( x_n \to 0 \) implies \( U(x_n) \to 0 \).

Suppose it is not the case; then there exists an \( \varepsilon_0 > 0 \) and a sequence \( x_n \to 0 \) for which \( \| U(x_n) \| > \varepsilon_0 \). The conditions \((\Sigma_4)\) and \((b_3)\) imply the existence of a sequence \( \{\lambda_n\} \) of indices and of a sequence \( \{x_{k_n}\} \) of elements satisfying the conditions (ii) and (iii) of \((\Sigma_2)\), the condition \( 3^* \), p. 270, and the following one:

\[
\| U(x_{k_n}) \| > \varepsilon_0 / 2 \quad \text{for} \ n = 1, 2, \ldots .
\]

From these conditions it follows that the partial sums of the series \( \sum_{n=1}^{\infty} \lambda_n x_{k_n} \), where \( \lambda_n = 0 \) or 1, are bounded with respect to the

\(1^*\) This condition was introduced in my paper "Sur les opérations linéaires dans l'espace des fonctions bornées," Studia Mathematica 10 (1948), p. 60-86.
norm \|\|^*; hence by Theorem 2 the sequence of values
\[ U \left( \sum_{n=1}^{\infty} \lambda_n x_n \right) = \sum_{n=1}^{\infty} \lambda_n U(x_n), \]
where \( \lambda_n = 0 \) or 1, is bounded. It follows by (2) that \( U(x_n) \to 0 \), which is contradictory.

**Theorem 3.** If the space \( Y \) is a Banach space, and the space \( X \), satisfies the condition \((a)\), then the conditions \((b)\) and \((c)\) suffice jointly for an additive operation \( U(x) \) to be \((X, Y)\)-linear.

**Proof.** Write
\[ \sup_{y \in Y} |\eta(y)| = |y|_0. \]

The norms \( \| \cdot \| \) and \( \| \cdot \|_0 \) are equivalent by virtue of conditions \((f)\) and \((f_0)\), p. 267. We prove that
\[ x_n \to x \implies \|U(x)\|_0 \leq \lim_{n \to \infty} \|U(x_n)\|_0. \]

In the contrary case we must have for a sequence \( \{x_n\} \)
\[ \|U(x_n)\|_0 < \|U(x)\|_0 - \varepsilon. \]

But \((b)\) implies \( \eta(U(x_n)) \to \eta(U(x)) \) for every \( \eta \in Y' \); hence choosing \( \eta \in Y' \) so that
\[ |\eta(U(x))| > \|U(x)\|_0 - \varepsilon/2 \]
we get
\[ \|U(x)\|_0 - \varepsilon > \|U(x_n)\|_0 > |\eta(U(x_n))| \to |\eta(U(x))|, \]
which is impossible. We complete the proof by applying Theorem 3.

*(Reçu par la Rédaction le 20, 3, 1950).*