Generalized inverses in $C^*$-algebras II

by

ROBIN HARTE (Belfast) and MOSTAFA MEKHTA (Lille)

Abstract. Commutativity and continuity conditions for the Moore-Penrose inverse and the "conorm" are established in a $C^*$-algebra; moreover, spectral permanence and $B^*$-properties for the conorm are proved.

Suppose $A$ is a ring, with identity 1 and invertible group $A^{-1}$ (more generally, an "additive category"); then an element $a \in A$ will be called regular if it has a generalized inverse in $A$, $b \in A$ for which

$$a = aba.$$  

It is clear that both products

$$ba = (ba)^2 \quad \text{and} \quad ab = (ab)^2$$

are idempotents of $A$; in the presence of an involution $*: A \to A$ we can also enquire whether or not they are self-adjoint: when $(ba)^* = ba$ and $(ab)^* = ab$ then (provided also $b = bab$) the generalized inverse is called a Moore-Penrose inverse for $A$. If this exists then ([7], Theorem 5) it is uniquely determined, and lies in the double commutant of $a$ and $a^*$; when $A$ is a $C^*$-algebra then ([7], Theorem 6) every regular element has a Moore-Penrose inverse. We write $a^+$ for the Moore-Penrose inverse of $a \in A$; thus

$$a = aa^+a; \quad a^+ = a^+aa^+; \quad (a^+a)^* = a^+a; \quad (aa^+)^* = aa^+.$$  

By the uniqueness it is clear that

$$a = (a^+)^* = (a^+)^+.$$  

We recall also that, in a $C^*$-algebra $A$, necessary and sufficient for an element $a \in A$ to be regular (and hence have a Moore-Penrose inverse) is ([7], Theorems 2 and 8) that the range ideal be closed:

$$aA = cl aA.$$  

1991 Mathematics Subject Classification: Primary 46L05.
In this note we enquire for which elements \( a \in \mathbb{A}a \) the Moore Penrose inverse \( a^+ \) is a continuous function of \( a \); this leads us to introduce the "conorm" of a normed algebra element.

If \( 0 \neq T : X \to Y \) is a bounded linear operator between normed spaces then its reduced minimum modulus is given by
\[
\gamma(T) = \inf \{ \|Tx\| : \text{dist}(x, T^{-1}(0)) \geq 1 \}.
\]

For \( T = 0 \) this suggests (Kato [9], Ch. IV, §5) \( \gamma(T) = \infty \), although Apostol [1] prefers \( \gamma(0) = 0 \). For example if \( X \) and \( Y \) are both complete then ([9], Theorem IV.5.2)
\[
\gamma(T) > 0 \iff T(X) = \text{cl} T(X);
\]
if \( T \) is invertible then
\[
\gamma(T) = \|T^{-1}\|^{-1}.
\]

When \( A \) is a normed algebra then an element \( a \in A \) acquires a "left" and a "right" conorm, the reduced minimum modulus of the operators \( L_a \) and \( R_a \) of multiplication by \( a \) on the normed space \( A \):

1. Definition. The (left) conorm of an element \( a \in A \) in a normed algebra \( A \) is given by
\[
\gamma(a) = \gamma^L_a(a) = \inf \{ \|xa\| : \text{dist}(x, a^{-1}(0)) \geq 1 \},
\]
where
\[
a^{-1}(0) = L_a^{-1}(0) = \{ x \in A : ax = 0 \}
\]
is the right annihilator of \( a \) in \( A \).

Similarly, the minimum modulus of the right multiplication \( R_a \) gives a "right conorm" for \( a \in A \):
\[
\gamma^R_a(a) = \inf \{ \|xa\| : \text{dist}(x, a^{-1}(0)) \geq 1 \},
\]
where
\[
a^{-1}(0) = R_a^{-1}(0) = \{ x \in A : xa = 0 \}.
\]

Whether or not the algebra is complete, the conorm of a regular element is positive:

2. Theorem. If \( a \) and \( b \) are elements of a normed algebra \( A \) then
\[
0 \neq a = aba \Rightarrow 1 \leq \|b\| \gamma(a) \leq \|ba\| \|ab\|.
\]
If in particular \( 0 \neq a \in \mathbb{A}a \) is regular in a \( C^\ast \)-algebra \( A \) then
\[
\|a^+\| \gamma(a) = 1.
\]

Proof. Suppose first that \( T \in BL(X, Y) \) has a complemented null space in \( X \), and that \( P = P^2 \in BL(X, X) \) satisfies \( P^{-1}(0) = T^{-1}(0) \); then we may define \( T^\ast : P(X) \to \text{cl} T(X) \) by setting
\[
T^\ast(Px) = T x \quad \text{for each } x \in X.
\]
Now we claim
\[
\|T^\ast\| \leq \|T\| \leq \|P\| \|T^\ast\|
\]
and
\[
\gamma(T^\ast) \leq \|P\| \gamma(T).
\]
For (2.4) argue
\[
\sup \frac{\|Tx\|}{\|Px\|} = \sup \frac{\|TPx\|}{\|Px\|} \leq \sup \frac{\|Tx\|}{\|x\|} = \|T\| \leq \|P\| \sup \frac{\|Tx\|}{\|Px\|};
\]
for (2.5) observe
\[
\text{dist}(x, T^{-1}(0)) = \text{dist}(Px, T^{-1}(0)) \leq \|Px\| \leq \|P\| \text{dist}(x, T^{-1}(0)).
\]
If in particular \( T = STS \) has a generalized inverse \( S \in BL(Y, X) \) then we may take \( P = ST \), and apply also the analogue of (2.4) with \( S \) and \( Q = TS \) in place of \( T \) and \( P \); then (0.8) gives
\[
\gamma(T^\ast) = \frac{1}{\|S\|^2}.
\]
Now (2.4), (2.5) and (2.7) together give (2.1). When \( a = aba \) is a normed algebra \( A \) then (2.1) applies with \( T = L_a \) and \( S = L_b \); to deduce (2.2) we need only show that if \( b = a^+ \) then
\[
\|aa^+\| = \|a^+a\| = 1,
\]
giving \( \|ST\| = \|TS\| = 1 \) in (2.1). But if \( p^* = p = p^2 \in A \) then \( \|p\| = \|p^*p\| = \|p\|^2 \).

In a \( C^\ast \)-algebra, the conorm can be represented as a sort of "spectral radius", and hence acquires "\( B^\ast \)" characteristics. Recall that the spectrum of a linear algebra element \( a \in A \) is given by
\[
\sigma(a) = \sigma_A(a) = \{ \lambda \in \mathbb{C} : a - \lambda \notin A^{-1} \},
\]
and the spectral radius by
\[
|a|_\sigma = \sup \{ |\lambda| : \lambda \in \sigma(a) \}.
\]
Note that in general
\[
|a|_\sigma = \sigma(a) \Rightarrow |(a - \lambda)^{-1}|_\sigma \text{ dist}(\lambda, \sigma(a)) = 1.
\]
When \( A \) is a complex Banach algebra then the spectral radius is less than or equal to the norm: for normal elements of a \( C^\ast \)-algebra there is equality. This includes "positive" elements ([10], Théorème 1.5):
3. **Theorem.** If $0 \neq a \in A$ is positive in a $C^*$-algebra $A$ then

\[
\gamma(a) = \inf(\sigma(a) \setminus \{0\}).
\]

**Proof.** We make two claims: there is implication

\[
\inf(\sigma(a) \setminus \{0\}) > 0 \Rightarrow \gamma(a) \geq \inf(\sigma(a) \setminus \{0\})
\]

and

\[
\gamma(a) > 0 \Rightarrow \inf(\sigma(a) \setminus \{0\}) \geq \gamma(a).
\]

Towards (3.2) suppose $0 < \lambda < \inf(\sigma(a) \setminus \{0\})$ and write $b = (a - \lambda)^{-1}$; then

\[
a - \lambda
\]

and $b$ are both positive in $A$, so that ([6], Theorem 9.9.4; [8], Theorem 2.2) for arbitrary $z \in A$,

\[
|abz| = \|(x + \lambda bx^*)^*(x + \lambda bx)\| \geq \|z^*x^*bz\| = \|\lambda bx\|^2.
\]

Since $bA = A$ this gives $\gamma(a) \geq \lambda$ and hence (3.2). Towards (3.3) note that if $\gamma(a) > 0$ then by (0.7) the ideal $aA$ is closed and hence $a \in aAa$ is regular, and has a Moore-Penrose inverse $a^+ \in A$. Since $a$ is also positive and hence normal it actually commutes ([7], Theorem 10) with $a^+$, and is thus “simply polar” ([7], Theorem 9; [6], Definition 7.3.5). In particular, $a$ is the direct sum of the ideals $aA$ and $a^{-1}(0)$, and the restriction to $aA$ of the multiplication $L_a$ is invertible (inverse given by restricting $L_{a^+}$), while the restriction to $a^{-1}(0)$ is zero. If $0 < |\lambda| < \gamma(a)$ therefore both restrictions, and hence $a - \lambda$, are invertible, giving (3.3). \n\hfill \blacksquare

By the spectral mapping theorem it follows that

\[
\gamma(a)^2 = \gamma(a^2).
\]

Theorem 3 gives the corresponding result for arbitrary elements ([10], Théorème 1.6):

4. **Theorem.** If $0 \neq a \in A$ is a non-zero $C^*$-element then

\[
\gamma(a)^2 = \inf(\sigma(a^*a) \setminus \{0\}),
\]

and hence

\[
\gamma(a)^2 = \gamma(a^*a) = \gamma(aa^*) = \gamma(a^*)^2.
\]

Also

\[
\gamma(a) = \gamma(R_a);
\]

if $A \subseteq B$ for a $C^*$-algebra $B$ then

\[
\gamma_B(a) = \gamma_B(a).
\]

**Proof.** Recalling the square root equality

\[
\|ax\| = \|(a^*a)^{1/2}x\| \text{ for each } x \in A,
\]

together with (3.4), gives the first equality in (4.2), and hence (4.1), since Theorem 3 applies to the positive element $a^*a$. Since in general

\[
1 - ba \in A^{-1} \Leftrightarrow 1 - ab \in A^{-1}
\]

there is equality

\[
\sigma(aa^*) \setminus \{0\} = \sigma(a^*a) \setminus \{0\}.
\]

This with Theorem 3 gives the second equality in (4.2), and the third is just the first applied to $a^*$. Equality (4.3) follows from $\gamma(a^*) = \gamma(a)$, and finally the spectral permanence (4.4) is (4.1) together with the corresponding property of the spectrum $\sigma$.

In the special case $A = BL(X, X)$ of operators, (4.1) is given by Apostol ([11], page 280). Theorem 4 shows that “spectral permanence” in $C^*$-algebras extends to regularity and in particular the Moore-Penrose inverse:

\[
\gamma(a) = \frac{\|a\|}{\|a\|} \Rightarrow a \in aAa + a^+ \in A.
\]

This is an improvement on Theorem 5 of [7], which gives (4.8) for $W^*$-algebras; (4.8) of course follows also from the formulae of Groetsch ([4]; [3], Corollary 1, §2.2). Another simple corollary is that (if $a \neq 0$)

\[
\gamma(a) = \|a\| \Rightarrow \text{ partial isometry}.
\]

To investigate the continuity of the Moore-Penrose inverse and the conorm, we need a simple observation:

5. **Theorem.** If $0 \neq a \in Aa$ and $b \in bAb$ have generalized inverses in a $C^*$-algebra $A$ then there is equality

\[
b^+ - a^+ = -(b^+ + b + a)a + a + b + (b^+ - a^+)(1 - aa^+) + (1 - b^+)(b^* - a^+)(1 - a^+),
\]

and, if $a \neq 0 \neq b$, implication

\[
\|b^+ - a^+a\| < 1 \Rightarrow |\gamma(b) - \gamma(a)| \leq \|b - a\|.
\]

**Proof.** Towards (5.1) write

\[
b^+ - a^+ + b^+(b - a)a = b^+(1 - aa^+) - (1 - b^+)a^+, \quad \text{and use (0.3) and (0.4) to see that}
\]

\[
b^+b - (b^+ - a^+)a = b^+(1 - aa^+)
\]

and

\[
(1 - b^+)(b^* - a^+)(1 - aa^+) = -(1 - b^+)a^+.
\]

For (5.2) write $c = (1 + a^+a - a^+b^+)^{-1}$ and argue

\[
\|bx\| \geq \|aa^+a^+x\| - \|b - a\| \|a^+a^+\| \geq (\gamma(a) - \|b - a\|) \text{dist}(x, b^{-1}(0)),
\]

since, by (2.6) and (2.8),
\[ \text{dist}(a^+b^+a, a^{-1}(0)) = \|a^+b^+a\| = \|b^+\text{b}\| = \text{dist}(x, b^{-1}(0)). \]
This gives \( \gamma(b) \geq \gamma(a) - \|b - a\| \), which is half of (5.2), and similarly the other half. ■

Various conditions are equivalent to the convergence of a sequence of Moore-Penrose inverses ([11], Théorème 2.2):

6. **Theorem.** If \( 0 \neq a \in \langle AAa \rangle \) and \( 0 \neq a_n \in \langle AAa \rangle \) are regular in a \( C^* \)-algebra \( A \), with \( \|a_n - a\| \to 0 \), then the following are equivalent:

\[
\begin{align*}
(6.1) & \quad \lim_{n \to \infty} \|a_n^+ - a^+\| = 0; \\
(6.2) & \quad \gamma(a_n) \to \gamma(a); \\
(6.3) & \quad \sup_n \|a_n^+\| < \infty.
\end{align*}
\]

**Proof.** If (6.1) holds then both sequences of projections also converge:

\[
\begin{align*}
(6.4) & \quad a_n^+a_n \to a^+a; \\
(6.5) & \quad a_n a_n^+ \to aa^+;
\end{align*}
\]

Conversely, if either (6.4) or (6.5) is valid then (5.2) gives convergence (6.2) for the conorms. If (6.2) holds then (since \( \gamma(a) > 0 \)) the sequence \( \gamma(a_n) \) is bounded below, and (6.3) follows from the equality (2.2). Finally, (6.1) follows from (6.3) and the equality (5.1). ■

In the finite-dimensional case (matrix algebra), Theorem 6 recovers a result of Penrose ([11], Theorem 3.5), which says that, if \( (a_n) \) converges to \( a \), then \( (a_n^+) \) converges to \( (a^+) \) if and only if eventually \( \text{rank}(a_n) = \text{rank}(a) \). We can also see that the only normal elements at which the Moore-Penrose inverse is continuous are invertible: more generally, if the conorm \( \gamma \) is continuous at \( a \in A \) then, using (6.4) and (6.5),

\[
(6.6) \quad 0 \neq a \in \langle AAa \rangle \text{ and } a \in \text{cl}(A^{-1}) \Rightarrow a \in A^{-1}.
\]

We can improve Theorem 6: the regularity of \( a \) actually follows from the convergence \( a_n \to a \) of a regular sequence \( (a_n) \) satisfying the condition (6.3),

7. **Theorem.** The conorm is upper semi-continuous on \( A \setminus \{0\} \):

\[
\|a_n - a\| \to 0 \Rightarrow \limsup_n \gamma(a_n) \leq \gamma(a).
\]

**Proof.** We claim that for each \( k > 0 \),

\[
\{a \in A \mid \gamma(a) \geq k\} \text{ is closed in } A,
\]

and show first that the analogue of (7.2) holds in the subset \( A^+ \) of positive elements of \( A \). If \( a \in A^+ \) then \( \gamma(a) \geq k \) if and only if

\[
(7.3) \quad |0, a| \subseteq C \setminus \sigma(a)
\]

and hence by (2.11),

\[
(7.4) \quad 0 < \lambda < k \Rightarrow \|(a - \lambda^{-1}\| = \|(a - \lambda)^{-1}\| \leq \frac{1}{\min(\lambda, k - \lambda)}.
\]

If, more generally, \( a \in A^+ \) lies in the closure of the set of positive elements satisfying (7.3) then whenever \( 0 < \lambda < k \) there is \( b \in A^+ \) with \( b - \lambda \in A^{-1} \) and \( \|b - a\| < \min(\lambda, k - \lambda) \), giving

\[
a - \lambda = (b - \lambda)(1 + (b - \lambda)^{-1}(a - b)) \in A^{-1}.
\]

This means that (7.3) holds for \( a \), (7.2) follows: the set of elements \( a \in A \) with \( \gamma(a) \geq k \) is the counterimage, under the continuous mapping \( x \mapsto x^+x \), of the set of positive elements \( b \in A^+ \) with \( \gamma(b) \geq k^2 \). Now (7.1) is clear. ■

In the special case \( A = BL(X, X) \) of operators, (7.3) is given by Apostol ([1]), Corollary 1.2). Theorem 7 gives the improved version of Theorem 6:

8. **Theorem.** If \( a \in A \) and \( (a_n) \) in \( A \) with \( a_n \to a \) and \( a_n \in \langle AAa \rangle \) for each \( n \in \mathbb{N} \) then

\[
\liminf_n \|a_n^+\| < \infty \Rightarrow a \in \langle AAa \rangle,
\]

and hence

\[
(8.2) \quad \sup_n \|a_n^+\| < \infty \Rightarrow \exists a^+ = \lim_n a_n^+.
\]

**Proof.** Observe, using (2.2),

\[
\gamma(a) \geq \limsup_n \gamma(a_n) \geq 1/\liminf_n \|a_n^+\| > 0.
\]

This makes \( a \in A \) regular, giving (8.1); for (8.2) apply Theorem 6. ■

We shall call an element \( a \in A \) semi-invertible if it has either a left inverse or a right inverse; for the full algebra of bounded operators on Hilbert space these are the only non-trivial continuity points of the conorm. We look first at general normed spaces:

9. **Theorem.** The reduced minimum modulus is continuous on the open sets of bounded below and of almost open operators between a pair of normed spaces. If \( T : X \to Y \) is a bounded linear operator between normed spaces for which

\[
(9.1) \quad \gamma(T) > 0 \text{ and } T^{-1}(0) \neq \{0\} \text{ and } \text{cl}T(X) \neq Y,
\]

then \( T \) is not a continuity point of the reduced minimum modulus \( \gamma \).
Theorem 10. If \( a \in \mathfrak{a} \mathfrak{a} \) is a regular element of a \( C^* \)-algebra \( A \) then the following are equivalent:

\[
\begin{align*}
(10.1) & \quad a^+ = a^+ a; \\
(10.2) & \quad a^{-1}(0) = a^{-1}(0); \\
(10.3) & \quad a_{-1}(0) = a_{-1}(0); \\
(10.4) & \quad a \in A^{*}a^*; \\
(10.5) & \quad a \in a^* A^{-1}.
\end{align*}
\]

Proof. The argument divides neatly in two: if \( a \in \mathfrak{a} \mathfrak{a} a \) in a \( C^* \)-algebra then

\[
(10.6) \quad a^* \in A^{-1} a^* \quad \text{and} \quad a^* \in a^* A^{-1};
\]

if \( a = aba \in A \) with no restriction on \( A \) then there is implication

\[
(10.7) \quad ba^2 = a = a^2 b \iff ba = ab,
\]

while if in addition \( b = bab \) then

\[
(10.8) \quad a \in Ab \Rightarrow b^{-1}(0) \subseteq a^{-1}(0) \Rightarrow a = a^2 b
\]

and

\[
(10.9) \quad ab = ba \iff a \in A^{-1} b \quad \text{and} \quad a \in b A^{-1}.
\]

These are quickly checked: for example \( a a^+ = (a a^+)^* = a^+ a^* \), giving \( a^* = a^* a a^+ \in \mathfrak{a} a \mathfrak{a}^+ \), and hence

\[
(10.10) \quad a^* = (a^* a + 1 - a^+ a) a^+ + a^+ a a^+ + 1 - a^+ a = (a^* a + 1 - a^+ a)^{-1},
\]

giving (10.6). For (10.7) note that if \( aba = a \) with \( ba = ab \) then \( ba^2 = aba = a \); conversely, if \( a = a^2 b = a^2 b \) then \( ba^2 = ab \). The first implication of (10.8) is immediate; if \( b^{-1}(0) \subseteq a^{-1}(0) \) and \( b = bab \) then \( 1 - ab \in a^{-1}(0) \), giving the second. Finally, if \( ba = ab \) then

\[
(10.11) \quad a = (a^2 + 1 - a^+ a) b \quad \text{with} \quad b^2 + 1 - a^+ a = (a^2 + 1 - a^+ a)^{-1},
\]

giving (10.9).

There are also “one-sided” versions of Theorem 10, combining (10.6) and (10.8). It is sufficient for example for

\[
(10.12) \quad a = a^+ a^2
\]

that \( a \) be either left invertible, or quasinormal ([5], Problem 108) in the sense that

\[
(10.13) \quad a a^* = a^* a^2;
\]
equivalently ([5], \( a \) has a commuting “polar decomposition”.)
On continuity properties of functions with conditions on the mean oscillation

by

HUGO AIMAR and LILIANA FORZANI (Santa Fe)

Abstract. In this paper we study distribution and continuity properties of functions satisfying a vanishing mean oscillation property with a lag mapping on a space of homogeneous type.

Since the initial works by F. John and L. Nirenberg and J. Moser in 1961, the study of regularity of functions with properties on their mean oscillation over balls was developed by S. Campanato, G. Meyers, S. Spanne and A. P. Calderón. Extensions from the euclidean setting to spaces of homogeneous type were considered by N. Burger, R. Macías and C. Segovia and one of the authors.

In 1967, J. Moser in his paper on Harnack’s inequality for parabolic equations introduces a BMO type condition with a time lag. In 1985, E. Fabes and N. Garofalo, applying an extension of Calderón’s method as stated by U. Neri obtained a John–Nirenberg type lemma for this parabolic case. In 1988 one of us proved an extension of these results to the setting of spaces of homogeneous type that can be applied to degenerate parabolic equations. Related results come from the analysis of one-sided maximal functions and weights; in a recent paper F. Martí-Reyes and A. de la Torre prove a John Nirenberg type lemma for one-sided BMO functions.

In this paper we study distribution and continuity property of functions satisfying a vanishing mean oscillation property with a lag mapping on a space of homogeneous type.

1. Main results. Let $X$ be a set. A symmetric function $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ is a quasi-distance on $X$ if $d(x, y) = 0$ iff $x = y$ and there exists a constant $K$ such that $d(x, z) \leq K[d(x, y) + d(y, z)]$ for $x, y, z \in X$. The ball with center $x \in X$ and radius $r > 0$ is the set $B(x, r) = \{y \in X : d(x, y) < r\}$. We shall say that a positive measure $\mu$ defined on a $\sigma$-algebra contain-