On sequences of operations (II)

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In this part ¹) we deal with the sequences of linear operations in spaces which are more general than the Banach spaces.

Terminology and notation.

A space $X$, $Y$, ... will denote linear spaces.

If a space $X$ is a limit space of Fréchet, i.e., a notion $a$ of limit (called also notion of convergence) is defined in it, satisfying the usual postulates of Fréchet, we shall denote the convergence of a sequence $(x_n)$ to $x_a$, according to the notion of limit $a$, by writing $x_n \rightarrow^a x_a$, or $(a) \lim x_n = x_a$; then the sequence $(x_n)$ will be called $a$-convergent to $x_a$.

The linear space $X$ provided with the notion $a$ of limit will be denoted by $X_a$.

It may happen that in the same space $X$ several notions of limit $a, \beta, ...$ will be distinguished.

The convergences $a$ and $\beta$ are called equivalent (in symbols: $a = \beta$) if $x_n \rightarrow^a x_a$ implies $x_n \rightarrow^\beta x_a$ and conversely. The convergence $a$ will be called non-narrower than the convergence $\beta$ if $x_n \rightarrow^\beta x_\beta$ implies $x_n \rightarrow^a x_a$; if the convergence $a$ is non-narrower than the convergence $\beta$, and if $a \neq \beta$, then $a$ will be called narrower than $\beta$.

We say that the space $X_a$ has the property $P$ (e.g., satisfies a postulate) instead of saying that the convergence $a$ in $X$ has this property.

¹) This notation has been introduced by Orlicz (14), p. 61.

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The sequence $(x_n)$ will be termed $a$-bounded if, given any sequence $(\theta_n)$ of real numbers, $\theta_n \rightarrow 0$ implies $\theta_n x_n \rightarrow 0$.

A sequence which is not $a$-convergent or not $a$-bounded will be said to be $a$-divergent or $a$-unbounded respectively.

A set $D$ will be called dense in $X_a$ if for every element $x \in X$ there exists a sequence $(x_n)$ of elements of $D$ which is $a$-convergent to $x$.

In the sequel we suppose that all the notions of limit have the property that both the addition and the multiplication by real numbers are continuous in both variables. All the spaces with such a notion of limit will be called $A$-spaces.

An operation $U(x)$ from a $A$-space $X_a$ to a $A$-space $Y_b$ will be called $(X_a, Y_b)$-continuous at $x_0$ if $x_n \rightarrow^a x_0$ implies $U(x_n) \rightarrow^b U(x_0)$. If this continuity holds at any point $x$ of the space, $U(x)$ will be simply said to be $(X_a, Y_b)$-continuous.

Any operation $U(x)$ satisfying the equation

$$U(ax + by) = aU(x) + bU(y)$$

for real $a$ and $b$ will be called additive.

An additive $(X_a, Y_b)$-continuous operation will be called $(X_a, Y_b)$-linear. If $Y_b$ is the space of real numbers with the usual notion of limit, any $(X_a, Y_b)$-linear operation will be termed a $(X_a)$-linear functional.

In the case when the $a$-convergence is strong (i.e., equivalent with the convergence according to the norm) in a $F^a$-space $X$, we denote the space $X_a$ by $X$, and omit the symbol $a$ (except in section 2.2).

Contents.

This part is concerned with the problem under what conditions the following statements hold:

I. Let $U(x)$ be the limit of a sequence $(U_n(x))$ of $(X_a, Y_b)$-linear operations $\beta$-convergent everywhere. Then $U(x)$ is $(X_a, Y_b)$-linear.

II. Let $(U_n(x))$ be a sequence of $(X_a, Y_b)$-linear operations $\beta$-bounded for any $x$, and $\beta$-convergent in a set $D$ dense in $X_a$.

Then this sequence is $\beta$-convergent everywhere.

III. Let $(U_n(x))_{n=1,2,...}$ be a sequence of $(X_a, Y_b)$-linear operations, and suppose that, given any $p$, there exists an element $x_p$
such that the sequence \( \{U_p(x_p)\}_{p=1,2,...} \) is \( \beta \)-divergent. Then there exists an element \( x_0 \) such that the sequences \( \{U_p(x_0)\}_{p=1,2,...} \) are \( \beta \)-divergent for \( p=1,2,... \).

III. Let \( \{U_p(x_p)\}_{p=1,2,...} \) be a sequence of \( (X_q,Y_q) \)-linear operations, and suppose that, given any \( p \), there exists an element \( x_p \) such that the sequence \( \{U_p(x_p)\}_{p=1,2,...} \) is \( \beta \)-unbounded for \( p=1,2,... \). Then there exists an element \( x_0 \) such that the sequences \( \{U_p(x_0)\}_{p=1,2,...} \) are \( \beta \)-unbounded for \( p=1,2,... \).

In order to point out what spaces are referred to in I, II, III and IV we shall sometimes denote these statements also by I\((X_e,Y_e)\), II\((X_e,Y_e)\), III\((X_e,Y_e)\) and IV\((X_e,Y_e)\) respectively.

This problem was investigated in 1933 by Mazur and Orlicz \cite{12} in connection with the spaces conjugate to \( B_k \)-spaces. Kantorovich \cite{9} has proved the truth of I\((X_e,Y_e)\) and of II\((X_e,Y_e)\) in the case of \( X_e \) and \( Y_e \) being Kantorovich spaces. Finally, Fichtenholz \cite{7} has shown that I holds in a concrete \( A \)-space which is not a Banach space.

In this paper first two groups of postulates concerning the notion of limit will be analysed, and several examples of spaces satisfying some of them will be given. Then it will be shown that, contrarily to the spaces considered by Banach, in general \( A \)-spaces theorems I, II, III and IV are independent of one another.

Finally, it will be shown what sets of previously considered postulates are sufficient for I, II, III, or IV to be true. The theorems contain the results of the authors mentioned above.

1. Postulates. Let \( X_e \) be a \( A \)-space. We will need the following two groups of postulates \((a_0)\) and \((a_0')\), denoting the elements of \( X_e \), \( \lambda \), and \( \delta \) — real numbers, and \( \{a_n\}, \{p_n\}, \{r_n\} \) and \( \{q_n\} \) — sequences of indices, i.e. increasing sequences of positive integers:

(a) If \( x_n \to 0 \), then there exists \( \lambda_n \to \infty \) and \( \{a_n\} \) such that \( \lambda_n x_n \to 0 \).

(a') If \( x_n \to 0 \) and \( \lambda_n \to 0 \), then there exists a subsequence \( \{x_{n_k}\} \) such that the series \( \sum_{k=1}^{\infty} \lambda_{n_k} x_{n_k} \) is \( \alpha \)-convergent.

\( ^* \) This postulate constitutes a slight modification of a postulate of Fichtenholz \cite{8}, p. 196.

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(a) If \( x_n \to 0 \), then there exists a subsequence \( \{x_{n_k}\} \) such that the series \( \sum_{k=1}^{\infty} x_{n_k} \) is \( \alpha \)-convergent.

(b) \( \{x_n\} \) being an arbitrary sequence there exists a sequence \( \{\theta_n\} \) of numbers, all different from 0, such that \( \sum_{n=1}^{\infty} |\theta_n| < \infty \) implies the \( \alpha \)-convergence of the series \( \sum_{n=1}^{\infty} \lambda_n \theta_n x_n \).

(c) If the sequence \( \{x_n\} \) is not \( \alpha \)-convergent to 0, then there exists a subsequence \( \{x_{n_k}\} \) such that every subsequence of \( \{x_{n_k}\} \) is not \( \alpha \)-convergent to 0.\(^*\)

(d) If \( (a) \lim x_{n_p} = x_p \) for \( p = 1, 2, ... \), and, given any sequence \( \{q_n\} \) of indices the sequence \( \{x_{n_p}\} \) is \( \alpha \)-bounded, then the sequence \( \{x_p\} \) is \( \alpha \)-bounded.

(e) If \( (a) \lim x_{n_p} = x_p \) for \( p = 1, 2, ... \), and, given any sequence \( \{q_n\} \) of indices, \( x_{n_p} \to 0 \), then \( x_p \to 0 \).

(f) If \( (a) \lim x_{n_p} = 0 \) for \( p = 1, 2, ... \), and, given any sequence \( \{q_n\} \) of indices, the sequence \( \{x_{n_p}\} \) is \( \alpha \)-bounded, then there exists a subsequence \( \{q_n\} \) of indices such that \( x_{n_p} \to 0 \).

(g) If \( (a) \lim x_{n_p} = 0 \) for \( p = 1, 2, ... \), and \( \lambda_n \to 0 \), then there exists a sequence of indices \( \{q_n\} \) such that

\[ (a') \lim_{p \to \infty} \sum_{n=1}^{p} \lambda_{q_n} x_{q_n} = 0, \]

where \( \{q_n\} \) is an arbitrary sequence of zeros and ones, and \( \omega_p \gg p \).

(h) If, given any pair \( \{p_n\} \) and \( \{q_n\} \) of sequences of indices, \( x_{n_p} \to x_p \), then the sequence \( \{x_n\} \) is \( \alpha \)-convergent.

It is obvious that (a') implies (a), and (b) implies (b'). The pair of postulates (a) and (a') implies (a').

\( ^* \) Introduced by Mazur and Orlicz \cite{12}.

\(^{9} \) Introduced by Fichtenholz \cite{8}, p. 199.
2.2. The two-norms convergence. Let in a linear space $X$ two $F^*$-norms $||x||$ and $||x||^*$ be defined, satisfying the following condition:

$$(a) \quad ||x|| \to 0 \quad \text{implies} \quad ||x||^* \to 0.$$ 

Denote by $x$ and $x^*$ respectively the convergences generated by the norms $||x||$ and $||x||^*$. A sequence $\{x_n\}$ is said to be $\gamma$-convergent if it is $\gamma$-bounded and $||x_n - x||^* \to 0$.

The convergence $\gamma$ will be termed the two-norms convergence.

2.2.1. The $\gamma$-convergence satisfies the postulates $(a)$, $(a_1)$, $(b_1)$, $(b_2)$ and $(b_3)$.

Proof. We only prove that the condition $(b_3)$ is satisfied. Suppose $(\gamma)$ $\lim x_n = 0$ for $p=1,2,\ldots$ and $\lambda_n \to 0$; hence

$$\lim_{n \to \infty} ||\lambda_n x_n|| = 0 \quad \text{for} \quad p=1,2,\ldots$$

We easily construct by the diagonal method a sequence $\{x_n\}$ of indices such that

$$||\lambda_n x_n|| < 1/2^n \quad \text{for} \quad p=1,2,\ldots \text{ and } n=p,p+1,\ldots;$$

$\varepsilon_i$ being zeros or ones, it follows

$$\left| \sum_{i=p}^{n} \lambda_{i} x_{i} \right| < 1/2^n + 1/2^{n+1} + \ldots + 1/2^n,$$

and by $(a_1)$

$$(\gamma) \lim_{p \to \infty} \sum_{i=p}^{n} \lambda_{i} x_{i} = 0.$$

The $\gamma$-convergence does not in general satisfy the postulate $(a)$. A more precise result is the following:

2.2.2. If the $\gamma$-convergence satisfies the postulate $(a)$, then it is equivalent to the $\nu$-convergence.

Proof. It is sufficient to prove that $x_n \xrightarrow{\gamma} x_0$ implies $||x_n - x_0|| \to 0$. Suppose it is not the case; then there exists an $\varepsilon > 0$ and a sequence of indices $\{k_n\}$ such that $||x_{k_n} - x_0|| \geq \varepsilon$.

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*This convergence is a generalization of the notion of limit considered by Fichtenholz [6] in some concrete spaces.

*This theorem has been formulated in a slightly different form by Fichtenholz [6], p. 209.*
by (a) there exists a sequence \( \{k_0\} \) extracted from \( \{k_n\} \) and a sequence of numbers \( \lambda_n \to \infty \) such that \( \lambda_n(x_{k_0} - x_k) \to 0 \); hence

\[
\|x_k - x_{k_0}\| = \|x_k - x_{k_0}\|_0 = 0,
\]

which is impossible.

It is obvious that \( \gamma \)-boundedness is equivalent to \( \gamma \)-boundedness.

2.2.3. If \( \|x\| \) is a \( F \)-norm, the \( \gamma \)-convergence satisfies the postulates (a0) and (a3).

Proof. We prove only that (a3) is satisfied. Let \( x_n \to 0 \) and \( \lambda_n \to 0 \); it follows \( \|\lambda_n x_n\| \to 0 \). It is sufficient to choose \( n_0 \) so that \( \|\lambda_n x_n\| < 1/2^j \).

If the norms \( |x| \) and \( |x|^* \) are both of \( F \)-type, the condition (a3) implies by a theorem of Banach ([3], p. 41) that the convergences \( \gamma \) and \( \gamma^* \) (and hence \( \gamma \) also) are equivalent.

The following particular case of two-norms convergence is important. Let \( \|x\| \) be a \( F \)-norm. Consider the space \( X \) with the norm \( \|x\|^* \); it can be completed (by addition of new elements) to a \( F \)-space \( X^* \) without altering the norm; let \( \|x\|^* \) denote this norm in \( X^* \). Let us formulate the following conditions:

(a0) If \( x_n \in X, x_n \to x \), and \( \|x_n - x\|^* \to 0 \), then \( x_n \to x \).

(a0) If \( x_n \in X, x_n \to x \), and \( \|x_n - x\|^* \to 0 \), then \( \lim \|x_n\| = \|x\| \).

The conditions (a0), (a2), and (a3) being satisfied the two-norms convergence will be termed strong and will be denoted by \( \gamma \).

We need the following lemma.

2.2.4. Let \( X \) be a \( F \)-space. A necessary and sufficient condition for the sequence \( \{x_n\} \) to be bounded is the boundedness of the sequence \( \{x_n - x_k\} \), where \( \{p_n\} \) and \( \{q_n\} \) are any sequences of indices.

Proof. The necessity is obvious. Suppose now the condition satisfied. It follows that, given any \( \varepsilon > 0 \), there exists a \( \delta > 0 \) and a \( M > 0 \) such that \( |x_k - x_l| < \delta \) and \( p, q > M \) implies \( \|x_k - x_l\| < \varepsilon \).

Suppose the sequence \( \{x_n\} \) unbounded, then there exists a sequence \( \lambda_n \to 0 \) such that \( \lambda_n x_k \to x \) where \( n_0 \to \infty \). Choose \( \delta > 0 \) and \( M > 0 \) such that \( |x| < \delta \) and \( p, q > M \) imply \( \|x\| < \varepsilon/3 \).

There exists a \( P \) such that \( \|\lambda_n x_k\| < \varepsilon/3 \) for \( n > P \). Hence \( k \) being sufficiently large we have

\[
\|\lambda_n x_k\| < \|\lambda_n (x_k - x_p)\| + \|\lambda_n x_p\| < \varepsilon/3 < \varepsilon,
\]

which is impossible.

2.2.5. The \( \gamma \)-convergence satisfies the postulates (b0), (b3) and (b4).

Proof. The first prove that (b3) is satisfied. Let \( (\gamma^0) x_n \to x \) for \( p = 1, 2, \ldots \). Suppose that \( q \to \infty \) implies \( (\gamma^0) x_n \to x \), and let \( \lambda \to 0 \). It follows that, given any \( \varepsilon > 0 \), we have \( \|\lambda x_n\| < \varepsilon \) for \( p, q \) sufficiently large. Since

\[
\lim_{\varepsilon \to 0} \|\lambda x_n\|^* = 0 \quad \text{for} \quad p=1, 2, \ldots,
\]

we get by (a3)

\[
\|\lambda x_n\|^* < \varepsilon \quad \text{for} \quad p > P(\varepsilon);
\]

hence \( \|\lambda x_n\| \to 0 \). It is obvious that \( \|x\|^* \to 0 \); it follows \( x_n \to 0 \).

We prove (b3) other. Suppose that \( p_n \to \infty, q_n \to \infty \) imply \( (\gamma) x_n \to x \). By 2.2.4, the sequence \( \{x_n\} \) is \( \gamma \)-bounded.

Suppose the sequence \( \{x_n\} \) unbounded; then there exists a sequence \( \lambda_n \to 0 \) such that \( \|\lambda_n x_n\| \to \infty \).

2.3. Kantorovich spaces. We recall here the definition of an important class of spaces.

Let \( X \) be a linear semi-ordered space, i. e. one in which an asymmetrical and transitive relation \( x_1 \preceq x_2 \) is defined for certain pairs of elements. Suppose that for these inequalities the usual arithmetical laws hold. Suppose further that \( X \) is a conditional \( \alpha \)-lattice, i. e. given any sequence \( \{x_n\} \) such that \( x_n \preceq x \) or \( x_n \preceq x \), respectively, there exists an element \( x \), denoted by \( \sup x_n \) (or \( \inf x_n \), respectively), such that

\[
\begin{align*}
\text{1}^a & \quad x_n \preceq x \quad \text{for} \quad n = 1, 2, \ldots \quad \text{for} \quad n = 1, 2, \ldots \quad \text{for} \quad n = 1, 2, \ldots \quad \text{for} \quad n = 1, 2, \ldots \quad \text{for} \quad n = 1, 2, \ldots
\end{align*}
\]

Adding the ideal elements \(-\infty\) and \(-\infty\) as usual we can define the lower and the upper limit by the formulae:

\[
\begin{align*}
\liminf_{n \to \infty} x_n &= \inf_{\varepsilon > 0} \lim_{n \to \infty} x_n - \varepsilon, \\
\limsup_{n \to \infty} x_n &= \lim_{\varepsilon \to 0} \inf_{n \to \infty} x_n + \varepsilon,
\end{align*}
\]

\[
\begin{align*}
\liminf_{n \to \infty} x_n &= \inf_{\varepsilon > 0} \lim_{n \to \infty} x_n - \varepsilon, \\
\limsup_{n \to \infty} x_n &= \lim_{\varepsilon \to 0} \inf_{n \to \infty} x_n + \varepsilon.
\end{align*}
\]
The sequence \( \{x_n\} \) is called \( \kappa \)-convergent if
\[
\lim_{n \to \infty} x_n = \lim_{n \to \infty} x_n = x \neq \pm \infty;
\]
the common value of these elements will be denoted by \( \lim x_n \).

A linear space with a \( \kappa \)-convergence is called a Kantorovich space.

Kantorovich has shown [8], p. 134) that

2.3.1. The \( \kappa \)-convergence satisfies the postulate (b).

Write \(|x| = \sup \{|x_1 - x_2|, x_3, \ldots\} \); then \( (\kappa) \lim_{n \to \infty} x_n = x_0 \) implies \( \lim x_n = x \).

Adding three supplementary postulates [8], p. 137) we get special Kantorovich spaces called regular in the sense of Kantorovich. It follows easily from Kantorovich’s results that

2.3.2. The \( \kappa \)-convergence in regular Kantorovich spaces satisfies the postulates (a), (b), (c), (d), (e), (f) of (b).

The regular Kantorovich spaces have also the following properties:

(i) A sequence \( \{x_n\} \) is \( \kappa \)-convergent to \( x \) if and only if there exists an element \( x' \neq \pm \infty \) such that, given any \( \varepsilon > 0 \), the inequality \( |x_n - x| < \varepsilon \) holds for \( n \) sufficiently large;

(ii) A necessary and sufficient condition for the sequence \( \{x_n\} \) to be \( \kappa \)-bounded is the existence of an element \( x \) such that \( |x_n - x| < \varepsilon \) for \( n = 1, 2, \ldots \).

From the results of Kantorovich it follows that the star-convergence corresponding to the \( \kappa \)-convergence has the following property:

2.3.3. The \( \kappa \)-convergence in regular Kantorovich spaces satisfies the postulates (a), (b), (c), (d), (e), (f) of (b).

2.4. Weak convergences. Let \( X \) be a Banach space, \( \Omega \) be a set of linear functionals over \( X \) and \( \Omega \) be said to be fundamental (Blich, [14], p. 66), if there exist positive numbers \( C \) and \( \varepsilon \) such that
\[
\sup_{f \in \Omega} |f| \leq C, \quad \sup_{f \in \Omega} |f(x)| > \varepsilon |x|.
\]

A sequence \( \{x_n\} \) will be termed \( \omega \)-convergent if \( \ell(x_n) \to \ell(x_0) \) for each \( \ell \in \Omega \). The \( \omega \)-convergence will be called also the \( \omega \)-weak-convergence. If \( \Omega \) is identical with the set of all linear functionals over \( X \), the \( \omega \)-convergence will be denoted by \( \sigma \) and termed the \( \omega \)-weak-convergence.

2.4.1. The \( \omega \)-convergence satisfies the postulates (b), (b), (b), (b), (b), (b).

Proof. We only prove that (b) is satisfied. Suppose that
\[
(\omega) \lim_{n \to \infty} x_n = x_p \quad \text{for} \quad p = 1, 2, \ldots,
\]
and that \( q_p \to \infty \) implies \( x_{n_p} \to \infty \). Suppose that the sequence \( \{x_n\} \) does not converge to 0. Then there exists a functional \( \ell \in \Omega \) such that \( \lim_{p \to \infty} |\ell(x_{n_p})| = \varepsilon > 0 \). Given any \( p \), choose \( q_p \) so that
\[
q_p > p \quad \text{and} \quad |\ell(x_{n_p}) - \ell(x_p)| < \varepsilon/2;
\]
we get
\[
|\ell(x_{n_p}) - \ell(x_p)| - |\ell(x_{n_p}) - \ell(x_p)| < \varepsilon/2,
\]
hence \( \lim_{p \to \infty} |\ell(x_{n_p})| > \varepsilon/2 \). This is, however, impossible since \( x_{n_p} \to \infty \).

2.4.2. If the \( \omega \)-convergence satisfies the postulate (a), it is equivalent to the convergence generated by the norm.

Proof. It suffices to prove that \( x_n \to 0 \) implies \( \|x_n\| \to 0 \). The \( \omega \)-convergence implies the boundedness of the sequence of norms (Banach [3], p. 80). Let \( \{x_n^*\} \) be any subsequence of the sequence \( \{x^*_n\} \). By (a) there exist \( x^* \) and a sequence \( \{x_n\} \) of indices such that \( \lambda_n x_n \to x^* \); hence \( \lim_{n \to \infty} \|x_n^*\| = 0 \) and \( \|x_n^*\| \to 0 \). From this we easily infer that \( \|x_n\| \to 0 \).

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It is obvious that

2.4.3. The $\sigma$-convergence satisfies the postulates (a), (a) and (b),
(b), (b'), (b).

2.5. $\textbf{R}_e$-spaces and their conjugate spaces. Let $X$ be a $\textbf{R}_e$-space
(MAZUR and OULCEZ [11], p. 185), and let $\{x_n\}$ be the sequence of
pseudo-norms determining the metrics in this space. The space
$X$ may be considered as a $F$-space with the norm

$$||x|| = \sum_{n=1}^{\infty} \left( \frac{1}{2^n} + \frac{1}{3^n} \right)^{1/n} |x_n|.$$

Hence

2.5.1. The space $X$ satisfies all the postulates (a), (a).

The space conjugate to $X$ is the set $\mathcal{E}$ of all linear functional $\xi(x)$ defined on $X$ (the addition of elements and their
multiplication being defined in the usual manner). MAZUR and
OULCEZ [12] have introduced the $\gamma$-convergence, called the
strong convergence, as follows: $\lim_{n \to \infty} \xi_n = \xi_0$ means that there
exists a $r > 0$ such that $\xi_n(x) < r$ uniformly in the
sphere $||x|| < r$.

2.5.2. The space $X$ is isomorphic with a Banach space, or $\mathcal{E}$ is not (topologically)
isomorphic with any complete metric space.

It is easy to show that

2.6. Convergence almost everywhere. Let $X$ be a linear
space of real measurable functions $x = x(t)$ defined on an interval
$I = (a,b)$. Let two functions equal almost everywhere be considered
as one element of the space. Call $\pi$-convergence the
convergence almost everywhere.

2.6.1. The sequence $\{x_n\}$ is $\pi$-bounded if and only if
lim $||x_n|| < \infty$ almost everywhere [10].

Proof. The necessity only requires proof. Writing

$$x_n(t) = \max(|x_1(t)|, |x_2(t)|, \ldots, |x_n(t)|),$$

it is sufficient to prove that $\lim_{n \to \infty} x_n(t) < \infty$. Suppose it is not

true. Since $x_n(t) \leq \cdots < x_1(t)$, we have $\lim_{n \to \infty} x_n(t) = 0$ in a set of
positive measure, and by the theorem of Egoroff, there exists a
set $H$ of positive measure such that $\inf_{t \in H} x_n(t) = a_n \to 0$. Putting
$\delta_n = \pi^{-n}$ we have $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_n \to 0$. The $\pi$-boundedness
of the sequence $\{x_n\}$ implies the existence of a set $R \subset H$ such
that $|R - H| = 0$ and $\delta_n x_n(t) \to 0$ in $R$. Let $t \in R$; choose $N$ to
have $|\delta_n x_n(t)| < 1$, for $n > N$, and put

$$K = \max(|x_1(t)|, |x_2(t)|, \ldots, |x_N(t)|).$$

There exists a $k, n$ such that $x_n(t) = x_k(t); \quad$ hence

$$|x_n(t)| < K \quad \text{if} \quad k \leq N; \quad \text{otherwise} \quad |\delta_n x_n(t)| = |\delta_n x_n(t)| < |\delta_n x_n(t)| < 1.$$

In both cases $|x_n(t)| \leq \max(K, \delta_n) = \max(K, \delta_n)$. This leads to
contradiction: $1 < a_n \leq \delta_n$ for $n$ sufficiently large.

It is easy to prove that

2.6.2. The $\pi$-convergence satisfies the postulates (a), (b), (b),
(b), (a), (b), and (b).

If the space $X$ is composed of all measurable functions, then
(a), (a), (a), (a) and (b) are also satisfied.

Denote by $c_0(t)$ the characteristic function of the set $E$, and
consider the following condition:

(p) $E$ being any interval in $I$. and $x = x(t)$ being any element
of $X$ the function $x(t) c_0(t)$ belongs to $X$.

2.6.3. The condition (p) being satisfied, every $(X_\pi)$-linear function is
identically equal to 0.

Proof. Let $\xi(x)$ be a $(X_\pi)$-linear functional, and $\xi(x) \neq 0$.
We can suppose $\xi(x) > 0$. Dividing $I$ into two intervals
$A_1, A_2$ we have either $\xi(x(t) c_2) \geq \gamma/2$ or $\xi(x(t) c_2) > \gamma/2$; hence
there exists a element $x_1 = x_1(t)$ such that $\xi(x_1) > \gamma/2$ and
$x_1(t) = 0$ in a set of measure greater than $|I|/(1 - 1/2)$. Continuing
this process we obtain a sequence $\{x_n\}$ such that $\xi(x_n) > 2^n/2^n
and $x_n(t) = 0$ in a set of measure greater than $|I|/(1 - 1/2^n)$. Setting
$x_n = 2^n x_n$, we have $x_n > 0$ and $\xi(x_n) > n$, in contradiction to
the $(X_\pi)$-linearity of $\xi(x)$.

It is well known that $\pi$-convergence is identical with asymptotic
convergence.
2.7. The space \( M \). This space is composed of the measurable and essentially bounded functions \( x = x(t) \) defined on an interval \((a, b)\). Two equivalent functions are considered as one element of the space. This space with the norm \( \|x\| = \text{esssup}_{x \in (a, b)} |x(t)| \) is a Banach space.

Consider the following definitions of limit:

(i) \( \lim_{n \to \infty} x_n = x_0 \) means that \( \|x_n - x_0\| < K \) for \( n = 1, 2, \ldots \) and \( x_n(t) \to x_0(t) \) a.e.

(ii) \( \lim_{n \to \infty} x_n = x_0 \) means that \( \|x_n - x_0\| < K \) for \( n = 1, 2, \ldots \) and \( x_n(t) \to x_0(t) \) almost everywhere.

Denoting \( X^* \) the space of Lebesgue integrable functions in \((a, b)\) and putting \( \|x\|^* = \int |x(t)| \, dt \), we can easily prove that \( y^* \)-convergence is a strong two-norms convergence. Hence

2.7.1. The space \( M^* \) satisfies the postulates \( (a)_1 \), \( (a)_2 \), \( (b)_1 \), \( (b)_2 \), \( (b)_3 \) and \( (b)_4 \).

As may be easily proved, the space \( M^* \) does not satisfy the postulates \( (a)_3 \) and \( (a)_4 \).

The space \( M \) is a Kantorovich space (\textit{ibid.}, p. 156) corresponding to the following partial ordering: \( x_1 < x_2 \) means that \( x_1(t) < x_2(t) \) almost everywhere. The space \( M \) is regular in the sense of Kantorovich. It is easy to prove that

2.7.2. The space \( M^* \) satisfies the postulates \( (a)_1 \), \( (a)_2 \), \( (b)_1 \), \( (b)_2 \), \( (b)_3 \) and \( (b)_4 \), but does not satisfy the postulates \( (a)_3 \), \( (a)_4 \) and \( (b)_5 \).

2.7.3. The space \( M^* \) satisfies the postulates \( (a)_1 \), \( (a)_2 \), \( (b)_1 \), \( (b)_2 \), \( (b)_3 \) and \( (b)_4 \), but does not satisfy the postulates \( (a)_3 \), \( (a)_4 \), \( (b)_5 \) and \( (b)_6 \).

\( \text{esssup} \{x(t)\} \) denotes the greatest lower bound of the numbers \( k \) for which the set \( E = \{t \mid x(t) > k\} \) is of measure 0.

\( \lim_{n \to \infty} x_n(t) \) denotes the asymptotic limit of the sequence \( x_n(t) \).

2.8. The space \( M^* \). This space is composed of the real functions \( x = x(t) \) defined in an interval \((a, b)\), two non-identical functions being considered as different elements of the space. This space with the norm \( \|x\| = \text{esssup}_{x \in (a, b)} |x(t)| \) is a Banach space.

Consider instead of the \( x \)-convergence the same modified by omission of the word „almost” in its definition (ii).

The space \( M^* \) with so modified \( x \)-convergence is a Kantorovich space corresponding to an analogous definition of partial ordering as in the case of the space \( M \).

2.8.1. The space \( M^* \) satisfies the postulates \( (a)_1 \), \( (a)_2 \), \( (b)_1 \), \( (b)_2 \), \( (b)_3 \) and \( (b)_4 \), but does not satisfy the postulates \( (a)_3 \), \( (a)_4 \) and \( (b)_5 \).

\textbf{Proof.} We prove only that \( (b)_4 \) is not satisfied. Represent the interval \( I_0 = [a, b] \) as the sum \( I_0 = \sum I_i \) of open on the right and disjoint intervals, such that \( I_{i+1} \) adheres to \( I_i \) at the right.

Continue the same process with every one of the intervals \( I_i \), and so on. Given any finite sequence \( a_1, \ldots, a_p \) of positive integers, we obtain thus an interval \( I_{a_1, \ldots, a_p} \), open on the right and such that

(i) \( I_{a_1, \ldots, a_p} = \sum \delta I_{a_1, \ldots, a_p} \),

(ii) if \( a_1 - b_1 \neq 0 \) then \( I_{a_1, \ldots, a_p} \cap I_{b_1, \ldots, b_p} = \emptyset \),

(iii) \( I_{a_1, \ldots, a_p} \) adheres to \( I_{b_1, \ldots, b_p} \) at the right.

Put

\[
x_p(t) = \begin{cases} 1 & \text{for } t \in \bigcap_{a_1, \ldots, a_p} I_{a_1, \ldots, a_p} \\ 0 & \text{elsewhere}. \end{cases}
\]

Obviously \( |x_p(t)| < 1 \), and \( \lim x_p(t) = 0 \) for each \( t \); hence \( (a)_4 \) is satisfied.

Let now \( q \to \infty \); the sequence \( \{x_p(q)\} \) is \( x \)-bounded, however, it is not \( x \)-convergent to 0, since the sequence \( \{x_p(q)\} \) does not converge to 0 everywhere. In fact, \( x_p(q) = 1 \) for \( t \in I_{a_1, \ldots, a_p} \), hence \( \lim x_p(t) = 1 \) for any \( t \in \bigcap I_{a_1, \ldots, a_p} \). It is obvious that \( E \neq 0 \).
2.9. The space \( V^* \). This space is composed of all the functions \( x = x(t) \) which are equivalent to functions of bounded variation in \( \langle a, b \rangle \); two equivalent functions of \( V^* \) are considered as one element of the space. Denote by \( \text{ess var} x(t) \) the greatest bound of total variation \( \text{var} x^*(t) \) of the functions \( x^*(t) \) equivalent to \( x(t) \). A function \( x(t) \) belongs to \( V^* \) if and only if \( \text{ess var} x(t) < \infty \).

2.9.1. If \( \text{ess var} x_n(t) < K \) for \( n = 1, 2, \ldots \), and \( \lim x_n(t) = x_0(t) \), then \( \text{ess var} x_0(t) \leq K \).

Proof. Choose a function \( x^*_n(t) \) of bounded variation equivalent to \( x_n(t) \) and such that \( \text{var} x^*_n(t) \leq \text{ess var} x_n(t) + 1/n \). There exists a subsequence \( \{x^*_n(t)\} \) convergent to \( x_0(t) \) almost everywhere. By the theorem of Hales, the sequence \( \{x^*_n(t)\} \) contains a uniformly convergent subsequence \( \{x^*_m(t)\} \). The function \( x^*_m(t) = \lim x^*_m(t) \) is equivalent to \( x_0(t) \); moreover

\[
\text{var} x^*_m(t) \leq \text{ess var} x^*_m(t) \leq K;
\]

hence \( \text{ess var} x_0(t) \leq K \).

If we introduce in \( V^* \) the norm by the formula

\[
||x|| = \text{ess sup} |x(t)| + \text{ess var} x(t),
\]

\( V^* \) becomes a Banach space.

Now we introduce the following convergence:

\( y' \lim x_n = x_0 \) means that \( \text{ess var} x_n(t) \leq K \) for \( n = 1, 2, \ldots \), and \( \lim x_n(t) = x_0(t) \).

The \( y' \)-convergence is equivalent to a strong two-norms convergence. To see this it suffices to denote by \( \hat{X}^* \) the space of the integrable functions and to put \( ||x||^* = \int |x(t)| dt \); the conditions \((n_0)\) and \((n_0)\) follow from 2.9.1. Hence

2.9.2. The \( y' \)-convergence satisfies the postulates \((a)_1\), \((a)_2\), \((b)_1\), \((b)_2\), \((c)_1\), \((c)_2\), \((d)_1\), \((d)_2\) and \((b)_3\) but does not satisfy \((a)_3\), \((a)_4\), as easily can be seen.

\(^{10}\) considered first by Orlicz \cite{[13]}.

\(^{11}\) see, for instance, \cite{[16]}, p. 80.

2.10. The space \( L \). In the well-known space \( L \) of Lebesgue integrable functions in \( \langle a, b \rangle \) consider following notions of convergence:

(i) \((a)\lim x_n = x_0\) means that \( x_n(t) \rightarrow x_0(t) \) almost everywhere, and that there exists an integrable function \( x_0(t) \) such that \( |x_n(t) - x_0(t)| < 1/n \) for \( n = 1, 2, \ldots \).

(ii) \((a)\lim x_n = x_0\) means that \( \int_a^b x_n(t) dt \rightarrow \int_a^b x_0(t) dt \) uniformly in the interval \( \langle a, b \rangle \).

(iii) \((a)\lim x_n = x_0\) means that \( \int_a^b x_n(t) dt \leq K \) for \( n = 1, 2, \ldots \) and \( 0 \leq s \leq 1 \), that \( \int_a^b x_n(t) dt \rightarrow \int_a^b x_0(t) dt \), and that \( \lim x_n(t) dt = \int_a^b x_0(t) dt \).

Kantorovich has shown \( \langle 8 \rangle \), p. 156) that the space \( L \) is a regular Kantorovitch space corresponding to the same partial ordering as the space \( M \). Hence

2.10.1. The \( x \)-convergence satisfies the postulates \((a)_1\), \((a)_2\), \((a)_3\), \((b)_1\), \((b)_2\), \((b)_3\), \((b)_4\), \((b)_5\) and \((b)_6\), but does not satisfy \((b)_7\), as easily can be seen.

\( \gamma \)-convergence is the convergence generated by the norm

\[
||x||^* = \max_{0 \leq c < \infty} \int_a^b |x(t)| dt;
\]

this space, however, is a \( B^\circ \)-space and not a Banach space. This follows from the proposition:

2.10.2. The space \( L \) satisfies the postulates \((a)_1\), \((a)_2\), \((b)_1\), \((b)_2\), \((b)_3\), \((b)_4\), \((b)_5\) and \((b)_6\), but does not satisfy the postulates \((a)_3\), \((a)_4\), \((b)_7\), \((b)_8\) and \((b)_9\).

Proof. \((a)_1\) follows by formula

\[
||x||^* = \int_a^b |x(t)| dt.
\]

We prove now that \((a)_2\) is not satisfied. Denote by \( u(t) \) the function equal to the distance between the number \( t \) and the set of all integers. It is well-known \( \langle 13 \rangle \) that the function

\[
\sum_{n=1}^\infty u(n^2 t)
\]

is continuous and nowhere differentiable. It is easy to prove also that, given any sequence of indices \( \{n_i\} \),
the series $\sum_{n=0}^{\infty} u(4^n t)$ represents a function with the same properties. Put
$$u(t) = \begin{cases} 1 & \text{for } n \leq t < n + 1/2, \\ -1 & \text{for } n + 1/2 < t < n + 1, \end{cases}$$
$$x_n(t) = \frac{u(4^n t)}{2^n}.$$

In order to establish the proposition suppose for instance that $a = 0, b = 1$; then $\int_a^b x_n(t) dt = \frac{u(a)}{2^n}$; hence $\|x_n\|^* \to 0$.

Put $\lambda_n = 2^{-n}$, and let $\{a_n\}$ be any sequence of indices. We prove that the series $\sum_{n=1}^{\infty} x_n x_{a_n}$ is not $\eta$-convergent. In fact, in the contrary case there would exist a function $x_0(t) \in L$ such that
$$\int_0^1 \sum_{n=1}^{\infty} x_n x_{a_n}(t) dt \to \int_0^1 x_0(t) dt$$
uniformly in $s$ as $m \to \infty$. On the other hand $\int_0^1 \sum_{n=1}^{\infty} x_n x_{a_n}(t) dt \to \sum_{n=1}^{\infty} u(4^{a_n} s)$, and this would imply the differentiability almost everywhere of the function $\sum_{n=1}^{\infty} u(4^{a_n} s)$, which is impossible.

To see that $\{b_n\}$ is not fulfilled, note that the sequence constructed above fulfils the condition of Cauchy and is not convergent.

The following theorems can be proved:

2.10.3. The space $L_2$ satisfies the postulates $(a_n)$, $(b_n)$, $(b_1)$, $(b_2)$ and $(b_3)$, but does not satisfy the postulates $(a_1)$, $(a_2)$, $(a_3)$, $(b_4)$.

2.10.4. The general form of the ($L_2$)-linear functionals is
$$\ell(x) = \int_a^b x(t) h(t) dt,$$
where $h(t)$ is an arbitrary function of bounded variation.

The general form of the ($L_2$)-linear functionals is (1) with $h(t)$ absolutely continuous.

2.11. The space $L(X)$. This space is composed of the functions $x = x(t)$ from a real interval $(a, b)$ to a Banach space $X$, integrable in the sense of Bocerna [5], p. 205; two equivalent functions are considered as one element of the space.

Introducing in $L(X)$ the norm by the formula $\|x\| = \int_a^b \|x(t)\| dt$ we get a Banach space.

Consider the following notion of convergence:

$(a) \lim x_n = x_0$ means that $x_n(t) \to x_0(t)$ almost everywhere, and there exists a real integrable function $\gamma(t)$ such that $\|x_n(t) - \gamma(t)\| \leq \varepsilon(t)$ for $n = 1, 2, \ldots$.

It is easy to see that $x_n \to x_0$ if and only if the sequence $\{x_n(t) - x_0(t)\}$ as elements of the space $L$ is $\ast$-convergent to 0, i.e. if there exists an integrable real function $\omega(t)$ such that, given any $\varepsilon > 0$, the inequality $|x_n(t) - x_0(t)| \leq \varepsilon \omega(t)$ holds for any $n$ sufficiently large. It follows:

2.11.1. The space $L_1(X)$ satisfies the postulates $(a_1)$, $(a_2)$, $(a_3)$, $(a_4)$, $(b_1)$, $(b_2)$, $(b_3)$ and $(b_4)$, but does not satisfy the postulate $(b_5)$, as easily may be seen.

2.12. The spaces $L_1$ and $L^1(X)$, In the well-known space $L_1$ of functions integrable with the $p$-th power ($p > 1$) denote by $\sigma$ the weak convergence, by $\ast$ the convergence almost everywhere, and by $\times$ the convergence defined as follows:

$(a) \lim x_n = x_0$ means that $x_n(t) \to x_0(t)$ almost everywhere, and that there exists a function $\gamma(t) \in L^1$ such that $|x_n(t) - \gamma(t)| \leq \varepsilon(t)$ for $n = 1, 2, \ldots$.

The space $L_1$ is a regular Kantorovich space [8], p. 156, corresponding to the same partial ordering as the space $M$. Hence

2.12.1. The $\ast$-convergence satisfies the postulates $(a_1)$, $(a_2)$, $(a_3)$, $(a_4)$, $(b_1)$, $(b_2)$, $(b_3)$, $(b_4)$ and $(b_5)$, but does not satisfy the postulate $(b_6)$, as easily may be seen.

It is easy to prove that

2.12.2. The $\times$-convergence satisfies the postulates $(a_1)$, $(a_2)$, $(a_3)$, $(b_1)$, $(b_2)$ and $(b_4)$, but does not satisfy the postulates $(a_4)$, $(a_5)$, $(b_3)$, $(b_4)$.
The $\sigma$-convergence may be characterized as follows (see Banach [3], p. 155):

\[(o) \lim_{n \to \infty} x_n = x_0 \text{ if and only if } \|x_n - x_0\| < K \text{ for } n = 1, 2, \ldots, \text{ and } \int x_n(t) \, dt \to \int x_0(t) \, dt \text{ for every } s.
\]

2.12.3. The $\sigma$-convergence satisfies the postulates $(a_1)$, $(a_2)$, $(b_1)$, $(b_2)$, $(b_3)$, $(b_4)$, but does not satisfy the postulates $(a_3)$ and $(a_4)$.

Proof. We prove only $(b_4)$. Suppose that $(o) \lim x_n = x_0$ for $p = 1, 2, \ldots,$ and that $q \to \infty$ implies the $\sigma$-boundedness of the sequence $\{x_n\}$. It follows from a theorem of Banach (§3, p. 80) that $\|x_n\| \leq M$ with $M$ independent of $p$ and $q$. Denote by $\mathcal{E}$ the conjugate space to $L^p$. If $\mathcal{E}$ is separable, let $\{\xi_j\}$ be a sequence of elements of $\mathcal{E}$ dense everywhere. Choose $q_j$ to have $|\xi_j(x_n)| < 1/p$ for $j = 1, 2, \ldots, p$ and denote by $\zeta_j(\xi)$ the $(\mathcal{E})$-linear functional of the form $\zeta_j(\xi) = \xi(x_n)$. The inequality

$$|\xi(x_n)| \leq \|\xi\| \|x_n\|$$

implies the boundedness of the sequence $\{\zeta_j(\xi)\}$ for every $\xi$; moreover, $\lim_{n \to \infty} \zeta_j(\xi) = 0$ for $i = 1, 2, \ldots$ Hence by the theorem of Banach-Stinespring (§3, p. 79)

$$\zeta_j(\xi) \to 0 \text{ for any } \xi \in \mathcal{E}, \text{ i.e. } x_{\xi_j} \to 0.$$

Let $X$ denote a Banach space. By $L^p[X]$ we will be denoted the space of the functions $x = x(t)$ from a real interval $(a, b)$ to the space $X$, integrable in Bochner sense with the $p$-th power. Introducing in $L^p[X]$ the norm by the formula

$$\|x\| = \left(\int |x(t)|^p \, dt\right)^{1/p},$$

we get a Banach space.

Let $\sigma$-convergence be defined as follows:

\[(o) \lim_{n \to \infty} x_n = x_0 \text{ means that } x_n(t) \to x_0(t) \text{ almost everywhere, and there exists a function } \gamma(t) \in L^p \text{ such that } \|x_n(t) - \gamma(t)\| < \gamma(t) \text{ for } n = 1, 2, \ldots, \text{ and } \int x_n(t) \, dt \to \int x_0(t) \, dt \text{ for every } s.
\]

As in 2.11 we can prove that

2.12.4. The space $L^p(X)$ satisfies the postulates $(a_1)$, $(a_2)$, $(a_3)$, $(a_4)$, $(b_1)$, $(b_2)$, $(b_3)$, $(b_4)$ and $(b_5)$, but does not satisfy the postulate $(b_6)$.

2.13. The space $H^p$. This space consists of the functions $x = x(t)$ satisfying in $(a, b)$ the Hölder condition

$$|x(t_1) - x(t_2)| \leq M|t_1 - t_2|^p,$$

where $0 < p \leq 1$. Two functions are considered as one element of the space if and only if they are identical. If we introduce the norm

$$\|x\| = |x(0)| + \sup_{0 \leq t_1 < t_2 \leq a} \frac{|x(t_1) - x(t_2)|}{t_1 - t_2},$$

$H^p$ becomes a Banach space.

Consider the following convergence:

\[(o') \lim x_n = x_0 \text{ means that } \|x_n - x_0\| \leq K \text{ for } n = 1, 2, \ldots, \text{ and } x_n(t) \to x_0(t) \text{ uniformly in } [a, b].
\]

Putting $X^* = C[a, b]$ and $\|x\|^* = \max_{0 \leq t \leq b} |x(t)|$ we easily see that $\gamma$ is a strong two-norms convergence. This follows from the lemma:

2.13.1. Let $x_n(t) \in H^p$ and $\|x_n\| \leq K$ for $n = 1, 2, \ldots,$ and suppose that the sequence $\{x_n(t)\}$ converges in a set dense in $(a, b)$. Then there exists an element $x_0(t) \in H^p$ such that $x_n(t) \to x_0(t)$ uniformly in $(a, b)$, and that $\|x_0\| \leq K$.

Proof. The hypothesis implies the uniform equicontinuity of the sequence $\{x_n(t)\}$ in $(a, b)$. By the theorem of Arzelà $\{x_n(t)\}$ converges uniformly to a continuous function $x_0(t)$. The remaining part of the lemma follows by passing to the limit in the formula

$$|x_n(t) - x_0(t)| \leq \frac{|x_n(t_1) - x_n(t_2)|}{t_1 - t_2} \leq K,$$

valid for $a \leq t_1 < t_2 \leq b$.

\[\Box\]
2.15.2. The \( \gamma \)-convergence satisfies the postulates \((a_0), (a_1), (b_1), (b_2), (b_3), (b_4), (b_5), (b_6), (b_7)\), but does not satisfy the postulates \((a_2)\) and \((a_3)\), as easily may be seen.

We can easily prove that if \( \| x_n \| < K \) for \( n = 1, 2, \ldots \), and lim as \( x_n(t) \to x(t) \) uniformly in \(<a, b>\). Hence we can replace in the definition of \( \gamma \)-convergence the condition

\[
x_n(t) \to x(t)
\]

uniformly in \(<a, b>\) by lim as \( x_n(t) = x(t) \).

2.14. The space \( \mathcal{S} \). Let \( s \) denote the space of the sequences \( x = \{ x^n \} \) of real numbers. Introducing the \( n \)-th pseudonorm by formula \( |x^n| = |x^n|_n \), we easily see that \( s \) is a \( B^\varepsilon \)-space. It is known that the general form of the \( (s) \)-linear functionals is

\[
\xi(x) = \sum_{n=1}^{\infty} \xi^n x^n
\]

where \( \xi^n = 0 \) for \( n > N \). Given any \( \xi(x) \), we denote by \( b(\xi) \) the greatest \( n \) for which \( \xi^n \neq 0 \).

Let \( \mathcal{G} \) be the space conjugate to \( s \); \( \mathcal{G} \) consists of the sequences \( \xi = \{ \xi^n \} \) such that \( \xi^n = 0 \) for almost every \( n \). Mazur and Oracz [12] have shown that the \( (s) \)-convergence in \( \mathcal{G} \) may be characterized as follows: if \( \xi^n = \{ \xi^n \} \), and \( \xi^n = \{ \xi^n \} \), then

\[
(\text{a}) \lim b(\xi^n) = K \text{ for } n = 1, 2, \ldots , \text{ and that}
\]

\[
\lim b(\xi^n) = \xi^n
\]

for \( n = 1, 2, \ldots \).

2.14.1. The \( \gamma \)-convergence satisfies the postulates \((a_0), (a_1), (a_2), (b_1), (b_2), (b_3), (b_4), (b_5), (b_6), (b_7)\), but does not satisfy the postulates \((a_0)\) and \((a_2)\).

2.14.2. The general form of the \( (\mathcal{G}) \)-linear functionals is

\[
\xi(x) = \sum_{n=1}^{\infty} \xi^n x^n
\]

with arbitrary \( \xi^n \).

\[\textbf{2.15. The space }L\] In the space \( L \) of the sequences \( x = \{ x^n \} \) such that \( |x^n| = \sum_{n=1}^{\infty} |x^n| < \infty \) consider the following notions of convergence: if \( x_n = \{ x^n \} \) and \( x_n = \{ x^n \} \), then

\[
(\text{a}) \lim x_n = x \text{ means that } \lim x^n_n = x^n \text{ for } i = 1, 2, \ldots \,
\]

and there exists an element \( z = \{ z^n \} \) such that \( z^n_n \leq x^n \) for \( n = 1, 2, \ldots \), and \( i = 1, 2, \ldots \).

\[
(\text{b}) \lim x_n = x \text{ means that } \lim \sum_{n=1}^{\infty} x^n_n = \sum_{n=1}^{\infty} x^n_n \text{ uniformly with respect to } m.
\]

Kantorovich [18, p. 105] has shown that the space \( L \) is a regular Kantorovich space corresponding to the following partial ordering: \( x_n = \{ x^n \} \leq x = \{ x^n \} \) means that \( x^n_n \leq x^n \) for \( i = 1, 2, \ldots \). Hence

2.15.1. The \( \gamma \)-convergence satisfies the postulates \((a_0), (a_1), (a_2), (b_1), (b_2), (b_3), (b_4), (b_5), (b_6), (b_7)\), but — as easily seen — does not satisfy the postulate \((b_3)\).

The \( \gamma \)-convergence is generated by the norm

\[
\| x \| = \sup_{i=1}^{\infty} \sum_{n=1}^{\infty} |x^n|
\]

this norm is, however, a \( B^\varepsilon \)-norm. It is easy to show that

2.15.2. The \( \gamma \)-convergence satisfies the postulates \((a_0), (a_1), (b_1), (b_2), (b_3), (b_4), (b_5), (b_6), (b_7)\), but does not satisfy the postulates \((a_0), (a_1), (a_2)\), and \((b_3)\).

2.16. The independence of postulates. The examples of the spaces \( M, M^*, \mathcal{G}, \mathcal{E}, \) and \( L \) show that the postulates \((a_0), (a_1), (a_2)\) and \((a_3)\) are independent of one another. By the properties of the spaces \( M, M^*, \mathcal{G}, \mathcal{E}, \) and \( L \) it follows that the postulates \((b_0), (b_1), (b_2)\) and \((b_3)\) do not follow from the remaining of the postulates \((b_1)\)-(b_3). The problem of the independence of \((b_3)\) remains open.

3. Independence of statements \( I, II \) and \( III \). Let \( a \) and \( \beta \) denote the convergence generated by the norm in the \( P \)-spaces \( X \) and \( Y \) respectively. Then \( I, Y(X, Y), II(Y, X) \) and \( III(Y, X) \) are true for \( i = 1 \) and \( 2 \). It is not the case in general \( A \)-spaces.

3.1. Theorem. In general \( A \)-spaces the statements \( I, II \) and \( III \) are independent of one another.
Proof. I' does not follow from II' and III'\(^{17}\). Put \(X = L\) and \(Y = S_{0}\). By a theorem of BANACH ([4], p. 32) it follows that II' (\(L, S_{0}\)) is true. We prove later on (see Part IV of this paper, Theorem 3.2) that the statement III' (\(L, S_{0}\)) is true.

The statement I' (\(L, S_{0}\)) is however false. In fact, denote by \(s_{n}(x)\) the \(n\)-th partial sum of the Fourier development of the function \(x = x(t)\), and by \(U_{n}(x)\) the \(n\)-th polynomial of Fejér corresponding to this function. It is obvious that \(s_{n}(x)\), and hence \(U_{n}(x)\), are \((L, S_{0})\)-linear operations. By the classical Fejér-Lebesgue theorem \(\lim_{n \to \infty} U_{n}(x) = x\) for each \(x \in L\). The limit operation \(U(x) = x\) is however not \((L, S_{0})\)-linear since the convergence in mean does not imply the convergence almost everywhere.

I' does not follow from I' and III'. Put \(X = M_{p}\), and let \(Y_{p}\) be the space \(R\) of the reals. The truthfulness of I' (\(M_{p}, R\)) has been proved by OHLICZ\(^{18}\), and the truthfulness of III' (\(M_{p}, R\)) follows from the results of section 8. The statement I' (\(M_{p}, R\)) however is not true. FICHTENHOLZ ([6], p. 199) has shown that the general form of the \((M_{p}, R)\)-linear functionals is

\[
\xi(x) = \frac{1}{2} \int x(t) \cdot h_{\xi}(t) \, dt,
\]

where \(h(t)\) belongs to \(L\). Denote by \(D\) the class of the step-functions; this set is dense in \(M_{p}\). Define the function \(h_{\xi}(t)\) as follows:

\[
h_{\xi}(0) = \frac{2}{4\pi}, \quad h_{\xi}(\frac{1}{4\pi}) = \frac{1}{4\pi}, \quad h_{\xi}(\frac{1}{2\pi}) = -\frac{1}{2\pi}, \quad h_{\xi}(\frac{3}{4\pi}) = -\frac{3}{4\pi}, \quad h_{\xi}(1) = 0,
\]

and \(h_{\xi}(t)\) is linear in the intervals \((0, 1/4\pi), (1/4\pi, 2/4\pi), (2/4\pi, 3/4\pi)\) and \((3/4\pi, 1)\). Put

\[
\xi_{\ast}(x) = \frac{1}{2} \int x(t) \cdot h_{\xi}(t) \, dt;
\]

it is a \((M_{p}, R)\)-linear operation. Since

\[
\int_{0}^{1} |h_{\xi}(t)| \, dt = 1/2 \quad \text{and} \quad \lim_{s \to 0} \int_{s}^{1} |h_{\xi}(t)| \, dt = 0 \quad \text{for} \quad 0 < s \leq 1,
\]

it follows that

\[
|\xi_{\ast}(x)| \leq \int |x(t)| |h_{\xi}(t)| \, dt \leq \frac{1}{2} \|x\| \quad \text{for every} \quad x \in M_{p}.
\]

Moreover,

\[
\lim_{s \to 0} \int_{s}^{1} x(t) \cdot h_{\xi}(t) \, dt = 0 \quad \text{for each} \quad x \in D.
\]

The sequence \(\{\xi_{\ast}(x)\}\) is however not convergent in the whole of \(M_{p}\), since the set-functions \(q_{\ast}(x) = \int x(t) \, dt\) are not equi-absolutely continuous.

III' does not follow from I' and III'. Put \(X_{\ast} = \mathbb{S}_{\ast}\) and \(Y_{\ast} = R\). The truthfulness of the statements II' (\(\mathbb{S}_{\ast}, R\)) and II' (\(\mathbb{S}_{\ast}, R\)) follows from a theorem of MAXIA and OHLICZ\(^{18}\). The general form of \((\mathbb{S}_{\ast})\)-linear functionals being \(\xi = \sum \xi_{n} \cdot \xi_{n}\) with \(\xi = \{\xi_{n}\}\) and arbitrary \(\xi_{n}\), write

\[
\xi_{n}(x) = q \cdot \xi_{n}.
\]

Putting

\[
\xi_{n} = \begin{cases} 1 & \text{for} \quad i = k, \\ 0 & \text{for} \quad i \neq k \end{cases}
\]

we see that the sequence \(\{\xi_{n}(x)\}_{n=1,2,\ldots}\) of \((\mathbb{S}_{\ast}, R)\)-linear operations is unbounded (hence divergent) for \(\xi = \xi_{\ast}\). No element \(\hat{\xi} = \{\hat{\xi}_{n}\}\), however, exists for which the sequences \(\{\xi_{n}(x)\}_{n=1,2,\ldots}\) would be divergent simultaneously, since \(\xi_{\ast}(x) = 0\) for \(p > (\xi_{\ast})\).

4. Some sufficient conditions for I' and II'. In the following lines \(X_{\ast}\) and \(Y_{\ast}\) are \(A\)-spaces, and \(U(x), U_{\ast}(x), U_{\ast}(x)\) and \(U_{\ast}(x)\) operations from \(X_{\ast}\) to \(Y_{\ast}\). \(U_{\ast}(x)\) being any sequence of \((X_{\ast}, Y_{\ast})\)-linear operations, following conditions will be useful:

\((Q)\) If the sequence \(\{U_{\ast}(x)\}\) is \(\beta\)-convergent everywhere, then \(x_{\ast} \to 0\) implies \(U_{\ast}(x_{\ast}) \to 0\).

\((Q)\) If the sequence \(\{U_{\ast}(x)\}\) is \(\beta\)-bounded everywhere, then \(\alpha\)-boundedness of the sequence \(x_{\ast}\) implies \(\beta\)-boundedness of the sequence \(U_{\ast}(x_{\ast})\).

\((Q)\) If the sequence \(\{U_{\ast}(x)\}\) is \(\beta\)-bounded everywhere, then \(x_{\ast} \to 0\) implies \(U_{\ast}(x_{\ast}) \to 0\).

\(^{17}\) See [14]: this follows also from the results of section 6 of the present paper.
It is obvious that (Qₜ) implies (Qₜ), and that (Qₜ) implies (Qₚ). If the space $Xₚ$ satisfies the postulate $(aₚ)$, and $Yₚ$ satisfies the postulate $(bₚ)$, then (Qₚ) implies (Qₚ).

We use in the following considerations the property of additive operations which are $(Xₚ, Yₚ)$-continuous at one point of being $(Xₚ, Yₚ)$-linear. Hence, to prove that an additive operation is $(Xₚ, Yₚ)$-linear it is sufficient to prove its $(Xₚ, Yₚ)$-continuity at $x=0$.

4.1. If the space $Yₚ$ satisfies the postulate $(bₚ')$ and the condition $(Qₚ)$ is satisfied, then theorem $I'(Xₚ, Yₚ)$ holds.

Proof. Let $(U_{a}\{x\})$ be a sequence of $(Xₚ, Yₚ)$-linear operations, $β$-convergent to $U(x)$ everywhere, and let $xₚ \xrightarrow{β} 0$. Since $(β)\lim U_{a}(xₚ) = U(xₚ)$ for $p=1, 2, ..., \text{ and } (Qₚ)$ is satisfied by hypothesis, $qₚ \xrightarrow{β} \infty$ implies $(β)\lim U_{a}(xₚ) = 0$, and $(bₚ')$ implies

$$(β)\lim U(xₚ) = 0.$$  

Thus $U(x)$ is $(Xₚ, Yₚ)$-continuous at $x=0$.

4.2. If the space $Xₚ$ satisfies the postulate $(aₚ)$, the space $Yₚ$ satisfies the postulates $(bₚ)$ and $(bₚ)$, and the condition $(Qₚ)$ is satisfied, then $I'(Xₚ, Yₚ)$ holds.

Proof. Suppose it is not the case. Then there exists a sequence $(U_{a}\{x\})$ of $(Xₚ, Yₚ)$-linear operations and a sequence $\{xₚ\}$ such that $xₚ \xrightarrow{β} 0$, $U_{a}(xₚ) \xrightarrow{α} U(x)$ everywhere, and $U_{a}(xₚ)$ is not $α$-convergent to 0. By $(bₚ)$ we may suppose that every subsequence of $(U_{a}(xₚ))$ is not $β$-convergent to 0; $(aₚ)$ implies the existence of sequences $\{λₚ\}$ and $\{ηₚ\}$ such that $λₚ \xrightarrow{β} \infty$ and $ηₚ = λₚ xₚ \xrightarrow{β} 0$.

By $(Qₚ)$, $qₚ \xrightarrow{β} \infty$ implies the $β$-boundedness of the sequence $(U_{a}(yₚ))$. Since $(β)\lim U_{a}(yₚ) = U(yₚ)$ for $k=1, 2, ..., \text{ the postulate } (bₚ)$ implies the $β$-boundedness of the sequence $(U(yₚ))$, in particular $λₚ^{-1}U(yₚ) = U(xₚ) \xrightarrow{β} 0$, which is impossible.

4.3. If the space $Yₚ$ satisfies the postulates $(bₚ')$, $(bₚ')$, $(bₚ')$, and if the condition $(Qₚ)$ is satisfied, then $I'(Xₚ, Yₚ)$ holds.

Proof. Let $D$ be a set dense in $Xₚ$, and $(U_{a}(xₚ))$ a sequence of $(Xₚ, Yₚ)$-linear operations $β$-bounded everywhere and $β$-convergent in $D$.

Suppose there exists an element $xₚ \in D$ such that the sequence $(U_{a}(xₚ))$ is $β$-divergent.

By $(bₚ)$, there exist two sequences $pₙ \xrightarrow{β} \infty$ and $qₙ \xrightarrow{β} \infty$ such that the sequence $(U_{a}(xₚ) - U_{a}(xₚ))$ does not $β$-converge to 0. By $(bₚ)$ we can suppose that every subsequence of $U_{a}(xₚ)$ has the same property. There exist elements $xₚ \in D$ such that $xₚ \xrightarrow{β} xₚ$. We then have $(β)\lim (U_{a}(xₚ) - U_{a}(xₚ)) = 0$ for $m=1, 2, ..., \text{ and, by } (Qₚ)$, $nₚ \xrightarrow{β} \infty$ implies the $β$-boundedness of the sequence

$$(U_{a}(xₚ) - U_{a}(xₚ)).$$

By $(bₚ)$ there exists a sequence $nₚ \xrightarrow{β} \infty$ such that

$$(U_{a}(xₚ) - U_{a}(xₚ)) \xrightarrow{β} 0.$$  

By the additivity of $U_{a}(xₚ)$ we have

$$(U_{a}(xₚ) - U_{a}(xₚ) = U_{a}(xₚ) - U_{a}(xₚ) + U_{a}(xₚ) - U_{a}(xₚ) + U_{a}(xₚ) - xₚ).$$  

By $(Qₚ)$ the first and the last term of the right-hand side $β$-converges to 0. The second term $β$-converges also to 0, as we have already seen. Hence $U_{a}(xₚ) - U_{a}(xₚ) \xrightarrow{β} 0$, which is impossible.

5. The condition $(Qₚ)$. We now analyse more precisely the condition $(Qₚ)$. Note that the postulate $(bₚ)$ implies the following consequence:

5.1. If $(β)\lim yₚ = yₚ$ for $p=1, 2, ..., \text{ and } λₚ \xrightarrow{β} 0$, then there exist sequences $(qₚ)$ and $(qₚ)$ such that $qₚ \xrightarrow{β} \infty$, implies

$$(β)\lim λₚ^{-1} yₚ = 0.$$  

5.2. Theorem. If the space $Xₚ$ satisfies the postulate $(aₚ)$ and the space $Yₚ$ satisfies the postulates $(bₚ)$, $(bₚ)$, and $(bₚ)$, then the condition $(Qₚ)$ is satisfied.
Proof. Suppose the contrary. Then there exists an everywhere $p$-bounded sequence $\{U_n(x)\}$ of $(X, Y)^*\text{-linear operations}$, and an $\alpha$-bounded sequence $\{x_n\}$, for which the sequence $\{U_n(x_n)\}$ is not $p$-bounded. Hence there exists a sequence $\theta_n \to 0$ such that $\theta_n U_n(x_n)$ is not $p$-convergent to 0, and, by $(b)$, such that every partial sequence of it has the same property. Thus we can assume that $\theta_n \to 0$. Put

$$
\tau_n = \sqrt[p]{v_{\theta_n}}, \quad y_{\tau_n} = U_n\left(\sqrt[p]{v_{\theta_n}} x_n\right).
$$

Since $\sqrt[p]{v_{\theta_n}} \to 0$, we have

$$
(\beta) \lim_{n \to \infty} y_{\tau_n} = (\beta) \lim_{n \to \infty} U_n\left(\sqrt[p]{v_{\theta_n}} x_n\right) = 0,
$$

and by $(b)$ there exists a sequence of indices $\{s_n\}$ such that

$$
(\beta) \lim_{n \to \infty} \sum_{p=0}^{\infty} \sqrt[p]{v_{\theta_n}} U_n\left(\sqrt[p]{v_{\theta_n}} x_n\right) = (\beta) \lim_{n \to \infty} \sum_{p=0}^{\infty} s_n U_n\left(v_{\theta_n} x_n\right) = 0,
$$

$\varepsilon_n$ being zeros or ones and $\omega_p \geq p$. Arrange the elements of the form

$$
e_0 x_n + e_1 x_n + \ldots + e_{\omega_p} x_n,
$$

where $e_0 = 0$ or 1 and $n_1, n_2, \ldots, n_{\omega_p}$ in a sequence $\{x_n\}$, and put $y_{e_i} = \sqrt[p]{v_{\theta_n}} U_n(x_n)$. Since $(\beta) \lim_{n \to \infty} y_{e_i} = 0$ for $p=1, 2, \ldots$, Theorem 5.1 implies the existence of two sequences of indices $r_n$ and $t_n$ such that $n_p = s_n$ and $n_p = \omega_p$ implies

$$
(\beta) \lim_{n \to \infty} U_n(x_n) = 0
$$

(it is sufficient to put $t_n = s_n$). We now construct a sequence $\{z_n\}$ extracted from $\{x_n\}$ as follows: put $z_0 = x_0$ and suppose $z_1, \ldots, z_{k-1}$ determined. Choose $M$ so that all the elements of the form

$$
e_0 z_{n_1} + e_1 z_{n_2} + \ldots + e_{\omega_p} z_{n_{\omega_p}}.
$$

with $e_0 = 0$ or 1, and $i = 1, 2, \ldots, k-1$, appear in the sequence $z_1, z_2, \ldots, z_M$, and put

$$
(\beta) \lim_{n \to \infty} U_n(x_n) = 0,
$$

and formula $(2)$ implies

$$
(\beta) \lim_{k \to \infty} \sum_{k=1}^{n_k} e_k U_n(z_{n_k} x_{n_k}) = 0.
$$

In $(5)$ and $(6)$ the sequence $\{e_k\}$ may be chosen arbitrarily; hence they remain true if we replace the sequence $\{r_n\}$ by any subsequence $\{r_n'\}$. From $(a)$ follows the existence of such a sequence for which the series $\sum_{k=1}^{\infty} r_n z_{n_k} x_{n_k}$ is $\alpha$-convergent; let $x_0$ be its limit.

The operation $U_2(x)$ is $(X, Y)^*\text{-linear}$; hence

$$
U_2(x) = \sum_{r=1}^{\infty} U_2(r_n x_i),
$$

the series being $p$-convergent. By $(5)$ and $(6)$ the sequence

$$
\left[\sum_{r=1}^{\infty} U_2(r_n x_i)\right]
$$

is $p$-bounded. We have

$$
U_2(z_n x_n) = (\alpha) U_2(z_n x_n) + r_n U_2(r_n x_{n_k} + \ldots + r_{n_1} x_{n_1})
$$

$$
+ t_n \sum_{k=1}^{\infty} U_2(r_n x_{n_k}).
$$

In this formula the two last terms of the right-hand side are $p$-convergent to 0. The first term however is not $p$-convergent to 0. It follows that the sequence $\{U_2(x_n)\}$ is $p$-bounded, contrary to hypothesis.

In particular, the condition $(Q_2)$ is satisfied in all the cases, if the $\alpha$-convergence is strong two-norms convergence, or weak convergence in a Banach space, or strong convergence in a space conjugate to a $\ell_p$-space, or $\alpha$-convergence in a Kantorovich space, and if the $p$-convergence is convergence generated by norm in a $\ell_p$-space, or strong two-norms convergence, or $\alpha$-convergence in a Kantorovich space.
It follows from 5.2 that
5.5. If the space $X_\alpha$ satisfies the postulates $(a_s)$ and $(a_\alpha)$, and the space $Y_\beta$ satisfies the postulates $(b_1), (b_2)$ and $(b_3)$, then the condition (Q3) is satisfied.

6. General sufficient conditions for $P$ and $P'$. From sections 4 and 5 we get the following.

6.1. Theorem. If the space $X_\alpha$ satisfies the postulates $(a_\alpha)$ and $(a_\alpha)$, and the space $Y_\beta$ satisfies the postulates $(b_1), (b_2)$ and $(b_3)$, then $P(X_\alpha, Y_\beta)$ is true 31).

In particular, $P(X_\alpha, Y_\beta)$ is true in all the cases of the $\alpha$-convergence being

1. convergence generated by norm in a $F$-space,
2. $\alpha$-convergence in a Kantorovich space,
3. $\alpha$-convergence in the space $L(X)$ or $L^\alpha(X)$,
4. strong convergence in a space conjugate to a $B_\alpha$-space,

and of the $\beta$-convergence being

1. convergence generated by norm in a $F^\beta$-space,
2. weak convergence in a Banach space,
3. strong two-norms convergence,
4. $\alpha^\beta$-convergence in a Kantorovich space 32).

6.2. Theorem. If the space $X_\alpha$ satisfies the postulates $(a_\alpha)$ and $(a_\alpha)$, and the space $Y_\beta$ satisfies the postulates $(b_1), (b_2), (b_3)$, then $P(X_\alpha, Y_\beta)$ is true 33).

In particular, $P'(X_\alpha, Y_\beta)$ holds in the case of $\alpha$-convergence being one of the convergences (1)-(4) and of $\beta$-convergence being the convergence (III), or

1. the convergence generated by norm in a $F$-space,
2. the $\alpha^\beta$-convergence in a Kantorovich space, under the supplementary hypothesis of $(b_3)$ being satisfied.

7. Special sufficient conditions for $P'$ and $P''$. In some more specialized cases we can give other sufficient conditions for $P'$ and $P''$ to hold.

7.1. The case of $\beta$-convergence being two-norms convergence. Suppose the space $X_\alpha$ to satisfy the postulate $(a_\alpha)$, and $\beta$-convergence to be a strong two-norms convergence. Let $\{U_s(x_s)\}$ be a sequence of $(X_s, Y_\beta)$-linear operations $\beta$-convergent to $U(x)$ everywhere. By Theorem 5.2, $x_\alpha \to 0$ and $q_\alpha \to \infty$ imply $\beta$-boundedness of the sequence $\{U_s(x_s)\}$. Since $Y_\alpha$ satisfies the postulate $(b_3)$, and $(\beta)\lim U(x_s) = U(x)$ for $p = 1, 2, \ldots$, the sequence $U(x_s)$ is $\beta$-bounded.

Hence:

7.1.1. Suppose the space $X_\alpha$ to satisfy the postulate $(a_\alpha)$, and the $\beta$-convergence to be strong two-norms convergence in $Y$. Denote by $\beta$ the convergence generated by the norm $\|y\|\beta$ in $Y$. If $P(X_\alpha, Y_\beta)$ holds, then $P(X_\alpha, Y_\beta)$ holds also.

Orelicz [14], p. 78 has shown the truthfulness of $P(M_s, Y_\beta)$, the $\beta$-convergence being convergence generated by the norm in a $F^\beta$-space. Hence:

7.1.2. $P(M_s, Y_\beta)$ holds in the case of the $\beta$-convergence being strong two-norms convergence 34).

For the sake of completeness we give here the proof of this theorem of Orelicz.

We may suppose that the space $Y$ is a $F$-space. It suffices to prove that $(Q_3)$ is satisfied. Denote by $X_0$ the set of all the elements of $M_s$ for which $\|x\| < 1$. Introduce the distance in $X_0$ by the formula $d(x_1, x_2) = \|x_1 - x_2\| \beta$. We easily verify that $X_0$ is a complete metric space. We define the addition in $X_0$ in the usual manner, but only for such elements $x_1, x_2$ for which $\|x_1 + x_2\| < 1$; it is easy to see that $X_0$ is a pseudogroup of Saks [1], p. 15.

Let $\{U_s(x_s)\}$ be a sequence of $(M_s, Y_\beta)$-linear operations convergent everywhere to $U(x)$. The operations $U_s(x) = U_s(x_s)X_0$, are additive and continuous in the pseudogroup of Saks $X_0$; it follows easily from [1], p. 16, that the condition $(Q_3)$ is satisfied.

31) This has been proved by Mazur and Orelicz [12] for the case of $\beta$-convergence being the convergence generated by norm.
32) The cases (2) and (IV) have been proved by Kantorovich [9], p. 237.
33) This has been proved by Mazur and Orelicz [12] for the case of $\beta$-convergence being the convergence generated by norm.
34) Fichtenholz [17], p. 222 has shown that $P(M_s, M_s)$ is true.
It is easy to show that

7.1.3. If the space \( X \) satisfies the postulates \((a)\) and \((a')\), and \(\beta\)-convergence is the weak convergence in a weakly complete \(\beta\) Banach space or in a Banach space, then \(I' (X, \Phi)\) holds.

7.2. The case of functionals. Let \( R \) be the space of the reals with the usual definition of convergence.

7.2.1. If the space \( X \) satisfies the postulate \((a')\), then \(I' (X, \Phi)\) is true \([8]\).

Proof. Let \( \xi (x) \) be the limit of a convergent sequence \( \xi_n (x) \) of \((X, \Phi)\)-linear functionals. Suppose \( \xi (x) \) is not \((X, \Phi)\)-linear. Then there exists a sequence \( x_n \to 0 \) and an \( \varepsilon > 0 \) such that \( |\xi (x_n)| > \varepsilon \).

We can simply suppose that \( |\xi (x_n)| > \varepsilon \). By \((a')\) there exists a sequence of indices \( n_k \) such that the series \( \sum_{n_k} x_n \) is \(\alpha\)-convergent.

Put

\[
\sum_{n=1}^{N} x_{n_k},
\]

Since \( \lim x_{n_k} = \xi (x) \) for \( k=1, 2, \ldots \), there exists for each \( k \) a \( m_k \) such that \( |\xi (x_{m_k}) - \xi (x)| < 1 \). Hence

\[
|\xi (x)| > |\xi (x)| - |\xi (x_{m_k}) - \xi (x)| = \sum_{k=1}^{m_k} |\xi (x_{m_k}) - \xi (x)| = \sum_{k=1}^{m_k} \xi (x_{m_k}) = \xi (x) + \sum_{k=1}^{m_k} \xi (x_{m_k}) > \varepsilon - k \varepsilon .
\]

This is however impossible, since the \(\alpha\)-boundedness of the sequence \( \sum_{n=1}^{N} x_{n_k} \) implies by 5.2 the boundedness of the sequence \( \{\xi (x_{n_k})\} \).

7.2.2. If the space \( X \) satisfies the postulate \((a')\), and \( \Phi \) is the space of reals, then the condition \((Q)\) is satisfied.

Proof. Suppose the contrary; then there exists a sequence \( \{\xi_n (x)\} \) of \((X, \Phi)\)-linear functionals, convergent to \( \xi (x) \), and a sequence \( \{x_n\} \) such that \( x_n \to 0 \), and that \( \xi_n (x_n) \to \varepsilon \) for an \( \varepsilon > 0 \) and for a sequence \( \{m_n\} \) of indices.

We now construct a subsequence \( \{n_i\} \) extracted from \( \{m_n\} \) as follows: put \( n_1 = m_1 \) and suppose \( n_1, \ldots, n_{m_i-1} \) defined; choose then \( n_i \) so that

\[
|\xi (x_{n_i}) - \xi (x)| < 2^{-i\varepsilon} \quad \text{for} \quad i=1, 2, \ldots, k-1 \quad \text{and} \quad p > n_i .
\]

\[
|\xi (x_{n_i}) - \xi (x)| < 2^{-i\varepsilon} \quad \text{for} \quad i=1, 2, \ldots, k-1 .
\]

This is possible, since \( \xi (x) \to \xi (x) \) everywhere, and since \( \xi_n (x) \) and \( \xi (x) \) are \((X, \Phi)\)-linear. Thus the sequence \( \{n_i\} \) is defined by induction.

By \((a')\) we can suppose that the series \( \sum_{n=1}^{N} x_{n_k} \) is \(\alpha\)-convergent. Hence, for \( k \) sufficiently large,

\[
|\xi (x_{n_k}) - \xi (x)| = \sum_{n=1}^{N} |\xi (x_{n_k}) - \xi (x)| > \sum_{n=1}^{N} |\xi (x_{n_k}) - \xi (x)| - \sum_{n=1}^{N} |\xi (x_{n_k}) - \xi (x)| > \varepsilon - \frac{\varepsilon}{2^k}.
\]

and on the other hand \( \xi (x) \to \xi (x) \), which is impossible.

7.3. The condition of Fichtenholz. We shall say the \(\beta\)-convergence in \( Y \) satisfies the condition of Fichtenholz \([9]\) if \( y_{n_k} \to y \) is equivalent to \( \eta (y_k) \to \eta (y) \) for every \((Y, \Phi)\)-linear functional \(\eta (y)\).

7.3.1. If the \(\beta\)-convergence in \( Y \) satisfies the condition of Fichtenholz and \( I' (X, \Phi) \) holds, then \( I' (X, \Phi) \) holds also.

Proof. Let \( U (x) \) be a sequence of \((X, \Phi)\)-linear operations \(\beta\)-convergent to \( U (x) \), and let \( \eta (y) \) be any \((Y, \Phi)\)-linear functional. Put \( \eta (x) = \eta (U (x)) \). The functionals \( \eta (x) \) are \((X, \Phi)\)-linear, and \( \eta (x) \to \xi (x) \to \xi (x) \implies \eta (U (x)) \to \eta (U (x)) \).

We can prove similarly that

7.3.2. If the condition \((Q)\) is satisfied for any sequence of \((X, \Phi)\)-linear functionals, and the \(\beta\)-convergence satisfies the condition of Fichtenholz, then the condition \((Q)\) is satisfied for any sequence of \((X, \Phi)\)-linear operations.

\([\ast]\) Fichtenholz ([9], p. 197) has called regular a convergence satisfying this condition; we prefer to call it by the name of its author.
7.3.3. If the $\beta$-convergence in $Y$ satisfies (a) and the condition of Fichtenholz, and $\Pi(X_\alpha, Y)$ holds, then $\Pi(X_\alpha, Y_\beta)$ holds also.

Fichtenholz has shown ([6], p. 198) that the $x$-convergence in the space $M^*$ satisfies the condition of Fichtenholz. Hence, if $\Pi(X_\alpha, M^*)$ or $\Pi(X_\alpha, M^*)^*$ is true, then $\Pi(X_\alpha, M^*)$ or $\Pi(X_\alpha, M^*)^*$ is also true respectively.

It follows e. g. that $\Pi(M_\alpha, M_\beta)$ holds.

7.4. Theorem II in Kantorovitch spaces. Kantorovitch has shown ([9], p. 539) that $\Pi(X_\alpha, Y_\beta)$ is true if both $X_\alpha$ and $Y_\beta$ are regular Kantorovitch spaces. We give a slight generalization of this result.

Suppose the space $X_\alpha$ satisfies the postulates (a) and (b), and $Y_\beta$ is a regular Kantorovitch space. The $x^*$-convergence satisfies then the postulates (a), (b), (b)' (b), and (b)'.

An operation $U(x)$ from $X_\alpha$ to $Y$ will be said to be $(X_\alpha, Y_\beta)$-quasilinear if it is $(X_\alpha, Y_\beta)$-continuous and satisfies the conditions

$$(7) \quad |U(x+y)| \leq |U(x)| + |U(y)|, \quad |U(x)| \leq \sum |U(x)|_i.$$

It is easy to prove that $U(x)$ being any $(X_\alpha, Y_\beta)$-quasilinear operation, and the series $\sum x_i$, being $\alpha$-convergent with the sum $x_\alpha$, we have

$$|U(x+y)| \geq |U(x)| + |U(y)|, \quad |U(x)| \leq \sum |U(x)|_i.$$

7.4.1. If the sequence $U(x)_\alpha$ of $(X_\alpha, Y_\beta)$-quasilinear operations is $x^*$-bounded everywhere, and the sequence $\{x_i\}$ is a $\alpha$-bounded, then the sequence $(U(x)_\alpha)$ is $x^*$-bounded.

Proof. This theorem may be proved very much like the theorem 5.2. We must first only extract from the sequence $\{x_n\}$ a sequence $\{x_n\}$ such that the series $\sum |U(x_n)|$ be convergent for $p=1,2, \ldots$. This can be done by the diagonal method. Then the proof goes on like that of the theorem 5.2. In the final evaluations inequalities (7) will be used.

This definition resembles a notion introduced by Mazur and Orlicz ([10], p. 157).

Since the space $X_\alpha$ satisfies the postulate (a), we deduce from 7.4.1 as in section 7 that 7.4.2. If the sequence $\{U(x)_\alpha\}$ of $(X_\alpha, Y_\beta)$-quasilinear operations is $x^*$-convergent to $U(x)$ everywhere, then the operation $|U(x)|$ is $(X_\alpha, Y_\beta)$-continuous.

7.4.3. Theorem. Suppose the space $X_\alpha$ satisfies the postulates (a) and (b), and the space $Y_\beta$ is a regular Kantorovitch space. Let $\{U(x)_\alpha\}$ be a sequence of $(X_\alpha, Y_\beta)$-linear operations, $x^*$-bounded everywhere and $x$-convergent in a set $D$, dense in $X_\alpha$. Then this sequence is everywhere $x$-convergent.

Proof. Write

$$V(x) = \sup_{\alpha} |U(x)_\alpha|, \quad W(x) = \lim_{\alpha} U(x)_\alpha - \lim_{\alpha} U(x).$$

The operations $V(x)$ are obviously $(X_\alpha, Y_\beta)$-quasilinear, and $V(x) = |U(x)_\alpha|$. The sequence $(V(x))$ is $x^*$-bounded everywhere and non-decreasing. Hence ([8], p. 132) it is $x$-convergent everywhere. Let $V(x)$ be the limit of this sequence. By 7.4.2 $V(x)$ is $(X_\alpha, Y_\beta)$-continuous. The inequalities

$$|W(x)| < 2V(x), \quad \|W(x) - |W(y)|\| < |W(x-y)|$$

imply further the $(X_\alpha, Y_\beta)$-continuity of $W(x)$. By hypothesis $W(x) = 0$ and $W(x) = 0$ everywhere.

Theorem 7.4.3 remains true if we replace the space $Y_\beta$ by $L(X_\alpha)$ or $L_\beta(X_\alpha)$.

8. Theorems III$\alpha$ and III$\beta$. We now give conditions for $X_\alpha$ and $Y_\beta$ which are sufficient for the truthfulness of III$\alpha(X_\alpha, Y_\beta)$ and III$\beta(X_\alpha, Y_\beta)$. We restrict our considerations to the case of the $\beta$-convergence being convergence generated by the norm in a $F$-space or strong two-norms convergence.

8.1. Theorem. If the space $X_\alpha$ satisfies the postulate (a), and if $\beta$ is the convergence generated by norm in the space $Y_\beta$, then III$\alpha(X_\alpha, Y_\beta)$ and III$\beta(X_\alpha, Y_\beta)$ are true.

Proof. Let $\{U(x)_\alpha\}$ be a sequence of $(X_\alpha, Y_\beta)$ linear operations. Suppose that, given any $p$, there exists an element $x_\alpha$ such that the sequence $\{U(x))_{p=1,2, \ldots} \beta\}$ is $\beta$-divergent. By (a) there exists a sequence $\{\theta_\alpha\}$ of numbers different from 0, such
that $\sum |\lambda_i| < \infty$ implies the $\alpha$-convergence of the series $\sum \lambda_i \theta_i x_i$.

The sequence $(U_p(x_i)_{\alpha\in T})_{\alpha}$ is obviously $\beta$-divergent for $p=1,2,...$. Write, $x=[\lambda_i]$ being any element of the space $I$,

$$U_p^\alpha(x)=U_p\left(\frac{\sum \lambda_i}{\alpha} \theta_i x_i \right), \quad U_p^\beta(x)=U_p\left(\sum \lambda_i \theta_i x_i \right).$$

The last operations are $(I,Y)$-linear, and $\beta \lim U_p^\beta(x)=U_p^\beta(x)$ everywhere. Hence $U_p^\alpha(x)$ also are $(I,Y)$-linear. Put $x_\delta=[\delta_i]_{\alpha=1,2,...}$, $\delta_{\alpha} \theta_i$ denoting the delta of Kronecker. The sequence $U_p^\alpha(x_\delta)_{\alpha=1,2,...}$ is $\beta$-divergent for $p=1,2,...$. Since $\Pi^I(I,Y)$ holds (10), there exists an element $x_\delta=[\delta_i]_{\alpha=1,2,...}$ such that the sequences $U_p^\alpha(x_\delta)_{\alpha=1,2,...}$ are $\beta$-divergent for $p=1,2,...$. It follows that the sequences $U_p^\alpha(x_\delta)_{\alpha=1,2,...}$ are $\beta$-divergent for $p=1,2,...$, $x_\delta$ being the element $\sum \lambda_i \theta_i x_i$.

The proof of $\Pi^I(I,Y)$ to hold is similar.

8.2. Theorem. If the space $X_\delta$ satisfies the postulate $(a_\delta)$, and if the $\beta$-convergence is strong two-norms convergence, then $\Pi^I(I,Y)$ and $\Pi^I(I,Y)$ are true.

Proof. We first prove $\Pi^I(I,Y)$. Let $(U_p(x))_{\alpha=1,2,...}$ be a sequence of $(X_\delta,Y)$-linear operations. Suppose, given any $p$, there exists an element $x_\delta$ such that the sequence $\sum U_p(x_\delta)_{\alpha=1,2,...}$ is $\beta$-unbounded. Choosing the sequence $[\delta_i]$ like in the proof of Theorem 8.1, define the sequences $U_p^\alpha(x)$ and $U_p^\alpha(x)$ by (8). These operations are $(I,Y)$-linear. This is obvious for $U_p^\alpha(x)$, and for $U_p^\alpha(x)$ it follows from $U_p^\alpha(x) \geq U_p^\alpha(x)$ and from Theorem 6.1. Now, we can finish the proof as that of Theorem 8.1.

To prove $\Pi^I(I,Y)$ let $(U_p(x))_{\alpha=1,2,...}$ be a sequence of $(X_\delta,Y)$-linear operations. Suppose, given any $p$, there exists an element $x_\delta$ such that the sequence $\sum U_p(x_\delta)_{\alpha=1,2,...}$ is $\beta$-divergent. Thus the sequence of sequences $\sum U_p(x_\delta)_{\alpha=1,2,...}$ can be decomposed in two, $\sum U_p(x_\delta)_{\alpha=1,2,...}$ and $\sum U_p(x_\delta)_{\alpha=1,2,...}$, the first of which is divergent in $Y^\alpha$ for $x=x_\delta^\alpha$, and the other is unbounded in $Y$ for $x=x_\delta^\beta$. Putting, as in the proof of Theorem 8.1,
Linear operations in Saks spaces (I)

by

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This paper deals with metric spaces composed of elements of the unit sphere of a linear normed space, the metric of which is defined (see 1.3) by means of another norm, not necessarily homogeneous. The spaces of this kind may be considered as pseudolinear in a certain sense, and some investigations of Banach spaces can be adapted to the spaces of this kind 1).

1.1. Let $X$ be a linear space. A functional $|x|$ defined in $X$ will be called a $B$-norm if it satisfies the following conditions:

(a) $|x|=0$ if and only if $x=0$,

(b) $|x+y| < |x| + |y|$, 

(c) $|θx| = |θ||x|$, $θ$ being any real number.

Each functional $|x|$ satisfying the above conditions (a), (b) and the following one:

(c') if the sequence $|θ_n|$ of real numbers tends to 0 and $|x - x_n| \to 0$, then $|θ_n x_n - 0 x_n| \to 0$

will be said to be a $F$-norm.

Any functional $|x|$ satisfying the conditions (b) and (c), or (b) and (c'), will be termed a $B$- or $F$-pseudonorm respectively.

A Banach space or a Fréchet space is a linear space $X$ provided with a $B$- or $F$-norm (i.e. Banach norm or Fréchet norm) respectively and such that the distance

$d(x, y) = |x - y|$

makes $X$ a complete metric space.

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1) The results of this paper were presented September 26th 1948 at the VI Polish Mathematical Congress in Warsaw. The second part of the present paper (to appear) will deal with investigation of sequences of operations and with applications of the results of part I.