On an estimate for the norm of a function of a quasihermitian operator

by

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Abstract. Let $A$ be a closed linear operator acting in a separable Hilbert space. Denote by $\text{co}(A)$ the closed convex hull of the spectrum of $A$. An estimate for the norm of $f(A)$ is obtained under the following conditions: $f$ is a holomorphic function in a neighbourhood of $\text{co}(A)$, and for some integer $p$ the operator $A^p - (A^*)^p$ is Hilbert–Schmidt. The estimate improves one by I. Gelfand and G. Shilov.

1. Introduction. Notations. Let $H$ be a separable Hilbert space, and let $A$ be a closed linear operator acting on $H$ with domain $D(A)$. Then $A$ is called quasihermitian if $D(A) \subseteq D(A^*)$ and the imaginary component $A_I = (A - A^*)/2i$ is completely continuous. Denote by $\text{co}(A)$ the closed convex hull of the spectrum $\sigma(A)$ of $A$. In this paper we obtain an estimate for the norm of $f(A)$ if $f$ is a holomorphic function in a neighbourhood of $\text{co}(A)$, and $A$ is a quasihermitian operator with

\begin{equation}
A_I \in C_2
\end{equation}

where $C_2$ is the Hilbert–Schmidt ideal [9]. Moreover, this estimate is generalized to the case

\begin{equation}
A^p - (A^*)^p \in C_2
\end{equation}

for some integer $p$.

Singular integral and integral-differential operators are examples of operators which satisfy (1.1) and (1.2).


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Below we generalize and improve the results from [6, 7] and also supplement Carleman’s estimate for the resolvent $R_\mu(A)$ of $A \in C_2$ [4, Ch. XI]. Let $I$ denote the identity operator in $H$, and let

$$v(A) = \left[ |A|_2^2 - \sum_{k=1}^{\infty} |\text{Im} \mu_k(A)|^2 \right]^{1/2} \sqrt{2}$$

where $|B|_2$ is the Hilbert–Schmidt norm of a Hilbert–Schmidt operator $B$, and $\mu_1(A), \mu_2(A), \ldots$ are all nonreal eigenvalues of $A$ counted with their multiplicity. If $A$ is a normal operator, then $v(A) = 0$ (see [8]).

We define $f(A)$ by

$$f(A) = -\frac{1}{2\pi i} \int_I f(\mu) R_\mu(A) d\mu + f(\infty) I$$

where $I$ is a smooth contour encircling $\sigma(A)$.

2. Main result

THEOREM 1. Let $A$ be a quasihermitian operator satisfying (1.1) and let $f$ be a holomorphic function in a neighbourhood of $\sigma(A)$. Then

$$||f(A)|| \leq \sum_{k=0}^{\infty} \sup_{\mu \in \sigma(A)} |f^{(k)}(\mu)| \frac{|v(A)|^k}{(k!)^{3/2}}.$$ (2.1)

First we prove a few lemmata.

LEMMA 1. Let the imaginary part $A_j$ of a quasihermitian operator $A$ belong to the Miatke ideal $C_\varphi$ [10, Ch. 4.3], i.e.

$$\sum_{k=1}^{\infty} (2k-1)^{-1} \mu_k(A_j) < \infty \quad (\mu_k(A_j) \in \sigma(A_j)).$$

Then there is an orthogonal resolution of the identity $E(t)$ ($-\infty < t < \infty$), a normal operator $N$ and a Volterra (completely continuous quasinilpotent) operator $V$ such that for all $t \in (-\infty, \infty)$

$$NE(t) = E(t)N,$$ (2.2)

$$E(t)V E(t) = VE(t),$$ (2.3)

$$A = N + V.$$ (2.4)

Proof. As is shown in [2],

$$A = \int_{-\infty}^{\infty} h(t)dP(t) + i \int_{-\infty}^{\infty} P(t)A_j dP(t).$$ (2.5)

Here $P(t)$ is an orthogonal resolution of the identity and $h$ is a nondecreasing scalar-valued function. The second integral in (2.5) is the limit in the $C_\varphi$-norm of the sums

$$\frac{1}{2} \sum_{k=1}^{n} (P(t_k) + P(t_{k-1})) A_j \Delta P_k = S_n + U_n$$

$$t_k = \frac{2k(n)}{n}, \quad \Delta P_k = P(t_k) - P(t_{k-1}), \quad -\infty < t_0 < t_1 < \ldots < t_n < \infty$$

where

$$U_n = \sum_{k=1}^{n} P(t_{k-1})A_j \Delta P_k, \quad S_n = \sum_{k=1}^{n} \Delta P_k A_j \Delta P_k.$$ (2.6)

The sequence $\{S_n\}$ is norm convergent by Lemma 1.5.1 of [10]. We denote its limit by $S$. By Theorem 2.5.2 of [9, p. 77], each $S_n$ belongs to $C_\varphi$. According to Theorem 3.5.1 of [9, p. 113], so does $S$. It is clear that the $P(t)$ ($-\infty < t < \infty$) are projectors of $H$ onto invariant subspaces of the selfadjoint operator $S$. We arrive at (2.2) when $E(t) = P(t)$ and $N = \int_{-\infty}^{\infty} h(t)A_j dP(t) + iS$. Further, $U_n$ is a nilpotent operator: $(U_n)^n = 0$. The sequence $\{U_n\}$ converges in the $C_\varphi$-norm because so do the second integral in (2.5) and $\{S_n\}$. We denote the limit by $U$. Then $U$ is a Volterra operator by Lemma 2.17 of [3]. From (2.5) we obtain (2.4).

By Neumann’s theorem [1, p. 314] there exists a bounded scalar-valued function $\varphi$ such that $S = \int_{-\infty}^{\infty} \varphi(t) dE(t)$ since $E(t)S = SE(t)$. Hence

$$N = \int_{-\infty}^{\infty} \varphi(t) dE(t)$$ (2.7)

where $\varphi = h + i\psi$.

DEFINITION 1. Suppose there are an orthogonal resolution of the identity $E(t)$, a scalar-valued function $\varphi$ and a Volterra operator $V$ such that (2.3), (2.4) and (2.7) hold. Then we call $E(t)$, $N$, $V$ and (2.4) a spectral function, a diagonal part, a nilpotent part and a triangular representation of $A$, respectively.

Our definition of spectral function is analogous to the corresponding definitions in [2, 3].

LEMMA 2. Let a bounded operator $A$ have a triangular representation and a spectral function $P(t)$ which consists of $n < \infty$ projectors $0 = P_0 < P_1 < \ldots < P_n = I$. Suppose its nilpotent part $V$ is in $C_2$. Then

$$||f(A)|| \leq \sum_{k=0}^{n-1} \sup_{\mu \in \sigma(A)} |f^{(k)}(\mu)| \frac{|V^{(k)}|_{L^2}}{(k!)^{3/2}}$$ (2.8)

for every function $f$ holomorphic in a neighbourhood of $\sigma(A)$. 


Proof. (2.7) has the form $N = \sum_{k=1}^{n} \varphi_k \Delta P_k$ in this case. Here $\varphi_k$ ($k = 1, \ldots, n$) are eigenvalues of $A$. Let $\{e_j(m)\} \ (m = 1, 2, \ldots)$ be an orthonormal basis in $\Delta P_j H$. We set

$$a_{ij}(m) = (Ae_i(m), e_j(m)), \quad N_m = \sum_{j=1}^{n} a_{ij}(m)e_i(m),$$

$$V_m = \sum_{1 \leq i < j \leq n} a_{ij}(m)(e_i(m), e_j(m)), \quad A_m = N_m + V_m.$$ 

Clearly, $a_{ij}(m) = 0$ when $i > j$. The operators $A, N, V$ and $f(A)$ are the orthogonal sums of $A_m, N_m, V_m$ and $f(A)$ ($m = 1, 2, \ldots$), respectively. Therefore

$$(2.9) \quad \sigma(A_m) \subseteq \sigma(A), \quad |V_m|_2 \leq |V|_2, \quad \max_m \|f(A_m)\| = \|f(A)\|.$$ 

Let $B$ be an $(n \times n)$-matrix. We apply the estimate [7]

$$\|f(B)\| \leq \sum_{k=0}^{n-1} \sup_{\mu \in \sigma(B)} |f^{(k)}(\mu)| |V_\mu|^k \|V_\mu\|^{3/2}$$

where $V_\mu$ is the nilpotent part of $B$. Hence

$$\|f(A_m)\| \leq \sum_{k=0}^{n-1} \sup_{\mu \in \sigma(A_m)} |f^{(k)}(\mu)| |V_\mu|^k \|V_\mu\|^{3/2}$$

since $A_m$ is a finite-dimensional operator and $V_m$ is its nilpotent part. From this and from (2.9) we obtain (2.8). \qed

**Corollary 1.** Under the conditions of Lemma 2,

$$\|f(A)\| \leq \sum_{k=0}^{n-1} \sup_{\mu \in \sigma(A)} |f^{(k)}(\mu)| |V_\mu|^k \|V_\mu\|^{3/2}.$$ 

This follows from Lemma 2 and the equality

$$(2.10) \quad \nu(A) = |V|_2,$$

which is proved in [8, p. 164].

**Lemma 3.** Let $A$ admit a triangular representation, and let $N$ be its diagonal part. Then $\sigma(A) = \sigma(N)$.

**Proof.** By (2.4),

$$R_\mu(A) = R_\mu(N)(I + VR_\mu(N))^{-1}.$$ 

Now, $VR_\mu(A)$ ($\mu \notin \sigma(A)$) is a Volterra operator by Corollary 2 of Theorem 17.1 of [3, p. 121]. Hence

$$(I + VR_\mu(N))^{-1} = \sum_{k=0}^{\infty} (VR_\mu(N))^k (-1)^k,$$

i.e.

$$R_\mu(A) = R_\mu(N) \sum_{k=0}^{\infty} (VR_\mu(N))^k (-1)^k,$$

which clearly implies our assertion. \qed

**Proof of Theorem 1.** (2.3), (2.4) and (2.7) hold by Lemma 1. Define

$$V_n = \sum_{k=1}^{n} P(t_k - 1) V \Delta P_k, \quad N_n = \sum_{k=1}^{n} \varphi(t_k) \Delta P_k, \quad B_n = N_n + V_n.$$ 

First, suppose that $A$ is bounded. Then $\{B_n\}$ strongly converges to $A$. By (1.3), $\{f(B_n)\}$ strongly converges to $f(A)$. The inequality

$$(2.11) \quad \|f(A)\| \leq \sup_n \|f(B_n)\|$$

follows from the Banach–Steinhaus theorem. Since the spectral function of $B_n$ consists of $n < \infty$ projectors, Lemma 2 yields

$$\|f(B_n)\| \leq \sum_{k=0}^{n-1} \sup_{\mu \in \sigma(B_n)} |f^{(k)}(\mu)| |V_{\mu}|^k \|V_{\mu}\|^{3/2}.$$ 

By Lemma 3, $\sigma(B_n) = \sigma(N_n)$. Clearly, $\sigma(N_n) \subseteq \sigma(N)$. Hence, $\sigma(B_n) \subseteq \sigma(A)$. By Theorem 3.6.3 of [9, p. 119], $|V_n|_2 \to |V|_2$ as $n \to \infty$. (2.11) holds by (2.10) and (2.11).

Now, let $A$ be an unbounded operator. Let $Q_n = P(n) - P(-n)$. Then $AQ_n$ is bounded for each $n < \infty$. We have $(A - \mu I)^{-1} Q_n (AQ_n - \mu Q_n) = Q_n$. By (1.3), $\|f(AQ_n)\| \to \|f(A)\|$. Moreover, $AQ_n$ is a restriction of $A$ onto its invariant subspace. Hence $\sigma(AQ_n) \subseteq \sigma(A)$. Now, we obtain by (2.1) the estimate

$$(2.12) \quad \|f(AQ_n)\| \leq \sum_{k=0}^{n-1} \sup_{\mu \in \sigma(AQ_n)} |f^{(k)}(\mu)| |V_{\mu}|^k \|V_{\mu}\|^{3/2}.$$ 

Since $|VQ_n|_2 \to |V|_2$ and $v(AQ_n) = |VQ_n|_2$ by (2.10), we have $v(AQ_n) \to v(A)$ as $n \to \infty$. From this and from (2.12) we get (2.1). \qed

Theorem 1 is sharp: (2.1) turns into the equality $\|f(A)\| = \sup_{\nu(A)} |f(\mu)|$ if $A$ is a normal operator and $\sup_{\nu(A)} |f(\mu)| = \sup_{\nu(A)} |f(\mu)|$. 

Corollary 2. Let $A$ satisfy (1.2) and let $g(\lambda) = f(\lambda^{1/p})$ be an analytic function on $\text{co} (A^p)$. Then
\[
\|f(A)\| \leq \sum_{k=0}^{\infty} \sup_{k \in \text{co} (A^p)} |g^{(k)}(\mu)| \frac{v(A^p)^k}{(k!)^{3/2}}.
\]

Corollary 3. Let $A$ satisfy (1.1). Then
\[
\|\exp(At)\| \leq \exp[\alpha(A)t] \sum_{k=0}^{\infty} \frac{v(A)^k}{(k!)^{3/2}} t^k \quad (t \geq 0)
\]
where $\alpha(A) = \sup \text{Re } \sigma(A)$.

Corollary 4. Let $A$ satisfy (1.1) and let $\sigma(A) = [a, b]$ ($-\infty < a < b < \infty$). Then
\[
\|R_{\mu}(A)\| \leq \sum_{k=0}^{\infty} \frac{v(A)^k}{d(\mu, A)^{k+1}}
\]
where $d(\mu, A)$ is the distance between $\sigma(A)$ and $\mu$ on the complex plane.

This supplements Carleman's estimate [4, Ch. XI] and also generalizes the author's estimate [8] in the case $\text{Im } \sigma(A) = 0$.

3. Perturbation of the spectrum

Lemma 4. Let $A$, $B$ be linear operators acting in a Banach space and suppose
\begin{align*}
(3.1) & \quad q = \|A - B\| < \infty, \\
(3.2) & \quad \|R_{\mu}(A)\| \leq b(d(\mu, A)^{-1})
\end{align*}
where $b(y)$ is an increasing function of $y > 0$. Then $\sup \{\text{dist}(\lambda, \sigma(A)) : \lambda \in \sigma(B)\} \leq 1/\psi(q^{-1})$, where $\psi$ is the inverse function to $b$: $\psi(b(y)) = y$.

Proof. We have $R_{\mu}(A) - R_{\mu}(B) = R_{\mu}(B - A)R_{\mu}(A).$ Let $q\|R_{\mu}(A)\| < 1$. Then $\|R_{\mu}(B)\| \leq \|R_{\mu}(A)\|(1 - q\|R_{\mu}(A)\|)^{-1}$, hence $\mu \not\in \sigma(B)$. Therefore $1 \leq q\|R_{\mu}(A)\| \leq b(d(\mu, A)^{-1})$ if $\mu \in \sigma(B)$. This implies $d(\mu, A) \leq 1/\psi(q^{-1})$ for each $\mu \in \sigma(B)$.

Lemma 4 and Corollary 4 give:

Corollary 5. Let $A$ satisfy (1.1) and (3.1), and suppose $\sigma(A) = [a, b]$, $-\infty \leq a < b \leq \infty$. Then
\[
\sup \{\text{dist}(\lambda, \sigma(A)) : \lambda \in \sigma(B)\} \leq 1/\psi_A(q^{-1})
\]
where $\psi_A$ is the inverse function to $b_A(y) = \sum_{k=0}^{\infty} \frac{v(A)^k}{\sqrt{k!}} y^{k+1}$.

Let $A = A^*$. Then $v(A) = 0$, $b_A(y) = y$. In this case under the condition (3.1), $\text{dist}(\sigma(B), \sigma(A)) \leq q$, i.e. (3.3) generalizes the well-known result for selfadjoint operators with $\sigma(A) = [a, b]$ [12, Ch. V].

Remark. Schwarz's inequality gives
\[
b_A(y)^2 \leq \sum_{j=0}^{\infty} \frac{(1/2)^j y^2}{k!} \sum_{k=0}^{\infty} \frac{(yu(A))^2 k}{k!} \quad \Rightarrow \quad 2y^2 \exp[2v(A)^2/y^2].
\]
By Corollary 4 under (1.1) and $\sigma(A) = [a, b]$ we have
\[
\|R_{\mu}(A)\| \leq \sqrt{2} d(\mu, A)^{-1} \exp[v(A)^2/d(\mu, A)^2].
\]

4. Nonlinear perturbation of a linear semigroup. Consider the equation
\[
\frac{du}{dt} = Au + F(u, t) \quad (0 \leq t \leq \infty)
\]
where $A$ is a linear operator in $H$ and $F$ maps $H \times [0, \infty)$ into $H$.

A solution of the Cauchy problem for (4.1) is a continuously differentiable function $u : [0, \infty) \to D(A)$ which satisfies (4.1) and an initial condition $u(0) = u_0 \in D(A)$. Assume
\[
\|F(x, t)\| \leq q\|x\| \quad \text{for each } x \in D(A) \text{ and } t \geq 0.
\]

Theorem 2. Let $x(t)$ be a solution of the Cauchy problem for (4.1) under the conditions (1.1), (4.2), $\alpha(A) < 0$ and
\[j \equiv \sum_{k=0}^{\infty} \frac{v(A)^k}{|\alpha(A)|^{k+1}} < 1/q.
\]
Then
\[
\|x(t)\| \leq a\|x(0)\|(1 - qj)^{-1} \quad (t \geq 0, \quad a = \text{const}).
\]

Proof. We have by (4.1)
\[
x(t) = \exp[At]x(0) + \int_0^t \exp[A(t - s)]F(x(s), s) \, ds
\]
(see [11, p. 53]). This implies
\[
\|x(t)\| \leq \|\exp[At]x(0)\| + \int_0^t \|\exp[A(t - s)]\|q\|x(s)\| \, ds.
\]
By Corollary 3,
\[
\|\exp[At]\| \leq a \quad (t \geq 0),
\]

\[ \int_0^t \| \exp(A(t-s)) \| \, ds \leq \int_0^\infty \| \exp(As) \| \, ds \]
\[ \leq \int_0^\infty \exp[\alpha(A)t] \sum_{k=0}^\infty \frac{t^k \nu(A)^k}{(k!)^{3/2}} \, dt = j \quad (t \geq 0). \]

Hence, \( \max_{t \geq 0} \| \pi(t) \| \leq \alpha \| \pi(0) \| + \max_{t \geq 0} \| \pi(t) \| j \) and we arrive at (4.3). \( \square \)

References


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On molecules and fractional integrals on spaces of homogeneous type with finite measure

by

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Abstract. In this paper we prove the continuity of fractional integrals acting on non-homogeneous function spaces defined on spaces of homogeneous type with finite measure. A definition of the molecules which are used in the $L^p$ theory is given. Results are proved for $L^p$, $H^p$, BMO, and Lipschitz spaces.

1. Definitions and statement results. We shall follow the definitions and notation of [GV], and we assume that the reader is familiar with that paper. In the present paper $(X, \delta, \mu)$ is a normal space of homogeneous type of finite measure and of order $\gamma$, $0 < \gamma \leq 1$. In this case the diameter of the space is finite and will be denoted by $D$. We may and will assume that $\mu(X) = 1$.

For the sake of completeness we will repeat the definitions of normality and order. $(X, \delta, \mu)$ is a normal space if there are positive constants $A_1$ and $A_2$ such that for all $x$ in $X$

(1.1) $A_1r \leq \mu(B_r(x))$ if $0 < r \leq R_x$, 
(1.2) $\mu(B_r(x)) \leq A_2r^\gamma$ if $r > R_x$,

where $B_r(x)$ denotes the ball of radius $r$ and center $x$, and where $R_x = \inf\{r > 0 : B_r(x) = X\}$, and $r_x = \sup\{r > 0 : B_r(x) = \{x\}\}$ if $\mu(\{x\}) \neq 0$, and $r_x = 0$ if $\mu(\{x\}) = 0$. Note that sup$\{R_x : x \in X\} = D < \infty$, that (1.1) holds for $0 < r < 2D$ with constant $A_1/2$ instead of $A_1$, and that (1.2) holds for $r = r_x$ if $r_x \neq 0$. The space $(X, \delta, \mu)$ is said to be of order $\gamma$, $0 < \gamma \leq 1$, if there exists a positive constant $M$ such that for every $x, y, z$ in $X$,

$$|\delta(x, z) - \delta(y, z)| \leq M \delta(x, y)^\gamma (\max\{\delta(x, z), \delta(y, z)\})^{1-\gamma}.$$ 

We will consider on $(X, \delta, \mu)$ the following function spaces and norms. If $0 < p \leq \infty$ then $L^p$ and $\|f\|_p$ have their usual meaning. For a measurable

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