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References


On the best constant in the Khinchin–Kahane inequality

by

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Abstract. We prove that if \( r_1 \) is the Rademacher system of functions then

\[
\left( \int \left\| \sum_{i=1}^n x_i r_i(t) \right\|^2 dt \right)^{1/2} \leq \sqrt{2} \int \left\| \sum_{i=1}^n x_i r_i(t) \right\| dt
\]

for any sequence of vectors \( x_i \) in any normed linear space \( F \).

Introduction. The classical result of Khinchin [3] states that for each \( p, q > 0 \) there exists a constant \( c_{p,q} \) such that for any real numbers \( x_1, \ldots, x_n \),

\[
(1) \quad \left( \int \left| \sum_{i=1}^n x_i r_i(t) \right|^p dt \right)^{1/p} \leq c_{p,q} \left( \int \left| \sum_{i=1}^n x_i r_i(t) \right|^q dt \right)^{1/q}.
\]

The smallest constant \( c_{p,q} \) will be denoted by \( C_{p,q}^R \). Obviously, \( C_{p,q}^R = 1 \) for \( p \leq q \), but it took some effort to calculate the other best constants. The especially interesting case \( p = 2, q = 1 \) was first solved by S. J. Szarek [4], who proved \( C_{2,1}^R = \sqrt{2} \). A similar proof was given by U. Haagerup [1] who also found \( C_{p,2}^R \) and \( C_{2,q}^R \) for each \( p > 0 \). A simple and elementary proof that \( C_{2,1}^R = \sqrt{2} \) was also presented by B. Tomaszweski [6].

J.-P. Kahane [2] generalized the result of Khinchin to sequences \( x_1, \ldots, x_n \) in a normed linear space \( F \), replacing in (1) the absolute value by the norm in \( F \). Let \( C_{p,q} \) denote the smallest constant in the vector-valued inequalities, over all normed linear spaces \( F \). It is of interest to know if the constants are the same in the vector and real cases. As far as we know the best result for \( p = 2 \) and \( q = 1 \) was first obtained by B. Tomaszweski [5], who proved that \( C_{2,1} \leq \sqrt{3} \). In this paper we show that \( C_{2,1} = \sqrt{2} \); we think that our proof is simpler than the others known for real numbers.

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Notation. For $\sigma = (\sigma_1, \ldots, \sigma_n) \in \{0,1\}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n), \eta = (\eta_1, \ldots, \eta_n) \in \{-1,1\}^n$ and $x_1, \ldots, x_n \in F$ let us define

- $|\sigma| = \sum_{i=1}^{n} \sigma_i$,
- $\alpha^{\sigma} = \prod_{i=1}^{n} \alpha_{i}^{\sigma_i}$ (where $x_i = 1$ for any $x \in \mathbb{R}$),
- $-\varepsilon = (-\varepsilon_1, \ldots, -\varepsilon_n)$,
- $X_{\varepsilon} = \|\sum_{i=1}^{n} \varepsilon_i x_i\|$, 
- $d(\varepsilon, \eta) = \text{card}\{i : \varepsilon_i \neq \eta_i\}$.

We will denote by $E(\cdot)$ the mean value of $(\cdot)$.

Results. We will prove the following theorem:

**Theorem 1.** Let $S = \sum_{i=1}^{n} \varepsilon_i x_i$, where $\varepsilon_i$ are independent Bernoulli random variables and $x_i$ are vectors of a normed linear space $F$. Then

$$E\|S\|^2 \leq \sqrt{2}E\|S\|.$$  

The constant $\sqrt{2}$ is the best possible.

**Proof.** Differentiating in $t$ both sides of the equality

$$t^2 \prod_{i=1}^{n} (1 + t^{-1} \alpha_i) = \sum_{\sigma \in \{0,1\}^n} t^{2 - |\sigma|} \alpha^{\sigma}$$

and setting $t = 1$ we get

$$2 \prod_{i=1}^{n} (1 + \alpha_i) - \sum_{j=1}^{n} \alpha_j \prod_{i=1, i \neq j}^{n} (1 + \alpha_i) = \sum_{\sigma \in \{0,1\}^n} (2 - |\sigma|) \alpha^{\sigma}.$$ 

Hence setting $\alpha_i = \varepsilon_i \eta_i$ and summing over $\varepsilon$ and $\eta$ we obtain

$$2 \prod_{i=1}^{n} (1 + \varepsilon_i \eta_i) - \sum_{j=1}^{n} \varepsilon_j \eta_j \prod_{i=1, i \neq j}^{n} (1 + \varepsilon_i \eta_i) = E_{\varepsilon, \eta \sim \{-1,1\}^n} X_{\varepsilon} X_{\eta}$$

$$= \sum_{\varepsilon, \eta \sim \{-1,1\}^n} \sum_{\sigma \in \{0,1\}^n} (2 - |\sigma|) \sum_{\sigma' \in \{0,1\}^n} \varepsilon^{\sigma} \eta^{\sigma'} X_{\sigma'} X_{\sigma}$$

$$= \sum_{\sigma \in \{0,1\}^n} (2 - |\sigma|) \left( \sum_{\varepsilon \sim \{-1,1\}^n} \varepsilon^{\sigma} X_{\varepsilon} \right)^2 \leq 2 \left( \sum_{\varepsilon \sim \{-1,1\}^n} X_{\varepsilon} \right)^2.$$ 

The last inequality holds because $X_{\varepsilon} = X_{-\varepsilon}$ for each $\varepsilon$, so that obviously

$$\sum_{\varepsilon \sim \{-1,1\}^n} X_{\varepsilon} = 0$$

for each $\sigma$ with $|\sigma| = 1$.

Since $\prod_{i=1}^{n} (1 + \varepsilon_i \eta_i) \neq 0$ if $\varepsilon = \eta$, and $\prod_{i=1}^{n} \varepsilon_i \eta_i (1 + \varepsilon_i \eta_i) \neq 0$ if $\varepsilon_i = \eta_i$ for all $i \neq j$, the left-hand side of (2) is equal to

$$2^{n+1} \sum_{\varepsilon \sim \{-1,1\}^n} X_{\varepsilon}^2 - 2^{n+1} \sum_{\varepsilon \sim \{-1,1\}^n} X_{\varepsilon}^2 + 2^{n+1} \sum_{\varepsilon, \eta \sim \{-1,1\}^n, d(\varepsilon, \eta) = 1} X_{\varepsilon} X_{\eta}$$

we arrive at

$$2^n \sum_{\varepsilon \sim \{-1,1\}^n} X_{\varepsilon}^2 + 2^{n-1} \sum_{\varepsilon \sim \{-1,1\}^n} X_{\varepsilon} \left( \sum_{\eta \sim \{-1,1\}^n, d(\varepsilon, \eta) = 1} X_{\eta} - (n-2) X_{\varepsilon} \right) \leq 2 \left( \sum_{\varepsilon \sim \{-1,1\}^n} X_{\varepsilon} \right)^2.$$ 

By the triangle inequality for each fixed $\varepsilon$ we get

$$(n-2) X_{\varepsilon} \leq \sum_{\eta \sim \{-1,1\}^n, d(\varepsilon, \eta) = 1} X_{\eta}.$$ 

So inequality (3) yields

$$2^n \sum_{\varepsilon \sim \{-1,1\}^n} X_{\varepsilon}^2 \leq 2 \left( \sum_{\varepsilon \sim \{-1,1\}^n} X_{\varepsilon} \right)^2.$$ 

Dividing by $2^{2n}$ we get

$$E\|S\|^2 \leq 2 (E\|S\|)^2.$$ 

To see that the constant $\sqrt{2}$ is the best possible it suffices to take $n = 2$, $x_1 = x_2 \neq 0$.

**Remark 1.** If we replace in the above proof $X_{\varepsilon}$ by $X_{\varepsilon}^p$ and use the inequality

$$X_{\varepsilon} \leq \frac{1}{n-2} \sum_{\eta \sim \{-1,1\}^n, d(\varepsilon, \eta) = 1} X_{\eta} \leq \frac{n-2}{n} \left( \frac{1}{n} \sum_{\eta \sim \{-1,1\}^n, d(\varepsilon, \eta) = 1} X_{\eta}^p \right)^{1/p}$$

we will obtain, for $p \in [1,2]$,

$$E\|S\|_{(2p)^{1/2p}} \leq (1 - p/2)^{-1/2p} (E\|S\|_p)^{1/p}$$

but we do not think that the above constants are optimal for $p > 1$.

**Remark 2.** Since for each bounded real random variable $X$ the function $f(r) = r \ln(E|X|^{r})$ is convex, Theorem 1 yields that for all $q \in (0,1]$ and $p \in [0,2]$ with $q \geq p$ the following inequality holds:

$$E\|S\|_p^{1/p} \leq 2^{1/q-1/p} (E\|S\|_q)^{1/q}$$

and the constants $2^{1/q-1/p}$ are optimal.
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