An example of a generalized completely continuous representation of a locally compact group

by

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Abstract. There is constructed a compactly generated, separable, locally compact group $G$ and a continuous irreducible unitary representation $\pi$ of $G$ such that the image $\pi(C^*(G))$ of the group $C^*$-algebra contains the algebra of compact operators, while the image $\pi(L^1(G))$ of the $L^1$-group algebra does not contain any nonzero compact operator. The group $G$ is a semidirect product of a metabelian discrete group and a "generalized Heisenberg group".

In [6] the following theorem was proved. Let $\pi$ be an irreducible continuous unitary representation of a connected Lie group $G$ such that $\pi(C^*(G))$ contains the algebra of compact operators, i.e., $\pi$ is a generalized completely continuous representation in our terminology (apparently this notion is used in different ways in the literature). Then the image of $L^1(G)$ under $\pi$ contains orthogonal projections of rank one. After the efforts at proving this result it is hard to imagine that a corresponding theorem is true for general locally compact groups $G$. There is even no evidence why in general $\pi(L^1(G))$ should contain nonzero compact operators if $\pi(C^*(G))$ does. However, to my best knowledge there is no example in the literature where such a pathology occurs. It is the purpose of this note to provide such an example. Clearly, such groups cannot be connected, but still they will be compactly generated and separable. In [3], Guichardet constructed an example of a discrete group and a generalized completely continuous representation $\pi$ of this group such that the image of the finitely supported functions under $\pi$ does not contain nonzero compact operators. In some sense, my example is an extension of his.

The basis of the construction is a discrete group $S$ acting automorphically on a locally compact abelian group $H$: there is given an homomorphism $\varphi : S \rightarrow \text{Aut}(H)$. Moreover, it is assumed that $H$ contains compact open subgroups. Fix one of them and call it $K$. Later $S$, $H$ and $K$ will be specified. The duality between the Pontryagin dual $\hat{H}$ and $H$ is denoted by

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\( (\chi, h) = \chi(h) \) for \( h \in H \), \( \chi \in \hat{H} \). The group \( S \) acts on \( \hat{H} \) as well, \( \langle s \chi, h \rangle = (\chi, \varphi(s)^{-1}(h)) \). Using the duality one may form the "generalized Heisenberg group" \( N = H \times \hat{H} \times \mathbb{T} \) with multiplication law
\[
(h, \chi, t)(h', \chi', t') = (h + h', \chi \chi', t' \chi, \varphi(s)(h')^{-1}).
\]
Observe that the abelian group \( H \) is written additively, while the groups \( \hat{H} \) and \( \mathbb{T} \) are written multiplicatively. Associated with \( \varphi \) there is an homomorphism \( \psi : S \to \text{Aut}(N) \) given by
\[
\psi(s)(h, \chi, t) = (\varphi(s)(h), s \chi, st).
\]
Then one may form the semidirect product \( G \) of \( S \) and \( N \), i.e., as a topological space \( G \) is the direct product \( S \times H \times \hat{H} \times \mathbb{T} \), and the multiplication is defined as
\[
(1) \quad (s, h, \chi, t)(s', h', \chi', t') = (ss', \varphi(s)^{-1}(h) + h', ss^{-1} \chi \chi', t' \chi - s(\varphi(s)(h'))^{-1}).
\]
Later we shall consider representations of \( G \) which coincide on the central subgroup \( T \) with the identity map. Hence we define \( \gamma : T \to \mathbb{T} \) by \( \gamma(t) = t \), and we denote by \( L^1(G)_{\gamma} \) the involutive convolution algebra of all \( L^1 \)-functions \( f \) on \( G \) satisfying \( f(x) = \overline{\gamma(t)}f(x) \) for all \( x \in G \) and \( t \in T \) where, of course, \( T \) is identified with \( \{e\} \times \{0\} \times \{1\} \times \mathbb{T} \). The algebra \( L^1(N)_{\gamma} \) is defined similarly; it acts by convolution on \( L^1(G)_{\gamma} \). Moreover, \( S \) acts on \( L^1(N)_{\gamma} \) by \( f'(x) = f(\varphi(s)(x)) \) for \( s \in S \), \( x \in N \) and \( f \in L^1(N)_{\gamma} \).

In [5] it was shown that \( L^1(N)_{\gamma} \) is a simple Banach algebra (this will be discussed in more detail later on) and that it contains "orthogonal projections of rank one". Using the chosen compact open subgroup \( K \) we are going to construct a particular projection \( p \) in \( L^1(N)_{\gamma} \) and to determine the algebra \( p * L^1(G)_{\gamma} \).

Associated with \( K \) there is a compact open subgroup of \( \hat{H} \), namely \((H/K)^{\gamma} \), the annihilator of \( K \). The Haar measures of \( H \) and \( \hat{H} \) are normalized so that \( K \) and \((H/K)^{\gamma} \) have measure one. The function \( p : N \to \mathbb{C} \) is defined by
\[
(2) \quad p(h, \chi, t) = \begin{cases} 1 & \text{if } h \in K \text{ and } \chi \in (H/K)^{\gamma}, \\ 0 & \text{otherwise}. \end{cases}
\]

To describe \( p * L^1(G)_{\gamma} \) we need a certain family \( q_s, s \in S \), of functions in \( L^1(N)_{\gamma} \). Let \( \delta : S \to \mathbb{R}_+^{\times} \) be the modular function of the action of \( S \) on \( H \), which is given by
\[
\int_H \delta_\chi(h) \ d\chi = \int_H f(y) \ dy
\]
for all, say, compactly supported continuous functions \( f \) on \( H \). Choosing \( f \)

to be the characteristic function of \( K \), one sees that
\[
\delta_\chi(h) = |\varphi(s)^{-1}(K)|^{-1} \cdots |\varphi(s)^{-1}(K)|
\]
where \( |X| \) denotes the Haar measure of a measurable subset \( X \) of \( H \). The same notation is used for measurable subsets of \( \hat{H} \). From this description of \( \delta \) one easily derives that
\[
(3) \quad \delta_\chi(h) = \#(\varphi(s)^{-1}(K)/\varphi(s)^{-1}(K) \cap K) \cdot \#(K/\varphi(s)^{-1}(K) \cap K)^{-1}
\]
\[
= \#(K/\varphi(s)(K) \cap K) \cdot \#(K/\varphi(s)^{-1}(K) \cap K)^{-1}
\]
\[
= |\varphi(s)^{-1}(K) \cap K| \cdot |\varphi(s)(K) \cap K|^{-1}
\]
for \( s \in S \).

Observe in passing that \( S \) acts via \( \psi \) on \( N \) in a measure-preserving way.
In particular, one has \( ||f^*||_1 = ||f||_1 \) for \( f \in L^1(N)_{\gamma} \), and \( S \) belongs to \( L^1(N)_{\gamma} \). By
\[
(4) \quad \chi_s(l + k) = \chi_l(k) \quad \text{if } l \in \varphi(s)^{-1}(K) \text{ and } k \in K.
\]
Then define \( q_s \in L^1(N)_{\gamma} \) by
\[
(5) \quad q_s(h, \chi, t) = \begin{cases} t^{-1} \chi_s(t^{-1/2}|K \cap \varphi(s)^{-1}(K)|) & \text{if } h \in \varphi(s)^{-1}(K) + K \text{ and } \chi \in (H/K \cap \varphi(s)^{-1}(K))^{\gamma}, \\ 0 & \text{otherwise}. \end{cases}
\]

For each \( s \in S \) the equality
\[
(6) \quad q_s = (q_{s^{-1}})^\ast
\]
holds true for the following reasons: By definition of the involution and the action, one has
\[
(q_{s^{-1}})(h, \chi, t) = q_{s^{-1}}(\varphi(s)(h), \varphi(s)(h), t^{-1}(\chi, h)^{-1}).
\]
This is zero unless \( \varphi(s)(h) \in K + \varphi(s)(K) \), which is equivalent to \( h \in \varphi(s)^{-1}(K) + K \), and \( (\chi, h)^{-1} \in (H/K \cap \varphi(s)(K))^{\gamma} \), which is equivalent to \( \chi \in H/K \cap \varphi(s)^{-1}(K) \). If the latter conditions are not satisfied, both functions \( (q_{s^{-1}}) \) and \( q_s \) vanish at \( (h, \chi, t) \). Suppose that the conditions are satisfied. Write \( h = l + k \) with \( l \in \varphi(s)^{-1}(K) \) and \( k \in K \). Then
\[
q_{s^{-1}}(\varphi(s)(h), \varphi(s)(h), t^{-1}(\chi, h)^{-1})
\]
\[
= t^{-1} \chi_s(l + k) \cdot (\varphi(s)(h), \varphi(s)(h)) |K \cap \varphi(s)(K)|^{-1}
\]
\[
= t^{-1} \chi_s(l + k) \cdot (\varphi(s)^{-1}(K))^{\gamma} \cdot (\varphi(s)(h), \varphi(s)(h)) |K \cap \varphi(s)(K)|^{-1}
\]
\[
= t^{-1} \chi_s(l + k) \cdot (\varphi(s)^{-1}(K))^{\gamma} \cdot |\varphi(s)(K) \cap K|^{-1}
\]
\[
= \delta_\chi(h)
\]
Since \( \varphi(s)(-l-k) = \varphi(s)(-l) + \varphi(s)(-k) \) with \( \varphi(s)(-l) \in K \) and \( \varphi(s)(-k) \in \varphi(s)(K) \) the middle term gives
\[
(s(x^{-1}))(l-k) = (s(x^{-1})), \varphi(s)(-l)) = (x^{-1}, -l).
\]

Hence
\[
(q_{s-1})^{*}(h, x, l = t^{-1}(x, l + k)^{-1}(x, l) |(s(x^{-1}))| |(s(x^{-1}))|^{-1} \delta(s)^{-1/2} |\{H/K \cap \varphi(s)^{-1}(K)\}|.
\]

which gives (5) in view of (3). Since \( q_{s} = p \) one has in particular
\[
p^{*} = p.
\]

The \( L^{1} \)-norm of \( q_{s} \) is easily computed:
\[
||q_{s}|| = \delta(s)^{-1/2} |\{H/K \cap \varphi(s)^{-1}(K)\}| |(s(x^{-1}))|^{-1} \delta(s)^{-1/2} |\{H/K \cap \varphi(s)^{-1}(K)\}|.
\]

From the exact sequence
\[
(H/K)^{\wedge} \rightarrow (H/K \cap \varphi(s)^{-1}(K))^{\wedge} \rightarrow (K/K \cap \varphi(s)^{-1}(K))^{\wedge}
\]

one reads off that
\[
|(H/K \cap \varphi(s)^{-1}(K))^{\wedge}| = |(K/K \cap \varphi(s)^{-1}(K))|^{-1},
\]

hence
\[
||q_{s}|| = \delta(s)^{-1/2} |\{H/K \cap \varphi(s)^{-1}(K)\}|^{-1}.
\]

Since \( \varphi(s)^{-1}(K) = K \) is isomorphic to \( \varphi(s)^{-1}(K)/K \cap \varphi(s)^{-1}(K) \) the number of \( \varphi(s)^{-1}(K) \) in \( K \) equals
\[
|\varphi(s)^{-1}(K)/K \cap \varphi(s)^{-1}(K)| = |\varphi(s)^{-1}(K)| - 1
\]

hence
\[
||q_{s}|| = \delta(s)^{-1/2} |\{K/K \cap \varphi(s)^{-1}(K)\}|^{1/2} = \delta(s)^{-1/2} |\{K/K \cap \varphi(s)^{-1}(K)\}|^{1/2}.
\]

(15) If \( u \) denotes the characteristic function of \( K \) then
\[
\widehat{q}_{u}(h, x) = \delta(s)^{-1/2} u(h, x) u(h, x),
\]

in particular, \( \widehat{p}(h, x) = \hat{q}_{u}(h, x) = u(u h, x) \), for \( h, x \in H \).

If \( h \notin \varphi(s)^{-1}(K) + K \) then \( \widehat{q}_{u}(h, x) = 0 \). Suppose now that \( h = l + k \) with \( l \in \varphi(s)^{-1}(K) \) and \( k \in K \). Then
\[
\widehat{q}_{u}(l + k, x) = \int_{(H/K \cap \varphi(s)^{-1}(K))^{\wedge}} d\chi \delta(s)^{-1/2} |\{H/K \cap \varphi(s)^{-1}(K)\}|^{-1} \delta(s)^{-1/2} \delta(s)^{-1/2} |\{H/K \cap \varphi(s)^{-1}(K)\}^{\wedge} = \delta(s)^{-1/2}.
\]

This integral is zero unless \( k + x \in K \cap \varphi(s)^{-1}(K) \). In that case one obtains
\[
\widehat{q}_{u}(l + k, x) = \delta(s)^{-1/2} |\{H/K \cap \varphi(s)^{-1}(K)\}|^{-1} \delta(s)^{-1/2} |\{H/K \cap \varphi(s)^{-1}(K)\}^{\wedge} = \delta(s)^{-1/2}.
\]
But for a given pair \((h,x)\) in \(H \times H\) the conditions: \(h \in \varphi(s)^{-1}(K) + K\)
and for some (any) decomposition \(h = l + k\) with \(l \in \varphi(s)^{-1}(K)\) and \(k \in K\)
the sum \(k + x\) lies in \(K \cap \varphi(s)^{-1}(K)\), are equivalent to: \(x \in K\) and \(k + x \in \varphi(s)^{-1}(K)\). Therefore, \(\tilde{q}_h\) is of the form as claimed in (15).

To prove \(q^*_h \ast q_x = p\) of (9) one first observes that since, by (13), one has 
\[
(q^*_h)^\wedge(h,x) = (\tilde{q}_h)^\wedge(h,x) = \tilde{q}_h(-h,x+h)^{-1}, \text{ it follows that } (q^*_h)^\wedge(h,x) = \\
\left(\delta(s)^{-1/2}u(x+h)u(\varphi(s)(x))\right) \text{ by (15). Using (13) one obtains }
\]
\[
(q^*_h \ast q_x)^\wedge(h,k) = \int_H (q^*_h)^\wedge(h+x,k-x)\tilde{q}_x(-x,k)\,dk
\]
\[
= \int_H \delta(s)^{-1}u(h+k)u(\varphi(s)(k-x))u(k)u(\varphi(s)(k-x))\,dk
\]
\[
= u(h+k)\int_H \delta(s)^{-1}u(\varphi(s)(x))^2\,dx
\]
\[
= \tilde{p}(h,k) \int_H u(x)^2\,dx = \tilde{p}(h,k).
\]

The proof for \(q_x \ast q^*_h = p\) is similar and omitted. To show \(p^* \ast q_x = q_x\) one first uses (14) to obtain
\[
p^\wedge(h,k) = \delta(s)^{-1}\tilde{p}(\varphi(s)(h),\varphi(s)(k))
\]
\[
= \delta(s)^{-1}u(\varphi(s)(k+h))u(\varphi(s)(h+k)).
\]

Hence by (13),
\[
(p^* \ast q_x)^\wedge(h,k) = \int_H p^\wedge(h+x,k-x)\tilde{q}_x(-x,k)\,dx
\]
\[
= \int_H \delta(s)^{-1}u(\varphi(s)(k-x))u(\varphi(s)(h+k))
\]
\[
\times \delta(s)^{-1/2}u(k)u(\varphi(s)(k-x))\,dk
\]
\[
= \delta(s)^{-1/2}u(k)u(\varphi(s)(h+k))\int_H \delta(s)^{-1}u(\varphi(s)(x))^2\,dx
\]
\[
= \tilde{q}_x(h,k).
\]

Also the proof of \(q_x \ast q^*_h \ast p\) is omitted as well as the proof of (11); they are straightforward calculations of the same type.

To show (10) one has to compute \(p^* \wedge * \tilde{f} * \tilde{p}\) for any \(\tilde{f} \in L^1(H, A(H))\).
First one observes that for \(\tilde{f} = \tilde{q}_h\) one has \(p^* \wedge * \tilde{q}_h \ast \tilde{p} = q^*_h\) by (9), hence
\[
p^* \ast L^1(N)_\gamma \ast p \ni Cq_x. \text{ Now let } \tilde{f} \text{ be arbitrary. By (13), }
\]
\[
(\tilde{f} \ast \tilde{p})(h,k) = \int_H \tilde{f}(h+x,k-x)u(k)u(k-x)\,dx,
\]
and by (16) and (13),
\[
(p^\wedge \ast \tilde{f} \ast \tilde{p})(h,k) = \int_H p^\wedge(h+y,k-y)(\tilde{f} \ast \tilde{p})(y-k)\,dy
\]
\[
= \delta(s)^{-1/2}u(\varphi(s)(h+k))u(k)\delta(s)^{-1/2}
\times \int_H \int_H u(\varphi(s)(k-y))u(k-x)\tilde{f}(x-y,k-x)\,dx\,dy.
\]
Substituting \(y' = k-y\) and \(x' = k-x\) yields
\[
(p^\wedge \ast \tilde{f} \ast \tilde{p})(h,k) = \delta(s)^{-1/2}u(\varphi(s)(h+k))u(k)\delta(s)^{-1/2}
\times \int_H \int_H u(\varphi(s)(y))u(x)\tilde{f}(y-x,k-x)\,dx\,dy,
\]
which is \(q^*_h(h,k)\) times a scalar independent of \(h\) and \(k\).

The map \(w : S \to \mathbb{R}\) defined by
\[
w(s) = \|q_s\|_1 = \#(K/K \cap \varphi(s)(K))^{1/2} : \#(K/K \cap \varphi(s)^{-1}(K))^{1/2}
\]
(cf. (8)) is clearly submultiplicative and greater than or equal to one, i.e.,
\[
\text{it is a weight function. For this notion see [7]. Therefore, }
\]
\[
\ell^1(S,w) = \left\{ f : S \to \mathbb{C} \mid \sum_{s \in S} |f(s)|w(s) < \infty \right\}
\]
is a subalgebra of the convolution algebra \(\ell^1(S)\). Moreover, \(w(s^{-1}) = w(s)\), hence \(\ell^1(S,w)\) is an involutive subalgebra of \(\ell^1(S)\).

Using the foregoing notations and formulas one can show the following proposition.

**Proposition.** The map \(\ell^1(S,w) \to L^1(G)_\gamma, \phi \mapsto \tilde{\phi}\), given by \(\tilde{\phi}(s,h,\lambda,t) = \phi(s)q_h(\lambda,\cdot,t)\) is an isometric \(\ast\)-isomorphism from \(\ell^1(S,w)\) onto \(p \ast L^1(G)_\gamma \ast p\). If \(\pi\) is a continuous unitary representation of \(G\) in \(\mathcal{F}\) with \(\pi(t) = \tau(t)\) for \(t \in T\) then \(\pi\) yields involutive representations of \(L^1(G)_\gamma\) and of \(L^1(N)_\gamma\), also denoted by \(\pi\). The operator \(\pi(p)\) is a nonzero orthogonal projection onto \(\mathcal{F}\). The map \(\ell^1(S,w) \ni \phi \mapsto \pi(\phi(\tau)^{\wedge})\mathcal{F}\) is an involutive representation of \(\ell^1(S,w)\). It is obtained by integrating the unitary representation \(\pi^{\wedge}\) of \(G\) given by \(\pi^{\wedge}(s) = \pi(s)\pi(q_s)^{\wedge}\mathcal{F}\). The representation \(\pi\) is irreducible if \(\pi^{\wedge}\) is.

In case that \(\pi\) is irreducible the following equivalences hold true.
The algebra \( \pi(C^*(G)) \) contains the algebra of compact operators on \( \mathfrak{h} \) iff \( \pi^p(C^*(S)) \) contains the algebra of compact operators on \( \mathfrak{h}^p \). The algebra \( \pi(L^1(G)) \) contains a nonzero compact operator iff \( \pi^p(L^1(S_w)) \) does.

Proof. The equality \( \|\Phi\|_{L^1(G)} = \|\Phi\|_{L^1(S_w)} \) is an immediate consequence of the definitions. To prove the multiplicativity of \( \Phi \mapsto \Phi' \) let \( \Phi, \Psi \in L^1(S, w) \). Then
\[
(\Phi \ast \Psi'(r x) = (\Phi \ast \Psi)(r) q_\gamma(x) \quad \text{for } r \in S, \ x \in N,
\]
and
\[
(\Phi' \ast \Psi') (r x) = \sum_{s \in S} \int_{N} d y \Phi(r x s y) \Psi'((s y)^{-1})
\]
\[
= \sum_{s \in S} \int_{N} d y \Phi'(r s^{-1} x s y) \Psi'(s^{-1} s y^{-1} s^{-1})
\]
\[
= \sum_{s \in S} \int_{N} d y \Phi(r s) q_\gamma(s^{-1} s y) \Psi(s^{-1}) q_\gamma^{-1}(y^{-1})
\]
\[
= \sum_{s \in S} \Phi(r s) \Psi(s^{-1}) (q_\gamma r + q_\gamma^{-1}) (s^{-1} x s).
\]
But \( (q_\gamma r + q_\gamma^{-1}) (s^{-1} x s) = (q_\gamma r + q_\gamma^{-1})^* (s^{-1} x s) = (q_\gamma^{-1} r + q_\gamma^{-1}) (s^{-1} x s) = q_\gamma(x) \) by (11). Therefore,
\[
(\Phi' \ast \Psi')(r x) = \sum_{s \in S} \Phi(r s) \Psi(s^{-1}) q_\gamma(x) = (\Phi \ast \Psi)(r) q_\gamma(x)
\]
as desired.

The equality \( (\Phi')' = (\Phi')^* \) for \( \Phi \in L^1(S, w) \) is a sufficient consequence of (6), we omit the details. Hence \( \Phi \mapsto \Phi' \) is an isometric \(*\)-isomorphism from \( L^1(S, w) \) into \( L^1(G)_\gamma \).

To show that each \( \Phi' \) is contained in \( p \ast L^1(G)_\gamma \ast p \) it suffices to prove that \( p \ast \Phi' = \Phi' \) and \( \Phi' \ast p = \Phi' \) because \( p \ast p = p \) by (9). But
\[
(p \ast \Phi')(r x) = \int_{N} d y p(y) \Phi'(y^{-1} r x)
\]
\[
= \int_{N} d y p(y) \Phi'(r r^{-1} y^{-1} r x)
\]
\[
= \Phi(r) \int_{N} d y p(y) q_\gamma(r^{-1} y^{-1} r x)
\]
\[
= \Phi(r) \int_{N} d y p(y) q_\gamma(r^{-1}) q_\gamma^{-1}(y^{-1} r x r^{-1}) = \Phi(r) (p \ast q_\gamma^{-1}) (r x r^{-1})
\]
\[
= \Phi(r) (p \ast q_\gamma)(x) = \Phi(r) q_\gamma(x) \quad \text{by (9)}.
\]
Now suppose that \( \pi(C^*(G)) = \pi(L^1(G)) \) contains \( \mathcal{K}(\mathfrak{g}) \) where the closure is taken in the operator norm. In particular, for each \( T \in \mathcal{K}(\mathfrak{g}) \) (the latter space being considered in the most obvious way as a subset of \( \mathcal{K}(\mathfrak{g}) \)) there exists a sequence \( \{\gamma_n\} \) in \( L^1(G) \) such that \( \pi(\gamma_n) \) converges to \( T \). Then \( \pi(p \ast \gamma_n \ast p) \) converges to \( T \). If \( \gamma_n \in \mathcal{L}(S, \mathfrak{g}) \) is determined by \( \mathfrak{g}^n = p \ast \gamma_n \ast p \) then \( \pi(\mathfrak{g}^n) \) converges to \( T \).

Finally, suppose that \( \pi(C^*(S)) \) contains \( \mathcal{K}(\mathfrak{g}) \). Since \( \pi(\mathfrak{g}^n) = \pi(\mathfrak{g}^n) \), for each \( T \in \mathcal{K}(\mathfrak{g}) \) there exists a sequence \( \{\mathfrak{g}^n\} \) in \( \mathcal{L}(S, \mathfrak{g}) \) such that \( \pi(\mathfrak{g}^n) \) converges to \( T \). Then \( (\pi(\mathfrak{g}^n)) \) converges to \( T \), hence \( \pi(C^*(S)) \) contains \( \mathcal{K}(\mathfrak{g}) \). By the irreducibility of \( \pi \) it contains all of \( \mathcal{K}(\mathfrak{g}) \).

Remark. By a theorem of Green [2], the \( C^* \) hull \( C^*(G) \) of \( L^1(G) \) is isomorphic to the \( C^* \) tensor product of \( C^*(S) \) and \( \mathcal{K}(L^2(H)) \). This explains why irreducible or reducible generalized completely continuous representations of \( G \) correspond to those of \( \mathcal{K}(\mathfrak{g}) \) as long as the latter are equal to \( \gamma \) on \( T \).

To obtain the desired example the groups \( H, K, S \) and the homomorphism \( \phi : S \to \text{Aut}(H) \) are now specified. Let \( p \) be any prime number, denote by \( Q_p \) the field of \( p \)-adic numbers and by \( Z_p \) the ring of \( p \)-adic integers with its usual topology. We will mainly view \( Q_p \) and \( Z_p \) as locally compact abelian groups under addition; their multiplicative structure is used to define automorphisms.

Let \( H \) be the restricted direct product of copies of \( Q_p \) over the integers with respect to the compact open subgroup \( Z_p \), i.e.,

\[
H = \{ h : Z \to Q_p \mid h(j) \in Z_p \text{ for almost all } j \in Z \}.
\]

The subgroup \( K \) of \( H \) consisting of all maps \( h : Z \to Z_p \), which is isomorphic to \( Z_p^\infty \), is declared to be open in \( H \), and \( K \) is endowed with the product topology. This way \( H \) is a locally compact abelian group.

The group \( S \) is the semidirect product of \( A = \mathbb{Z} \) and \( B = \mathbb{Z}^{(\infty)} \), the direct sum over \( \mathbb{Z} \) of copies of \( \mathbb{Z} \). The multiplication in \( S = A \times B \) is given by

\[
(a, b)(a', b') = (a + a', b''),
\]

where the \( j \)th component of \( b'' \in \mathbb{Z}^{(\infty)} \) is defined by \( b'' = b_{j+1} + b_j \).

Finally, the homomorphism \( \varphi : S \to \text{Aut}(H) \) is defined by

\[
\varphi(a, b)(h)(j) = p^{b_{j+1}}h(j - a).
\]

Altogether, on \( G = A \times B \times H \times T \) endowed with the product topology, the general formula (1) gives a group multiplication

\[
(a, b, h, x, t)(a', b', h', x', t') = (a'' + b''', b''', h''', x''', t''').
\]

where \( a'' = a + a', b'' = b_{j+1} + b_j, h'' = \varphi(a', b)'^{-1}(h) + h', (x', x) = (x, \varphi(a', b)(x))x'(x) \) for \( x \in H \), and \( b'' = t'(x, \varphi(a', b)(x)) - h'(x) \).

Lemma 1. The locally compact group \( G \) is compactly generated and separable, i.e., it has a countable basis of the topology. The weight function \( w \) and the modular function \( \delta \) of the action of \( S = A \times B \) on \( H \) (compare (8) and (3)) are given by

\[
w(a, b) = p^{-\min\{\mathbb{Z}^{(\infty)}|b_j|\}}, \quad \delta(a, b) = p^{-\min\{\mathbb{Z}^{(\infty)}|b_j|\}} \text{ if } b = (b_j) \in B = \mathbb{Z}^{(\infty)}.
\]

Proof. Let \( b_j \in B \) be defined by \( b_0 = 1 \) and \( b_j = 0 \) for \( j \neq 0 \). It is evident that \( S \) is generated by \((1, 0)\) and \((0, b_0)\). Hence the subgroup \( L \) of \( G \) generated by the compact set \( K \times (K/H)^{\infty} \times T \cup \{(1, 0), (0, b_0)\} \) contains \( K \times (K/H)^{\infty} \times T \). Conjugating \( K \times (K/H)^{\infty} \times T \) by elements in \( B \) produces the whole of \( H \times T \). Hence \( L = G \).

The question of separability reduces at once to \( H \) and \( \tilde{H} \). But \( H \) is a countable extension of the compact metrizable group \( K = Z_p^\infty \), hence is separable. Moreover, the group \( H \) is a field. This can be seen as follows. The quotient \( Q_p/Z_p \) is isomorphic to \( \mathbb{Z}[1/p]/\mathbb{Z} \). The latter group can be identified with a subgroup of \( Q/\mathbb{Z} \) or of \( \mathbb{R}/\mathbb{Z} \) which is isomorphic to \( T \) in the usual manner. This way we find a canonical \( \kappa \in Q_p^\times \) with \( \kappa \in Z_p \). Then define \( H \times T \to B \) by

\[
((q_j), (r_j)) \rightarrow \prod_{j=-\infty}^{\infty} \kappa(q_j, r_j).
\]

It is easy to see that this pairing establishes an isomorphism from \( H \) onto \( \tilde{H} \).

The formulas for \( w \) and \( \delta \) follow at once from the fact that for \( n \in \mathbb{Z} \) the cardinality of \( p^nZ_p \cap Z_p \) is one for \( n \leq 0 \) and \( p^n \) for \( n \geq 0 \).

The Pontryagin dual \( \tilde{B} \) is isomorphic to \( \mathbb{T}^\mathbb{Z} \). Each \( z = (z_j) \in \mathbb{T}^\mathbb{Z} \) defines a character \( \eta_z \in \tilde{B} \) by

\[
\eta_z(b) = \prod_{j=-\infty}^{\infty} z_j^{b_j}.
\]

The character \( \eta_z \) extends to a character \( \eta_z \) of the subgroup \( \{0\} \times B \times \{0\} \times \tilde{H} \times T \) of \( G \) by \( \eta_z(0, b, 0, x, t) = t \eta_z(b) \). This character is induced to obtain a representation, say \( \pi_z \), of \( G \). The representation \( \pi_z \) can be realized in \( L^2(A \times H) \) where \( A \times H \) carries the product measure of the Haar measures on \( A = \mathbb{Z} \) and \( H \). One finds that

\[
\pi_z(a, b, h, x, t)z(a', h') = \delta(a, b)^{1/2}t(x, h_\varphi(a', b, h') - \varphi(a - a', 0)(h)) \times \xi(a', \varphi(0, b)(h') - \varphi(0, b')(h)) \times \delta(a, b)^{-1/2}t(x, h_\varphi(a', b, h') - \varphi(a - a', 0)(h))
\]

where \( \beta \in B \) is given by \( \beta_j = -b_{j+1} + a_j - a \).
With \( \pi_x \) there is associated (see the Proposition) a representation \( \pi^p_x \) of \( S = A \times B \) in \( \mathfrak{g}^p \). The space \( \mathfrak{g}^p = \pi^p_x(p)(L^2(A \times H)) \) is easily identified. More generally, we shall compute the operator \( \pi_x(q_a), s \in S \); for the definition of \( q_a \), see (5) and (15).

For \( \xi \in L^2(A \times H) \),

\[
\{ \pi_x(q_a)\xi \}(a', h') = \int dh \int d\xi \int dt \tilde{q}_s(h, \chi, t) \times \tilde{\xi}(h, h') = \int dh \tilde{q}_s(h, \varphi(a', 0)h')(h') \xi(a', h') - \varphi(-a', 0)(h)
\]

Substituting \( h'' = h' - \varphi(-a', 0)(h) \) yields

\[
\{ \pi_x(q_a)\xi \}(a', h') = \delta(s)^{-1/2} \varphi(a', 0)(h') \int d\xi u(\varphi(a', 0)(h')) \xi(a', h).
\]

In particular,

\[
\{ \pi_x(p)\xi \}(a', h') = u(\varphi(a', 0)(h')) \int d\xi u(\varphi(a', 0)(h')) \xi(a', h).
\]

One verifies easily that

(21) The map \( V: L^2(A) \to \mathfrak{g}^p \) defined by

\[
(V\cdot\xi)(a', h') = u(\varphi(a', 0)(h')) \xi(a')
\]

is unitary.

Transferring via \( V \) the representation \( \pi^p_x \) of \( S \) in \( \mathfrak{g}^p \) into the space \( L^2(A) \) one gets a representation \( \varrho_x \) of \( S \) in \( L^2(A) \) given by

(22) \[
\{ \varrho_x(a, b)\xi \}(a') = \eta_x((b_{j+1} - a_j)a)\xi(a - a) = \sum_{j=-\infty}^{\infty} b_{j+1} - a_j \xi(a' - a).
\]

This formula follows from the definitions of \( V \) and \( \pi^p_x, \pi^p_x(s) = \pi^p_x(s)\pi^p_x(q_a) \), and from the above determined structure of \( \pi^p_x(q_a) \). The easy computation is omitted. Of course, \( \varrho_x \) is nothing but \( \text{ind}_{\mathfrak{g}^p} \eta_x \) realized in \( L^2(A) \).

**Lemma 2.** The representation \( \pi_x \) of \( G, x \in \mathbb{T}^3 \), is irreducible if and only if the sequence \( x \) is not periodic, i.e., there is no positive integer \( m \) such that \( x_{j+m} = x_j \) for all \( j \in \mathbb{Z} \). If this condition is satisfied then \( \pi_x \) is a generalized completely continuous representation if and only if the \( A \)-orbit \( \Omega_x = \{ x_{j+a} \mid a \in A \} \) is locally closed in \( \mathbb{T}^2 \).

**Remark.** In order to establish the relation to the results in [3] we observe that the condition \( \mathfrak{a} \Omega_x \) is locally closed in \( \mathbb{T}^{2n} \) is equivalent to \( \mathfrak{a} \Omega_x \) is a discrete subset of \( \mathbb{T}^{2n} \) for the following reasons. Clearly, any discrete subspace is locally closed. If the \( A \)-orbit \( \Omega_x \) is locally closed then under the map \( a \mapsto x_a = (x_{j+1})_j \) the subspace \( \Omega_x \) is homeomorphic to \( A \) as the stabilizer group is trivial. Hence \( \Omega_x \) is discrete.

**Proof of Lemma 2.** By the Proposition the questions of whether \( \pi_x \) is irreducible or whether \( \pi_x \) is a generalized completely continuous representation, can be reduced to the corresponding questions for the representation \( \varrho_x \) of \( S \). In the latter case the answers are known (see [3]). We shall repeat here the essential arguments. This gives the opportunity to introduce some notations which will be needed later anyway.

If \( z \) is periodic, say \( z_{j+m} = z_j \) for all \( j \), then the operator \( M : \ell^2(A) \to \ell^2(A), (M\cdot\xi)(a') = \zeta(a' + m) \) commutes with \( \varrho_x(S) \), hence \( \varrho_x \) is not irreducible.

Now suppose that \( z \) is not periodic, and let \( U : \ell^2(A) \to \ell^2(A) \) be any intertwiner operator for \( \varrho_x \). Let \( \mathfrak{e}_0 \) be the “Dirac delta” in \( \ell^2(A) \), and let \( \varepsilon : = U\mathfrak{e}_0 \in \ell^2(A) \). From \( U\varphi(0, b) = \varphi(0, b)U \) it follows that \( \varrho(0, b) = \eta_b \varepsilon \) for all \( b \in B \). As \( z \) is not periodic the latter identity implies that \( \varepsilon \) is a scalar multiple of \( \mathfrak{e}_0 \), say \( \varepsilon = \lambda \mathfrak{e}_0 \). Since \( U \) commutes with the translates \( \varrho(a, 0) \), and since the translates of \( \mathfrak{e}_0 \) span \( \ell^2(A) \), one concludes that \( U = \lambda \mathfrak{1}_d \).

The \( L^1 \)-group algebra of the semidirect product \( S = A \ltimes B \) may be considered in the usual way as the \( L^1 \)-covariance algebra \( \ell^1(A, \ell^2(B)) \) (see [4]). Via Fourier transform the \( C^* \)-hull of \( \ell^2(B) \) is nothing but \( C^*(B) \), and \( C^*(S) \) is the \( C^* \)-covariance algebra \( C^*(A, C(B)) \). The \( L^1 \)-covariance algebra \( \ell^1(A, C(B)) \) lies halfway between \( \ell^1(S) \) and \( C^*(S) \); there are (normal-decreasing) embeddings

\[
\ell^1(A, \ell^2(B)) \to \ell^1(A, C(B)) \to C^*(A, C(B)).
\]

The representation \( \varrho_x \) yields representations of \( \ell^1(A, C(B)) \) and of \( C^*(A, C(B)) \), also denoted by \( \varrho_x \). The image \( \varrho_x(C^*(S)) \) contains nonzero compact operators if and only if there exist continuous functions \( \varphi \) on \( B = \mathbb{T}^2 \) such that \( \varphi \) is not identically zero on \( \mathfrak{a} \), but \( \varphi \) is zero on \( \mathfrak{a}_2 \setminus \Omega_x \), where \( \mathfrak{a}_2 \) denotes the closure of \( \mathfrak{a} \). Such functions exist precisely when \( \Omega_x \) is locally closed. In this case for \( g \in \ell^1(A, C(B)) \) the operator \( \varrho_x(g) \) is compact if and only if for all \( a \in A \) the function \( g(a) \in C(B) \) vanishes on \( \mathfrak{a}_2 \setminus \Omega_x \).

The proof of Lemma 2 is finished. It has also shown what we have to do further. We have to specify a locally closed \( A \)-orbit \( \Omega_x \) such that the above
condition on \( g \) is not satisfied for functions in the image of \( \ell^1(S, w) \) under the map \( \ell^1(A, \ell^1(B)) \rightarrow \ell^1(A, C(B)) \), unless \( \rho_2(g) = 0 \) (compare Proposition). To this end we need a little lemma on a particular decomposition of the integers.

**Lemma 3.** Let \( D \) be a countable set. There exists a decomposition \( Z = \bigcup_{d \in D} C_d \) of the set of integers with the following property: If \( n \) is any positive integer and if \( d_{-n}, \ldots, d_{-1}, d_0, \ldots, d_n \) are any elements in \( D \) then the intersection

\[
\bigcap_{j=-n}^n (C_{d_j} - j)
\]

is not empty (and hence infinite). In particular, all the sets \( C_d \) are infinite.

**Proof.** Let \( (D_n)_{n \in \mathbb{N}} \) be an increasing sequence of finite subsets of \( D \) with \( \bigcup_{n \in \mathbb{N}} D_n = D \). First we claim that for each \( n \in \mathbb{N} \) there exists a collection \( (C_d^{(n)})_{d \in D_n} \) of disjoint finite subsets of \( Z \) with the following properties:

\[
\begin{align*}
  & (i) \quad \left\{ \bigcup_{d \in D_n} C_d^{(n)} \right\} \cap \left\{ \bigcup_{d \in D_n} C_d^{(n)} \right\} = \emptyset, \\
  & (ii) \quad \text{if } d_{-n}, \ldots, d_0, \ldots, d_n \text{ are any elements in } D_n \text{ then } \bigcap_{j=-n}^n (C_{d_j}^{(n)} - j)
\end{align*}
\]

is not empty. It is easy to see that such collections exist because in (ii) there are only finitely many conditions to be fulfilled; and clearly for a given \( n \) the sets \( C_d^{(n)} \), \( d \in D_n \), can be chosen in the complement of the previously constructed finitely many finite sets.

Then for each \( d \in D \) choose an \( m \in \mathbb{N} \) with \( d \in D_m \) and put \( C_d' = \bigcup_{n \geq m} C_d^{(n)} \). The sets \( C_d' \), \( d \in D \), are pairwise disjoint. Finally, choose any family \( C_d, d \in D \), with \( C_d' \subseteq C_d \) for each \( d \) and \( Z = \bigcup_{d \in D} C_d \) (for instance \( C_d = C_d' \) for all \( d \in D \) except for a distinguished point \( d_0 \)). Such a family has the claimed property.

To see that for any given \( n \) and any given sequence \( d_{-n}, d_0, \ldots, d_n \) in \( D \) the intersection \( \bigcap_{j=-n}^n (C_{d_j} - j) \) is automatically an infinite set, let \( t \) be any positive integer, let \( m = n + t(2n+1) \), and define the sequence \( d_{-m}, \ldots, d_m \) in \( D \) by \( d'_k = d_k \) if \( k \equiv j \mod (2n+1) \) and \( |j| \leq n \). As \( \bigcap_{j=-n}^n (C_{d_j} - j) \neq \emptyset \), we may take a number \( y \) in this intersection. It is easily verified that then the numbers \( y + a(2n+1), e \in Z, \ |e| \leq t \), are contained in \( \bigcap_{j=-n}^n (C_{d_j} - j) \), hence the latter intersection contains at least \( 2t + 1 \) elements.

In particular, let \( D \) be a countable subset of \( T \) such that \( 1 \notin D \) and that the closure \( \overline{D} \) equals \( D \cup \{1\} \). For each \( d \in D \) choose \( r_d > 0 \) such that

\[
\{ z \in \mathbb{C} \mid |z - d| \leq 2r_d \} \cap \overline{D} = \{d\}.
\]

Let \( Z = \bigcup_{d \in D} C_d \) be a decomposition according to Lemma 3. Choose \( z = (z_j) \in T^Z \) with the following properties:

(24) The map \( Z \ni j \mapsto z_j \in T \) is injective.

(25) If \( j \in C_d \) then \( 0 < |z_j - d| < r_d \).

(26) For each \( d \in D \) and each \( r > 0 \) the set \( \{ j \in C_d \mid |z_j - d| \geq r \} \) is finite.

These conditions imply that \( d \) is the only cluster point of \( \{ z_j \mid j \in C_d \} \), that \( \overline{D} \) and \( \{ z_j \mid j \in Z \} \) are disjoint, and that \( \overline{D} \cup \{ z_j \mid j \in Z \} \) is a closed subset of \( T \).

**Lemma 4.** Let \( z \in T^Z \) be as above and let \( \Omega = \Omega_z \) be its orbit under the “shift group”, i.e., \( \Omega = \{ (z_{j+k}) \mid a \in Z \} \). Then \( \Omega \) is locally closed in \( T^Z \) and the closure \( \overline{\Omega} \) equals \( \Omega \cup \overline{D} \), which is a disjoint union since \( \overline{D} \) and \( \{ z_j \mid j \in Z \} \) are disjoint subsets of \( T \).

**Proof.** To prove that \( \overline{\Omega} \) is contained in \( \overline{D} \) it is clearly sufficient to verify that any \( z = (z_j) \in \overline{D} \) is contained in \( \overline{D} \). To this end, let any \( n \in \mathbb{N} \) and \( \varepsilon > 0 \) be given. We have to show that there exists an \( a = A = Z \) such that

\[
|x_j + a - z_j| < \varepsilon \quad \text{for } |j| \leq n.
\]

By Lemma 3 the set \( A' = \{ a \in A \mid j + a \in C_a \text{ for } |j| \leq n \} \) is finite. By (26), for almost all \( a \in A' \) the inequalities (27) are true.

To prove conversely that \( \overline{\Omega} \) is contained in \( \overline{D} \cup \overline{D} \), let \( x \) be a given point in \( \overline{\Omega} \). Since \( \overline{D} \cup \{ z_j \mid j \in Z \} \) is closed in \( T \), each \( x_k \) is contained in this set. If each \( x_k \) is contained even in \( \overline{D} \) we are done. So, assume that there is a \( k_0 \in Z \) with \( x_{k_0} = x_{j_0} \) for some \( j_0 \). We have to show that then \( x \in \Omega \). By applying a suitable element in the shift group we may suppose that \( x_{j_0} = x_{j_0} \) for some \( j_0 \), and our claim reduces to \( z = x \). Given \( j_0 \) from the properties (24)–(26) of \( z \) it follows that there exists an \( \varepsilon_0 > 0 \) such that

\[
|x_j - x_{j_0}| < \varepsilon_0 \quad \text{implies} \quad j = j_0.
\]

Then take any \( j \in Z \) and any \( \varepsilon, 0 < \varepsilon < \varepsilon_0 \). Since \( x \in \overline{\Omega} \) there is an \( a = a(j, \varepsilon) \in A \) such that

\[
|x_j + a - z_j| < \varepsilon \quad \text{and} \quad |x_{j_0 + a} - x_{j_0}| < \varepsilon.
\]

As \( x_{j_0} = x_{j_0} \) from (28) we deduce that \( a = 0 \), hence \( |z_j - x_j| < \varepsilon \). Since \( x \) and \( j \) were arbitrary, we conclude that \( z = x \).

The known structure of \( \overline{\Omega} \) yields \( \overline{\Omega} \setminus \Omega = \overline{D}^Z \), which is a closed subset of \( T^Z \). Therefore, \( \Omega \) is locally closed.

**Theorem.** Let \( G = A \times B \times H \times \hat{H} \times T \) be the group as constructed above (see in particular (17) and (18)), and let \( z \in T^Z \) be a point as above.
Then the continuous unitary representation \( \pi_a \) of \( G \) (see (20), (17) and Lemma 1) is irreducible and \( \pi_a(C^*(G)) \) contains the algebra of compact operators, while \( \pi_a(L^1(G)) \) contains no compact operator except for zero.

**Proof.** By Lemma 2, since clearly \( z \) is not periodic and since the \( A \)-orbit \( \Omega_z \) of \( z \) is locally closed by Lemma 4, \( \pi_a \) is an irreducible generalized completely continuous representation. To prove that \( \pi_a(L^1(G)) \) contains no nonzero compact operator, by the Proposition it is sufficient to show the corresponding property for \( \varphi_a(\ell(S,w)) \). By what we have seen in the proof of Lemma 2, the operator \( \varphi_a(f), f \in \ell(S,w) \subset \ell(S) \), is compact if and only if for all \( a \in A \) the function \( g_a \in C(T^2) \) defined by

\[
g_a(x) = \sum_{b \in B} f(ab) \eta_a(b)^{-1} = \sum_{b \in B} f(ab) \prod_{j=-\infty}^{\infty} x_j^{-b_j}
\]

vanishes on \( \Omega_z \). Hence we have to show that if \( f \) satisfies this condition, then \( \varphi_a(f) = 0 \). We claim that even better: \( f \) is then necessarily identically zero.

From the structure of \( w \), \( w(ab) = p \sum_{j=-\infty}^{\infty} |k_j| \) (compare Lemma 1); it follows easily that the series \( \sum_{b \in B} f(ab) \prod_{j=-\infty}^{\infty} x_j^{-b_j} \) converges not only for \( x \in T^2 \), but also for \( x \in Y \), where \( Y \) denotes the annulus \( \{ y \in C \mid p^{-1/2} \leq |y| \leq p^{1/2} \} \). Define \( g_a(x), x \in Y \), to be the sum of this series. For \( n \in \mathbb{N} \) let \( i^{(n)}(y, y_0, \ldots, y_n) \) be the canonical embedding from \( Y^{2n+1} = \{(y, y_0, \ldots, y_n) \mid y_k \in Y \text{ for } |k| \leq n \} \) into \( Y^2 \), i.e.,

\[
i^{(n)}(y, y_0, \ldots, y_n)_j = \begin{cases} y_j & \text{if } |j| \leq n, \\ 1 & \text{if } |j| > n. \end{cases}
\]

The function \( g_a \circ i^{(n)} \) is continuous on \( Y^{2n+1} \) and analytic in the interior \( Y^{2n+1} \). Since \( g_a \) vanishes on \( \Omega_z \setminus \Omega \), \( D \) (see Lemma 4) we conclude that \( g_a \circ i^{(n)} \) vanishes on the subset \( D^{2n+1} \) of \( Y^{2n+1} \). As \( g_a \circ i^{(n)} \) is analytic this yields that \( g_a \circ i^{(n)} \) is identically zero. In particular, \( g_a \) vanishes on \( i^{(n)}(T^{2n+1}) \). Since \( \bigcup_{n \in \mathbb{N}} i^{(n)}(T^{2n+1}) \) is dense in \( T^2 \), it follows that \( g_a \) is identically zero. Hence for each \( a \in A \) the function \( b \mapsto f(ab) \) is identically zero and, therefore, \( f \) is identically zero.

**References**

