On differentiation of vector-valued functions

by

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In recent years a series of papers appeared, dealing with the problem of differentiation of vector-valued functions. The most interesting problem was perhaps to inquire under what hypotheses the weak differentiability implies the strong one. The most complete results in this direction obtained Petrić [7].

In the present paper 1) further remarks on this subject will be added, generalizing 2) some results of my paper [1] and of the paper of Petrić.

In §1 preliminary definitions are given, and the main result of this paper is formulated. In §§2 and 3 the lemmas are grouped, upon which the principal theorems contained in §4 are based. Finally, in §5 some applications to Analysis are given.

§1. Preliminary considerations. $X$ denotes a Banach space, $\|x\|$ the norm of the element $x$ of $X$, $\xi$ the space conjugate to $X$, and $\xi(x)$ the elements of $\xi$.

By functions I mean in this paper the vector-valued functions, i.e., functions from an arbitrary fixed interval $J$ or from a set $E$ of reals to the space $X$; for these functions the symbols $x(t)$, $y(t)$ and $z(t)$ are reserved. Real-valued functions will be denoted by $f(t)$.

The limit of $\varphi(t)$ as $t$ tends to $t_0$ by values of the set $P$ will be denoted by $\lim_{t \to t_0^+} \varphi(t)$.

1) whose results were in part presented September 22th, 1949, to the VI Polish Mathematical Congress in Warsaw.

2) The author is indebted to Professor W. Orlicz for having called his attention to the possibility of such a generalization.
The symbol $|E|$ will denote, as usually, the Lebesgue measure of the set $E$, and $|E|_n$ will denote the outer measure of the same set.

A subset $E_0$ of the set $E$ will be called fundamental for $X$ if, given any $\varepsilon > 0$ and $x_0 \in X$, there exist elements $\xi_1, \xi_2, \ldots, \xi_n \in E_0$ and real numbers $a_1, a_2, \ldots, a_n$ such that

$$|\xi| = 1, \quad \|\xi(x)\|_n = \|x\| - \varepsilon \text{ for } \xi = a_1\xi_1 + a_2\xi_2 + \ldots + a_n\xi_n.$$  

In § 4, § 5 and § 6 $E_0$ will stand for an arbitrary but fixed set fundamental for $X$; in § 5 this set will be specialized to concrete cases.

We will deal with the following notions of differentiability of vector-valued functions:

A function $x(t)$ will be said to be **strongly** differentiable at $t_0$ to $x_0$, if the expression

$$\lim_{h \to 0} \frac{x(t_0 + h) - x(t_0) - x_0}{h}$$  

(1)

 tends to 0 when $h \to 0$; the element $x_0$ will be called the **strong derivative** of $x(t)$ at $t_0$ and denoted by $x'(t_0)$.

A function $x(t)$ will be said to be $\xi_0$-**weakly** differentiable at $t_0$ to $x_0$, if for every $\xi \in E_0$ the expression

$$\xi \left( \lim_{h \to 0} \frac{x(t_0 + h) - x(t_0) - x_0}{h} \right)$$  

(2)

 tends to $\xi(x_0)$ when $h \to 0$; the element $x_0$ will be termed the $\xi_0$-**weak derivative** of $x(t)$ at $t_0$, and denoted by $x'_\xi(t_0)$.

The function $x(t)$ will be said to be **approximately strongly** differentiable at $t_0$ to $x_0$, if the expression (1) tends approximately to 0 when $h \to 0$; in this case the element $x_0$ will be termed the **strong approximate derivative** of $x(t)$ at $t_0$, and written $x'_{a\xi}(t_0)$.

It is obvious that the elements $x'(t_0)$, $x'_\xi(t_0)$ and $x'_{a\xi}(t_0)$ are uniquely determined, if existing. If the function $x(t)$ is differentiable at any point of a set $E$ to the element $y(t)$ according to any one of the above definitions, I shall say that $x(t)$ is **differentiable** in the respective sense in $E$ to $y(t)$.

The definition of the differentiability a.e. (almost everywhere) in $E$ is obvious.

I shall also consider another notion of differentiability, which is not so closely related to the behaviour of the considered functions at particular points.

The function $x(t)$ will be said to be $\xi_0$-**pseudodifferentiable** to $y(t)$ in the set $E$, if for every $\xi \in E_0$ there exists a set $H_1$ depending on $\xi$, such that

(i) $|E - H_1| = 0$,

(ii) $\frac{d}{dt} \xi(x(t)) \approx \xi(y(t))$ at any point of $H_1$.

In this case the function $y(t)$ will be termed the $\xi_0$-**weak pseudoderivative** of $x(t)$ in $E$, and denoted by $\dot{x}'(t_0)$.

If, given any $\xi \in E_0$, there exists for the function $x(t)$ a set $H_1$ satisfying (i) and such that $\xi(y(t))$ is the approximate derivative of $\xi(x(t))$ at any point of $H_1$, the function $x(t)$ will be said to be $\xi_0$-**approximately pseudodifferentiable** to $y(t)$ in $E$; the function $y(t)$ will be termed the $\xi_0$-**approximate pseudoderivative** of $x(t)$ in $E$, and written $\dot{x}'_{a\xi}(t_0)$.

The above definitions are due essentially to Petrus [7].

A function $x(t)$ will be said to be **essentially separably valued**, or briefly e.s.v., in $E$, if there exists a set $H$ such that $|H| = 0$, the set $E \setminus (y = x(t), t \in E \setminus H)$ being separable.

The main result of this paper is included in the following:

**Theorem 1.** Let the function $x(t)$ be $\xi_0$-weakly differentiable in a set $E$, and let $\dot{x}'(t_0)$ be e.s.v. in $E$. If

$$\lim_{h \to 0} \left\| \frac{x(t_0 + h) - x(t_0)}{h} \right\|_n < \infty$$

at any point of $E$, then $x(t)$ is strongly differentiable a.e. in $E$ to $\dot{x}'(t_0)$.

**Corollary 1.** Under the hypotheses of Theorem 1 there exists a set $H \subset E$ such that $|E - H| = 0$ and

$$\frac{d}{dt} \xi(x(t)) = \xi(\dot{x}'(t_0))$$

for every $t \in H$ and every functional $\xi(x)$, linear on $X$.  

\(\xi\) i.e., the set of the $y$'s satisfying the conditions in \{ \}.
The hypothesis of $x'(t)$ being e.s.v. cannot be removed in Theorem 1. Glaeser [4], p. 205] has given an example of a function $x\in sAC$, (see §2) everywhere $\mathbb{R}$-weakly differentiable ($E_0$ being a fundamental set), and nowhere strongly differentiable, but the $E_0$-weak derivative of this function is not e.s.v., as may be easily verified.

**§ 2. Lemmas.** A function $x(t)$ will be said to be $sAC$ (strongly absolutely continuous) on $E$, if for every $\varepsilon > 0$ there corresponds a $\delta > 0$ such that, given any finite sequence $[(a_i, b_i)]$ of non-overlapping intervals the endpoints of which belong to the set $E$, $\sum_i |a_i - b_i| < \delta$ implies $\sum_i \|x(a_i) - x(b_i)\| < \varepsilon$.

**Lemma 1.** If for a function $x(t)$ the inequality
$$\lim_{h \to 0} \sup_{t \in E} \left\| \frac{x(t+h) - x(t)}{h} - x'(t) \right\| < \infty$$
holds at any point of a set $E$, then $E$ can be decomposed in a sequence of sets on each of which the function $x(t)$ is $sAC$.

The well-known proof of this Lemma in case of $X$ being the set of real numbers $\mathbb{R}$ can be easily applied to the case of $X$ being a Banach space.

Let $E$ be a closed set with the bounds $a$ and $b$, and let $[(a_n, b_n)]$ be the sequence of open intervals contiguous to $E$, contained in the closed interval $[a, b]$. Then, $x(t)$ being any function defined in a set $H \supseteq E$, I shall denote by $x(t)$ or, if necessary, by $x(t; E)$ the function coinciding with the function $x(t)$ on the set $E$ and linear on the intervals $(a_n, b_n)$, i.e. the function defined by the formula
$$\hat{x}(t) = \begin{cases} x(t) & \text{for } t \in E, \\ x(a_n) + \frac{x(b_n) - x(a_n)}{b_n - a_n} (t - a_n) & \text{for } t \in (a_n, b_n). \end{cases}$$

**Lemma 2.** If the function $x(t)$ is $sAC$ on a closed set $E$, so is the function $\hat{x}(t; E)$.

The easy proof is left to the reader.  

\footnote{See, for instance, Saks [8], p. 239.}

The function $x(t)$ is said to fulfill the condition (l) at $t_0$, if there exists a constant $M$ such that for any $t \in J$
$$\|x(t) - x(t_0)\| \leq M|t - t_0|.$$  

**Lemma 3.** If the function $x(t)$ fulfills at any point of the set $E$ the condition (l) and is strongly approximately differentiable in $E$, then $x(t)$ is strongly differentiable a.e. in $E$.

**Proof.** Denote by $T_n$ the set of the elements $t_n$ at which (3) holds with $M \leq n$. Each of these sets is closed, and $E \subseteq \bigcup_{n=1}^{\infty} T_n$. Denote by $D_n$ the set of the points of density of the set $T_n$. By Density Theorem it is sufficient to prove that $x'(t)$ exists at any point of the set $ED_n$. Let $t_n \in ED_n$, and write $x_n = x'(t_n)$. There exists a set $P$ for which $t_n$ is a point of outer density such that
$$\lim_{t \to t_0} \|x(t) - x(t_n) - x_n\| = 0.$$  

Hence
$$\lim_{t \to t_0} \|x(t) - x(t_n) - x_n\| = 0,$$
and we easily see that $t_n$ is a point of outer density of the set $PD_n$. Let $t \to t_0$, and, to fix ideas, suppose that $t < t_0$. Then
$$\frac{|(t_0, t_0) \times (t_0, t)|}{t_0 - t} = \delta_0 \to 1.$$  

In the interval $(t, t_0)$ there must exist at least one point $\tau_0$ such that $\tau_0 \in PD_n$ and
$$\frac{|(t, \tau_0) \times (\tau_0, t_0)|}{t_0 - t} \leq \frac{1}{\delta_0(1 - \frac{1}{t_0 - t})},$$
for in the contrary case the interval $[(t, t_0) \times (t_0, t)]$ would be contained in the set $(t, \tau_0) \times (t_0, t)$, and hence
$$\frac{|(t_0, t_0) \times (t_0, t)|}{t_0 - t} \leq (t_0 - t_0)|t_0 - t| \delta_0(1 - \frac{1}{t_0 - t}),$$
contrarily to (4). Since
$$\frac{x(t) - x(t_n)}{t_0 - t} = \frac{x(t) - x(t_0)}{t_0 - t_0} + \frac{x(t_0) - x(t_n)}{t_0 - t_0} \to 0,$$
and
\[ \frac{|x(t_i) - x(t_j)|}{|t_i - t_j|} \leq n_i, \]
we get
\[ \frac{x(t_i) - x(t_j)}{t_i - t_j} = \frac{x(t_i) - x(t_j)}{t_i - t_j} \to \frac{x(t) - x(t)}{t - t} \to \frac{x(t) - x(t)}{t - t} \to 0 \]
as \( i \to \infty \). Hence
\[ \lim_{h \to 0} \frac{|x(t_i + h) - x(t_j)|}{h} \leq \infty \]
at any point of \( E \), then the function \( x(t) \) is approximately strongly differentiable to \( y(t) \) a.e. in \( E \).

**Proof.** We can suppose without loss of generality that the space \( X \) is separable. By Lemma 6 there exists a sequence \( \{y_n(t)\} \) of simple functions converging to \( y(t) \) a.e. in \( E \). The set \( E_1 \) of the points at which \( y^*(t) = \lim y_n(t) \) exists and (5) holds is, as may be easily seen, measurable, and \( |E - E_1| = 0 \). By a theorem of Banach ([2], p. 124) there exists a sequence \( \{\xi_n\} \) of elements of \( E_1 \), weakly dense in \( E \). The set \( E_1 \) of the linear combinations with rational coefficients of the \( \xi_n \)'s is fundamental for \( X \) and we easily observe that \( x(t) \) is \( E \)-approximately pseudodifferentiable to \( y^*(t) \) a.e. in \( E_1 \).

Let \( E_0 \) be the set of points at which
\[ \lim_{h \to 0} \frac{|x(t_i + h) - x(t_i) - y^*(t)|}{h} = 0 \]for any \( \xi \in \mathbb{E}_1 \).

The set \( E_0 \) is measurable, and \( |E - E_0| = 0 \). We shall prove that \( x(t_i) - y^*(t) \) exists a.e. in \( E_0 = E_1 \).

By Lemma 1 the set \( E_0 \) can be represented as a sum \( \sum_{n=1}^{\infty} \text{H}_n \) of sets on each of which \( x(t) \) is sAC. Since the sets \( K_n \) of all points of outer density of the set \( \text{H}_n \) are measurable, the sets \( E_0, K_n \) are measurable; moreover \( |\text{H}_n - K_n| = 0 \). Hence
\[ |E_0 - \sum_{n=1}^{\infty} E_n K_n| = 0. \]
The function $x(t)$ being measurable, it can be proved similarly as for real-valued functions that for almost every $t \in E$, there exists a set $E_t$ for which $t$ is a point of density, and
\[
\lim_{t \to t^+} x(y) = x(t).
\]

Thus, any point of $K_n$, being a point of outer density for $H_n$, and $x(t)$ being sAC on $H_n$, we can easily prove that there exists a set $R_n$ of measure 0 such that $x(t)$ is sAC on $L_n = E_n K_n - R_n$.

It is sufficient to prove that $x(t)$ is approximately strongly differentiable to $y^*(t)$ at almost every point of $L_n$. Let $\varepsilon > 0$ be arbitrary, $F$ being any closed set such that $F \subseteq L_n$ and $|L_n - F| < \varepsilon$, put $x(t) = x(t;F)$. The function $x(t)$ is evidently e.s.v. in $J$ and is strongly differentiable to $c_0 = \text{const.}$ in any interval $J$, contiguous to $F$. Since for every $\xi \in \mathcal{E}$, the real-valued function $\xi(x(t))$ is sAC on $J$, the derivative $\frac{d}{dt} \xi(x(t))$ exists a.e. in $J$; moreover, $\frac{d}{dt} \xi(x(t)) = \xi(y^*(t))$ a.e. in $F$. Thus $x(t)$ is $\mathcal{E}$-approximately pseudodifferentiable to $y^*(t)$ in $F$. By Lemma 4, $x(t)$ is strongly differentiable a.e. in $F$; it follows that at almost any point $t \in F$
\[
x'(y) = x'(t;F) = y^*(t) = y(t).
\]

The number $\varepsilon > 0$ being arbitrary, the above relation holds a.e. in $L_n$.

Theorem 1 is an immediate consequence of the following

**Theorem 3.** Let the function $x(t)$ be $\mathcal{E}$-approximately pseudodifferentiable in $E$ to a function $y(t)$ e.s.v. in $E$, and let
\[
\lim_{h \to 0} \frac{|x(t+h) - x(t)|}{h} = \infty
\]
at any point $t \in E$. Then $x(t)$ is strongly differentiable to $y(t)$ a.e. in $E$.

**Proof.** It is sufficient to prove that, given any point $t \in E$, there exists an interval $I = (a, b)$ including the point $t$, in which $x'(t)$ exists a.e. in $E$. By (6) there exist for any $t \in E$ two numbers $M(t)$ and $\delta(t)$ such that
\[
|t - t'| < \delta(t) \implies \|x(t) - x(t')\| \leq M(t)|t - t'|.
\]

Write $I = (t_3 - \delta(t), t_3 + \delta(t))$; the function $x(t)$ is then bounded on $I$, i.e., $\|x(t)\| \leq A$.

Let $t'$ be any point of the set $IE$. We easily observe that $t \in I$ implies $\|x(t) - x(t')\| \leq [M(t') + \frac{2A}{\delta(t)}]|t - t'|$.

Thus the function $x(t)$ fulfills the condition (1) on $IE$. Hence $x(t)$ is continuous in $IE$. We can suppose that $x(t)$ is measurable in $IE$. Applying Theorem 2, we see that $x'(t)$ exists a.e. in $IE$, and by Lemma 3 also $x'(t)$ exists a.e. in $IE$.

Any function $\mathcal{E}$-weakly differentiable will be now simply said to be weakly differentiable.

**Theorem 4.** Let $x(t)$ be weakly differentiable in a set $E$ to $y(t)$. Then $x(t)$ is strongly differentiable a.e. in $E$ to $y(t)$.

**Proof.** Since, given any $t \in E$,
\[
x'(t) = \lim_{h \to 0} \frac{\xi(x(t+h) - x(t))}{h} = \xi(y(t)),
\]
we see, applying a theorem of Banach ([2], p. 80), that the condition (6) is satisfied at any point of $E$. The function $y(t)$ being e.s.v. by a theorem of Pettis ([6], theorem 1.2), we can apply Theorem 5 to get the conclusion.

A function $x(t)$ is said to be Lipschitzian, if
\[
\|x(t_1) - x(t_2)\| \leq M|t_1 - t_2|
\]
with $M$ non depending on $t_1$ and on $t_2$.

A Banach space $X$ will be said to have the property (D), if every Lipschitzian function from $J$ to $X$ is strongly differentiable a.e. in $J$.

Examples of spaces with the property (D) are furnished by the uniformly convex, the reflexive, and the locally weakly compact spaces (Pettis ([7], p. 262).

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7 This theorem is a slight generalization of theorem 2.9 of Pettis ([7], p. 262).

8 This condition has been introduced by Pettis ([5], p. 427).
Theorem 5. Let the space $\mathcal{X}$ have the property (D). If for a function $x(t)$ the inequality
\begin{equation}
\lim_{h \to 0} \left\| x(t+h) - x(t) \right\| < \infty
\end{equation}
is satisfied at any point of a set $E$, then the strong derivative $x'(t)$ exists a.e. in $E^1$.

Proof. It can be easily shown that the set of the points at which (7) holds is measurable. Hence we can suppose that the set $E$ is so. By Lemma 1 there exists a sequence $\{E_n\}$ of sets on each of which $x(t)$ is sAC, and such that $E = \bigcup_{n=1}^{\infty} E_n$. Since $x(t)$ is continuous at any point of $E$, the function $x(t)$ is sAC on $EE_n$. It follows that the sets $E_n$ may be supposed to be measurable.

Let $n$ be fixed. Given an arbitrary $\varepsilon > 0$, denote by $F$ a closed set for which $F \subseteq E_n$ and $|E_n - F| < \varepsilon$. Write $x(t) = \chi(t, F)$. This function is sAC by Lemma 2. Hence $x'(t) = x'_n(t)$ exists a.e. in $F$ by a theorem of Pettis [5], theorem 7.

§ 5. Applications. Consider first as the space $\mathcal{X}$ the space $c$ composed of the convergent sequences $x = \{a_n\}$ with the norm $\|x\| = \sup_{n \in \mathbb{N}} |a_n|$. This space is separable. Any convergent sequence of real-valued functions $f_n(t)$ defined in $J$ may be considered as a function $x(t)$ from $J$ to $c$.

The functions $f_n(t)$ are said to be equidifferentiable at $t_0$, if the derivatives $f'_n(t_0)$ exist, and if for every $\varepsilon > 0$ there exists an $\eta > 0$ such that

$$|h| < \eta \implies \left| f_n(t_0 + h) - f_n(t_0) - h f'_n(t_0) \right| < \varepsilon$$

for $n = 1, 2, \ldots$

It is easy to see that the strong differentiability of $x(t)$ at $t_0$ is then equivalent to the convergence of the sequence $f'_n(t_0)$ together with its equidifferentiability at $t_0$.

Consider the set $\mathcal{E}_a$ composed of the functionals
\begin{equation}
\xi_n(x) = a_n, \quad \xi_k(x) = a_k, \quad \ldots
\end{equation}

1) This result may be considered as a generalization of Denjoy's relations to vector-valued functions.

The set $\mathcal{E}_a$ is fundamental for the space $c$. The $\mathcal{E}_a$-weak differentiability at $t_0$ is equivalent to the convergence of the sequence $f'_n(t_0)$. Since the functional $\xi(x) = \lim_{n \to \infty} a_n$ is linear in $c$, we get, applying Theorem 1 and Corollary 1, the following

Theorem 7. Let $f_n(t)$ be a convergent sequence of real-valued functions, and let the derivatives $f'_n(t)$ exist in a set $E$. If the sequence $\{f'_n(t)\}$ converges in $E$, and

$$\lim_{h \to 0} \sup_{n=1, h} \left| f_n(t+h) - f_n(t) \right| < \infty$$

at every point of $E$, then the functions $f_n(t)$ are equidifferentiable a.e. in $E$; moreover,

$$\frac{d}{dt} \left( \lim_{n \to \infty} f_n(t) \right) = \lim_{n \to \infty} f'_n(t)$$
a.e. in $E$.

In a similar manner we can apply Theorem 1 and Corollary 1 to the space $\mathcal{P}$ of the sequences $x = \{a_n\}$ such that $\|x\|^2 = \sum a_n^2 < \infty$, considering as $\mathcal{E}_a$ the set of the functionals (8).

We easily get

Theorem 8. If the real-valued functions $f_n(t)$ are differentiable at every point of the set $E$, and satisfy the conditions

$$\sum_{n=1}^{\infty} f_n^2(t) < \infty,$$

and

$$\lim_{h \to 0} \frac{1}{h^2} \sum_{n=1}^{\infty} \left| f_n(t+h) - f_n(t) - h f'_n(t) \right|^2 < \infty$$
at every $t \in E$,

then there exists a set $H$ such that $|E - H| = 0$, and

$$\lim_{n \to \infty} \sum_{n=1}^{\infty} \left| f_n(t+h) - f_n(t) - h f'_n(t) \right|^2 = 0$$
in $H$;

moreover, $\sum a_n^2 < \infty$ and $t \in H$ imply $\frac{d}{dt} \left( \sum_{n=1}^{\infty} f_n(t) \right) = \sum_{n=1}^{\infty} f'_n(t)$.

9) This condition may be replaced by the following one:

$$h_n \to 0 \implies \lim_{n \to \infty} \frac{f_n(x+h_n) - f_n(x)}{h_n} < \infty.$$
Remarque au travail „Sur les bases statistiques“

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Les termes et les notations employés dans la suite sont les mêmes que dans mon travail précédent 1). Parmi les résultats de ce travail se trouve une estimation de la valeur de $L(F, \eta)$ pour un système $F$ de deux fonctions continues périodiques $f_1(x)$ et $f_2(x)$ à périodes incommensurables, ce symbole désignant un nombre positif tel que tout intervalle de longueur $L(F, \eta)$ contient au moins une $\eta$-presse-période commune de $f_1(x)$ et $f_2(x)$. L'estimation en question fait l'objet du théorème II.

Le but de cette remarque est d'en donner une démonstration plus simple et qui permet même d'en améliorer la thèse 2). En conséquence, la thèse du théorème III, qui donne une estimation de $L(F, \eta)$ pour un cas spécial et dont la démonstration est basée sur le théorème II, est susceptible d'une amélioration analogue.

Montrons d'abord un lemme concernant la répartition mod 1 de la suite $(n\theta)$, où $\theta$ est un nombre irrationnel fixé.

Lemme. Soit $I$ un sous-intervalle de longueur $\beta$ de l'intervalle demi-ouvert $(0, 1)$. Soient $q$ un nombre naturel et $p$ un entier, tels que $|q\theta - p| < \beta$. Soit $Q_j$ la suite croissante de tous les entiers non-négatifs tels que $R(\theta Q_j) > I$. Alors

\[ |Q_{i+1} - Q_i| < \frac{1}{|q\theta - p|} + 1 \quad (i = 1, 2, \ldots) \]

Démonstration. La distance entre les points $R(kq\theta)$ et $R((k+1)q\theta)$ $(k=0, 1, 2, \ldots)$, prise le long du plus petit arc de la circonférence $C$ de périmètre 1, est égale à $\min \{R(q\theta), 1 - R(q\theta)\}$.

\[ 1) \text{ Voir Studia Mathematica 10 (1948), p. 120-139.} \]

\[ 2) \text{L'idée de cette simplification a été suggérée par K. Florek.} \]