A weighted vector-valued weak type \((1, 1)\) inequality and spherical summation

by

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Abstract. We prove a weighted vector-valued weak type \((1, 1)\) inequality for the Bochner–Riesz means of the critical order. In fact, we prove a slightly more general result.

1. Introduction. For a nonnegative function \(w\) on \(\mathbb{R}^n\) \((n \geq 2)\), let \(L^p_w(\mathbb{R}^n) = \{f : \|f w^{1/p}\|_p = \|f\|_{p,w} < \infty\}\) be the weighted \(L^p\) space and let \(L^{1,\infty}_w\) be the weighted weak \(L^1\) space. We write for \(f \in L^{1,\infty}_w\),

\[
\|f\|_w^* = \sup_{\lambda > 0} \lambda w(\{x : |f(x)| > \lambda\}),
\]

where \(w(E) = \int_E w\). Next for \(R > 0\) let

\[
S^\delta_R(f)(x) = \int_{\mathbb{R}^n} f(\xi)(1 - |\xi|^2)^{\delta - R^2} e^{i2\pi\xi \cdot x} d\xi
\]

be the Bochner–Riesz means of order \(\delta\). In this note we shall prove a weighted vector-valued version of Christ [1, Theorem 1].

**Theorem 1.** Let \(w(x) = |x|^\beta, -n < \beta \leq 0\), and let \(\alpha = (n - 1)/2\) be the critical index. Then for a sequence \(\{R_k\}\) of positive numbers, we have

\[
\|\left( \sum |S^\delta_{R_k}(f_k)|^2 \right)^{1/2} \|_w^* \leq c \left( \sum |f_k|^2 \right)^{1/2} \|_1,w.
\]

See [2, 3, 4, 10] for related results. We shall prove a more general result. Following [3], we consider a sequence \(\{T_k\}\) of bounded linear operators on \(L^2\) such that there exists a sequence \(\{K_k\}\) of kernels satisfying

\[
(T_k(f), g) = \iint g(x)f(y)K_k(x - y) dy \, dx
\]

for \(f, g \in C_0^\infty\) with disjoint supports. Furthermore, we assume the following.

(1.1) The operators \(T_k\) are bounded on \(L^2_w\) and \(\sup_k \|T_k\|_{2,w} = c_1 < \infty\), where \(\| \cdot \|_{2,w}\) denotes the operator norm.

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(1.2) The kernels $K^k$ can be written in polar coordinates as

$$K^k(r, \theta) = r^{-n} \Omega^k(r, \theta),$$

where $\sup_{r, \theta, k} (\Omega^k(r, \theta) + |b_k \Omega^k(r, \theta)|) = c_2 < \infty$.

Then we can obtain a weighted vector-valued version of a special case of
[3, Theorem 4].

Theorem 2. Let $w(x) = |x|^\beta, -n < \beta \leq 0$, and $\{T_k\}$ be as above. Then there exists a constant $c$ depending only on $c_1, c_2, n$ and $w$ such that

$$\left\| \left( \sum_k |T_k(f_k)|^2 \right)^{1/2} \right\|_{L^1} \leq c \left\| \left( \sum_k |f_k|^2 \right)^{1/2} \right\|_{L^1}.$$  

Theorem 1 immediately follows from Theorem 2. In the rest of this note, we consider only a weight $w$ as in Theorems 1 and 2. As a consequence of Theorem 1 for $R_k = 2^k$, by a standard argument we have the following.

Corollary 1. Define

$$\sigma(f)(x) = \left( \sum_{k \in \mathbb{Z}} |S^{\alpha+1}_k f (x) - S^\alpha_k f (x)|^2 \right)^{1/2}.$$  

Then $\sigma(f)(x) \leq c \|f\|_{H^\alpha}$, where $H^\alpha$ denotes the weighted Hardy space (see [14]).

Here we give a sketch of the proof. First we note that there are $\hat{\phi}, \hat{\psi} \in C_0^\infty$ such that $\hat{\phi}(0) = \hat{\psi}(0) = 0$ and

$$S^{\alpha+1}_k f (x) - S^\alpha_k f (x) = f * \phi_R + S^\alpha_k (f * \psi_R),$$

where $g_n(x) = R^n g(Rx)$. Then we have

$$\sigma(f) \leq \left( \sum |f * \varphi_{2^k}|^2 \right)^{1/2} + \left( \sum |S^\alpha_k (f * \psi_{2^k})|^2 \right)^{1/2}.$$  

By Chebyshev’s inequality, Theorem 1 and the Littlewood-Paley inequality for $H^\alpha$, we obtain the assertion of Corollary 1.

By Corollary 1 we have the following.

Corollary 2. Let $S^\alpha_k (f)(x) = \sup_{|\xi| \leq 2^k} |\hat{S}_k f (\xi)|$. Then

$$\|S^\alpha_k (f)\|_{L^1} \leq c \|f\|_{H^\alpha}.$$  

The inequality $S^\alpha_k (f) \leq S^{\alpha+1}_k (f) + \sigma(f)$ proves the corollary. From this we obtain almost everywhere convergence of the lacunary Bochner–Riesz means for $H^\alpha$. See [13] and also [7], [8], [16, Chap. XV]. We can prove in the same way a continuous analogue of Theorem 1 where $L^2$ is replaced by $L^2((0,\infty), dR/R)$. Using this, we obtain the following similarly to Corollary 1.

Corollary 3. Let $f \in H^\alpha_w$. Then $\|\sigma(f)(x)\|_{L^1} \leq c \|f\|_{H^\alpha}$, where

$$\sigma(f)(x) = \left( \int_0^\infty |S^{\alpha+1}_k (f)(x) - S^\alpha_k (f)(x)|^2 \frac{dR}{R} \right)^{1/2}.$$  

See [6] for the pointwise equivalence between $\sigma$ and other square functions.

The proof we shall give below is a combination of arguments of Christ–Rubiño de Francia [3] and Hofmann [5]. Theorems 1 and 2 and their corollaries for $w = 1$ can be found in [9].

2. Outline of proof of Theorem 2. Let $L^\infty_{\lambda}(L^2)$ be the space of $\lambda^\infty$-valued functions $f = (f_k)$ such that $f_k \in L^\infty_{\lambda}$, where $\cdot_\lambda$ denotes the $\lambda^\infty$ norm. We also write $\|f\|_{\lambda, w} = (\int |f|^2 w \, dx)^{1/p}$ for the norm of $f \in L^\infty_{\lambda}(L^2)$ and when $w = 1$ this norm is denoted by $\| \cdot \|_\lambda$ (this will not cause any confusion).

Let $f = (f_k) \in L^\infty_{\lambda}(L^2) \cap L^2_{\lambda}(L^2)$ and $\lambda > 0$. We use a Calderón–Zygmund decomposition, i.e., a collection $\{Q\}$ of nonoverlapping dyadic cubes and a decomposition $f = g + b$, $b = \sum b_Q$, with the following properties:

$$w(\bigcup Q) \leq c \|f\|_{L^2}/\lambda,$$

$$\|b_Q\|_1 \leq c \lambda|Q|, \quad \int b_Q = 0, \quad b_Q \text{ is supported on } Q.$$  

Define $S(f) = (T_k(f_k))$. Then by (1.1), $S$ is bounded on $L^2_{\lambda}(L^2)$. Thus by (2.1) we have

$$w(\{ |S(g)|_2 > \lambda \}) \leq \lambda^{-2} \|S(g)^2\|_{L^2_{\lambda}} \leq c \lambda^{-2} \|g\|^2_{L^2_{\lambda}} \leq c \lambda^{-2} \|f\|_{L^2},$$

so that, by (2.2), Theorem 2 follows from

$$\|w(\{ x \in \mathbb{R}^n \cap E^* : |S(b)(x)|_2 > \lambda \}) \|_1 \leq c \|f\|_{L^2},$$

where $E^* = \bigcup Q^*$ with $Q^*$ denoting the cube with the same center as $Q$ and with sidelength $2^{10+n}$ times that of $Q$.

Let $\eta \in C_0^\infty$ be radial $(\eta(x) = \eta_0(|x|))$, nonnegative and such that $\text{supp}(\eta) \subset \{ 1/4 \leq |x| \leq 4 \}$ and $\sum_{j \in \mathbb{Z}} \eta(2^{-j} x) = 1$ for $x \in \mathbb{R}^n \setminus \{ 0 \}$. Define $K_j(x) = (\eta(2^{-j} x)) K^k(x)$. Then to obtain (2.4) it is sufficient to prove that

$$\left\| \sum_j K_j * B_{j-\delta} \right\|_{L^2_{\lambda}} \leq c \lambda^{-\delta} \|f\|_{L^2}$$

for all $\delta > n + 4$ with some $\delta > 0$, where $B_j = \sum_{|\xi| = 2^j} b_Q$, the convolution is defined by $f * g(x) = (f_k * g_k(x))$ for $f = (f_k), g = (g_k)$ and by our
construction of the exceptional set $E^*$ we may assume that $s > n + 4$. (See [3].)

Now using the Schwarz inequality, we see that

$$\left\| \sum_j K_j * B_{j-s} \right\|_{L^2, w}^2 \leq c \sum_j \left| (K_j * B_{j-s})_w \right| + c \sum_j \sum_i \left| (K_j * B_{j-s}, K_i * B_{i-s})_w \right|,$$

where $(,)_w$ denotes the inner product of the Hilbert space $L^2_w(\mathbb{R}^d)$. Let $K_j = (K_f^j)$, $B_i = (B_{i}^f)$. Then

$$\langle K_j * B_{j-s}, K_i * B_{i-s} \rangle = \sum_k \int K_j^k * B_{j-s}(x) \overline{K_i^k} * B_{i-s}(x) w(x) \, dx$$

$$= \sum_k \int K_j^k(x-y) B_{j-s}^*(y) dy \int \overline{K_i^k}(x-z) B_{i-s}^*(z) w(x) \, dx$$

$$= \sum_k \int B_{j-s}(y) \int \overline{K_i^k}(x-y) K_j^k(x-z) w(x) \, dx dy$$

$$= \sum_k \int B_{j-s}(y) \int \overline{K_i^k}(x-y) w(x) \, dx dy$$

$$= \int \langle B_{j-s}(y), B_{i-s} * L_{i}^f(y) \rangle_2 dy,$$

where $K^k(x) = K_f^k(-x)$, $w_i(x) = w(x+y)$, $L_{i}^f(x) = (K_f^i \overline{w}_y * K_f^i)(x)$ and $\langle , \rangle_2$ denotes the inner product in $L^2$.

Next, let $B_{1, j-s} = \sum_b Q_b$, where $b Q$ ranges over the collection of those $b Q$ which satisfy supp$(b Q) \subset \{ 2^{j-3} \leq |x| \leq 2^{j+3} \}$ and $|Q| = 2^n(j-s)$. Then following Hofmann [5], we make a decomposition

$$B_{j-s} = B_{1, j-s} + B_{2, j-s}.$$

We note that since $s > n + 4$, if $B_{2, j-s} = \sum b Q$, then each $Q$ is contained in $\{ 2^{j-4} \leq |x| \leq 2^{j+4} \}$. We shall prove (2.5) for $B_{1, j-s}$ and $B_{2, j-s}$ separately. By the above expression of $(K_j * B_{1,j-s}, K_i * B_{i-s})_w$ and the inequality $\sum_j \left\| B_{j-s} \right\|_{L^2, w} \leq c \| f \|_{L^2, w}$, for this it is sufficient to prove the following results.

**Lemma 1.** Let $y \in \text{supp}(B_{1, j-s})$. Then

$$\left\| B_{1, j-s} * L_{i}^f(y) \right\|_2 \leq c \lambda 2^{-s \epsilon} w(y).$$

**Lemma 2.** Let $y \in \text{supp}(B_{2, j-s})$. Then

$$\left\| B_{2, j-s} * L_{i}^f(y) \right\|_2 \leq c \lambda 2^{-s \epsilon} w(y).$$

**Lemma 3.** Let $y \in \text{supp}(B_{1, j-s})$. Then

$$\left| B_{1, j-s} * L_{i}^f(y) \right|_2 \leq c \lambda 2^{-s \epsilon} w(y).$$

**Lemma 4.** Let $y \in \text{supp}(B_{2, j-s})$. Then

$$\left| B_{2, j-s} * L_{i}^f(y) \right|_2 \leq c \lambda 2^{-s \epsilon} w(y).$$

We observe that by dilation invariance, to prove these lemmas we may assume that $j = 0$. Thus in the following sections, we shall give the proofs only for $j = 0$, and then we shall use a (vector-valued) version of [3, Lemma 6.1].

Let $E = (E^k)$ and $F_i = (F_{i}^k)$ be kernels which can be written in polar coordinates as

$$E^k(r, \theta) = r^{-n} \rho^k(r, \theta) \eta_0(r), \quad F_{i}^k(r, \theta) = r^{-n} \rho_{i}^k(r, \theta) \eta_0(2^{-i} r).$$

We assume that

$$(2.6) \sup_{r, \theta} | \rho^k(r, \theta) | + | \partial_r \rho^k(r, \theta) | \leq 1 \quad \text{uniformly in } k,$$

$$(2.7) \sup_{r, \theta} | \rho_{i}^k(r, \theta) | + | \partial_r \rho_{i}^k(r, \theta) | \leq 1 \quad \text{uniformly in } k.$$

Then we have the following (see [3, Lemma 6.1]).

**Lemma 5.** Let $x \in \mathbb{R}^n \setminus \{ 0 \}$, $|h| < |x|/2$. Then

(a) $|E * F_i(x + h) - E * F_i(x)|_\infty \leq c |h|^{1/2}$ (i \leq -10),

(b) $|E * F_0(x + h) - E * F_0(x)|_\infty \leq c |h|^{1/2} |x|^{-3/2}$.

We shall give a sketch of the proof in §7 for completeness.

**3. Proof of Lemma 1.** Let $\zeta \in C_0^\infty(\mathbb{R})$ be nonnegative and such that $\zeta(r) = 1$ if $1/4 \leq r \leq 4$ and supp$(\zeta) \subset \{ 1/6 \leq r \leq 6 \}$. We define

$$K^\zeta(x) = (K_f^\zeta(x) \overline{w}_y(x)) = (r^{-n} \omega_{i}^\zeta(r, \theta) \eta_0(r)), $$

where $\omega_{i}^\zeta(r, \theta) = \overline{K_{i}^\zeta(r, \theta) r^{-\theta} \zeta(\theta) \eta_0(\theta)}$. Then $L_{i}^\zeta(x) = K^\zeta * K_{i}(x)$.

**Sublemma 1.** Let $y \in \text{supp}(B_{1, j-s})$. Then

(a) $\sup_{k, r, \theta} | \omega_{i}^\zeta(r, \theta) | \leq c |y|^\beta,$

(b) $\sup_{k, r, \theta} | \partial_r \omega_{i}^\zeta(r, \theta) | \leq c |y|^\beta.$

**Proof.** If $y \in \text{supp}(B_{1, j-s})$, then $|y| \leq 2^{-3}$ or $|y| \geq 2^{3}$. Thus for $r \in [1/5, 6]$, we have $|y - r \theta| \approx |y/1|$, so that

$$|y - r \theta| \leq c \max(|y/1|, 1) \leq c |y|^\beta.$$
Combined with (1.2), this proves (a). Similarly we have
\[ |\partial_\theta \omega^\delta_r(r, \theta) | \leq c (|y - r \theta| + |y - r \theta|^{\beta - 1}) \zeta(r) \]
\[ \leq c \max(|y|, 1)^{\beta} + c \max(|y|, 1)^{\beta - 1} \leq c \max(|y|, 1)^{\beta} \leq c |y|^{\beta}, \]
proving (b).

By Lemma 5 and Sublemma 1 we have the following.

**Sublemma 2.** Let \( y \in \text{supp}(B_{1, -s}) \), \( x \in \mathbb{R}^n \setminus \{0\} \) and \( |h| < \frac{|x|}{2} \). Then
\[
\begin{align*}
(4.1) & \quad |L^0_{y} (x + h) - L^0_{y} (x)|_{\infty} \leq \alpha y (|y|) |y|^{2 - s} |h|^{1/2} \quad (i \leq -10), \\
(4.2) & \quad |L^0_{y} (x + h) - L^0_{y} (x)|_{\infty} \leq \alpha y (|y|) |h|^{1/2} |x - y|^{-s/2}.
\end{align*}
\]

Now we prove Lemma 1. Denote by \( c_Q \) and \( d(Q) \) the center and the diameter of a cube \( Q \), respectively. Then for \( s > n + 4 \) and \( y \in \text{supp}(B_{1, -s}) \), we have
\[
\sum_{i \leq -10} \left| \int B_{1, -s} (x) L^0_{y} (y - z) \, dz \right|_{2}^2 = \sum_{i \leq -10} \left| \sum_{Q \subset Q} \int b_Q (z) L^0_{y} (y - z) \, dz \right|_{2}^2,
\]
where \( \int f(z) g(z) \, dz = \int f(z) g(z) \, dz \) for \( f = (f_k), \ g = (g_k) \). By Sublemma 2(a), (2.3) and Minkowski's inequality, this is majorized by
\[
\sum_{i \leq -10} \sum_{Q \subset Q} \left| \int b_Q (z) L^0_{y} (y - z) - L^0_{y} (y - c_Q) \right|_{2}^2 \leq c \sum_{i \leq -10} \sum_{Q \subset Q} \int \left| b_Q (z) \right| |z - c_Q|^{1/2} w(y) |y|^{2 - s/2} \, dz
\]
\[
\leq c \lambda \int w(y)^{2 - s/2} \sum_{Q \subset Q} \left| Q \right| \leq \lambda \int w(y)^{2 - s/2} \, w(y),
\]
where in the last summation, \( Q \) ranges over a family of nonoverlapping dyadic cubes contained in \( \{ x : |x - y| < 100 \} \). This completes the proof of Lemma 1.

**4. Proof of Lemma 2.** Let \( \mu, \nu \in C^\infty_0 (\mathbb{R}^n) \) be radial, nonnegative and such that \( \mu(x) + \nu(x) = 1 \) for all \( x \in \mathbb{R}^n \), supp(\( \mu \)) \( \subset \{ x : |x| \leq 1 \} \) and \( \mu(x) = 1 \) if \( |x| \leq 1/2 \). Let
\[
\omega^\delta_r(x) = w(x + y) \mu(2^s (x + y)) \quad \text{and} \quad w^\delta_r(x) = w(x + y) \nu(2^s (x + y))
\]
with \( \delta > 0 \) which will be specified later. We decompose \( L^0_{y} \) as \( L^0_{y} = M^0_{y} + N^0_{y} \), where
\[
M^0_{y} (x) = (\tilde{K}^0_{y} w^0_{y} * K^1_{y} (x)), \quad N^0_{y} (x) = (\tilde{K}^0_{y} w^1_{y} * K^1_{y} (x)).
\]
Let \( y \in \text{supp}(B_{2, -s}) \). We note that \( |y| \approx 1 \). Thus in order to prove Lemma 2 it is sufficient to prove
\[
\sum_{i \leq -10} |B_{2, i - s} * M^0_{y} (y)| \leq c \lambda \int w(x) \mu(2^s x) \, dx \| B_{2, i - s} \|_1
\]
and
\[
\sum_{i \leq -10} |B_{2, i - s} * N^0_{y} (y)| \leq c \lambda \int w(x) \nu(2^s x) \, dx \| B_{2, i - s} \|_1.
\]
First we prove (4.1). Since \( |x| \leq 2^{i+4} \) if \( x \in \text{supp}(B_{2, i - s}) \), we have
\[
|B^2_{2, i - s} * (\tilde{K}^0_{y} w^0_{y} * K^1_{y} (y))| = \int B_{2, i - s} (x) \int |y| - |x| \leq c 2^{i}
\times w(x - y) \mu(2^s (x - y)) K^1_{y} (y - x) \, dx \, dz \leq c 2^{-i} \int |B_{2, i - s} (x)| \, dx \int w(x) \mu(2^s x) \, dx.
\]
Thus by Minkowski's inequality we have
\[
|B_{2, i - s} * M^0_{y} (y)| \leq c 2^{-i} \int w(x) \mu(2^s x) \, dx \| B_{2, i - s} \|_1 \leq c \lambda \int w(x) \mu(2^s x) \, dx,
\]
where we have used
\[
\| B_{2, i - s} \|_1 \leq c \lambda \int |Q| \leq c \lambda 2^{ni},
\]
which holds since in the last summation \( Q \) ranges over a family of nonoverlapping dyadic cubes contained in \( \{ 2^{i-4} \leq |x| \leq 2^{i+4} \} \). Thus
\[
\sum_{i \leq -10} |B_{2, i - s} * M^0_{y} (y)| \leq c \lambda \sum_{i \leq -10} \int w(x) \mu(2^s x) \, dx \leq c \lambda \sum_{i \leq -10} \int |x| \mu(2^s x) \, dx \leq \lambda \sum_{2^{i-4} \leq |x| \leq 2^{i+4}} \int |x| \mu(2^s x) \, dx \leq c \lambda \int \nu(2^s x) \sum_{2^{i-4} \leq |x| \leq 2^{i+4}} 2^{s} \delta \nu(2^{s} 2^{i+4}) \leq c \lambda 2^{-s} \lambda 2^{s(n+\beta)},
\]
which proves (4.1).

Next we prove (4.2). Let
\[
J^\nu (x) = (\tilde{K}^0_{y} (x) w(x - y) \nu(2^s (x - y))) = (r^{-n} \sigma^\nu_r (x, \theta) \eta_r (\theta)),
\]
where $\sigma_k^2(r, \theta) = \overline{p_k}(r, -\theta) - r^\beta |y - r\theta|^{\beta_0} |2^{s_\theta} |y - r\theta| |\zeta(r), \nu_0(|x|) = \nu(x)$ and $\zeta$ is as in (3). Then $N_{00}^\psi(z) = J_0^\psi K_1^\psi(z)$. In order to apply Lemma 5 we use the following obvious estimates.

**Sublemma 3.** Let $y \in \text{supp}(B_{2,-s})$. Then
(a) $\sup_{k, r, \theta} \sigma_k^2 (r, \theta) \leq c \delta^{s_\theta},$
(b) $\sup_{k, r, \theta} \{\vartheta \sigma_k^2 (r, \theta) \leq c \delta^{s_\theta - \beta + 1} \delta^s.$

By Lemma 5 and Sublemma 3 we have the following.

**Sublemma 4.** Let $y \in \text{supp}(B_{2,-s}), x \in \mathbb{R}^n \setminus \{0\}$ and $|h| < |x|/2$. Then
(a) $|N_{00}^\psi(z + h) - N_{00}^\psi(x)|_\infty \leq c \delta^{1/2 s_\theta - \beta + 1} \delta^s$ \quad \left( \begin{array}{ll} (i) \leq 10, \end{array} \right.$
(b) $|N_{00}^\psi(x + h) - N_{00}^\psi(x)|_\infty \leq c |h|^{1/2} |x|^{-3/2} \delta^{s_\theta - \beta + 1} \delta^s.$

We first see that
\[
\sum_{i \leq -10} |B_{2,-s} * N_{00}^\psi(y)|_2 \leq \sum_{i} \sum_{|q - y| < d(Q)} \int b_Q(x) N_{00}^\psi(y - z) \, dz \bigg|_2 \\
+ \sum_{i} \sum_{|q - y| \geq d(Q)} \int b_Q(x) N_{00}^\psi(y - z) \, dz \bigg|_2 \\
= I + II, \quad \text{say}.
\]

By Sublemma 3(a) we have $\sup_{x} |N_{00}^\psi(x)|_\infty \leq c \delta^{s_\theta - \beta}. \quad \text{Thus by Minkowski's inequality and (2.3) we see that}$
\[
I \leq c \delta^{s_\theta - \beta} \sum_{i} \sum_{|q - y| < d(Q)} \int |b_Q(x)|_2 \, dz \\
\leq c \lambda \delta^{s_\theta - \beta} \sum_{i} \sum_{|q - y| < d(Q)} |Q| \\
\leq c \lambda \delta^{s_\theta - \beta} \sum_{i} 2^{n(1-s_\theta)} \leq c \lambda \delta^{s_\theta - \beta} 2^{-n s_\theta}.
\]

Next, using Sublemma 4(a), (2.3) and Minkowski's inequality, we have
\[
II \leq \sum_{i} \sum_{|q - y| \geq d(Q)} \int b_Q(x) (N_{00}^\psi(y - z) - N_{00}^\psi(y - c_Q)) \, dz \bigg|_2 \\
\leq c \sum_{i} \sum_{Q} \int |b_Q(x)|_2 |x - c_Q|^{1/2} \delta^{s_\theta - \beta + 1} \delta^s \, dz \\
\leq c \lambda \delta^{s_\theta - \beta} 2^{s_\theta - \beta + 1} \delta^s \sum_{i} |Q| \leq c \lambda \delta^{s_\theta - \beta} 2^{s_\theta - \beta + 1} \delta^s,
\]
where the last inequality follows as in the proof of Lemma 1. Combining the estimates for $I$, $II$ and taking $\delta$ small enough, we obtain (4.2).

5. Proof of Lemma 3. Let $y \in B_{2,-s}$. Then
\[
|B_{2,-s} * L_{00}^\psi(y)|_2 \leq \sum_{|c - y| < d(Q)} \left| \int b_Q(x) L_{00}^\psi(y - z) \, dz \right|_2 \\
+ \sum_{|c - y| \geq d(Q)} \int b_Q(x) L_{00}^\psi(y - z) \, dz \\
= I + II, \quad \text{say}.
\]

By Sublemma 1(a) we have $\sup_{x} |L_{00}^\psi(x)|_\infty \leq cw(y). \quad \text{Thus by Minkowski's inequality we see that}$
\[
I \leq cw(y) \sum_{|c - y| < d(Q)} \|b_Q\|_2 \leq c \lambda \omega(y) \sum_{Q} \|Q| \leq c \lambda \omega(y) 2^{-s n}.
\]

Next by Sublemma 2(b), (2.3) and Minkowski's inequality, we have
\[
II \leq \sum_{|c - y| \geq d(Q)} \int b_Q(x) (L_{00}^\psi(y - z) - L_{00}^\psi(y - c_Q)) \, dz \bigg|_2 \\
\leq c \sum_{Q} \int \left| b_Q(x) \right||\omega(y)|z - c_Q|^{1/2} |c_Q - y|^{-3/2} \, dz \\
\leq c \lambda \omega(y) 2^{-s n} \sum_{Q} |Q| |c_Q - y|^{-3/2}.
\]

If $|c_Q - y| \geq d(Q)$, we have $|c_Q - y| = |x - y|$ for $x \in Q$. Thus
\[
II \leq c \lambda \omega(y) 2^{-s n} \sum_{Q} \int |x - y|^{-3/2} \, dx \leq c \lambda \omega(y) 2^{-s n} \int_{B} \left| x \right|^{-3/2} \, dx,
\]

where $B$ is a fixed bounded set. Combining the estimates for $I$ and $II$, we obtain the conclusion of Lemma 3.

6. Proof of Lemma 4. Let $M_{00}^\psi$ and $N_{00}^\psi$ be as in (4.1). Then to obtain Lemma 4, it is sufficient to prove the following estimates for $y \in B_{2,-s}$:

(6.1) $|B_{2,-s} * M_{00}^\psi(y)|_2 \leq c \lambda 2^{-s}$,

(6.2) $|B_{2,-s} * N_{00}^\psi(y)|_2 \leq c \lambda 2^{-s}$.

We first prove (6.1). As in the proof of (4.1) we see that
\[
|B_{2,-s} * M_{00}^\psi(y)|_2 \leq c \int \left| w(x) \mu (2^{s_\theta}) \, dx \right| / |B_{2,-s}|_1 \\
\leq c \lambda 2^{-s_\theta (n + \beta)} \sum |Q| \leq c \lambda 2^{-s_\theta (n + \beta)},
\]
since in the last summation $Q$ ranges over a family of cubes contained in a fixed bounded set. This proves (6.1).
Next we prove (6.2). First we have
\[
|B_{2,-2} \ast N_{00}(y)|_{2} \leq \left| \sum_{|c_{0}-y|<d(Q)} \int b_{Q}(z)N_{00}(y-z)\,dz \right|_{2} + \left| \sum_{|c_{0}-y|\geq d(Q)} \int b_{Q}(z)N_{00}(y-z)\,dz \right|_{2} = I + II, \quad \text{say.}
\]
Since \( \sup_{x} |N_{00}(x)|_{\infty} \leq c_{2}^{-s/2} \) by Sublemma 3(a), using Minkowski’s inequality and (2.3), we see that
\[
I \leq c_{2}^{-s/2} \sum_{|c_{0}-y|<d(Q)} \|b_{Q}\|_{1} \leq c_{2}^{-s/2} \sum_{|c_{0}-y|<d(Q)} |Q| \leq c_{2}^{-s/2} 2^{-s}. \]

Next by Sublemma 4(b), (2.3) and Minkowski’s inequality, arguing as in §5 we have
\[
II = \left| \sum_{|c_{0}-y|\geq d(Q)} \int b_{Q}(z)(N_{00}(y-z) - N_{00}(y-c_{0}))\,dz \right|_{2} \\
\leq c_{2} \sum_{|Q|\geq d(Q)} \int |b_{Q}(z)| \left| z - c_{0} \right|^{1/2} |y - c_{0}|^{-3/2} 2^{(\theta-1)\delta} \,dz \\
\leq c_{2} 2^{-s/2} 2^{(\theta-1)\delta} \sum_{|Q|\geq d(Q)} |Q| |y - c_{0}|^{-3/2} \leq c_{2} 2^{-s/2} 2^{(\theta-1)\delta}.
\]
Combining the estimates for \( I \) and \( II \) and taking \( \delta \) small enough, we obtain (6.2).

7. Sketch of proof of Lemma 5. We fix \( k \) and write \( E = E^{k}, F_{1} = F_{1}^{k}, \Phi = \Phi^{k}, \Psi = \Psi^{k} \). Then
\[
(7.1) \quad E \ast F_{1}(z) = c \int_{0}^{\infty} \int_{0}^{\infty} (\Phi_{s} \ast \sigma_{s})(x)_{\eta_{0}}(r)_{\eta_{0}}(2^{-s})_{r} \,dr \,ds, \]
where \( \Phi_{s}(\theta) = \Phi(\theta, \theta), \Psi_{s}(\theta) = \Psi(s, \theta) \) and \( \sigma_{r} \) denotes the uniform surface probability measure of the sphere \( \{ x : |x| = r \} \). By (2.6) and (2.7) we have the following result of [3] (see [3, Lemma 6.2]).

**Sublemma 5.** Let \( r \geq s \) and \( r \in [1/4, 4] \). Then \( (\Phi_{s} \ast \sigma_{s})(x) = 0 \) if \( |x| \leq r - s \) or \( |x| > r + s \), and if \( r - s < |x| < r + s \) we have
\[
|\Phi_{s} \ast \sigma_{s}(x)| \leq c(|x|(r + s - |x|)(|x| - r + s))^{-1/2}, \quad \nabla |\Phi_{s} \ast \sigma_{s}(x)| \leq c(|x|(r + s - |x|)(|x| - r + s))^{-3/2}. \]

When \( r \geq s \), by a straightforward computation we see that \( \sigma_{r} \ast \sigma_{s}(x) = c_{2} r^{-s-2} \sum_{s-n+2}^{\infty} |x|^{-n+2} ((r + s)^{2} - |x|^{2})^{2} ((x^{2} - (r - s)^{2})^{n-3/2} \text{ if } r - s < |x| < r + s, \text{ and } \sigma_{r} \ast \sigma_{s}(x) = 0 \text{ otherwise.}\) From this, Sublemma 5 follows when \( \Phi_{r} = \Phi_{s} = 1. \) The proof of the general case is similar. We omit the details.

We can prove (a) and (b) of Lemma 5 similarly by using Sublemma 5. Here we only give the proof of (b). First we may assume that \( |x| < 100 \) and \( |\theta| < 10^{-10} |x| \) since \( E \ast F_{0} \) is bounded and supported in \( \{|x| \leq 10\} \). Put \( G(r, s, x, h) = (\Phi_{s} \ast \sigma_{s})(x + h) - (\Phi_{r} \ast \sigma_{r})(x) \). Then let
\[
\int_{r}^{\infty} \int_{s}^{\infty} G(r, s, x, h)_{\eta_{0}}(r)_{\eta_{0}}(s)_{\nu_{0}}(s) \,dr \,ds = \int_{r}^{\infty} \int_{s}^{\infty} G(r, s, x, h)_{\eta_{0}}(r)_{\eta_{0}}(s)_{\nu_{0}}(s) \,dr \,ds
\]
and put
\[
J_{1}(s) = \int_{r}^{\infty} \int_{s}^{\infty} G(r, s, x, h)_{\eta_{0}}(r) \,dr, \quad J_{2}(s) = \int_{r}^{\infty} \int_{s}^{\infty} G(r, s, x, h)_{\eta_{0}}(r) \,dr.
\]
Then since \( \text{supp}(\Phi_{s} \ast \sigma_{s}) \subset \{|x| - r - s \leq |x| \leq |r + s| \} \geq s \), we have \( I_{1} = \int_{1/4}^{4} (J_{1}(s) + J_{2}(s))_{\eta_{0}}(s) \,ds \). By Sublemma 5, for \( s \in [1/4, 4] \) we see that \( J_{1}(s) \) is dominated by
\[
c_{2} \int_{1/4}^{4} \left( \left| x + h \right| - |r + s - |x| + h| \right) \left( |x + h| - r + s \right)^{-1/2} \right. \left. + \left( |x| - |r + s - |x| | \right) \left( |x| - r + s \right)^{-1/2} \,dr. \!
\]
By a direct computation, this is bounded by
\[
c_{2} |x|^{-1/2} \int_{|r| < 5|s|} \frac{|r|^{-1/2} \,dr}{|r| < 5|s|} \leq c |x|^{-1/2} |h|^{1/2}. \]
Next by Sublemma 5 and the mean value theorem, via a direct computation, for \( s \in [1/4, 4] \) we see that \( J_{2}(s) \) is bounded by
\[
c_{2} |h| \int_{s-2}^{s+2} \int_{|r| < 5|s|} \frac{|r|^{-1/2} \,dr}{|r| < 5|s|} \leq c |h|^{1/2} |x|^{-3/2}. \]
Collecting the results we have \( I_{1} \leq c |h|^{1/2} |x|^{-3/2} \). We obtain the same estimate for \( I_{2} \). Since these estimates are uniform in \( k \), by (7.1) we obtain Lemma 5(b).
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References


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The converse of the Hőlder inequality and its generalizations
by
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Abstract. Let (Ω, Σ, μ) be a measure space with two sets A, B ∈ Σ such that 0 < μ(A) < 1 < μ(B) < ∞ and suppose that ϕ and ψ are arbitrary bijections of [0,∞) such that ϕ(0) = ψ(0) = 0. The main result says that if
\[ \int \phi(x) \, d\mu \leq \phi^{-1} \left( \int \phi \circ x \, d\mu \right) \psi^{-1} \left( \int \psi \circ x \, d\mu \right) \]

for all μ-integrable nonnegative step functions x, y then ϕ and ψ must be conjugate power functions.

If the measure space (Ω, Σ, μ) has one of the following properties:
(a) μ(A) ≤ 1 for every A ∈ Σ of finite measure;
(b) μ(A) ≥ 1 for every A ∈ Σ of positive measure,
then there exist some broad classes of nonpower bijections ϕ and ψ such that the above inequality holds true.

A general inequality which contains integral Hőlder and Minkowski inequalities as very special cases is also given.

Introduction. Let (Ω, Σ, μ) be a measure space. Denote by S = S(Ω, Σ, μ) the linear space of all μ-integrable step functions x : Ω → R and by S_+ the set of all x ∈ S such that x : Ω → R_+ where R_+ = [0,∞). One can easily verify that for every bijective function ϕ : R_+ → R_+ such that ϕ(0) = 0 the functional p_ϕ : S_+ → R_+ given by the formula
\[ p_ϕ(x) = \phi^{-1} \left( \int \phi \circ x \, d\mu \right) \quad (x \in S_+) \]

is well defined. In a recent paper [8] the author proved the following converse of Minkowski's inequality.

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