Moreover, let

\[ \lambda \varphi = \lambda \varphi, \]

where \( \varphi \neq 0 \). Let us suppose that the element \( \varphi \) is linearly dependent on the elements \( \varphi_1, \ldots, \varphi_n \), i.e., \( \varphi = \alpha_1 \varphi_1 + \ldots + \alpha_n \varphi_n \). Applying the operator \( \lambda I - \varphi \) to both sides of this equality, we obtain by (6.1),

\[ \alpha_1(\lambda - \lambda_1)\varphi_1 + \ldots + \alpha_n(\lambda - \lambda_n)\varphi_n = 0. \]

Hence there exists an element \( \varphi, \mu < \nu \), linearly dependent on the elements \( \varphi_1, \ldots, \varphi_n \). Repeating these arguments we finally obtain \( \alpha = 0 \), contradicting the assumption \( \varphi \neq 0 \). Hence the elements \( \varphi_1, \ldots, \varphi_n \) are linearly independent. We denote by \( X \), the linear space spanned by these elements. By Theorem 1.11, the spaces \( X \) are closed and Euclidean. Since

\[ X \neq X_{++}, \quad X \subseteq X_1 \subseteq \ldots \subseteq X_{n+1} \subseteq \ldots, \]

we conclude from Theorem 1.10 that there exists a \( y \) such that

\[ y \in X, \quad y \notin X_{++} \cap U \]

(6.2)

Here \( U \) is a neighborhood of zero transformed by the operator \( T \) in a precompact set. Since \( y \in X \), formula (6.1) and the definition of the space \( X \), imply

\[ \lambda y \notin (\lambda y - T(\lambda y + U)), \]

i.e., \( \lambda y \notin (\lambda y + T(\lambda y + U)) \). By Theorem 1.9, there exists a neighborhood of zero \( V \) such that \( V \cap U \). Hence \( \lambda y \notin (\lambda y + V) \). On the other hand, formula (6.2) implies \( \lambda y \notin T \). Applying Theorem 1.3 we conclude that the sequence \( \{\lambda\} \) is finite. 

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**PART C**

**LINEAR OPERATORS IN BANACH SPACES**

In Chapter I, Part A, we have shown a deep connection between the theory of linear equations in linear spaces and the properties of quasi-Fredholm ideals and Fredholm ideals in paraalgebras of operators. In § 5, B IV, we proved that the ideal \( T(X \equiv Y) \) of compact operators is a Fredholm ideal in the paraalgebra \( B(X \equiv Y) \) of continuous operators. In this part we shall investigate quasi-Fredholm and semi-Fredholm ideals in paraalgebras of operators over Banach spaces. We shall also deal with perturbations with a small norm.

Chapter I is of an auxiliary character: notions and theorems given here will be necessary in further considerations.

In Chapter II we shall investigate ideals of operators over Banach spaces. In particular, we shall deal with classes of operators which are proved in Chapter V to be semi-Fredholm ideals (positive or negative).

Chapter III contains the theory of perturbations with a small norm.

In Chapter IV we give elements of the spectral theory, in particular the theorem on the continuity of projections of a spectral decomposition.

Chapter V contains the general theory of perturbations of operators over Banach spaces. All the results of this chapter may be transferred without changes to the case of locally bounded spaces with a total family of functionals (see paper [5] by the present authors).
CHAPTER I

BANACH SPACES

§ 1. Definition of a Banach space. We say that a linear metric space \( X \) is a normal space if it is locally bounded and locally convex, i.e., if there exists a neighbourhood of zero \( V \) which is bounded and convex. One may suppose without loss of generality that \( V \) is a balanced neighbourhood, i.e., \( aV \subset V \) for \( |a| \leq 1 \) (see Theorem 2.1, B I).

We write

\[
|z| = \inf \left\{ t > 0 : \frac{z}{t} \in V \right\}
\]

for an arbitrary \( z \in X \). Since the neighbourhood \( V \) is convex and balanced, \( ||z|| \) is a pseudonorm. In § 9, B I, we have shown that this pseudonorm is a norm determining the topology. Hence one can say that the norm in normed spaces may be given by means of the norm \( \varepsilon (x, y) = ||x - y|| \) (see § 2, B I) possessing the following properties:

(i) \( |z| = |a||z| \) (homogeneity),
(ii) \( |z + y| \leq |z| + |y| \) (triangle inequality),
(iii) \( |z| = 0 \) if and only if \( z = 0 \).

On the other hand, if the metric in a space \( X \) is given by means of a norm satisfying conditions (i)-(iii), then the set

\[
V_\varepsilon = \{ z \in X : |z| < \varepsilon \}
\]

is open, bounded, convex and balanced.

A complete normed space is called a Banach space. Hence a space \( X \) is a Banach space if and only if it is a locally bounded \( B_1 \)-space.

A closed linear subset of a Banach space is a Banach space. Such linear subsets will be called subspaces of a Banach space.

The following spaces are Banach spaces:

\[
C_0 (\mathcal{O}), \; C_0 (\mathcal{O}, \Sigma), \; \mathcal{D}_p (\mathcal{O}, \Sigma, \mu), \; M (\mathcal{O}, \Sigma, \mu), \; H_p (\mathcal{O})
\]

(see § 3 and § 6, B I).

To prove this it is sufficient to remark that the norms appearing in the definitions of the above spaces are homogeneous norms (i.e., satisfy condition (i)).

§ 1. Definition of a Banach space 211

The following theorem is a special case of Theorem 5.4, B I:

THEOREM 1.1. A normed space \( X \) is complete if and only if \( \sum_{n=1}^{\infty} |x_n| < +\infty \), \( x_n \in X \), implies the convergence of the series \( \sum_{n=1}^{\infty} x_n \) to an element \( x \) of that space.

§ 2. Continuous operators and continuous functionals in Banach spaces. Since the norm in a Banach space is homogeneous, an operator \( A : L_p (X \to Y) \) is continuous if and only if

\[
||A|| = \sup_{x \neq 0} \frac{||Ax||}{||x||} < +\infty
\]

(see Corollary 1.5, B II).

The space of all linear continuous operators which map a Banach space \( X \) into a Banach space \( Y \) provided with bounded convergence will be denoted by \( B(X \to Y) \), as before (see § 1, B II). Let us remark that in the spaces \( B(X \to Y) \) the topology of bounded convergence and the topology of convergence in the norm of the operator are equivalent (see § 1, B II). Evidently, \( B(X \to Y) \) is a normed space and the norm of an operator is defined by means of formula (2.1).

THEOREM 2.1. If \( X \) and \( Y \) are Banach spaces, then \( B(X \to Y) \) is a Banach space.

Proof. It is sufficient to prove the space \( B(X \to Y) \) to be complete. Let \( (A_n) \) be a fundamental sequence in \( B(X \to Y) \), i.e., suppose that for every positive number \( \varepsilon \) there exists a natural number \( N \) such that \( ||A_n - A_m|| < \varepsilon \) for \( n > N \). Hence \( ||A_n - A_m|| < \varepsilon ||x|| \) for every \( x \in X \). But \( Y \) is a complete space. Thus the limit \( \lim_{n \to \infty} A_n x = A x \) exists for every \( x \in X \). Obviously,

\[
||A_n x - A x|| < \varepsilon ||x||
\]

If \( x, y \in X \), then

\[
A(x + y) = A x + A y = A_n x + A_n y = A_n (x + y) + (A_n - A) y
\]

Hence

\[
||A(x + y) - A x - A y|| < \varepsilon ||x|| + ||y||
\]

Since \( \varepsilon \) is an arbitrary positive number, the additivity of the operator \( A \) follows.

If \( t \) is a scalar, we have

\[
A(tx) = t(Ax) = A_n (tx) + t(A_n - A)x
\]

Applying formula (2.3) we conclude that

\[
||A(tx) - tA x|| < \varepsilon (||x|| + ||t|| ||x||)
\]

The number \( \varepsilon > 0 \) being arbitrary, \( A \) is a homogeneous operator.
Applying formula (2.1) we get
\[ \|Ax\| \leq c \|x\| + \|Ax_N\| \leq (c + \|A_N\|)\|x\| . \]

Hence the operator \( A \) is bounded. Consequently, it is continuous. Moreover, formula (2.2) implies
\[ \|A - A_N\| \leq \|A - A_N\| + \|A_N - A\| < 2e \quad \text{for} \quad n > N . \]

Since \( e \) is an arbitrary positive number, this proves that the sequence \( \{A_n\} \) is convergent to the operator \( A \) in the norm. 

If \( Y \) is a one-dimensional space of complex or real numbers (depending on the field of scalars in the space \( X \), we shall write briefly \( X^\ast = B(X \to Y) \). The elements of the space \( X^\ast \) are called continuous linear functionals, and the space \( X^\ast \) itself is called the conjugate space or the adjoint space.

It follows from Theorem 2.1 that \( X^\ast \) is a Banach space. The norm of a continuous linear functional \( f \) is defined according to formula (2.1)
\[ \|f\| = \sup_{x \in X^\ast} |f(x)| . \]

(2.3)

In case of Banach spaces the Hahn-Banach theorem (Theorem 8.1, B 1) can be formulated in the following manner:

**Theorem 2.3.** (Hahn, Banach.) If \( X_0 \) is a subspace of a Banach space \( X \) and \( f_0 \in X_0^\ast \), then there exists a functional \( f \in X^\ast \) such that \( f(x) = f_0(x) \) for \( x \in X_0 \) and
\[ \|f\| = \|f_0\| . \]

**Corollary 2.3.** If \( X \) is a Banach space, then to every \( x \in X \) there exists a functional \( f \in X^\ast \) such that \( \|f\| = 1 \) and \( f(x) = \|x\| \).

This corollary implies

**Corollary 2.4.** If \( X \) is a Banach space, then
\[ \|x\| = \sup_{f \in X^\ast} |f(x)| . \]

Every operator \( A \in B(X \to Y) \) induces a conjugate operator \( A^\ast \) (see § 1, A III) which maps the space \( Y^\ast \) into the space \( X^\ast \).

**Theorem 2.5.** If \( X \) and \( Y \) are Banach spaces and \( A \in B(X \to Y) \), then \( A^\ast \in B(Y^\ast \to X^\ast) \) and \( \|A^\ast\| = \|A\| . \)

**Proof.** We have
\[ \|A^\ast\| = \sup_{f \in Y^\ast} \|f(A^\ast x)\| = \sup_{f \in Y^\ast} \sup_{x \in X} |f(Ax)| = \sup_{x \in X} \sup_{f \in Y^\ast} \|Ax\| = \|A\| . \]

§ 2. Operators and functionals in Banach spaces

We say that an operator \( A \in B(X \to Y) \) is an **embedding** of a space \( X \) in a space \( Y \) if it is one-to-one and continuous together with its inverse. In other words, an operator \( A \) is an embedding if there exists a positive number \( c \) such that \( \|Ax\| \geq c\|x\| . \) An operator \( A \in B(X \to Y) \) is called an **epimorphism** if it maps the space \( X \) onto the whole space \( Y \).

We denote the space \( (X^\ast)^\ast \) conjugate to \( X \) by \( X^\ast \). Evidently, every element \( x \in X \) can be treated as a functional \( f_x \) on the space \( X^\ast \), where \( f_x(\xi) = \xi(x) \), and we have
\[ \|f_x\| = \sup_{\xi \in X^\ast} |\xi(x)| \leq \|x\| . \]

On the other hand, by the Hahn-Banach theorem, there exists a functional \( \xi_x \in X^\ast \), \( \|\xi_x\| = 1 \), such that \( \xi_x(x) = \|x\| \). Hence
\[ \|x\| = \xi_x(x) = \sup_{\xi \in X^\ast} |\xi(x)| = \|f_x\| . \]

The last inequality shows that the space \( X \) is embedded in the space \( X^\ast \). This embedding is called the **natural embedding**. We denote it by \( \kappa \). If \( \kappa \) is an epimorphism, the space \( X \) is called a **reflexive space**. Hence a space \( X \) is reflexive if \( \kappa X = X^\ast \). Identifying elements \( f_x \) and \( x \) we may write \( X^\ast \) = \( X \) in case of a reflexive space. Since the space \( X \) is complete, the image \( \kappa X \) is closed.

Evidently, a space conjugate to a reflexive space is also reflexive. Indeed, we have
\[ (X^\ast)^\ast = X^{\ast\ast} = (X^{\ast\ast})^\ast = X^\ast . \]

A subspace of a reflexive space is reflexive, since the isomorphism \( \kappa \) between spaces \( X \) and \( X^{\ast\ast} \) is also an isomorphism on every subspace of \( X \).

If \( X_0 \) is a subspace of a reflexive space \( X \), then the quotient space \( X/X_0 \) is also reflexive. Indeed, the conjugate space \( (X/X_0)^\ast \) is the space of all functionals \( \xi \) such that \( \xi(x) = 0 \) for \( x \in X_0 \). It follows from the reflexivity of the space \( X \) that for every functional \( f \in X^{\ast\ast} \) there exists an element \( x_f \) such that \( f(\xi) = \xi(x_f) \). If \( f \in X_0^{\ast\ast} \) and
\[ \xi(\xi(x)) \quad \text{and} \quad f(\xi) = \xi(x_f) \quad \text{for} \quad \xi \in (X/X_0)^\ast , \]
then \( \xi(x_f - x_0) = 0 \). Hence the functional \( f \) is determined by the cosets from \( X/X_0 \).

**Theorem 2.6.** An operator \( A \in B(X \to Y) \) is an **embedding** if and only if the operator \( A^\ast \in B(Y^\ast \to X^\ast) \) is an **epimorphism**.

**Proof.** Necessity. Let us suppose that the operator \( A \) is an embedding, and let \( \xi \in X^\ast \). Let \( \eta \in X_0^{\ast\ast} \) and \( \eta = \xi(A^\ast x) \). By the Hahn-Banach theorem, the functional \( \eta \) can be extended to the whole space. Hence \( A^\ast \eta = \xi(A^\ast Ax) = \xi(x) \). Thus \( A^\ast \) is an epimorphism.
Sufficiency. Let $A^+$ be an epimorphism, and let $a \in X$. There exists a functional $\xi \in X^+$, $\|\xi\| = 1$, such that $\xi(a) = \|a\|$. But the operator $A^+$ is an epimorphism. Hence there exists a functional $\eta \in X^+$ satisfying the conditions $\xi = A^+(\eta)$ and

$$\|\xi\| = \|\xi(a)\| = \|\xi(Ad)\| \leq \|\|A^+\||\|d\|\|.$$  

Theorem 2.7. An operator $A \in B(X \to Y)$ is an epimorphism if and only if the operator $A^+ \in B(Y^+ \to X^+)$ is an embedding.

Proof. Necessity. Let $A$ be an epimorphism. $A$ induces an operator $[A]$ which is a one-to-one map of the quotient space $X/Z_A$ onto the space $Y$. By Banach’s theorem, the operator $[A]$ has an inverse. Hence $\|\xi\| \leq \|\|A^+\||\|\xi\|\|$, where $[\xi]$ is the coset induced by the element $\xi$, and $\|\xi\| = \inf_{x \in A} \|x + y\|$. Thus to every number $m > \|\|A^+\||$ there exists an element $\xi$ such that

$$A^+ = y \quad \text{and} \quad \|\xi\| = m \|y\|.$$  

Let $\eta \in X^+$ and $\xi = A^+(\eta)$. Then

$$\|\eta\| = \|\eta(Ad)\| = \|\xi(Ad)\| \leq \|\|A^+\||\|\xi\|\| \leq \|\|A^+\||\|\|\eta\|\|.$$  

Hence

$$\|\eta\| = \sup_{\xi \in \langle A \rangle} \|\eta(\xi)\| \leq m \|\|\eta\|\|.$$  

Sufficiency. Theorem 2.6 implies that the operator $A^+ \in B(Y^+ \to X^+)$ is an embedding. Hence its restriction to the subspace $X$ is also an embedding.

By the standard method of decomposition into a direct sum we obtain the following generalization of Theorem 2.6 and 2.7:

Theorem 2.8. An operator $A \in B(X \to Y)$ is a $\Phi_+$-operator if and only if the operator $A^+ \in B(Y^+ \to X^+)$ is a $\Phi_+$-operator.

Theorem 2.9. An operator $A \in B(X \to Y)$ is a $\Phi_+$-operator if and only if the operator $A^+ \in B(Y^+ \to X^+)$ is a $\Phi_+$-operator.

We now give examples of general forms of continuous linear functionals over some Banach spaces. The proofs require powerful methods of the measure theory and can be found in Dunford and Schwartz [1], Chapter IV.

Example 2.1. If $\Omega$ is a compact set, there exists a one-to-one correspondence between the conjugate space $[C(\Omega)]^*$ of the space $C(\Omega)$ and the space $\text{rea}\Omega$. This correspondence is given by the formula

$$f(\xi) = \int_{\Omega} \xi(\xi) d\mu.$$  

Moreover, we have $\|f\| = \|\xi\|$.

Example 2.2. If $1 < p < \infty$ and $1/p + 1/q = 1$, then the spaces $\langle Lp(\Omega, \Sigma, \mu) \rangle^*$ and $Lp(\Omega, \Sigma, \mu)$ are isometrically isomorphic. This isomorphism is given by the equality

$$\pi^*(\xi) = \int y(t)\xi(\xi) d\mu,$$

where $\pi \in Lp(\Omega, \Sigma, \mu)$, $\xi \in \langle Lp(\Omega, \Sigma, \mu) \rangle^*$.

Example 2.3. If $\mu$ is a finitely additive positive measure, then the spaces $\langle Lp(\Omega, \Sigma, \mu) \rangle^*$ and $M(\Omega, \Sigma, \mu)$ are isometrically isomorphic. This isomorphism is given by the equality

$$\pi^*(\xi) = \int y(t)\xi(\xi) d\mu,$$

where $\pi \in Lp(\Omega, \Sigma, \mu)$, $\xi \in M(\Omega, \Sigma, \mu)$.

We quote without proof the following important theorem:

Theorem 2.2. To every space $M(\Omega, \Sigma, \mu)$ there exists a compact Hausdorff space $\Omega$ such that the spaces $M(\Omega, \Sigma, \mu)$ and $C(\Omega)$ are isomorphic.

The reader can find the proof of this theorem in the monograph by Dunford and Schwartz [1], Theorem V. 8.11.

§ 3. Weak convergence and weak topology. Let $X$ be a Banach space. The $X^*$-convergence and $X^*$-topology in $X$ (§ 10, B I) are called weak convergence and weak topology in $X$, respectively. A sequence is called weakly fundamental if it is $X^*$-fundamental.

We denote by $S(X)$ the closed unit ball in the space $X$:

$$S(X) = \{x \in X : \|x\| \leq 1\}.$$  

Theorem 3.1. (Goldstine [1]) If $X$ is a Banach space, then the set $\pi S(X)$ is dense in the ball $S(X^*)$ in the $X^*$-topology (as before, $\pi$ means the natural embedding of the space $X$ in the space $X^*$).

Proof. We denote by $S_1$, the $X^*$-closure of the set $\pi S(X)$. Since $S(X^*)$ is a $X^*$-closed set, we have $S_1 \subset S(X^*)$. Moreover, the set $S_1$ is convex. We shall prove that $S_1 = S(X^*)$. If there exists an element $x^+ \in S(X^*)$ such that $x^+ \in S_1$, Corollary 8.4, B I, implies the existence of an $X^*$-continuous linear functional $f$ defined on $X^*$ and of two constants $c$ and $\epsilon > 0$ such that $\|f(y)\| \leq c$ for $y \in S_1$, $\|f(x^+\|) \geq \pi + c$. By Theorem 10.1, B I, there exists an element $x^+ \in X^*$ satisfying the equality $f(x^+) = f(x^+) = f(x^+)$ if $x^+ \in X^*$. Since $S(X^*)$ is a $X^*$-closed set, we have $\pi x^+ = \pi x^+ = \pi x^+ = \pi x^+ = \pi x^+$, contradicting the inequality $\pi x^+ \leq c$ for $x \in S(X)$. Hence $\|x^+\| \leq c$ and $\|x^+\| \leq c \|x^+\| \leq c$, contradicting the inequality $\pi x^+ \leq c + c$. Thus every element $x^+ \in S(X^*)$ belongs to $S_1$.  


Corollary 3.2. If \( x \) is the natural embedding of a Banach space \( X \) in the space \( X^* \), then the set \( xX \) is dense in \( X^* \) in the \( X^* \)-topology.

Proof. The \( X^* \)-closure of the set \( xX \) is a subspace of the space \( X^* \). By Theorem 3.1, it contains the ball \( S(X^*) \). Hence it immediately follows that the \( X^* \)-closure of \( X \) contains every point of the space \( X^* \).

Theorem 3.3. If \( X \) is a Banach space, then the ball \( S(X^*) \) is compact in the \( X^* \)-topology of the space \( X^* \).

Proof. By definition, \( S(X^*) = \{ f \in X^*: ||f|| \leq 1 \} \). By Theorem 10.3, B I, it follows that \( S(X^*) \) is compact in the \( X^* \)-topology.

Let \( X \) and \( Y \) be Banach spaces. An operator \( A \in L_0(X \to Y) \) is called weakly continuous if the inverse image of an open set in the \( X^* \)-topology is a set open in the \( Y^* \)-topology.

If the space \( X \) is conjugate to a Banach space \( X_\infty \), then the \( X_\infty \)-convergence in the \( X^* \)-topology are called the weak convergence of functionals and the weak topology of functionals, respectively. If a sequence of functionals is \( X_\infty \)-fundamental, it is called a weak fundamental sequence of functionals.

A set \( E \subseteq X \) is called weakly compact if it is compact in the \( X^* \)-topology. \( E \subseteq X \) is called conditionally weakly compact if its closure in the weak topology is weakly compact. Finally, a set \( E \subseteq X \) is called weakly precompact if it is precompact in the weak topology (see § 1, B IV).

Theorem 3.4. (Berezin [1]) A Banach space \( X \) is reflexive if and only if the unit ball \( S(X) \) is weakly compact.

Proof. Let \( X \) be a reflexive Banach space and let \( x \) be the natural embedding of the space \( X \) in the space \( X^* \). Then \( x \) and \( x^{-1} \) are isometries. Moreover, \( x \) maps the ball \( S(X) \) onto the ball \( S(X^*) \). By the definition of topology, \( x \) is a homeomorphism of the ball \( S(X) \) with its \( X^* \)-topology onto the ball \( S(X^*) \) with its \( X^* \)-topology. By Theorem 3.3, the ball \( S(X) \) is weakly compact.

Conversely, let the ball \( S(X) \) be weakly compact. Since \( x \) is a homeomorphism between \( S(X) \) and \( xS(X) \) in the \( X^* \)-topology defined on the sets \( S(X) \) and \( xS(X) \), the set \( xS(X) \) is compact. Since the set \( xS(X) \) is closed in its \( X^* \)-topology and \( S(X) \) is dense in the ball \( S(X^*) \) (by Theorem 3.1), we have \( xS(X) = S(X^*) \). Consequently, \( x(X) = X^* \), i.e. the space \( X \) is reflexive.

Corollary 3.4. A Banach space \( X \) is reflexive if and only if every bounded, weakly closed set \( E \subseteq X \) is weakly compact.

The following theorem is a simple consequence of Theorem 10.1, B I.

Theorem 3.5. If \( X \) and \( Y \) are Banach spaces and \( A \in B(X \to Y) \), then

1. the operator \( A \) is weakly continuous,
2. \( A \) transforms \( X^* \)-convergent sequences in \( Y^* \)-convergent sequences,
3. the operator \( A^* \) is continuous if we provide \( Y^* \) with the \( Y^* \)-topology, and \( X^* \) with the \( X^* \)-topology,
4. the operator \( A^* \) transforms \( Y^* \)-convergent sequences in \( X^* \)-convergent sequences.

Theorem 3.6. If the sequence \( (x_n) \) is weakly convergent or if it is a weakly convergent sequence of functionals, then it is bounded.

Proof. First, let us consider the second case, i.e., let \( x_n \) be functionals over a Banach space \( X_\infty \). By the Banach–Steinhaus Theorem (Theorems 2.1, B II, and 2.2, B II), all functionals \( x_n \) are equicontinuous. Hence the sequence \( (x_n) \) is bounded. The first case is reduced to the second one if we consider \( x_n \) to be functionals over the space \( X^* \).

In order to investigate weak convergence effectively, the following theorem is of importance:

Theorem 3.7. A sequence \( (x_n) \) is weakly fundamental (or is a weak fundamental sequence of functionals) if and only if it is bounded and \( S^* \)-convergent, where \( S^* \) is a dense subset of the space \( X^* \) (\( X^* \), respectively).

Proof. Necessity. Theorem 3.6 implies that the sequence \( (x_n) \) is bounded. Since the sequence \( (x_n) \) is convergent in the space \( X^* \) (and \( X^* \)), it is \( S^* \)-convergent.

Sufficiency. Let us write \( |x_n| = M \). Let \( \varepsilon \) be an arbitrary positive number, and let \( f \in X^* \) (\( f \in X^* \), respectively). Since \( S^* \) is a dense subset, there exists a functional \( f \) satisfying the inequality \( ||f|| < \varepsilon / M \).

Since the sequence \( (x_n) \) is \( S^* \)-convergent, there exists a number \( N \) such that

\[ |f(x_n) - f(x_m)| < \varepsilon / 2 \quad \text{for} \quad n, m > N. \]

Thus

\[ |f(x_n) - f(x_m)| \leq |f(x_n) - f(x_m)| + |f(x_m) - f(x_n)| + |f(x_n) - f(x_m)| < \varepsilon / 2 + \varepsilon / 2 + \varepsilon / 2 = \varepsilon. \]

We now give without proofs the following important theorems:

Theorem 3.8. (Berezin [1], Smal’ [1]) If \( E \) is a subset of a Banach space \( X \), then the following three conditions are equivalent:

1. every sequence \( (x_n) \subset E \) contains a subsequence \( (x_{n_k}) \) weakly convergent to an element \( x_0 \in X \),
2. every sequence \( (x_n) \subset E \) contains a subsequence \( (x_{n_k}) \) weakly convergent to an element \( x_0 \subset X \),
3. the set \( E \) is conditionally weakly compact.
The reader can find the proof of this theorem in the monograph by Dunford and Schwartz [1], Theorem V. 6.1.

**Theorem 3.9.** (Krein, Smulian [1].) Let \( X \) be a Banach space. A convex set \( B \subset X^+ \) is \( X \)-closed if and only if for every natural number \( n \) the set \( E \cap nB(X) \) is \( X \)-closed, \( B(X) \) denoting the closed unit ball in the space \( X^+ \).

The proof of this theorem is given in the monograph by Dunford and Schwartz [1], V.5.7.

Further considerations will require a characteristic of weakly compact sets in the space \( rca_0 = [O(O)]^+ \). We quote it without proof (for the proof see Dunford and Schwartz [1], IV.9.12):

**Theorem 3.10.** A set \( E \subset rca_0 = [O(O)]^+ \) is conditionally weakly compact if and only if there exists a non-negative measure \( \mu \) such that

\[
\lim_{\mu \rightarrow 0} \lambda(E) = 0 \quad \text{for all measures } \lambda \in E
\]

uniformly.

If a set \( E \) is conditionally weakly compact, then every sequence \( \{\mu_n\} \subset E \) is conditionally weakly compact. If every sequence \( \{\mu_n\} \subset E \) is conditionally weakly compact, one can choose a weakly compact subsequence. By the Eberlein–Smulian theorem 3.8, the set \( E \) is conditionally weakly compact. Hence the following theorem is a consequence of Theorem 3.10:

**Theorem 3.11.** A set \( E \subset rca_0 = [O(O)]^+ \) is conditionally weakly compact if and only if for every sequence of measures \( \{\mu_n\} \subset E \) there exists a non-negative measure \( \mu \) such that all measures \( \mu_n \) are \( \text{equicontinuous with respect to the measure } \mu \), i.e.

\[
\lim_{\mu \rightarrow 0} \mu_n(E) = 0 \quad \text{for } n = 1, 2, ...
\]

§ 4. Bases in Banach spaces. Let us remember that a basis of a complete linear metric space \( X \) is a sequence of elements \( e_n \in X \) such that every element \( x \in X \) can be written uniquely as the sum of a series

\[
x = \sum_{n=1}^{\infty} a_n e_n \quad (a_n \text{ are scalars})
\]

The fundamental properties of bases in linear metric spaces are given in § 5, II. Here we give further properties of bases in the case when \( X \) is a Banach space.

Evidently, if a Banach space \( X \) has a basis, \( X \) is separable. However, it is not known whether every separable Banach space has a basis (see Banach [2], p. 111). Only the following result is known:

**Theorem 4.1.** (Banach [3], p. 206.) Every infinitely dimensional Banach space \( X \) contains an infinitely dimensional subspace \( X_{\infty} \) with a basis.

Proof. Let \( \{e_n\} \) be an arbitrary sequence of positive numbers such that \( \sum_{n=1}^{\infty} e_n < +\infty \). We construct a sequence \( \{f_n\} \) by induction in such a manner that

\[
\|f_1 + \cdots + f_{n-1} e_{n-1}\| \leq (1 + e_n)\|f_1 + \cdots + f_{n-1} e_{n-1}\|
\]

for arbitrary scalars \( f_1, \ldots, f_{n-1} \).

As \( e_1 \) we may take an arbitrary element different from zero. Let us suppose that the elements \( e_1, \ldots, e_{n-1} \) are already defined. Let \( X_{n-1} \) be the space spanned by these elements. Since a ball in the conjugate space \( X_{n-1}^* \) is precompact, there exists a finite system of functionals \( f_1, \ldots, f_k, \|f\| = 1 \), satisfying the inequality

\[
\|x\| = \sup \|f(x)\| \leq (1 + e_n) \sup_{1 \leq i \leq k} \|f_i(x)\|
\]

Let us extend the functionals \( f_i \) to the whole space, leaving their norms unchanged. Let \( e_n \) be an arbitrary element different from zero and satisfying the conditions \( f_i(e_n) = 0 \) for \( i = 1, 2, \ldots, k \). Then

\[
\|f_1 + \cdots + f_{n-1} e_{n-1}\| \leq (1 + e_n) \sup_{1 \leq i \leq k} \|f_i(e_1 + \cdots + e_{n-1} e_{n-1})\|
\]

\[
= (1 + e_n) \sup_{1 \leq i \leq k} \|f_i(e_1 + \cdots + e_{n-1} e_{n-1})\|
\]

\[
\leq (1 + e_n)\|e_1 + \cdots + e_{n-1} e_{n-1}\|
\]

Hence we conclude that the sequence \( \{f_n\} \) satisfies condition 4.1. Thus

\[
x = \sum_{n=1}^{\infty} a_n e_n
\]

implies

\[
\|x\| \leq C\|x\|
\]

where

\[
C = \prod_{n=1}^{\infty} (1 + e_n)
\]

By Corollary 4.6, B I, it follows that the sequence \( \{e_n\} \) is a basis of the space \( X_{\infty} \) spanned by \( \{e_n\} \).

We say that a sequence \( \{f_n\} \) is strongly linearly independent if

\[
f_i \not\in \text{lin}(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots)
\]

**Theorem 4.3.** (Krein, Milman, Rutman [11].) Let \( X \) be a Banach space with a basis \( \{e_n\} \), \( \|e_n\| = 1 \). If a sequence \( \{f_n\} \) of strongly linearly independent elements of \( X \) satisfies the condition

\[
\sum_{n=1}^{\infty} \|f_n - e_n\| < +\infty
\]

then \( \{f_n\} \) is a basis of the space \( X_{\infty} = \text{lin}(\{e_n\}) \) equivalent to the basis \( \{e_n\} \) in \( X_{\infty} \).

Proof. Let \( K \) be the norm of the basis \( \{e_n\} \). Since \( f_1 \) are strongly linearly independent elements, one can omit in the proof a finite number
of elements \( f_n \) and \( e_n \). Hence we may suppose that \( C = 1/2K \) without loss of generality. Thus

\[
\left\| \sum_{k=1}^{n} t_k e_k \right\| - \left\| \sum_{k=1}^{n} |t_k| e_k \right\| \leq \sum_{k=1}^{n} \left| t_k e_k \right| + \left\| \sum_{k=1}^{n} |t_k| e_k \right\|,
\]

\[
\left\| \sum_{k=1}^{n} t_k e_k \right\| \leq \sup_{n} \left\| \sum_{k=1}^{n} t_k f_k \right\| \leq (K + \frac{1}{2}) \sum_{k=1}^{n} |t_k| e_k.
\]

Hence the elements \( f_n \) constitute a basis of the space \( X \) of elements of the form \( \sum_{n=1}^{\infty} a_n f_n \), equivalent to the basis \( \{e_n\} \).

§ 5. Unconditional convergence and unconditional bases. A series \( \sum_{n=1}^{\infty} x_n \) of elements of a Banach space \( X \) is said to be unconditionally convergent if the series \( \sum_{n=1}^{\infty} \lambda_n x_n \) is convergent for every bounded sequence \( \{\lambda_n\} \).

If a series \( \sum_{n=1}^{\infty} x_n \) is unconditionally convergent, then there exists a constant \( C \) such that

\[
\left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\| \leq C \sup_n |\lambda_n|.
\]

Indeed, let us suppose that such a constant \( C \) does not exist. We choose a sequence of indices \( \{n_k\} \) and bounded sequences \( \{\lambda_k\} \) satisfying the inequalities

\[
\sum_{k=0}^{\infty} |\lambda_k| = k + \sum_{k=0}^{n_k} |\lambda_n|,
\]

by induction. It is easily verified that if \( \lambda_i = \lambda_k \) for \( n_k < i \leq n_{k+1} \), then \( |\lambda_i| < 1 \) and the series \( \sum_{n=1}^{\infty} \lambda_n x_n \) is convergent.

**Theorem 5.1. (Orlicz [1].) If a series \( \sum_{n=1}^{\infty} x_n \) of elements \( x_n \in D[0,1] \) is unconditionally convergent, then the series \( \sum_{n=1}^{\infty} |x_n| \) is also convergent.**

Proof. Let there be given a sequence \( \{x_n\} \subset D[0,1] \) such that the series \( \sum_{n=1}^{\infty} x_n(t) \) is unconditionally convergent. Let \( r(t) \) be the Rademacher system on the interval \([0,1]\), i.e., the system of functions

\[
r(t) = \text{sgn} \sin 2\pi t r.
\]

Evidently,

\[
\int_{0}^{1} r(t) r(t) \, dt = 0.
\]

Hence

\[
\left( 2 \right) \int_{0}^{1} \left| \sum_{n=1}^{\infty} x_n(t) r_n(t) \right|^2 \, dt = \int_{0}^{1} \left| \sum_{n=1}^{\infty} x_n(t) r_n(t) \right|^2 \, dt = \sum_{n=1}^{\infty} |x_n(t)|^2.
\]

On the other hand, since the series \( \sum_{n=1}^{\infty} x_n(t) \) is unconditionally convergent, we have

\[
\int_{0}^{1} \left| \sum_{n=1}^{\infty} x_n(t) r_n(t) \right|^2 \, dt = C < +\infty \quad \text{for every} \ t.
\]

Hence, by formula (2.5).

\[
\sum_{n=1}^{\infty} |x_n| = \sum_{n=1}^{\infty} |x_n(t)|^2 \, dt = \int_{0}^{1} \sum_{n=1}^{\infty} |x_n(t)|^2 \, dt = \int_{0}^{1} \sum_{n=1}^{\infty} |x_n(t)|^2 \, dt \leq C.
\]

A basis \( \{e_n\} \) of a Banach space \( X \) is called an unconditional basis if the expansion of every element \( x \in X \) is unconditionally convergent.

We say that an unconditional basis \( \{e_n\} \), \( \|e_n\| = 1 \), is homogeneous if every subbasis \( \{e_n\} \) is equivalent to \( \{e_n\} \).

An unconditional basis \( \{e_n\} \), \( \|e_n\| = 1 \) is called block homogeneous if every sequence \( \{a_n\} \) of elements of the form

\[
a_n = \sum_{i=n+1}^{n_k} t_i e_i, \quad |a_n| = 1,
\]

where \( \{p_i\} \) is an increasing sequence, is a basis equivalent to the basis \( \{e_n\} \).

Standard bases in \( \mathbb{P} \) and \( \mathbb{Q} \) are block homogeneous. M. Zippin [1] showed that the existence of a block homogeneous basis in a space \( X \) implies that \( X \) is isomorphic either to the space \( \mathbb{P} \) or to the space \( \mathbb{Q} \).

A series \( \sum_{n=1}^{\infty} x_n \) of elements of a Banach space \( X \) is called weakly unconditionally convergent if the series \( \sum_{n=1}^{\infty} x^+(x_n) \) is unconditionally convergent for every functional \( x^+ \in X^+ \), i.e., if \( \sum_{n=1}^{\infty} |x^+(e_n)| \) is convergent for every \( x^+ \in X^+ \).
Evidently, if a series $\sum_{n=1}^{\infty} x_n$ is weakly unconditionally convergent, then the sequence $(x_n)$ is bounded (see Theorem 4.6).

**Theorem 5.2.** (Bessaga and Pełczyński [11]) Let there be given a Banach space $X$ and a weakly unconditionally convergent series $\sum_{n=1}^{\infty} x_n$ of elements of this space. Let $(x_n)$ constitute a basis of the space spanned by these elements, and let $\inf |x_n| > 0$. Then the basis $(x_n)$ is equivalent to the standard basis in the space $c_0$.

**Proof.** Let

$$Z_k = \{ x^* \in X^* : \sum_{n=1}^{\infty} |x^*(x_n)| \leq k \}.$$  

The sets $Z_k$ are closed and since the series $\sum_{n=1}^{\infty} x_n$ is weakly conditionally convergent, it follows that $X^* = \bigcup_{k} Z_k$. By Baire's theorem on categories, one of the sets $Z_k$ contains a ball $K$ with centre $x^*_0$ and radius $r$. Let $\|x^*\| \leq r$; then

$$\sum_{n=1}^{\infty} |x^*(x_n)| \leq \sum_{n=1}^{\infty} |(x^* - x^*_0)x_n| + \sum_{n=1}^{\infty} |x^*_0(x_n)| \leq k + C_1,$$

where $C_1 = \sum_{n=1}^{\infty} |x^*_n(x_n)|$.

Let $C = (k + C_1)/r$ and let $\|x^*\| < 1$. Then $\sum_{n=1}^{\infty} |x^*(x_n)| < C$. Thus we have

$$\sum_{n=1}^{\infty} |x^*(x_n)| < C \sup_{n} |x_n|$$

for an arbitrary sequence $(x_n)$ convergent to zero.

Hence the series $\sum_{n=1}^{\infty} t_n x_n$ is convergent for an arbitrary sequence $(t_n)$ convergent to zero. On the other hand, the inequality $\inf |x_n| > 0$ implies that if the series $\sum_{n=1}^{\infty} t_n x_n$ is convergent, then $t_n \to 0$.

**Theorem 5.3.** (Pełczyński, Singer [11]) Let $X$ be a Banach space with an unconditional basis $(e_n), \|e_n\| = 1$. Let us suppose that the spaces $X$ and $X^*$ have the following property: if $\sum_{n=1}^{\infty} y_n$ is an unconditionally convergent series of elements of any of the spaces $X$ and $X^*$, then $\sum_{n=1}^{\infty} \|y_n\|^2 < +\infty$. Thus the space $X$ is isomorphic to the space $c_0$.

**Proof.** Let $(e_n^*) \subset X^*$ be a sequence of basis functionals: $e_n^*(e_m) = \delta_{nm}$. Let

$$x = \sum_{n=1}^{\infty} t_n e_n, \quad x^* = \sum_{n=1}^{\infty} a_n e_n^*,$$

and let $\lambda = (\lambda_n)$ be any bounded sequence of numbers. If we write $x_n = \sum_{k=1}^{n} \lambda_k t_k e_k$, formula (5.1) implies

$$\|x_n\|^2 \leq C \sup_{n} |\lambda_n| \leq C \sup_{n} |\lambda_n| \left( \sum_{n=1}^{\infty} |\lambda_n|^2 \right)^{1/2},$$

where $C$ is a positive constant dependent only on the basis $(e_n)$.

Thus we obtain

$$\|x^*(x_n) = x^*_n(x)\), \quad \|x^*_n\| = \sum_{n=1}^{\infty} \lambda_n a_n e_n^*.$$

Hence formula (5.2) and the convergence of the series $\sum_{n=1}^{\infty} a_n e_n^*$ in the usual sense imply its unconditional convergence. But the basis functionals are uniformly bounded. Hence we have

$$\sum_{n=1}^{\infty} |a_n|^2 < C \|x^*\|^2,$$

by hypothesis.

Consequently, for every series $\sum_{n=1}^{\infty} t_n e_n$ such that $\sum_{n=1}^{\infty} |t_n|^2 < 1$ we have:

$$\|\sum_{n=1}^{\infty} t_n e_n\| = \sup_{x^* \in X^*, \|x^*\| = 1} \|x^*(\sum_{n=1}^{\infty} t_n e_n)\| \leq C \sup_{n} |t_n| \left( \sum_{n=1}^{\infty} |t_n|^2 \right)^{1/2},$$

$$= \sup_{x^* \in X^*, \|x^*\| = 1} \|x^*\| \left( \sum_{n=1}^{\infty} |t_n|^2 \right)^{1/2},$$

Hence the series $\sum_{n=1}^{\infty} t_n e_n$ is convergent. Thus the basis $(x_n)$ is equivalent to the standard basis of the space $c_0$.

**§ 6. Linear dimension.** We say that a space $X$ has a linear dimension not greater than the linear dimension of a space $Y$, $\dim X \leq \dim Y$, if $Y$ contains a subspace isomorphic to the space $X$. The linear dimension of a space $X$ is equal to the linear dimension of a space $Y$, $\dim X = \dim Y$, if $\dim X \leq \dim Y$ and $\dim Y \leq \dim X$.
The linear dimension of a space $X$ is smaller than the linear dimension of a space $Y$ if $\dim X \leq \dim Y$ and the inequality $\dim X \leq \dim Y$ does not hold.

**Theorem 6.1. (Banach [2]) If there exists a block homogeneous basis $\{e_n\}$ in a Banach space $X$, then $\dim X = \dim_X X_n$ for every infinite-dimensional subspace $X_n$ of the space $X$.

Proof. By definition, we have $\dim X_n \leq \dim X$. On the other hand, let $\{e_n\} \subset X_n$, $\|e_n\|_n = 1$, $x_n = \sum_{i=1}^{\infty} \alpha_i e_i$, be a sequence satisfying the condition $\lim r_n^n = 0$. By Theorem 5.8, B II, such a sequence exists, because $X_n$ is an infinite-dimensional space. Applying Theorem 5.7, B II, we find a subsequence $\{x_{n_k}\}$ of the sequence $\{e_n\}$ and an increasing sequence of indices $\{p_k\}$ satisfying the inequalities

$$\|x_{n_k} - x_k\|_n < 1/2^k,$$

where $x_k = \sum_{i=2^{k-1}+1}^{2^k} c_i e_i$.

Since the basis $\{e_n\}$ is block homogeneous, the bases $\{e_n\}$ and $\{x_k\}$ are equivalent. By Theorem 5.2, bases $\{e_n\}$ and $\{x_k\}$ are equivalent. Hence the space $X_n$ spanned by the elements $\{e_n\}$ and the space $X$ are isomorphic (Theorem 5.5, B II). But $X_n \subset X$. Hence $\dim X_n \leq \dim X_n$. ■

Let us remark that the following result can be deduced from the proof of Theorem 6.1: If $X_n \subset X$ is any subspace of a space $X$ with a block homogeneous basis $\{e_n\}$ and if $\{x_n\}$ is any sequence which is not compact, $\|x_n\|_n = 1$, then there exists a subsequence $\{x_{n_k}\}$ which constitutes a basis of the space $X_n$ spanned by $\{x_{n_k}\}$, and this basis is equivalent to the basis $\{e_n\}$.

**Theorem 6.2. (Banach [2]) $\dim P \leq \dim L^p[0,1]$.**

Proof. Let $e(t) = 2^n n_{2^{n-1}}(t,2^n,2^n,0)$. If $x(t) = \sum_{n=1}^{\infty} a_n e(t)$, then

$$\int_0^1 |x(t)|^p dt = \sum_{n=1}^{\infty} |a_n|^p.$$

Hence the space $X_n \subset L^p[0,1]$ spanned by the elements $e_n$ and the space $P$ are isomorphic. ■

**Theorem 6.3.** $\dim P \leq \dim L^p[0,1]$; $1 \leq p < +\infty$.

The proof of this theorem is based on two lemmas.

**Lemma 6.1.** If $x \in L^p[0,1]$, then $x \in L^{p'}[0,1]$ and $\|x\|_{p'} \leq \|x\|_p$ for $p' < p$.

Proof. It follows from Hölder's inequality that

$$\|x\|_{p'}^p = \int_0^1 |x(t)|^p dt \leq \left( \int_0^1 |x(t)|^{p/p'} dt \right)^{p/p'} \left( \int_0^1 1^{(p-1)(p'/p') - p'} dt \right) = C_{p,p'} \|x\|_p^p.$$
On the other hand, if \( f(t) = \sum_{k} a_k r_k(t) \) in the norm in the space \( L^p \), then for every \( n \)

\[
\sum_{k=1}^{n} a_k^2 = \frac{1}{n} \int_{t_0}^{t_0+1/n} \sum_{k=1}^{n} a_k r_k(t) \, dt < \left( \sum_{k=1}^{n} a_k^p r_k(t_0)^p \right)^{1/p} \leq \left( \frac{1}{n^{1/p} + 1/q} \right) \left( \sum_{k=1}^{n} a_k^q \right)^{1/q},
\]

where \( 1/p + 1/q = 1 \). Substituting \( q = p/(p-1) \) in the last inequality, dividing both sides by \( \sum_{k=1}^{n} a_k^2 \) and taking the limits as \( n \to \infty \), we obtain the so-called *Raikcmez inequality* (Kaczmaz and Steinhaus [1]):

\[
\sqrt{\sum_{k=1}^{n} a_k^2} \leq \left( \frac{1}{n^{1/p} + 1/q} \right) \left( \sum_{k=1}^{n} a_k^q \right)^{1/q}. \]

This completes the proof in the case of real-valued functions. Let \( L^p[0, 1] \) be the space of complex-valued functions. It is easily verified that the space of functions of the form \( a(t) + b(t) \), where \( a(t) \) and \( b(t) \) belong to the space of real-valued functions spanned by the Rademacher system, is isomorphic to the space \( L^p \) of sequences of complex numbers.

**§ 7. Projections in Banach spaces.** A subspace \( Y \) of a Banach space \( X \) is called a *projection of the space \( X \)* if there exists a continuous projection operator onto the subspace \( Y \), i.e., a continuous operator \( P \) such that \( P^2 = P \) and \( P x = x \) if and only if \( x \in Y \) (see § 1, B II).

If an operator \( P \) is a projection operator, then the conjugate operator \( P^* \) is also a projection operator.

If a subspace \( Y \) is a projection of the space \( X \), \( Y = \{ x \in X : P x = x \} \), then every direct sum of the subspace \( Y \) and a finite-dimensional subspace is a projection of the space \( X \).

Let us write \( Y_\perp = \{ x \in X : P x = 0 \} \).

The set \( Y_\perp \) is a complete subspace of the space \( X \); it is called the *direction of the projection*. Evidently, the space \( X \) can be written as a direct sum

\[
X = Y \oplus Y_\perp. \]

A Banach space \( X \) is called *subprojective* (Whitley [1]) if every infinite-dimensional subspace \( X \) of \( X \) contains an infinite-dimensional subspace \( X \), which is a projection of the space \( X \). Evidently, every subspace of a subprojective space is also subprojective (Whitley [1]).

A Banach space \( X \) is called *superprojective* if to every subspace \( N \) of an infinite codimension there exists a subspace \( M \) of an infinite codimension which is a projection of \( X \) and contains the subspace \( N \), (Whitley [1]).
We choose an index $n$ in such a manner that
\[ C_n = \sum_{n+1}^{\infty} \delta(Y_n, Y_n') < \frac{1}{N} \delta(Y_1, Y_1') \cdot \]

Let $Y_n$ denote the space spanned by the elements $e_{n+1}, e_{n+2}, \ldots$. Then
\[ \delta(Y_n, Y_n') < C_n \cdot \delta(Y_1, Y_1') \cdot \]

But
\[ \delta(Y_1, Y_1') < \delta(Y_n, Y_1') \cdot \]

whence
\[ \delta(Y_n, Y_n') > \delta(Y_n, Y_n') \cdot \]

Thus
\[ \delta(Y_n, Y_n') > 0 \cdot \]

Hence the subspace $Y_n'$ is a projection of the direct sum $Y_n \oplus Y_n'$. But this direct sum differs from the whole space by a finite-dimensional space only. The subspace $Y'$ differs from the subspace also by a finite-dimensional space. Thus $Y_n$ is a projection of the space $X$. \[ \Box \]

**Theorem 7.2.** (Whitley [11]) Every Banach space $X$ with a block homogeneous basis $(e_n)$ is subprojective.

**Proof.** By Theorem 5.7, B II, there exist a sequence $(x_n) = \{\sum_{i=1}^{n} \xi_i e_i\}$, $\|x_n\| = 1$, and an increasing sequence of numbers $(y_n)$ such that
\[ \frac{1}{y_n} < \frac{1}{y_n'} \quad \text{where} \quad y_n = \sum_{i=n+1}^{\infty} \xi_i e_i \cdot \]

Let us denote by $X_n$ the subspace spanned by the sequence $(y_n)$. Let $f_n$ be a functional satisfying the conditions $f_n(y_n)[y_n] = 1$, $y_n || \leq K$ (where $K$ is the norm of the basis), and $f_n(x_n) = 0$ for $x < y_n$ and $y_n' > y_n + 1$.

The operator $P_n = \sum_{i=1}^{\infty} f_n(x_n)[y_i][y_i]$ is a well-defined projection operator. Indeed, the block homogeneity of the basis implies the unconditional convergence of the series
\[ \sum_{i=n}^{\infty} \xi_i y_i[y_i][y_i] \quad \text{where} \quad \xi_n = [x_{n+1} - [x]_{y_n}] \cdot \]

However, $|f_n(x)| < K \|x_n\|$; hence the series defining the operator $P_n$ is convergent. Moreover, $P_n$ is a continuous operator. Hence the space $X_n$ is a projection of the space $X$. By Theorem 7.1, the space $X_n$ is a projection of the space $X$. \[ \Box \]

**Theorem 7.3.** (Kadec and Pełczyński [11]) Let $\{x_n\}$ be a sequence in the space $L^p[0, 1]$, $p > 1$, such that for every $\epsilon > 0$ there exists an index $n_0$ such that $x_n \notin M_p$. There exists a sequence $(x_n')$, where $x_n' = x_n + (x_n - x_n)(k_1, k_2, \ldots)$, satisfying the conditions
\[ (1) \text{ the sequence $(x_n', y_n)$ is a basis equivalent to the basis $(x_n) = \{[x_n]\}$ in the space $P'$,} \]

\[ (2) \text{ the sequence $(x_n', y_n)$ is a basis equivalent to the basis $(x_n) = \{[x_n]\}$ in the space $P'$,} \]

**Proof.** If $x \in L^p[0, 1]$, then the set function $A(x) = \int |x(t)|^p dt$ is absolutely continuous. Hence applying the assumptions and the properties of sets $M_p$, we may define a subspace $(x_n')$ of the sequence $(x_n)$ and a sequence $(A_n)$ of sets by induction, satisfying the conditions
\[ (7.2) \quad \int_{A_n} |x(t)|^p \quad dt > 1 - 4^{-1/n+1/2p} \quad (n = 1, 2, \ldots) \]

\[ (7.3) \quad \int_{A_n} \sum_{i=1}^{\infty} |x_i(t)|^p \quad dt < 4^{-1/n+1/2p} \quad (n = 1, 2, \ldots) \]

\[ \Box \]
We write $A_n' = A_0 \setminus \bigcup_{k=1}^\infty A_k,$

$$z_n(t) = \begin{cases} \begin{array}{ll}
\frac{x_n(t)}{||x_n(t)||} & \text{for } t \in A_n', \\
0 & \text{for } t \notin A_n',
\end{array} \end{cases} \quad \text{for } n = 1, 2, \ldots .$$

Evidently, $A_n' \cap A_m' = 0$ for $n \neq m.$ Hence the following inequalities hold for every $n$,

$$\left| \frac{x_n(t)}{||x_n(t)||} \right| - y_n \leq \int_{A_1 \setminus A_n} \frac{|x_n(t)|^p}{||x_n(t)||^p} \, dt$$

$$\leq \int_{A_1 \setminus A_n} \frac{|x_n(t)|^p}{||x_n(t)||^p} \, dt + \int_{A_1 \setminus A_n} \frac{|x_n(t)|^p}{||x_n(t)||^p} \, dt$$

$$< 4^{-\alpha(j+1)p} + \sum_{l=1}^\infty \int_{A_1 \setminus A_n} \frac{|x_n(t)|^p}{||x_n(t)||^p} \, dt$$

$$< 4^{-\alpha(j+1)p} + \sum_{l=1}^\infty 4^{-lp} < 4^{-\alpha p}.$$
Corollary 7.8. (Whitney [1]). If a Banach space $X$ is reflexive, then it is superprojective (subprojective) if and only if the conjugate space $X^*$ is superprojective (subprojective).

Corollary 7.9. (Whitney [1].) Spaces with block-homogeneous bases and spaces $L^p[0, 1]$ for $1 < p < 2$ are superprojective.

§ 8. Universality of the Space $C[0, 1]$. In § 6 we have investigated the properties of the linear dimension. The following question arises: does there exist a separable Banach space $X$ such that $\dim X < \dim X'$ for every separable Banach space $X'$? Such a space $X$ will be called a universal space.

Theorem 8.1. (Banach and Mazur [1]). The space $C[0, 1]$ is universal for all separable Banach spaces. Moreover, every separable Banach space is isometrically isomorphic to a subspace of the space $C[0, 1]$.

The proof of this theorem is based on the following lemma:

Lemma 8.1. Every closed set $N$ contained in the set

$$N_x = \{ x \in (a): x = (a_1, a_2, \ldots, a_n, \ldots), |a_i| \leq 1 \ (i = 1, 2, \ldots) \}$$

is a continuous image of a closed subset $P$ of the interval $[0, 1]$.

Proof. Let $x = (a)$ be an arbitrary point of the set $N$. If the coordinate $a_i$ of the point $x$ is non-negative, then we write it by means of its binary expansion

$$0, b_0, b_1, \ldots,$$

where $b_i$ are either 0 or 1. However, if a coordinate $a_i$ is negative, we write $a_i = -1 + y_i$ where $y_i$ is a non-negative number. Hence the number $a_i$ can be written symbolically in the form

$$1, a_1 a_2 a_3 \ldots,$$

where $a_k$ are the digits of the binary expansion of the number $y_i$. Hence every coordinate $a_k$ of the point $x$ is of the form

$$a_k = a_k a_k a_k \ldots,$$

where $a_m = 0$ or $a_m = 1$.

We now associate the number $y \in (0, 1]$ of the trasic expansion

$$y = 0, a_1 a_2 a_3 a_4 a_5 a_6 \ldots$$

with the point $x$ (all digits of the trasic expansion of the number $y$ being equal either to 0 or to 1). Conversely, with every number $y \in [0, 1]$ with a trasic expansion of the form (8.1), i.e. containing only digits 0 and 1, we associate a point $x \in (a)$ with coordinates $a_k$ as follows:

$$x = 0, a_1 a_2 a_3 \ldots \ (i = 1, 2, \ldots) .$$

We consider the set $P$ of numbers $y \in [0, 1]$ which have trasic expansions consisting only of digits 0 and 1 and which correspond to points

$y \in N$ of the space $(s)$. Moreover, if $y \in P$ is trasic, we take the trasic expansion of $y$ whose all digits are equal to zero with the exception of a finite number of digits. Then every number $y \in P$ has exactly one trasic expansion of the form (8.1). Hence the correspondence

$$x = \varphi(y), \quad x \in N, \quad y \in P$$

is a one-to-one map of $P$ onto $N$. We shall prove that the transformation $x = \varphi(y)$ is continuous. Let $y_\infty \to y_\infty$ where all digits in the trasic expansions of numbers $y_\infty$ and $y$ are equal either to 0 or to 1. The number of identical digits in the trasic expansions of $y_\infty$ and $y$ increases to $\infty$ as $n \to \infty$. Let

$$a_n = \varphi(y_\infty) \quad \text{and} \quad x = \varphi(y).$$

The construction of points $a_n$ and $x$ implies that the number of first identical digits in the coordinates $a_n$ of points $a_n$ and the corresponding coordinates $x_i$ of the point $x$ increases to $\infty$. Hence $a_n \to x_i$ as $n \to \infty$ for $i = 1, 2, \ldots$. By the definition of convergence in the space $(s)$, this implies $a_n \to x$. This proves the continuity of the transformation $\varphi$.

It remains to prove that $P$ is a closed set. Let a sequence $(y_n) \subset P$ be convergent to a number $y$. All numbers $y_n$ and the limit $y$ have trasic expansions which consist of digits 0 and 1 only. Since the operator $\varphi$ is continuous, we have $a_n \to x_n$, where $a_n = \varphi(y_n)$, $x = \varphi(y)$. But $y \in P$ implies $x_n \in N$, by definition. Since the set $N$ is closed, we have $x \in N$. Consequently, $y \in P$, and the lemma is proved.

Proof of Theorem 8.1. Let $X$ be a separable Banach space; by the convergence in the ball $S(X')$ we shall understand weak convergence of functionals. By Theorem 3.2, the ball $S(X')$ is compact. Let $a_1, a_2, \ldots$, $a_m, \ldots$ be the elements of a countable set dense in the ball $S(X)$. In order to prove the weak convergence of a sequence of functionals $(f(a_n))$ in $S(X')$ to a functional $f \in S(X')$ it is sufficient to show that

$$f(a_n) \to f(a_0) \quad \text{as} \quad n \to \infty \quad \text{for} \quad n = 1, 2, \ldots.$$

Consequently, if we associate the element $(f(a_0))$ of the space $(s)$ with the functional $f \in S(X)$, the convergence of a sequence of functionals to a limit belonging to the ball $S(X')$ is equivalent to the convergence of the sequence of the respective elements to the limit element in the space $(s)$. But

$$|f(a_n)| \leq 1$$

Hence the elements of the space $(s)$ which correspond to the functionals $f \in S(X')$ constitute a set $N$ satisfying the assumptions of Lemma 8.1, because the compactness of the ball $S(X')$ implies that the set $N$ is closed.
Hence the set $X$, and consequently, also the ball $B(X^*)$ are continuous images of a closed subset $P$ of the interval $[0, 1]$. Thus to every number $t \in P$ there corresponds a functional $f_t \in S(X^*)$, the set of all functionals $f_t$ is identical with the ball $B(X^*)$, and $(f_t)$ tends to $f_t$ weakly if $t_n \to t$.

Let $x$ be an arbitrary element of the space $X$. It follows from the definition of weak convergence of functionals that

$$f_t(x) \to f(x) \quad \text{as} \quad t_n \to t.$$  

Hence, if the element $x$ is fixed, the function $f_t(x)$ is a continuous function of the variable $t \in P$. This function will be denoted by

$$f_t(x) = g_t(x).$$

We extend the function $g_t(0)$ defined on the set $P$ to the component intervals of the set $[0, 1] \setminus P$ as a continuous function linear in each of these intervals. We obtain a continuous function $g_t(0)$ defined on $[0, 1]$, i.e. belonging to the space $C([0, 1])$. By the definition of the norm in the space $C([0, 1])$, we have

$$\|g_t\|_C = \sup_{t \in [0, 1]} |g_t(0)|.$$  

Since the function $g_t(0)$ is linear in each of the component intervals of the set $[0, 1] \setminus P$, the maximum of the function $g_t(0)$ on the interval $[0, 1]$ is equal to the maximum of $g_t(0)$ on the set $P$. Hence

$$\|g_t\|_C = \max_{t \in P} |g_t(0)|.$$  

On the other hand, if $t \in P$, the definition of the function $g_t$ implies

$$(g_t(0)) = |f_t(x)| \leqslant |f_t| \cdot \|x\|_X \leqslant \|x\|_X.$$  

Thus

$$\max_{t \in P} |g_t(0)| \leqslant \|x\|_X.$$  

Let an element $x \in X$ be given. There exists a functional $g \in S(X^*)$, such that $f_t = g$. Hence

$$f_t(x) = |f_t| \cdot \|x\|_X,$$

and

$$\max_{t \in P} |g_t(0)| \leqslant \|x\|_X.$$  

Inequalities (8.3) and (8.4) imply

$$\|g_t\|_C = \max_{t \in P} |g_t(0)| \leqslant \|x\|_X.$$  

It easily follows from the construction of the function $g_t(0)$ that the functions $g_t(0)$ and $f_t(0)$ correspond to elements $x \in X$ and $y \in X$ respectively, then the function $g_t(0) + f_t(0)$ corresponds to the element $x + y$, and the function $g_t(0) + f_t(0)$, to the element $ax$. Hence the function $g_t$ is a linear operator which maps the space $X$ onto a part of the space $C([0, 1])$, isomorphically. We obtain from formula (8.5)

$$\|x - y\|_X = \|g_t - g_s\|_C.$$  

Hence $g_t$ is not only an isomorphism but also an isometry.

**Corollary 8.2.** Let $(x_n)$ be a weakly convergent sequence of elements of a Banach space $X$ such that $\inf_{n \to \infty} |x_n| = \delta > 0$. There exists a subsequence

$$(x_{n_k})$$

which is a basis of the space spanned by $(x_{n_k})$.

**Proof.** By Theorem 8.1, the space $X$ can be treated as a subspace of the space $C([0, 1])$. The space $C([0, 1])$ has a basis $(e_n)$ (Example 6.2, B III). The weak convergence of the sequence $(x_n) = \sum_{k=1}^{\infty} \alpha_k e_k$ implies

$$\lim_{k \to \infty} \|x_n - x_m\|_C = 0 \quad \text{for} \quad n, m = 1, 2, ...$$

Hence, according to Theorem 5.7, B II and Theorem 5.3, one may extract a sequence $(x_{n_k})$ which is a basis of the space spanned by this sequence.

The following theorem is another consequence of Theorem 8.1.

**Theorem 8.3.** (Sobczyk [11].) If $X$ is a separable Banach space which contains a subspace $X_0$ isomorphic to the space $c_0$, then $X_0$ is a projection of the space $X$ and the norm of this projection operator is not greater than 2.

**Proof.** By Theorem 8.1, the space $X$ can be considered as a subspace of the space $C([0, 1])$. If there exists a projection of the whole space $C([0, 1])$ onto the subspace $X_0$ with a norm not greater than 2, there exists also a projection operator of the space $X$ onto the space $X_0$ with a norm not greater than 2.

Let $Y$ be a subspace of the space $C([0, 1])$ isometrically isomorphic with the space $c_0$. Let this isomorphism transform the functions $f_n (n = 1, 2, ...)$ into unit vectors in the space $c_0$. Since $\|f\| = 1$, there exist points $t_n$ such that $f_n(t_n) = 1$ ($n = 1, 2, ...$). Let $Z$ be the set of all cluster points of the sequence $(t_n)$. Evidently, $Z$ is a closed subset of the interval $[0, 1]$. An obvious property of unit vectors in the space $c_0$ yields $\|f_n \pm f_m\| = 1$ ($n \neq m; n, m = 1, 2, ...$). Hence

$$f_n(t_n) = \begin{cases} 0 & \text{if} \quad n \neq m, \\ 1 & \text{if} \quad n = m. \end{cases}$$
If \( t \in Z \), then \( f_n(t) = \lim_{k \to \infty} f_k(t_{n_k}) = 0 \) \((n = 1, 2, \ldots)\). Finally, applying the fact that \( \{f_n\} \) is a basis of the space \( Y \) we find that \( t \in Z \) implies \( y(t) = 0 \) for every \( y \in Y \). Let \( C_Z = C([0, 1]; Z) \), i.e., let \( C_Z \) be the subspace of the space \( C([0, 1]) \) made of all functions which vanish at each point of the set \( Z \). Let us write

\[
Tz = \sum_{n=1}^{\infty} x(t_n) [\text{sgn}(f_n)] f_n \quad \text{for} \quad z \in C_Z.
\]

Since \( x \in C_Z \), we have \( \lim_{n \to \infty} x(t_n) = 0 \). It follows from the definition of the functions \( f_n \) that \( T \) is a well-defined linear operator which maps the space \( C_Z \) onto the space \( Y \). Since \( Y \subset C_Z \) and the sequence \( \{f_n\} \) is a basis of the subspace \( Y \) and \( T(f_n) = f_n \), we conclude that \( T \) is a projection operator of the space \( C_Z \) onto the space \( Y \). Moreover,

\[
\|T\| = \sup_{z \in C_Z} \|T z\| = \sup_{n} \|x(t_n)\| = 1.
\]

In order to complete the proof it is sufficient to show that the space \( C_Z \) is a projection operator of the space \( C([0, 1]) \) with a norm not greater than 2. We extend the functions from the space \( C_Z \) to the set \([0, 1]\) \(Z\) linearly in the same manner as in the proof of Theorem 8.1. Let us remark that if we associate the function

\[
Q(x) = \begin{cases} 
  x(t) & \text{for} \ t \in Z, \\
  \text{linear on each of the component intervals of the set} \ [0, 1]\end{cases}
\]

with the function \( x \in C([0, 1]) \), this correspondence is a projection operator with a norm not greater than 2. 

§ 9. Separable Banach space as a continuous image of the space \( l \).

In view of Theorem 2.6, the next theorem can be considered to be dual to the theorem on the universal space.

**Theorem 9.1.** (Banach and Mazur [11]) Every separable Banach space \( X \) is a continuous image of the space \( l \).

**Proof.** Let \( (x_n) \) be a sequence dense in the ball \( S(X) \). Let \( A \in B(l \to X) \) be an operator of the form

\[
A((t_n)) = \sum_{n=1}^{\infty} t_n x_n.
\]

Since the sequence \( (x_n) \) is bounded, it is easily verified that the operator \( A \) is continuous. It remains to prove that \( A \) is an epimorphism. Let \( x \in S(X) \). By hypothesis the sequence \( (x_n) \) is dense. Hence one can choose a subsequence \( (x_{n_k}) \) and a sequence of numbers \( \{t_n\} \) such that

\[
\|x - \sum_{n=1}^{\infty} t_n x_n\| < \frac{1}{2^n}.
\]

Hence \( A((t_{n_k})) = x \), where

\[
t_n' = \begin{cases}
  0 & \text{for} \ n \neq n_k, \\
  t_{n_k} & \text{for} \ n = n_k.
\end{cases}
\]

**Remark 9.1.** The above theorem remains true without any changes also in the case of non-separable spaces. Only the space \( l \) is replaced by the space \( l(\Omega) \), where \( \Omega \) is a set of the same power as a dense set in the space \( X \).
CHAPTER II
PARAALGEBRAS OF OPERATORS OVER BANACH SPACES

§ 1. Fundamental properties of Banach algebras and paraalgebras. A Banach algebra is an algebra which is a Banach space such that multiplication of elements is continuous with respect to each variable separately.

Every Banach algebra $X$ without unity can be extended to a Banach algebra with unity if we add the unity to $X$ formally. Namely, we consider the algebra of all formal sums of the form $a + x$, where $a$ is an arbitrary scalar, $x$ is an arbitrary element of the algebra $X$, and $e$ denotes the unity (see Theorem 0.1). The norm of the element $a + x$ is defined as follows:

$$||a + x|| = |a| + ||x||.$$ 

Hence we may limit ourselves to the consideration of Banach algebras with a unity. In the sequel we shall assume that a Banach algebra has a unity.

The algebra $B(X)$ of all continuous operators which map the space $X$ into itself is an example of a Banach algebra. The norm in the algebra $B(X)$ is defined as the norm of operators. Let us remark that

$$||AB|| \leq ||A|| ||B||.$$ 

Indeed,

$$||AB|| = \sup_{||a|| \leq 1} ||AB(a)|| = \sup_{||a|| \leq 1} ||A(a)|| \leq ||A|| ||B||.$$ 

Evidently, the unity of the algebra $B(X)$ is the identity operator $I$.

A paraalgebra will be called a paraalgebra $B = \left( A_1, S_1, S_2, A_3 \right)$, where $A_1, A_2, S_1, S_2$ are Banach spaces and the multiplication of elements is continuous with respect to each variable separately.

The norm in a Banach paraalgebra $P$ is defined as a function defined on the set of that paraalgebra and such that

- $||ax|| = |a||x|$ for an arbitrary scalar $a$,
- $||x + y|| \leq ||x|| + ||y||$ if the operation $x + y$ is performable,
- $||x|| = 0$ if and only if $x = 0$ in one of the spaces $A_1, A_2, S_1, S_2$.

Evidently, norms in spaces $A_1, A_2, S_1, S_2$ define a norm in the paraalgebra and, conversely, a norm in the paraalgebra induces norms in spaces $A_1, A_2, S_1, S_2$.

Two norms $||\cdot||, ||\cdot||$ in a Banach paraalgebra $P$ are called equivalent if the corresponding norms induced in spaces $A_1, A_2, S_1, S_2$ are equivalent.

Arguing as in the case of Banach algebras, one may prove that every Banach paraalgebra can be embedded in a Banach paraalgebra with unities $e_i$ $(i = 1, 2)$. Therefore in the sequel we shall consider Banach paraalgebras with unities only.

The paraalgebra of operators $B(X \Rightarrow Y)$ is an example of a Banach paraalgebra.

**Theorem 1.1.** In every Banach paraalgebra $P = \left( A_1, S_1, A_2 \right)$ with unities $e_i \in A_i$ there is a norm $||\cdot||$ equivalent to the given norm $||\cdot||$ and such that

$$||xy|| \leq ||x|| ||y||,$$

$$||x|| = 1 \quad (i = 1, 2).$$

**Proof.** By Theorem 10.1, $A_1$, every paraalgebra $P$ can be represented as a paraalgebra of operators $P(X \Rightarrow Y)$, where $X = A_1 \times S_1$, $Y = A_2 \times S_2$; with every element $x$ we associate an operator $A_x$.

We define the following norms in spaces $X$ and $Y$:

$$||x||_0 = \max \{ ||x||, ||x|| \}, \quad x = (u, s), \quad u \in A_1, \quad s \in S_1,$$

$$||y||_0 = \max \{ ||y||, ||y|| \}, \quad y = (u, s), \quad u \in A_1, \quad s \in S_2.$$ 

By the continuity of multiplication it is easily verified that the operators $A_x$ are continuous. We define the norm of the operator $A_x$ as follows:

$$||A_x|| = \sup_{||x||_0 < 1} ||xy||_0.$$ 

But, by the definition of the norm, $||xy||_0 = \max \{ ||xy||, ||xy|| \}$, where $u \in A_1, \quad s \in S_1 (i = 1$ or $2)$ depending on the space to which $x$ belongs.

Let us remark that $A_x(y) = x(y) = (xy)s = (A_x y)s$. But if the operator $A$ satisfies the equality $(Axy) = (Ax,y)$, then

$$A(x) = (Ax,y),$$

where we write $Ax = x$, $e_i$ being a unity such that the operation $e_i y$ is performable. In other words, the operator $A$ generates an operator of multiplication by the element $x$, i.e. $A \in P(X \Rightarrow Y)$.

Using this fact we now show that the paraalgebra

$$P(X \Rightarrow Y) = \left( A(X), S_1(X \Rightarrow Y), A_2(Y) \right)$$

is equivalent to the paraalgebra $P$.
is complete with respect to the norm \( || \cdot || \), i.e. that all four spaces \( A_{i}(X), A_{j}(Y), S_{i}(X \rightarrow Y), S_{j}(X \rightarrow Y) \) are complete with respect to the norm \( || \cdot || \).
Indeed, let a sequence of operators \( (A_{i})_{C} P(X \rightarrow Y) \) be convergent to an operator \( A \) in the norm. Then
\[
A(x) = \lim_{n \to \infty} A_{n}(x) = \lim_{n \to \infty} (A_{n}x)y = (Ax)y
\]
for every \( x \) and \( y \). Hence \( A \in P(X \rightarrow Y) \) and the paralgebra \( P(X \rightarrow Y) \) is complete with respect to the norm \( || \cdot || \).

Let us remark that if \( \epsilon_{i} \) is a unity such that \( x \cdot \epsilon_{i} = x \), then
\[
||A|| = \sup_{i \geq 2} \left( |x - \epsilon_{i}| \right) = \frac{||x||}{||x||}\epsilon_{i}
\]

Hence the transformation of the paralgebra \( P(X \rightarrow Y) \) onto the paralgebra \( P \) is continuous in the sense that the transformation \( A_{i} \rightarrow x \) of each of the spaces \( A_{i}(X), A_{j}(Y), S_{i}(X \rightarrow Y), S_{j}(X \rightarrow Y) \) onto the respective spaces \( A_{i}, A_{j}, S_{i}, S_{j} \) is continuous. Since the spaces \( A_{i}, A_{j}, S_{i}, S_{j} \) are complete, Theorem 3.2, B I, implies that the inverse map is also continuous.

Thus, taking \( ||x|| = ||x|| \) we obtain a norm satisfying the theorem.

**Theorem 1.2.** If \( B \in B(X) \) and \( ||B|| < 1 \), then the element \( I - B \) is invertible.

Proof. We shall show that
\[
(I - B)^{-1} = \sum_{n=0}^{\infty} B^{n}
\]

Indeed, the series on the right-hand side of (1.1) is convergent.

Moreover,
\[
(I - B)^{n+1} = I - BR^{n-1} \quad \text{as} \quad n \to \infty.
\]

Hence formula (1.1) is an immediate consequence of the continuity of multiplication.

**Theorem 1.3.** If \( J \) is a proper left ideal (right ideal, two-sided ideal) in a Banach algebra \( X \), then the closure of \( J \) is also a proper left ideal (right ideal, two-sided ideal).

Proof. We prove the theorem for left ideals; the proof in the case of a right ideal or a two-sided ideal is analogous. Let \( y \in J \); then there exists a sequence \( y_{n} \rightarrow y \) in \( J \). If \( x \in X \), then \( xy_{n} \rightarrow xy \). Consequently, \( xy \in J \). But \( J \) is a proper ideal. By Theorem 1.2, \( J \) does not contain a certain neighbourhood of the element \( e \). Hence the ideal \( J \) also does not contain a certain neighbourhood of the element \( e \). Thus \( J \) is a proper ideal.

\section{Properties of Banach algebras and paralgebras}

**Theorem 1.4.** Every maximal left ideal (right ideal, two-sided ideal) in a Banach algebra \( P \) with unital \( J \) is closed.

Proof. Evidently, \( J \) is closed. By Theorem 1.2, \( e_{i} \in J \) \((i = 1, 2)\). Hence \( J \) is a proper ideal. Since \( J \) is maximal, we have \( J = \{ e \} \).

**Theorem 1.5.** A radical in a Banach algebra \( P = \left( A_{1}, A_{2}, A_{3} \right) \) is closed.

Proof. Let the element \( x \) belong to the closure \( \overline{R} \) of a radical \( R \). Let \( a \) and \( b \) be two elements of the paralgebra such that \( a \neq b \in A_{i} \) \((i = 1, 2)\). Since \( x \in \overline{R} \), there exists an element \( x_{0} \in R \) such that \( ||x - x_{0}|| < 1/||b|| \). By Theorem 1.2, the element \( e_{i} + a(x - x_{0})b \) is invertible.

Since \( x_{0} \in \overline{R} \), the element
\[
eq \left( e_{i} + a(x - x_{0})b \right) \left( e_{i} + a(x - x_{0})b \right)^{-1}a_{n}b = (e_{i} + a(x - x_{0})b)^{-1}a_{n}b
\]

is invertible. Hence \( x \in \overline{R} \).

Let a Banach algebra \( X \) be given. According to the definition of an analytic function with values in a linear metric space (see § 11, B I) we say that a function \( \psi(\lambda) \) defined in a domain \( G \) of the complex plane and having values in an algebra \( X \) is analytic if for every \( \lambda_{0} \in G \) the function \( \psi(\lambda) \) can be expanded in a power series
\[
\psi(\lambda) = \sum_{n=0}^{\infty} (\lambda - \lambda_{0})^{n} \psi_{n} \quad (\lambda_{0} \in X)
\]

in a neighbourhood of the point \( \lambda_{0} \).

Evidently, if the function \( \psi(\lambda) \) is analytic, then for every \( \lambda_{0} \in G \), a derivative of the function \( \psi(\lambda) \) exists at the point \( \lambda_{0} \):
\[
\psi'(\lambda_{0}) = \lim_{\lambda \to \lambda_{0}} \frac{\psi(\lambda + h) - \psi(\lambda)}{h}
\]

If a linear continuous functional \( f(\lambda) \) is defined on an algebra \( X \), then the function \( F(\lambda) = f(\psi(\lambda)) \) is an analytic function of the variable \( \lambda \). Indeed, the function \( F(\lambda) \) possesses a derivative:
\[
F'(\lambda) = \lim_{h \to 0} \frac{F(\lambda + h) - F(\lambda)}{h} = \lim_{h \to 0} \frac{f(\psi(\lambda + h)) - f(\psi(\lambda))}{h}
\]

Since the set of all linear continuous functionals defined on the algebra \( X \) is total, the analyticity of \( F(\lambda) \) implies the following

**Theorem 1.6.** If \( e \) is the unity of a Banach algebra \( X \), then for every \( e \in X \) there exists a number \( \lambda \) such that the element \( e - \lambda e \) is not invertible.

Equations in linear spaces
Proof. Let us suppose that such a \( \lambda \) does not exist, i.e., that the element \( z - \lambda e \) is invertible for every number \( \lambda \). Let us remark that the function \( (z - \lambda e)^{-1} \) is analytic, because

\[
\lim_{\lambda \to -\infty} (z - (\lambda + h)) e^{-z} = (z - \lambda e)^{-1}.
\]

Let \( f \) be an arbitrary linear continuous function defined on the algebra \( X \). Then the function \( F(\lambda) = f((z - \lambda e)^{-1}) \) is analytic. Moreover,

\[
\lim_{|\lambda| \to \infty} F(\lambda) = \lim_{|\lambda| \to \infty} f((z - \lambda e)^{-1}) = 0.
\]

Hence the function \( F(\lambda) \) is bounded. By the Liouville theorem, \( F(\lambda) \) is constant. But \( \lim_{|\lambda| \to \infty} F(\lambda) = 0 \), hence \( F(\lambda) = 0 \), i.e., \( f(z^{-1}) = F(0) = 0 \). Since the functional \( f \) is arbitrary, it follows that \( z^{-1} = 0 \), which is impossible.

If every element \( x \neq 0 \) of a ring \( X \) is invertible, then the ring is called a field. A Banach algebra which is a field is called a normed field.

Corollary 1.7. (Gelfand [1], Mazur [1]) A normed field over the field of complex numbers is a field of complex numbers.

Proof. Let us suppose that there exists a normed field \( \mathbb{K} \) different from the field of complex numbers. Then there exists an element \( z \in \mathbb{K} \) such that \( z^{-1} \neq 0 \), which is impossible by Theorem 1.6.

§ 2. Compact operators. As we have seen in § 2, B IV, the set \( T(X \to Y) \) of compact operators is a two-sided proper ideal in the paralgebra \( B(X \to Y) \). Example 3.1, Chapter IV, shows that in the general case this ideal is not necessarily closed. As a consequence of Theorem 3.2, B IV, we obtain the following:

Theorem 2.1. If \( X \) and \( Y \) are Banach spaces, then the ideal \( T(X = Y) \) is closed.

Theorem 2.2. If \( X \) and \( Y \) are Banach spaces and the operator \( T \in B(X \to Y) \) is compact, then the operator \( T^* \in B(Y^* \to X^*) \) is compact.

Proof. Let \( S(X) = \{z \in X: ||z|| < 1\} \). In order to show that \( T^* \) is compact it is sufficient to prove the compactness of the set \( T^*S(X^*) \), where \( S(X^*) = \{z^* \in X^*: ||z^*|| < 1\} \). The set \( T^*S(X^*) \) is compact by hypothesis. Hence there exists a finite system of points \( y_n = T z_n \in T^*S(X) \) \((n = 1, 2, \ldots, k)\) which is an \( \varepsilon \)-net. Moreover, since the set \( S(X) \) is bounded, there exists a system of functions \( z_{n}^* \) \((t = 1, 2, \ldots, k')\) such that for every \( z^* \in S(X^*) \) we have

\[
\inf_{t = t_{i} \in k'} \sup_{n} ||z^*(y_n) - z_{n}^*(y_n)|| < \varepsilon.\]

Hence for every \( y \in TS(X) \),

\[
||z^*(y) - z_{n}^*(y)|| < ||z^*(y) - z^*(y_n)|| + ||z^*(y_n) - z_{n}^*(y_n)|| + ||z_{n}^*(y_n) - z_{n}^*(y)|| < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.
\]

This means that for every \( z^* \in S(X^*) \) there exists an index \( i \) such that

\[
||T^*(z^* - z_n^*)|| = \sup_{x \in S(X)} ||T^*(z^* - z_n^*)x|| = \sup_{x \in S(X)} (||z^* - z_n^*|| ||x||) < \varepsilon.\]

Thus the points \( T^*x_n^* \) form an \( \varepsilon \)-net in the set \( T^*S(X^*) \). Since the number \( \varepsilon \) is arbitrary, the set \( T^*S(X^*) \) is compact. Hence the operator \( T^* \) is compact.

Corollary 2.3. If \( X \) and \( Y \) are Banach spaces, \( T \in B(X \to Y) \) and the operator \( T^* \in B(Y^* \to X^*) \) is compact, then the operator \( T \) is compact.

Proof. By Theorem 2.2, the operator \( T^* \in B(Y^* \to X^*) \) is compact. But the operator \( T \) is a restriction of the operator \( T^* \) to the space \( s \) \((\text{see } \S 1, 1\)) \( Y \). Hence \( T \) is compact.

The following theorem holds for algebras \( B(X) \) over Banach spaces with a block-homogeneous basis:

Theorem 2.4. (Godsberg, Markus, Feldman [1]) If a block-homogeneous basis exists in a Banach space \( X \), then the only proper closed twosided ideal contained in the algebra \( B(X) \) of linear continuous operators is the ideal \( T(X) \) of compact operators.

Proof. Let us suppose that a linear continuous operator \( T \) is not compact. By Theorem 6.8, B II, there exists a number \( \eta > 0 \) and a sequence \( \{e_n\} \), \( ||e_n|| = 1 \), of the form

\[
e_n = \sum_{k=1}^{\infty} \lambda_k e_k, \quad \text{where} \quad \lim_{n \to \infty} \lambda_k = 0.
\]

such that \( ||T e_n|| > \eta \).

By Theorem 5.7, B II, there exist a subsequence \( \{e_n\} \) and an increasing sequence of indices \( \{p_n\} \) such that

\[
||T e_n - \sum_{n=1}^{p_n} \lambda_{n} e_k|| < \frac{\eta}{2 ||T||}.
\]

Let

\[
\sum_{n=1}^{p_n} \lambda_{n} e_k.
\]
It is easily verified that \( \|Tx_n\| > \eta/2 \). Let \( y_k = Tx_n \). If \( y_k = \sum_{j=1}^{\infty} (\eta_j y_k) \), the definition of the elements \( \eta_j \) implies the equality

\[
\lim_{k \to \infty} \eta_j^p y_k = 0.
\]

Hence there exist a subsequence \( \{y_{k_j}\} \) and a sequence of indices \( \{p_j\} \) satisfying the inequality

\[
|y_{k_j} - \sum_{i=p_j+1}^{\infty} \eta_i^p y_{k_j}| < \frac{1}{2^n}.
\]

It follows from the block-homogeneity of the basis that the operator \( A \) satisfying the equalities

\[
A\epsilon_j = \eta_j^p, \quad (j = 1, 2, \ldots)
\]

is linear and continuous. Hence \( A \in B(X) \).

Let us denote by \( Y_n \) the space spanned by the elements \( \{y_k\} \). By Theorem 5.2, 1, the basis \( \{y_k\} \) is equivalent to the basis \( \{y_{k_j}\} \), where \( y_{k_j} = \sum_{i=p_j+1}^{\infty} \eta_i^p y_{k_j} \). Applying block-homogeneity we find that \( \{y_{k_j}\} \) is equivalent to the basis \( \{\epsilon_j\} \). Hence the operator \( B \) defined on \( Y_n \) by the equalities

\[
B y_{k_j} = \epsilon_j, \quad (j = 1, 2, \ldots)
\]

is linear and continuous. Arguing as in the proof of Theorem 7.2, 1, we can show that the space \( Y \) spanned by the elements \( \{y_{k_j}\} \) is a projection of the space \( X \). According to Theorem 7.1, 1, the space \( Y_n \) is also a projection of the space \( X \). Hence the operator \( B \) can be extended to the whole space, \( B \in B(X) \). Consequently, we have

\[
BTA\epsilon_j = \epsilon_j, \quad (j = 1, 2, \ldots), \quad \text{i.e.} \quad BTA = I.
\]

Thus the operator \( T \) cannot belong to any non-trivial ideal.

**Remark 2.1.** In the above proof we did not make use of the fact that the ideal \( I(X) \) is closed. Hence the result can be formulated as follows:

**Theorem 2.4.** Let \( X \) be a Banach space with a block-homogeneous basis. If an operator \( T \in B(X) \) is not compact and belongs to a certain two-sided proper ideal \( J \subset B(X) \), then \( J = B(X) \).

**§ 3. The ideal of compact operators over Banach spaces containing \( p \).**

In the last section we proved that if there is a block-homogeneous basis in a Banach space \( X \), then the algebra \( B(X) \) possesses only one closed two-sided proper ideal, namely the ideal \( I(X) \) of compact operators. The following question arises: Suppose \( B(X) \) contains only one closed two-sided proper ideal: does there exist a block-homogeneous basis in the space \( X \)? A complete answer is not known. However, there exist some partial results which we shall quote here.

Let \( 1 \leq p < +\infty \). We denote by \( p \) the space \( e_q \). Although this is not conventional, it will enable us to conduct the proofs in a uniform manner. The formula \( \sum_{n=1}^{\infty} |e_n|^p < +\infty \) or the series \( \{\sum_{n=1}^{\infty} |e_n|^p \} \) is convergent, will mean \( \lim_{n \to \infty} e_n = 0 \).

Let \( X \) be a Banach space with a basis \( \{e_n\} \). We denote by \( I_p \) the set of operators satisfying the condition: an operator \( A \in B(X) \) belongs to \( I_p \) if for an arbitrary sequence obtained from the basis by a linear transformation \( B \in B(X) \),

\[
y_n = Be_n,
\]

and for an arbitrary sequence \( \{e_n\} \) of coefficients of expansion of any element \( x \) in terms of the basis, \( x = \sum e_n e_n \), the series \( \sum_{n=1}^{\infty} |e_n A e_n|^p \) is convergent.

**Lemma 3.1.** The set \( I_p \) is a two-sided ideal in the algebra \( B(X) \).

**Proof.** We show that

(i) if \( A \in B(X) \), \( T \in I_p \) then \( AT \in I_p \) and \( TA \in I_p \),

(ii) if \( T_1, T_2 \in I_p \), then \( T_1 + T_2 \in I_p \).

In order to prove (i) we show that the series

\[
\sum_{n=1}^{\infty} |e_n A T e_n|^p \quad \text{and} \quad \sum_{n=1}^{\infty} |e_n T A e_n|^p
\]

are convergent. The first series is convergent because \( A \) is a continuous operator. The convergence of the second one follows from the equality

\[
\sum_{n=1}^{\infty} |e_n T A e_n|^p = \sum_{n=1}^{\infty} |e_n T e_n|^p,
\]

where \( 0 = A e_n = A e_n \), because the series \( \sum_{n=1}^{\infty} |e_n A e_n|^p \) is convergent.

In order to prove (ii) we must show that the series

\[
\sum_{n=1}^{\infty} |e_n (T_1 + T_2) e_n|^p
\]

is convergent. However, we have

\[
\sum_{n=1}^{\infty} |e_n (T_1 + T_2) e_n|^p \leq C \left( \sum_{n=1}^{\infty} |e_n T_1 e_n|^p + \sum_{n=1}^{\infty} |e_n T_2 e_n|^p \right),
\]
where \( C = \max_{|x|=1,|y|=1} |x+y|^{p} \), and the series on the right-hand side of this inequality are convergent, by the definition of the set \( I_{p} \).

The closure \( I_{p} \) of the ideal \( I_{p} \) is also an ideal.

We now prove the following

**Lemma 3.2.** If there exists an element \( x = \sum_{n=1}^{\infty} a_{n} e_{n} \) such that the series \( \sum_{n=1}^{\infty} ||a_{n}||_{p} e_{n} \) is divergent to infinity, then \( I \not\subset I_{p} \).

Proof. Let us suppose that the identity \( I \) belongs to \( I_{p} \). Then there exists an operator \( A \in I_{p} \) such that

\[ ||I - A|| < 1. \]

Hence the operator \( A = I - (I - A) \) is invertible. Let us write \( y_{n} = A^{-1} a_{n} \). Evidently,

\[ \sum_{n=1}^{\infty} ||a_{n} A y_{n}||^{p} = \sum_{n=1}^{\infty} ||a_{n} e_{n}||^{p}, \]

and the series \( \sum_{n=1}^{\infty} ||a_{n} A y_{n}||^{p} \) is divergent, in contradiction to the assumption \( A \in I_{p} \). Thus \( I \not\subset I_{p} \).

Let us denote by \( \mathcal{K}_{p} \) the set of operators satisfying the following condition: an operator \( A \in B(X) \) belongs to the set \( \mathcal{K}_{p} \) if for every sequence \( (a_{n}) \) such that the series \( \sum_{n=1}^{\infty} ||a_{n}||^{p} \) is convergent and for every sequence \( (y_{n}) \) which is the image of the basis by means of a linear transformation \( B \in B(X) \): \( y_{n} = B a_{n} \), the series

\[ \sum_{n=1}^{\infty} ||a_{n} A y_{n}||^{p} \]

is convergent.

**Lemma 3.3.** The set \( \mathcal{K}_{p} \) is a two-sided ideal.

Proof. We show that

(a) if \( A \in B(X) \), \( T \in \mathcal{K}_{p} \), then \( A T \in \mathcal{K}_{p} \), \( TA \in \mathcal{K}_{p} \),

(b) if \( T_{1}, T_{2} \in \mathcal{K}_{p} \), then \( T_{1} + T_{2} \in \mathcal{K}_{p} \).

The proof of condition (a) is identical with the proof of condition (i) in Theorem 4.1. In order to prove (b) let us remark that the series

\[ \sum_{n=1}^{\infty} ||a_{n} (T_{1} + T_{2}) y_{n}||^{p} = \sum_{n=1}^{\infty} ||a_{n} T_{1} y_{n}||^{p} + \sum_{n=1}^{\infty} ||a_{n} T_{2} y_{n}||^{p} \]

is convergent, as the sum of two convergent series.

We denote by \( \overline{\mathcal{K}}_{p} \) the closure of the ideal \( \mathcal{K}_{p} \). Evidently, \( \overline{\mathcal{K}}_{p} \) is an ideal.

§ 3. The ideal of compact operators

**Lemma 3.4.** If the series \( \sum_{n=1}^{\infty} ||a_{n}||^{p} \) is convergent and the series \( \sum_{n=1}^{\infty} ||a_{n}||^{p} \) is not convergent, then the identity \( I \) does not belong to \( \overline{\mathcal{K}}_{p} \).

Proof. Let us suppose that \( I \in \overline{\mathcal{K}}_{p} \). Then there exists an operator \( A \in \mathcal{K}_{p} \) such that

\[ ||I - A|| < 1. \]

Hence \( A \) possesses the inverse \( A^{-1} \). Let us write \( a_{n} = A^{-1} e_{n} \). By the assumption of the lemma, the series \( \sum_{n=1}^{\infty} ||a_{n}||^{p} \) is not convergent, contradicting the assumption \( A \in \mathcal{K}_{p} \). Hence \( I \not\in \mathcal{K}_{p} \).

Applying Lemmas 3.2 and 3.4 we prove the following

**Theorem 3.5.** Let \( (e_{n}) \) be an unconditional basis of a space \( X \), \( ||e_{n}|| = 1 \), and let \( X \) contain a subspace \( X_{0} \), spanned by the elements of the basis and such that expansions of elements belonging to \( X_{0} \) constitute the space \( p \), \( p > 1 \).

If the algebra \( B(X) \) contains only one closed two-sided proper ideal, then the coefficients of expansions with respect to the basis \( (e_{n}) \) form the space \( p \).

Proof. We show that the series \( \sum_{n=1}^{\infty} ||a_{n}||^{p} \) is convergent if and only if the series \( \sum_{n=1}^{\infty} ||a_{n}||^{p} \) is convergent.

Let us suppose that the series \( \sum_{n=1}^{\infty} ||a_{n}||^{p} \) is convergent and the series \( \sum_{n=1}^{\infty} ||a_{n}||^{p} \) is not convergent. By Lemma 3.4, \( I \not\in \mathcal{K}_{p} \). Hence the ideal \( \overline{\mathcal{K}}_{p} \not\neq B(X) \). Applying the remark following Theorem 6.1, I, we find that the space \( X_{0} \) contains a subspace \( X_{0} \), spanned by such vectors \( e_{n} \), of the basis that the coefficients of expansion of every element from the space \( X_{0} \) are summable with power \( p \). The projection operator \( P_{X_{0}} \) of the space \( X \) onto the space \( X_{0} \) belongs to the ideal \( \mathcal{K}_{p} \), but obviously \( P_{X_{0}} \) does not belong to the ideal of compact operators \( T(X) \). Hence it follows that there exists a proper closed two-sided ideal in the algebra \( B(X) \), different from the ideal \( T(X) \), which is a contradiction.

On the other hand, if we suppose that the series \( \sum_{n=1}^{\infty} ||a_{n}||^{p} \) is divergent, then the equality \( ||e_{n}|| = 1 \) implies that the series

\[ \sum_{n=1}^{\infty} ||a_{n}||^{p} = \sum_{n=1}^{\infty} ||a_{n}||^{p} \]

is divergent.

Lemma 3.2 implies \( I \not\in \mathcal{I}_{p} \). Hence \( \mathcal{I}_{p} \neq B(X) \).
The operator $P_{X'}$ belongs to the ideal $I_p$, but $P_{X'} \notin I_p$. Hence there exists a closed two-sided ideal in the algebra $B(X)$, which is proper, a contradiction. □

**Corollary 3.6.** Let $\{e_n\}$ be an unconditional basis of a space $X$, and let $X$ contain a subspace spanned by the elements of the basis and expansions of elements belonging to $X$ constitute the space $\mathcal{P}$. The spaces $X$ and $\mathcal{P}$ are isomorphic if and only if the sequences of coefficients $\{a_n\}$ form the space $\mathcal{P}$.

**Proof.** If the spaces $X$ and $\mathcal{P}$ are isomorphic, then there exists one proper closed two-sided ideal in the algebra $B(X)$ (Theorem 2.4). By Theorem 3.5, the series $\sum \|a_n\|_P$ is convergent.

Let us now suppose that the series $\sum \|a_n\|_X$ is convergent. We define new norms in the space $X$ as follows:

$$\|x\|_X = \|x\|_p + \|x\|_Y.$$ The sequence $\{a_n\}$, considered as a sequence of linear functionals, is convergent to the same element in the norms $\|a_n\|_p$ and $\|a_n\|_X$. By the Banach theorem (Theorem 3.2, B II), the norm $\|x\|_p\|_p$ is equivalent to either of the norms $\|x\|_p$ and $\|x\|_X$. Hence the norms $\|x\|_p$ and $\|x\|_X$ are equivalent. Consequently, the spaces $X$ and $\mathcal{P}$ are isomorphic. □

**Example 3.1.** We denote by $P_X^* (p_n \to p, p_n \geq 1)$ the space of all sequences $x = (x_n)$ of real or complex numbers such that

$$\varphi(x) = \sum_{n=1}^{\infty} \|x_n\|^p < +\infty.$$ We define a norm in this space as follows:

$$\|x\| = \inf_{p_n \to p} (\varphi(x_n) < 1).$$ It is easily verified that the standard basis in $P_X^*$ is an unconditional basis.

Let us consider a sequence $\{a_n\}$ such that

$$\|a_n\|_P - \|a_n\|_P < \frac{1}{2^n}.$$ The spaces $P^*$ (spanned by elements $\{a_n\}$) and $\mathcal{P}$ are isomorphic. Applying Corollary 3.6 we find that the spaces $P^*$ and $\mathcal{P}$ are isomorphic if and only if the series $\sum \|a_n\|^p$ is convergent.

In the special case $P = \mathcal{P}$ the assumptions of Theorem 3.5 can be weakened, namely:

**Theorem 3.7.** Let a Banach space $X$ with an unconditional basis $\{e_n\}$, $\|e_n\|^p = 1$, contain a subspace $Y$ as a projection, and let the spaces $Y$ and $\mathcal{P}$ be isomorphic. If there exists only one closed two-sided proper ideal in the algebra $B(X)$, then the coefficients of expansions with respect to the basis $\{e_n\}$ form the space $\mathcal{P}$.

**Proof.** Let us denote by $J_p(X)$ the set of operators $A$ such that the series $\sum \|Aa_n\|^p$ is convergent for every unconditionally convergent series $\sum a_n$. As in case of the ideal $I_p$, we prove $J_p(X)$ to be an ideal.

If there exists an unconditionally convergent series $\sum a_n$ such that $\sum \|a_n\|^p = +\infty$, then we prove $I \neq J_p(X)$ as in Lemma 3.2. By Theorem 5.1, the projection operator $P$ onto the space $Y$ belongs to the ideal $J_p(X)$. Evidently, $P$ is not a compact operator.

Let us consider the conjugate space $X^*$. Let $J_p$ denote the set of operators for which the conjugate operators belong to the ideal $J_p(X^*)$. Evidently, $J_p$ is a closed two-sided ideal. If there exists an unconditionally convergent series $\sum e_n$ in the conjugate space $X^*$ such that $\sum \|e_n\|^p = +\infty$, then $I \neq J_p(X^*)$. Hence $I \neq J_p$. On the other hand, we have $P \cap J_p$. Thus $T(X) \neq J_p \neq B(X)$.

Hence, if there exists only one closed two-sided proper ideal in the algebra $B(X)$, then the series $\sum \|a_n\|^p$ is convergent for every unconditionally convergent series $\sum a_n$. Moreover, the series $\sum \|a_n\|^p$ is convergent for every unconditionally convergent series $\sum e_n$ in the conjugate space $X^*$. Hence, by Theorem 5.3, $J_p$, the spaces $X$ and $\mathcal{P}$ are isomorphic. Moreover, the sequences of coefficients with respect to the basis belong to the space $P$. □

§ 4. Weakly compact operators. Let $X$ and $Y$ be Banach spaces and let $S(X)$ denote the unit ball in the space $X$. An operator $T \in B(X \to Y)$ is called weakly compact if the weak closure of the set $TS(X)$ is a compact set in the weak topology of the space $Y$.

Since a weakly compact operator $T \in B(X \to Y)$ may be treated as a compact operator in the space $B(X \to Y)$, where $Y$ is the space $Y$ provided with a weak topology, a linear combination of weakly compact operators is a weakly compact operator. Moreover, by Theorem 10.4 B I, every continuous operator is weakly continuous; hence the superposition of two operators $AB$ such that one of the operators is continuous and the
other one is weakly compact, a weakly compact operator. Hence it follows that the set \( W(X \in Y) \) of weakly compact operators constitutes a two-sided ideal in the paranalegbra \( B(X \in Y) \). This ideal is not necessarily a proper one. If the spaces \( X \) and \( Y \) are reflexive, then every operator belonging to the paranalegbra \( B(X \in Y) \) is weakly compact (see Corollary 4.2).

**Theorem 4.1.** An operator \( T \in B(X \in Y) \) is weakly compact if and only if \( T^{+*}X^{+*} \subset X^{*} \), where \( * \) denotes the natural embedding of the space \( Y \) into the space \( Y^{*} \).

**Proof.** We write briefly \( S = S(X) \) and \( S^{+*} = S(X^{+*}). \) Let us remark that, by Theorem 10.4, Chapter I, Part B, the operator \( T^{+*} \) is continuous both in the \( X^{+*} \)-topology of the space \( X^{+*} \) and in the \( Y^{*} \)-topology of space \( Y^{*} \). But the operator \( T^{+*} \) is an extension of the operator \( T \). Hence, denoting by \( S \), the \( X^{+*} \)-closed of the set \( S \), we obtain for closures in the \( Y^{*} \)-topology

\[
T^{+*}(S) \subset T^{+*}(\wedge S) = \wedge (TS) \subset \wedge (TS).
\]

If the operator \( T \) is weakly compact, then the set \( TS \) is compact in the \( Y^{*} \)-topology of the space \( Y \). Hence the set \( \wedge (TS) \) is compact. Consequently, it is also closed in the \( Y^{*} \)-topology of space \( Y^{*} \). We conclude from formula (4.1) that if \( T \) is a weakly compact operator, then

\[
T^{+*}(S) \subset \wedge (TS).
\]

But, by Theorem 3.1, I, we have \( S = S^{+*} \). Hence \( T^{+*}S^{+*} \subset \wedge (TS) \), and consequently,

\[
T^{+*}X^{+*} \subset X^{*}.
\]

Conversely, let us suppose that the operator \( T \in B(X \in Y) \) satisfies condition (4.2). By Theorem 10.4, B, I, the operator \( T^{+*} \) is continuous both in the \( X^{+*} \)-topology of the space \( X^{+*} \) and in the \( Y^{*} \)-topology of the space \( Y^{*} \), and the set \( S^{+*} \) is compact in the space \( X^{+*} \) (Theorem 3.1, I). Hence the set \( T^{+*}S^{+*} \subset X^{*} \) is \( Y^{*} \)-compact. Consequently, the \( Y^{*} \)-homeomorphic image \( \Lambda (TS) \) of the set \( TS \) is a subset of a \( Y^{*} \)-compact subset of \( Y \). Hence it follows that the \( Y^{*} \)-closure of the set \( \wedge (TS) \) is a \( Y^{*} \)-compact subset of the set \( X^{*} \), and the \( Y^{*} \)-closure of the set \( TS \) is a \( Y^{*} \)-compact subset of the space \( Y \).

**Corollary 4.2.** If either \( X \) or \( Y \) is a reflexive space, then every operator \( T \in B(X \in Y) \) is weakly compact.

**Proof.** Let \( T \in B(X \in Y) \). If the space \( Y \) is reflexive, then

\[
T^{+*}X^{+*} \subset X^{*} = \wedge X = X^{*},
\]

and if the space \( X \) is reflexive, then

\[
T^{+*}X^{+*} = T^{+*}X = T^{*}X \subset X^{*}.
\]

Hence in both cases we conclude from Theorem 4.1 that the operator \( T \) is weakly compact.

**Corollary 4.3.** The two-sided ideal \( W(X \in Y) \) of weakly compact operators is closed in the paranalegbra \( B(X \in Y) \).

**Proof.** If \( T \in T \) in the space \( B(X \in Y) \), then Theorem 2.1 implies \( \| T^{+*} \| = 0 \). If \( T \) is weakly compact operator, then \( T^{+*}X^{+*} \subset X^{*} \) for every \( X^{+*} \in Y^{*} \) (Theorem 4.1), and since the set \( \wedge X \) is closed in the topology of the space \( Y^{*} \), we obtain \( T^{+*}X^{+*} \subset X^{*} \) and the Corollary follows from Theorem 4.1 and from the fact that the set \( W(X \in Y) \) is a two-sided ideal in the paranalegbra \( B(X \in Y) \).

We shall now investigate operators conjugate to weakly compact operators.

**Lemma 4.4.** An operator \( T \in B(X \in Y) \) is weakly compact if and only if the conjugate operator \( T^{*} \) is continuous both in the \( X^{+*} \)-topology of the space \( X^{*} \) and in the \( Y^{*} \)-topology of the space \( Y^{*} \).

**Proof.** Necessity. Let us suppose that the operator \( T \) is weakly compact. By Theorem 4.1, to every \( x^{+*} \in X^{+*} \) there exists an element \( y \in Y \) such that

\[
x^{*}(T^{*}y) = (T^{+*}x^{+*})y^{*} = y^{*}(y), \quad y^{*} \in Y^{*}.
\]

Let \( U \) be the following neighbourhood of zero in the \( X^{+*} \)-topology of the space \( X^{*} \):

\[
U = \{ x^{+*} \in X^{+*} : |x^{*}(x^{+*})| < \varepsilon, \quad i = 1, 2, \ldots, n \}.
\]

Let

\[
V(\varepsilon) = \{ y^{*} \in Y^{*} : |y^{*}(y^{*})| < \varepsilon, \quad i = 1, 2, \ldots, n \},
\]

where \( y^{*} \) are elements satisfying the equality \( x^{*}(T^{*}y^{*}) = y^{*}(y^{*}) \). It is easily verified that \( T^{*}(V) \subset U \), and since neighbourhoods of the form \( V \) constitute a basis of neighbourhoods and \( V \) is a \( Y^{*} \)-neighbourhood of zero in the space \( Y^{*} \), the necessity of the condition is proved.

Sufficiency. Let \( x^{+*} \) be an arbitrary element of the space \( X^{+*} \). We show that the functional \( x^{+*} \in X^{+*} \) is continuous in the space \( X^{*} \) provided with the \( Y^{*} \)-topology. Let \( e \) be an arbitrary positive number and let \( U \) be a \( Y^{*} \)-neighbourhood of zero in the weak topology of the space \( X^{*} \), of the form

\[
U = \{ x^{*} \in X^{*} : |x^{*}(x^{+*})| < \varepsilon \}.
\]

By hypothesis, there exists a \( Y^{*} \)-neighbourhood \( V \) of zero in the \( Y^{*} \)-topology of the space \( X^{*} \) such that \( T^{*}V \subset U \), i.e.

\[
|x^{+*}(T^{*}y^{*})| < \varepsilon \quad \text{for } y^{*} \in V, \quad \text{whence } |(T^{+*}x^{+*})y^{*}| < \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, this implies the continuity of the functional \( y^{+*} = T^{+*}x^{+*} \) in the \( Y^{*} \)-topology. By Theorem 10.1, B, I, we have \( y^{+*} \in X^{*} \).
Hence $T^+X^+ \subset X$. By Theorem 4.1, the operator $T$ is weakly compact.

**Theorem 4.5.** (Gantmacher [1]) An operator $T^+$ conjugate to a weakly compact operator $T \in B(X \to Y)$ is weakly compact.

**Proof.** Since the closed unit ball $S(Y^*)$ in the space $Y^*$ is compact in the $Y$-topology (Theorem 3.3, I), Lemma 4.4 implies that the set $T^+S(Y^*)$ is compact in the $Y^+$-topology of the space $Y^+$. Thus the operator $T^+$ is weakly compact.

**Corollary 4.6.** (Gantmacher [1]) Let $T \in B(X \to Y)$. If the operator $T^+ \in B(Y^+ \to X^+)$ is weakly compact, then the operator $T$ is weakly compact.

**Proof.** By Theorem 4.5 the operator $T^+ \in B(Y^+ \to X^+)$ is weakly compact. Hence its restriction $T$ to the space $X$ is a weakly compact operator. But the space $X^+$ contains more elements than the space $X$. Hence the weak topology in the space $X^+$ created as a subset of the space $X^+$ is not coarser than the weak topology of the space $X$. Since the set $TS(X)$ is compact in the $Y^+$-topology of the space $X$, it is compact also in the $X^+$-topology.

§ 5. Semicompact operators. An operator $T \in B(X \to Y)$ is called a Kato operator (Kato [1]) or a semicompact operator if the following condition is satisfied: if the restriction of $T$ to a certain subspace $M \subset X$ is a homeomorphism, then this subspace is of a finite dimension. In other words, an operator $T \in B(X \to Y)$ is semicompact if the fact that there exists a number $\gamma > 0$ such that $\|Tm\| \geq \gamma \|m\|$ for all $m \in M \subset X$ implies that $M$ is of a finite dimension.

Every compact operator is semicompact. Evidently, the restriction of a semicompact operator to a space $X_0 \subset X$ is a semicompact operator again.

**Theorem 5.1.** Let $X$ and $Y$ be Banach spaces. An operator $T \in B(X \to Y)$ is semicompact if and only if for every infinite-dimensional subspace $M \subset X$ there exists an infinite-dimensional subspace $M_0 \subset M$ such that the restriction of the operator $T$ to the subspace $M_0$ is compact.

**Proof.** The sufficiency is immediate, since the operator $T$ cannot be a homeomorphism on the subspace $M_0$ and, consequently, on the subspace $M$ either.

**Necessity.** Let $M$ be an arbitrary infinite-dimensional subspace of the space $X$. By Banach's Theorem (see Theorem 2.1, I), $M$ contains an infinite-dimensional subspace $M_1$ with a basis $\{e_n\}$. Since $T$ is a semicompact operator, there exists a divergent sequence of indices $\{p_n\}$ and a sequence $\{x_n\} = \sum_{n=1}^{\infty} \alpha_n e_n$, such that $\|x_n\| = 1$ and $\|T x_n\| < 1/2^n$. We construct these sequences by induction. Let $x_1'$ be an arbitrary element such that $\|x_1'\| = 1$, but $\|T x_1'\| < 1/4$. Evidently, there exists an index $p_2$ such that $\|T x_2'\| < 1/2$, where $x_2' = \sum_{n=1}^{\infty} t_n e_n$ (see Theorem 4.1, B II). Let $x_3' = \sum_{n=1}^{\infty} t_n e_n$. Let us now suppose the numbers $p_2', \ldots, p_n'$ are chosen and the elements $x_1', \ldots, x_n'$ are already constructed. The space $M_{p_n'}$ spanned by the elements $e_{p_{n+1}'}, e_{p_{n+2}'}, \ldots$ is infinite-dimensional. Hence there exists an element $x_{n+1}' \in M_{p_{n+1}'}$ such that $\|x_{n+1}'\| = 1$, $\|T x_{n+1}'\| < 1/4^{n+1}$. There is an index $p_{n+1}$ for which $\|T x_{n+1}'\| < 1/4^{n+1}$.

Let $x_{n+1} = \sum_{i=n}^{\infty} t_i e_i$. The element $x_{n+1}$ and the number $p_{n+1}$ satisfy the induction hypotheses.

Let $M_n$ be the space spanned by the elements $x_n$ ($n = 1, 2, \ldots$). It follows from Theorem 4.6, A II, that the sequence $(x_n)$ is a basis of the space $M_n$. Let $x \in M_n$, $x = \sum_{i=n}^{\infty} t_i e_i$, then $\|T x\| < K \|x\|$ (Theorem 4.6, B II).

Hence $\sum_{i=n}^{\infty} |t_i| < K 2^{-i}$ for all $x \in M_n$, $\|x\| < 1$, and the basis expansions are uniformly convergent. This implies that the restriction of the operator $T$ to the subspace $M_n$ is compact.

**Corollary 5.2.** (Kato [1]) If $X$ and $Y$ are Banach spaces, then the sum $T_1 + T_2$ of two compact operators $T_1, T_2 \in B(X \to Y)$ is a semi-compact operator.

**Proof.** Let $M$ be an arbitrary infinite-dimensional subspace of the space $X$. It follows from the assumption that $M$ contains an infinite-dimensional subspace $M_1$ such that the restriction of the operator $T_1$ to the subspace $M_1$ is a compact operator. It follows from the assumption concerning the operator $T_2$ that there exists an infinite-dimensional subspace $M_2 \subset M$ such that the restriction of the operator $T_2$ to the subspace $M_2$ is a compact operator. Hence the restriction of the operator $T_1 + T_2$ to the subspace $M_1$ is a compact operator. This proves the operator $T_1 + T_2$ to be semicompact.

**Theorem 5.3.** (Kato [1]) If $X, Y, Z$ be Banach spaces. Let $B \in B(Y \to Z)$, $A \in B(X \to Y)$. If one of the operators $A, B$ is semicompact, then the superposition $AB$ is a semicom- pact operator.

**Proof.** Let $M$ be an arbitrary subspace of the space $X$. If $\|BA\| \geq \gamma \|z\|$ for $z \in M$, then $\|BA\| \geq \gamma \|A\| \|z\|$ for $z \in M$. Hence if $B$ is a semicompact operator, then $AB$ is also a semicompact operator.
Let us remark that if \( \|ABx\| \geq \gamma \|x\| \) for \( x \in M \), then \( \|ABx\| \geq \gamma \|B^{-1}Bx\| \). Hence, if \( A \) is a semicompact operator, then the image \( B(M) \) of the space \( M \) is of a finite dimension. But \( x \notin M \) and \( Bx = 0 \) imply \( x = 0 \). Thus \( \text{dim } M = \text{dim } B(M) < +\infty \).

We denote by \( S(X \to Y) \) and \( S(X = Y) \) the set of semicom pact operators in the space \( B(X \to Y) \) and in the paraalgebra \( B(X \equiv Y) \), respectively.

**Corollary 5.4.** If \( X \) and \( Y \) are Banach spaces, then the set \( S(X = Y) \) of semicom pact operators is an ideal in the paraalgebra \( B(X \equiv Y) \).

We shall show that this ideal is closed.

**Theorem 5.5.** If \( X \) and \( Y \) are Banach spaces, then the ideal \( S(X = Y) \) of semicom pact operators is closed in the paraalgebra \( B(X \equiv Y) \).

**Proof.** Let there be given a sequence of operators \( (T_n) \subset S(X \to Y) \) converging to an operator \( T \) in the norm. Let \( M \) be an arbitrary subspace of the space \( X \) such that there is a number \( \gamma > 0 \) satisfying the inequality \( \|T_n - T\| \geq \gamma \|x\| \). By hypothesis, there exists an index \( n_0 \) such that \( \|T_n - T\| \geq \gamma \|x\| \) for \( n < n_0 \).

Thus the assumption \( T_n \in S(X = Y) \) implies that \( M \) is of a finite dimension.

**Theorem 5.6.** (Goldberg and Thorp [11].) If the Banach space \( X \) (or \( Y \)) is reflexive, and the Banach space \( Y \) (or \( X \), respectively) does not contain any reflexive infinite-dimensional subspace, then every operator \( T \in B(X \equiv Y) \) is semicom pact.

**Proof.** If the operator \( T \) is not semicom pact, then it is a one-to-one map of some infinite-dimensional subspace \( M \) onto some infinite-dimensional subspace \( M \). But one of these spaces is reflexive, as a subspace of a reflexive space, and the other one is not reflexive by assumption, which is a contradiction.

Applying Theorem 5.6 one can give an example of a semicom pact operator such that the conjugate operator is not semicom pact.

**Example 5.1.** (Goldberg and Thorp [11].) Let the operator \( T \) be an epimorphism of the space \( X \) onto the space \( Y \) (see Theorem 10.1, 1). By Theorem 5.6, the operator \( T \) is semicom pact. By Theorem 2.6, \( T \) is an embedding of the space \( Y \) into the space \( X \). Hence \( T^* \) cannot be semicom pact.

The following example shows that the semicom pactness of the conjugate operator \( T^* \) does not always imply the semicom pactness of the operator \( T \).

**Example 5.2.** (Pieczyński [1]). Let \( T \) be the operator of natural embedding of \( L^2[0,1] \) into \( L^1[0,1] \), i.e. \( T(x(t)) = x(t) \). It is easily verified that the operator \( T^* \) is the natural embedding of the space \( X \) into the space \( L^2[0,1] \). The operator \( T \) is not semicom pact, for it is an isomorphism to the space of functions spanned by the Rademacher system (see Theorem 4.1, 1). On the other hand, \( T^* \) is a semicom pact operator.

Indeed, let \( X \) be a subspace for which the norms

\[
\|x\|^2 = \int_{\mathbb{R}} |x(t)|^2 \, dt \quad \text{and} \quad \|x\|_\infty = \sup_{t \in \mathbb{R}} |x(t)|
\]

are equivalent. If the space \( X \) is infinite-dimensional, then it contains an orthonormal sequence \( (a_n) \) with respect to the scalar product \((x, y) = \int_{\mathbb{R}} x(t) \, dy(t)\). Evidently, the sequence \( (a_n) \) tends to zero weakly. Let \( \varepsilon \) be an arbitrary positive number. By Luzin's theorem, to every \( n \) there is a closed set \( F_n \) of measure greater than \( 1 - \varepsilon \), such that the function \( a_n(t) \) is continuous on the set \( F_n \). Hence all functions \( a_n(t) \) are continuous on the set \( F = \bigcap_{n=1}^{\infty} F_n \). It is easily verified that the measure of the set \( F \) is greater than \( 1 - \varepsilon \). But the sequence \( (a_n) \) converges to zero weakly. Hence the continuity of the functions \( a_n(t) \) on the set \( F \) implies \( \lim_{n \to \infty} \|a_n(t)\| = 0 \) for \( t \in F \). By Egorov's theorem, there exists a set \( F_0 \subset F \) of measure greater than \( 1 - 2\varepsilon \) such that the sequence \( (a_n(t)) \) is uniformly convergent on the set \( F_0 \). The equivalence of norms \( \| \cdot \| \) and \( \| \cdot \|_\infty \) implies that the functions \( a_n(t) \) are uniformly bounded: \( |a_n(t)| < M \). Let \( n \) be an index such that

\[
|a_n(t)| < \varepsilon \quad \text{for} \quad t \in F_0
\]

implies

\[
|a_n|^2 = \int_{F_0} |a_n(t)|^2 \, dt + \int_{F_0} |a_n(t)|^2 \, dt \leq 2\varepsilon M^2 + \varepsilon .
\]

Since \( \varepsilon \) is arbitrary, this contradicts the orthonormality of the sequence \( (a_n) \).

However, by some additional assumptions, the semicom pactness of the operator \( T^* \) implies the semicom pactness of the operator \( T \).

**Theorem 5.7.** (Whitley [1]). Let \( T \in B(X \equiv Y) \), where \( X \) and \( Y \) are Banach spaces and \( Y \) is a subprojective space. If the operator \( T^* \in B(X^* \equiv X^*) \) is semicom pact, then the operator \( T \) is also semicom pact.

**Proof.** Let us suppose that the operator \( T \) is not semicom pact. There exists an infinite-dimensional subspace \( M \subset X \) mapped isomorphically onto the set \( T(M \subset Y) \) by means of the operator \( T \). But the space \( Y \) is subprojective; hence there exists an infinite-dimensional subspace \( Y_0 \subset M \) such that \( Y \) can be projected on this subspace by means of an operator \( F \). The operator \( F^* \) is defined on the subspace \( Y_0 \), is continuous, and is a one-to-one map of \( Y_0 \) onto a subspace \( X_0 \subset X \). Let
a manner that \( \{ e_i' \} \) and \( \{ e_i'' \} \) are bases equivalent to the basis \( \{ e_i \} \) in the space \( Z \). (See also Theorem 6.1, I.) Hence it follows that the operator \( T \) is a one-to-one map of the space \( X = \lim (e_i) \) onto the space \( X = \lim (e_i'') \) continuous in both directions. Hence \( T \) is not a semicompact operator, which is a contradiction.

An example of a space satisfying the assumptions of Theorem 3.10 yields the space \( P_n \), where \( p_n = 2 + 1/n \ln(n) \) \( (n = 1, 2, \ldots) \) (see § 4).

§ 6. Co-semicolon operators. Theorem 5.7 is true only if \( Y \) is assumed to be a subprojective space. In order to investigate conjugate operators one can proceed also in another way, defining a class of operators in a certain sense dual to the class of semicomponent operators. First we give another definition of a semicomponent operator:

An operator \( T \in B(\mathcal{X} \rightarrow \mathcal{Y}) \) is semicomponent in the case of the nonexistence of any infinite-dimensional Banach space \( E \) and any embeddings \( i_e \) and \( i_E \) of the space \( E \) into Banach spaces \( \mathcal{X} \) and \( \mathcal{Y} \), respectively, such that the diagram

\[
\begin{aligned}
\mathcal{X} & \xrightarrow{i_e} \mathcal{E} & \xrightarrow{i_E} \mathcal{Y} \\
\mathcal{X} & \xrightarrow{T} \mathcal{Y} &
\end{aligned}
\]

(6.1)

is commutative, i.e. \( T i_e = i_E \).

Let us change the direction of maps in this diagram, and let us replace embeddings by continuous epimorphisms \( h_x \) and \( h_Y \):

\[
\begin{aligned}
\mathcal{X} & \xleftarrow{i_e} \mathcal{E} & \xleftarrow{i_E} \mathcal{Y} \\
\mathcal{X} & \xleftarrow{T} \mathcal{Y} &
\end{aligned}
\]

(6.2)

We shall say that an operator \( T \in B(\mathcal{X} \rightarrow \mathcal{Y}) \) is co-semicolon compact (Pczerzi [1]) in the case of the nonexistence of any infinite-dimensional Banach space \( E \) and continuous epimorphisms \( h_x \) and \( h_Y \) such that the diagram (6.2) is commutative, i.e. \( h_Y T = h_X \).

In other words, an operator \( T \) is co-semicolon compact if for every subspace \( Y \subseteq \mathcal{Y} \) such that \( T \mathcal{X} + Y = \mathcal{Y} \), the subspace \( Y \) is of a finite defect.

Hence it follows immediately that the restriction of a co-semicolon component operator to a subspace \( X \subseteq \mathcal{X} \) is a co-semicolon component. Indeed, if \( T X + Y = \mathcal{Y} \), then \( T X + Y = \mathcal{Y} \). Hence the subspace \( Y \) is of a finite defect.
§ 6. Co-semicompact operators

The inequalities $\|T^*y^*_k\| \leq 2^{-k}\delta_k$ and $\|y_k\| = 2^{k-1}$ imply that the operator $A$ is compact. The operator $A$ is conjugate to an operator $B \in B(X \to Y)$ of the form

$$Bx = \sum_{k=1}^{\infty} T^* y_k^*(x) y_k.$$ 

Hence the operator $T^* - A = (T - B)^*$ is continuous both in the $X$-topology and in the $X$-topology. This implies that the set $Z = Z_{T^* - A}$ is closed in the $X$-topology. The operators $T^*$ and $A$ restricted to the set $Z$ are identical, and thus the operator $T^*$ restricted to the set $Z$ is compact. The space $Z$ is infinite-dimensional, because $y_k^* \in Z$ for $k = 1, 2, \ldots$.

**Lemma 6.3.** (Vladimirovski [1]) If the operator $T \in B(X \to Y)$ is not a $\Phi_-$-operator, then there exists a subspace $M \subset Y$ with infinite codimension such that the operator $\Phi_M T$ is compact.

**Proof.** Theorem 2.7 implies that the operator $T^*$ is not a $\Phi_+$-operator. As follows from Lemma 6.2, there exists an infinite-dimensional $Y$-closed subspace $M^\perp$ such that the restriction of the operator $T^*$ to the space $M^\perp$ is a compact operator. Let

$$M = \{y \in Y : y^* = 0, y^+ \in M^\perp\}.$$ 

Since the subspace $M$ is of the infinite codimension, the operator $\Phi_M T$ is compact (see Corollary 2.3).

**Proof of Theorem 6.1. Sufficiency.** Let us suppose that the operator $T \in B(X \to Y)$ satisfies the assumptions of the theorem and that we are given a commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow{k_x} & & \downarrow{k_Y} \\
B & \xrightarrow{B} & Y
\end{array}$$

where $k_x$ and $k_Y$ are continuous epimorphisms. Let

$$N = Z_{k_x} = \{y \in Y : k_x y = 0\}.$$ 

If the space $E$ is of an infinite dimension, then of course $\text{codim } N = \dim E = +\infty$. By hypothesis, there exists a subspace $M$ of the space $Y$ of an infinite codimension containing the set $N$ and such that the operator $\Phi_M T$ is compact. But the operator $\Phi_M T$ maps the space $X$ onto the quotient space $E|\Phi_M$, where the subspace $E|\Phi_M = \{y \in E : k_Y^*(y) \in M\}$ is of infinite codimension. This contradicts the assumption of the compactness of the operator $\Phi_M T$. 

---

We denote by $\mathcal{C}(X \to Y)$ the set of co-semicompact operators belonging to the space $B(X \to Y)$. Evidently, every compact operator is co-semicompact.

A theorem given below is dual to Theorem 5.1.

**Theorem 6.1.** (Vladimirovski [1]) An operator $T \in B(X \to Y)$ is co-semicompact if and only if for every subspace $M \subset Y$ of an infinite codimension there exists a subspace $M_k$ of an infinite codimension containing the subspace $M_k$ and such that the superposition of transformations $\Phi_{M_k} T$ is a compact operator, the transformation $\Phi_{M_k}$ being the map of the space $Y$ onto the quotient space $Y/M_k$ which associates with every element $y \in Y$ the coset to which $y$ belongs:

$$\Phi_{M_k} y = \{y + M_k\}.$$ 

The proof of this theorem is based on the following lemmas:

**Lemma 6.2.** (Vladimirovski [1]). Let $T \in B(X \to Y)$. If $T^* \in B(Y \to X)$ is not a $\Phi_+$-operator, then there is an infinite-dimensional $Y$-closed subspace $Z \subset Y$ such that the restriction of the operator $T^*$ to the space $Z$ is a compact operator.

**Proof.** We shall construct by induction two sequences, $(y_k) \subset Y$ and $(y_k^*) \subset X$, such that

$$y_k^*(y_j) = \delta_{kj}, \quad \|y_k^*\| = 1, \quad \|y_k\| \leq 2^{k-1}, \quad \|T^* y_k^*\| \leq 2^{-k}.$$ 

The existence of the elements $y_k, y_k^*$ follows trivially from the fact that the operator $T^*$ is not an embedding. Let us suppose that we have already defined the elements $y_1, \ldots, y_k, y_1^*, \ldots, y_k^*$. Let $Z_k = \{y_1, \ldots, y_k\}^\perp$ (see § 1, A III). Since $T^*$ is not a $\Phi_+$-operator, the restriction of $T^*$ to $Z_k$ is not an embedding into $X$. And there exists a functional $y_k^*$, such that $\|y_k^*\| = 1$ and $\|T^* y_k^*\| \leq 2^{-k+1}$. Let $y_{k+1} \in Y$ be an element such that

$$\|y_{k+1}\| = 2^{k+1} \text{ and } y_{k+1}(y_k) = 1.$$ 

Let

$$y_{k+1} = \delta_{k+1} - \sum_{i=1}^{k} y_i^*(y_{k+1}) y_i.$$ 

Obviously $y_k^*(y_j) = \delta_{kj}$ for $i, j = 1, 2, \ldots, k+1$. Moreover,

$$\|y_{k+1}\| \leq \|y_{k+1}\| \left(1 + \sum_{i=1}^{k} |y_i^*| \cdot |y_i|\right) \leq 2 \left(1 + \sum_{i=1}^{k} 2^{k-1}\right) \leq 2^{k+1} = 2^{k+1} - 2k+1.$$ 

Let us consider an operator $A \in B(Y \to X)$ defined in the following manner:

$$Ay = \sum_{k=1}^{\infty} y_k^*(y) T^* y_k^*.$$
Necessity. Let $T \in B(X \to Y)$ be a compact operator. Let $M \subset Y$ be an arbitrary subspace of infinite codimension. This implies that the operator $\Phi_M T$ is not a proper. Hence (Lemma 6.2) there exists a subspace $M \subset Y$ of infinite codimension such that the operator $\Phi_M T$ is compact. ■

Corollary 6.4. (Vladimirovskii [1]) The set $\mathcal{C}(X \to Y)$ of all co-semicompact operators contained in the algebra $B(X \to Y)$ is linear.

Proof. A co-semicompact operator multiplied by a scalar is again a co-semicompact operator. We have to show that the sum $T_1 + T_2$ of two co-semicompact operators $T_1$ and $T_2$ is a co-semicompact operator. Let $M \subset Y$ be an arbitrary subspace of infinite codimension. Theorem 6.1 implies that there exists a subspace $M \subset Y$ of finite codimension such that the operator $\Phi_M T_1$ is compact. Using Theorem 6.1 once more we find that there exists a subspace $M \subset Y$ of finite codimension such that the operator $\Phi_M T_2$ is compact. Obviously $\Phi_M (T_1 + T_2)$ is also compact. The fact that the subspace $M$ is arbitrary implies by Theorem 6.1 the compactness of the operator $T_1 + T_2$. ■

Theorem 6.5. (Pelecyński [1]) Let $X, Y, Z$ be Banach spaces. Let $A \in B(X \to Y)$ and $B \in B(Y \to Z)$. If one of those operators is co-semicompact, then the superposition $BA$ is also a co-semicompact operator.

Proof. Let us suppose that the superposition $BA$ is not a co-semicompact operator. Then there exist an infinite-dimensional Banach space $K$ and continuous epimorphisms $h_X$ and $h_Y$ such that the following diagram is commutative:

\[ X \xrightarrow{A} Y \xrightarrow{B} Z \]

\[ h_Y \circ h_X = h_Z \]

If we write $h_Y = h_Z B$, then $h_Y$ is a continuous epimorphism and the following diagram is commutative:

\[ X \xrightarrow{A} Y \xrightarrow{B} Z \]

\[ h_Y \circ h_X = h_Z \]

Thus neither $A$ nor $B$ can be a co-semicompact operator. ■

§ 6. Co-semicompact operators

Theorem 6.6. (Pelecyński [1]) If $X$ and $Y$ are Banach spaces, then the set $\mathcal{C}(X \to Y)$ of co-semicompact operators is closed in the space $B(X \to Y)$.

Proof. Let $T \in \mathcal{C}(X \to Y)$. By hypothesis, there exist continuous epimorphisms $h_X$ and $h_Y$ and an infinite-dimensional space $E$ such that the following diagram is commutative:

\[ X \xrightarrow{h_X} E \xrightarrow{h_Y} Y \]

But the operator $h_X \circ h_Y T$ is an epimorphism if and only if there exists a number $\varepsilon > 0$ such that

\[ \inf h_Y T(K_X) \subset \varepsilon K_Y, \]

where

\[ K_X = (x \in X : \|x\| < 1), \quad K_Y = (y \in Y : \|y\| < 1). \]

Let $\delta = \varepsilon/2 \|h_Y\|$. Evidently, $\sup \|h_Y(T - T_0)x\| < \varepsilon$. On the other hand, $h_Y(T_0 K_X) \subset h_Y(T - T_0)K_X$, whence $h_Y(T_0 K_X) \subset \frac{\varepsilon}{2} E$. Thus $T_0 \in \mathcal{C}(X \to Y)$ and the complement of the set $\mathcal{C}(X \to Y)$ is an open set. ■

Corollary 6.7. If $X$ and $Y$ are Banach spaces, then the set $\mathcal{C}(X \to Y)$ of all co-semicompact operators is a closed two-sided proper ideal in the paralgebra $B(X \to Y)$.

Theorem 6.8. (Pelecyński [1]) Let $X$ and $Y$ be Banach spaces. Let $T \in B(X \to Y)$. If the operator $T^* \in B(Y^* \to X^*)$ is semicompact (co-semicompact), then the operator $T$ is co-semicompact (semicompact, respectively).

Proof. This is an immediate consequence of the fact that an operator conjugate to an embedding (epimorphism) is an epimorphism (embedding) (Theorems 2.6 and 2.7, Chapter 1). Thus, if $T \notin \mathcal{C}(X \to Y)$ (co$\mathcal{C}(X \to Y)$, respectively), then $T^* \notin \mathcal{C}(Y^* \to X^*)$ (co$\mathcal{C}(Y^* \to X^*)$, respectively). ■

Corollary 6.9. Let $X$ and $Y$ be Banach spaces, and let $X$ be reflexive. If an operator $T \in B(X \to Y)$ is semicompact (co-semicompact), then the operator $T^*$ is co-semicompact (semicompact, respectively).

Proof. Since the space $X$ is reflexive, we have $T^{**} = T$. Thus Corollary 6.9 is an immediate consequence of Theorem 6.8.

The examples given by Pelecyński [1] show that Corollary 6.9 is not true if the space $X$ is not reflexive, i.e. there exist semicompact (co-semicompact) operators such that the conjugate operators are not co-semicompact (semicompact).

The following theorem is a generalization of Corollary 6.9:

Theorem 6.10. (Pelecyński [1]) If $X$ and $Y$ are Banach spaces
and if the operator $T \in B(X \to Y)$ is weakly compact, then the conjugate operator $T^*$ is semicompact.

**Proof.** Let us suppose that the operator $T^*$ is not semicompact. Then there exist an infinite-dimensional Banach space $E$ and embeddings $i_X$ and $i_Y$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
X^+ & \xrightarrow{T^*} & Y^+ \\
i_X & \downarrow & \downarrow i_Y \\
E & \xrightarrow{i_{E^*}} & E^*
\end{array}
\]

The operator $T^*$ is weakly compact, since it is conjugate to a weakly compact operator (Theorem 4.5). Hence it follows that the embedding $i_{E^*} = i_{E^*} \circ i_X$ is a weakly compact operator. Thus the closure of the unit ball in the space $X^+$ is a weakly compact set. By Eberlein's theorem (Theorem 3.4, I), the space $E^{**}$ is reflexive. Let us write $h_{X^*} = i_{e^{*}}$, $h_{Y^*} = i_{e^{*}}$. The operators $h_{X^*}$ and $h_{Y^*}$ are epimorphisms (Theorem 2.6, Chapter I) which map the spaces $X^{**}$ and $Y^{**}$ into the space $E$, respectively. Let $h_X$ be the restriction of the operator $h_{X^*}$ to the space $X$, and let $h_Y$ be the restriction of the operator $h_{Y^*}$ to the space $Y$. Since the space $E$ is reflexive, we have $h_X^* = h_{X^*}$, $h_Y^* = h_{Y^*}$. Thus, by Theorem 2.6, I, $h_X$ and $h_Y$ are epimorphisms and the following diagram is commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
h_X & \downarrow & \downarrow h_Y \\
E & \xrightarrow{i_{X^*}} & E^*
\end{array}
\]

But the operator $T$ is co-semicompact. Hence the space $E$ is of a finite dimension, which is a contradiction. □

**Theorem 6.11.** Let $T \in B(X \to Y)$, where $X$ and $Y$ are Banach spaces and $X$ is a superprojective space. If the operator $T^* \in B(Y^* \to X^*)$ is co-semicompact, then the operator $T$ is also co-semicompact.

**Proof.** Let us suppose that the operator $T$ is not co-semicompact. Then there exist an infinite-dimensional space $E$ and continuous epimorphisms $h_X$ and $h_Y$ of the spaces $X$ and $Y$ onto the space $E$, respectively, such that the following diagram is commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
h_X & \downarrow & \downarrow h_Y \\
E & \xrightarrow{i_{E^*}} & E^*
\end{array}
\]

Since the space $X$ is superprojective, there exists a subspace $M$ of the space $Y$ of infinite codimension which contains the set $Z_M = \{x \in X : l_X(x) = 0\}$ and is a projection of the space $X$. Let $P$ be the projection operator onto the subspace $M$. Let us investigate the conjugate diagram

\[
\begin{array}{ccc}
X & \xrightarrow{T^*} & Y \\
& \downarrow i_X & \downarrow i_Y \\
E & \xrightarrow{i_{E^*}} & E^*
\end{array}
\]

Let us remark that the operator $P^* i_{X^*}$ maps the set $E^*$ onto the set $E = (X|M)^*$. Hence the operator $P^* T^*$ maps the space $Y^*$ onto the set $E$. Thus we obtain the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
h_X & \downarrow & \downarrow h_Y \\
E & \xrightarrow{i_{X^*}} & E^*
\end{array}
\]

where $h_X = P^*$ and $h_Y = P^* T^*$. Consequently, the operator $T^*$ is not co-semicompact, which is a contradiction. □

**§ 7. Spaces with the Dunford-Pettis property.** A Banach space $X$ is said to have the Dunford-Pettis property if for every sequence $(a_n) \subset X$ weakly convergent to zero and for every sequence $(a_n^*) \subset X^*$ weakly convergent to zero (in the $X^{**}$-convergence) we have

\[
\lim_{n \to \infty} \rho_{X^*}(a_n) = 0.
\]

**Theorem 7.1.** If the space $X^*$ conjugate to a Banach space $X$ has the Dunford-Pettis property, then the space $X$ also has the Dunford-Pettis property.

**Proof.** Let $(a_n) \subset X$ and $(a_n^*) \subset X^*$ be sequences weakly convergent to zero. The sequence $(a_n)$ remains weakly convergent if we consider $a_n$ as elements of the space $X^*$. Thus $\lim \rho_{X^*}(a_n) = 0$. □

We do not know whether the converse theorem is true.

**Theorem 7.2.** (Grothendieck [4]) Let $X$ and $Y$ be Banach spaces, and let the space $X$ have the Dunford-Pettis property. Every weakly compact operator $T \in B(X \to Y)$ maps sequences weakly convergent to zero onto sequences convergent to zero in the norm.

**Proof.** Let $T \in B(X \to Y)$ be a weakly compact operator, and let $(a_n)$ be an arbitrary sequence of elements of the space $X$ weakly convergent
to zero and satisfying the inequality \( \limsup_{n} \|T_{\mathbf{x}_n}\| = \delta > 0 \). We take a sequence of functionals \( \{g_{n}\} \), \( \|g_{n}\| = 1 \), satisfying the equality \( g_{n}(T_{\mathbf{x}_n}) = \|T_{\mathbf{x}_n}\| = n_1, n_2, \ldots \). Let \( \mathbf{z}^*_n = T^*g_{n} \). Since \( T \) is a weakly compact operator, the operator \( T^* \) is also weakly compact. Hence we may assume without loss of generality that \( \{\mathbf{z}^*_n\} \) is a weakly convergent Cauchy sequence (for otherwise we could consider a suitable subsequence of that sequence). Then
\[
\limsup_{n} \mathbf{z}^*_n(a_n) = \limsup_{n} \left[ T^*g_{n}(a_n) \right] = \limsup_{n} \left[ T_{\mathbf{x}_n} \right] = \limsup_{n} \|T_{\mathbf{x}_n}\| = \delta.
\]
On the other hand, if \( \{\mathbf{z}^*_n\} \) is a weakly convergent Cauchy sequence and \( \{a_n\} \) is a sequence weakly convergent to zero, then \( \limsup_{n} \mathbf{z}^*_n(a_n) = 0 \).

Indeed, let \( \limsup_{n} \mathbf{z}^*_n(a_n) = \delta \). Then \( \{a_n\} \) is a sequence of indices such that \( \limsup_{n} \mathbf{z}^*_n(a_n) = \delta \). Let \( \{a_k\} \) be a subsequence of \( \{a_n\} \) satisfying the inequalities \( \mathbf{z}^*_n(a_k) \leq \delta/2 \). Such a subsequence exists because the sequence \( \{\mathbf{z}^*_n(a_n)\} \) is weakly convergent to zero. We can write:
\[
\mathbf{z}^*_n(a_n) = (\mathbf{z}^*_n(a_n) - \mathbf{z}^*_n(a_k)) + \mathbf{z}^*_n(a_k).
\]

Then
\[
\delta = \limsup_{n} \mathbf{z}^*_n(a_n) \leq \limsup_{n} \left[ (\mathbf{z}^*_n(a_n) - \mathbf{z}^*_n(a_k)) + \mathbf{z}^*_n(a_k) \right] \leq \delta/2.
\]

Thus \( \delta = 0 \), a contradiction.

**Corollary 7.3.** If a Banach space \( X \) has the Dunford–Pettis property, then every weakly compact operator transforms Cauchy sequences with respect to weak convergence in Cauchy sequences with respect to the norm.

**Proof.**

If \( \{a_n\} \) is a Cauchy sequence, then the sequence \( \{a_n - a_m\} \) weakly tends to zero. Thus the sequence \( \{a_n - a_m\} \) tends to zero in the norm.

Theorem 7.2 can be reversed even in a stronger form. Namely, an arbitrary space \( X \) can be replaced by the space \( \mathbb{F} \).

**Theorem 7.4.** (Pekka Kallinen [7]). If every operator \( T \in B(\mathbb{X} \to \mathbb{F}) \) transforms sequences weakly convergent to zero into sequences convergent to zero in the norm, the Banach space \( X \) has the Dunford–Pettis property.

**Proof.**

Let \( \{\mathbf{z}^*_n\} \) be an arbitrary sequence of elements weakly convergent to zero in the space \( X^* \). Let us consider the linear operator \( T^*(X \to \mathbb{F}) \) defined by means of the formula \( T^*(\mathbf{z}^*) = (\mathbf{z}^*_n(a_n)) \) for every \( \mathbf{x} \in X \).

We shall show that the operator \( T^* \) is weakly compact. Indeed, we have \( T^*(\mathbf{z}^*)_n = \mathbf{z}^*_n \), where \( \mathbf{z}^*_n \) denotes the \( n \)-th element of the basis of the space \( \mathbb{F} \), and the operator \( T^* \) is conjugate to \( T \). The operator \( T^* \) is weakly compact. Indeed, let \( \{g_n\} = \left\{ \mathbf{z}^*_n(a_k^n) \right\} \) be an arbitrary sequence of elements of the space \( \mathbb{F} \) such that \( \|g_n\| \leq 1 \). Applying the diagonal method, one can extract a subsequence \( \{\mathbf{z}^*_n\} \) from this sequence in such a manner that \( \lim_{n \to \infty} a_k^n = a_k \) for \( k = 1, 2, \ldots \). It is easily verified that the element
\[
y = \sum_{k=1}^{\infty} a_k^n e_k^n
\]
belongs to the space \( \mathbb{F} \) and its norm is not greater than 1. Let us consider the sequence \( \{T^*g_n\} = \left\{ \sum_{k=1}^{\infty} a_k^n e_k^n \right\} \). We show this sequence to be weakly convergent. Let \( f \) be an arbitrary functional from the space \( X^* \) and let \( \epsilon \) be an arbitrary positive number. Since the sequence \( \{\mathbf{z}^*_n\} \) is weakly convergent, there exists an index \( k_0 \) such that \( \mathbf{z}^*_n(a_k^n) < \epsilon/3 \) for \( n \geq k_0 \). Let \( k_0 \) be an index such that
\[
|a_n(a_k^n - a_k^0)| < \frac{\epsilon}{3} |\mathbf{z}^*_n| \quad \text{for} \quad k > k_0, \quad i < i_0.
\]

Then
\[
|f(T^*g_n - T^*g_0)| = \left| \sum_{i=1}^{\infty} (a_n(a_k^n - a_k^0))f(a_i^n) \right|
\leq \sum_{i=1}^{k_0} |a_n(a_k^n - a_k^0)||f(a_i^n)| + \sum_{i=k_0+1}^{\infty} |a_n(a_k^n - a_k^0)||f(a_i^n)|
\leq \frac{\epsilon}{3} |\mathbf{z}^*_n| T i_{i_0} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\]

Hence, by Theorem 3.8, the operator \( T^* \) is weakly compact. Applying Gantmacher's theorem (Corollary 4.6) we conclude that the operator \( T \) is weakly compact. Thus it follows from the assumptions that
\[
\lim_{n \to \infty} \|T_{\mathbf{x}_n}\| = \limsup_{n \to \infty} \|\mathbf{z}^*_n(a_n)\| = 0
\]
for every sequence \( \{\mathbf{z}^*_n\} \) weakly convergent to zero. This implies the equality \( \lim_{n \to \infty} \mathbf{z}^*_n(a_n) = 0 \).

**Theorem 7.5.** (Dunford and Pettis [1]; Grothendieck [4]). The space \( C(\Omega) \) possesses the Dunford–Pettis property.

**Proof.** If the sequence of elements \( \{\mathbf{z}^*_n\} \subset C(\Omega) \) is weakly convergent to zero, Theorem 4.7, \( I \), implies that this sequence is bounded: \( \mathbf{z}^*_n(\mathbf{a}) < M \) for every \( \mathbf{a} \). If the sequence of measures \( \{\mu_n\} \) in the conjugate space is weakly convergent to zero, then there exists a measure \( \nu \) such that all measures \( \mu_n \) are equicontinuous with respect to the measure \( \nu \), i.e., for every number \( \epsilon > 0 \) there exists a number \( \delta > 0 \) such that \( \|\mathbf{a} - \mathbf{b}\| < \delta \) implies \( \mu_n(\mathbf{b}) < \epsilon \) (see Theorem 3.10, \( I \)). Without loss of generality we may suppose that \( \mu_n(\mathbf{b}) < \epsilon \).
§ 8. Semicompact and co-semicompact operators in $C(\Omega)$

In this section a characterization of semicompact and co-semicompact operators with domain $C(\Omega)$ is given. All the results of this and of the next section belong to A. Pełczyński [1], [3].

In our further considerations we shall need the properties of the so-called unconditionally converging operators. An operator $T$ is called unconditionally converging if it transforms weakly unconditionally convergent series into unconditionally convergent series.

**Theorem 8.1.** Let $X$ be a Banach space. Let the conjugate space $X^*$ have the following property: every set $E \subset X^*$ such that

$$\limsup_{m \to \infty} x^*(x_m) = 0$$

for every weakly unconditionally convergent series $\sum_{n} a_n$, is conditionally weakly compact. Then every unconditionally converging operator $T \in B(X \to Y)$ is weakly compact.

Proof. Let $E \subset X^*$ be a bounded set. Let $\sum_{n} a_n$ be an arbitrary weakly unconditionally convergent series. The assumption regarding the operator $T$ implies that the series $\sum_{n} a_n$ is unconditionally convergent. Hence $T a_n \to 0$, and we obtain

$$\limsup_{m \to \infty} y^*(T a_n) = \limsup_{m \to \infty} (T^* y^*) x_m = 0.$$  

Hence the assumed property implies that the weak closure of the set $T E$ is a weakly compact set. Thus the operator $T^*$ is weakly compact. By Gantmacher's theorem (Corollary 4.6), the operator $T$ is weakly compact.

**Theorem 8.2.** Every unconditionally converging operator which transforms the space $C(\Omega)$ into an arbitrary Banach space $Y$ is weakly compact.

The proof of this theorem is based on the following lemma:

**Lemma 8.3.** Let $\{a_n\}$ be a bounded sequence of elements of the space $C(\Omega)\]$. \textit{If} $\limsup_{n \to \infty} |a_n| > 0$ \textit{then} $\limsup_{n \to \infty} |a_n| > 0$.

Proof. Define open sets $\mathcal{G}_r$, $\sigma_r (r = 0, 1, 2, ...)$, sequences of Borel sets $\{E_{r,n}\}$ and sequences of measures $\{\nu_r\}$ for $r = 1, 2, ...$


by induction in such a manner that the following conditions are satisfied:

1. The sequence \( (\mathcal{E}^n) \) is a subsequence of the sequence \( (\mathcal{E}^n_{k_n}) \);

2. \( E_{n} \cap E_{m} = \emptyset \) for \( n \neq m \) (\( n, m = 1, 2, \ldots \));

3. \( E \subset \bigcup_{k=1}^{\infty} \mathcal{G}_{k-1} \);

4. \( \mathbf{v}(E_{r}) > \delta_{r} \) where \( \delta_{r} = \delta - \sum_{k=1}^{r-1} \delta(E_{k}) \);

5. \( v_{r}(G_{i}) \geq \delta_{r} > \delta/2 \) and \( G_{r} \cap G_{t} = 0 \) for \( i < r (r = 1, 2, \ldots) \).

Let us write

\[ v_{n} = \mu_{n}; \quad E_{n} = E_{n} \quad \text{for} \quad n = 1, 2, \ldots \quad \text{and} \quad \mathcal{G}_{n} = 0, \quad v_{0} = 0. \]

Let us suppose that the sequences \( (\mathcal{E}^{n}) \), \( (\mathcal{E}^{n}) \), \( \mathcal{G}_{n-1} \), and \( v_{n-1} \) are already defined in such a manner that conditions (1), (2), and (3) are satisfied for \( 1 < n \leq k \). Let \( N = [2^{k+1} + 1] + 1 \), where \( C \) is the supremum of the measures \( \mu_{n} \) defined above. Since the measures \( \mu_{n} \) are regular, condition (4) implies the existence of closed subsets \( F_{i} \subset E_{n}^{(i)} \) such that \( |\mathcal{E}^{n}_{n}(T_{j})| > \delta_{n} \) for \( i = 1, 2, \ldots, N \).

Let \( \mathcal{P}_{n} = \bigcup_{k=1}^{N} \mathcal{G}_{k} \). Conditions (2b) and (3b) imply \( F_{i} \cap F_{j} = 0 \) for \( i \neq j \) and \( \mathcal{G}_{n} \subset E_{n} \). Hence (see § 3, B I) there exist open sets \( G_{j} \) such that \( G_{i} \supset F_{i} \) and \( G_{j} \cap G_{i} = 0 \) for \( i \neq j \) and \( i, j = 0, 1, \ldots, N \). Since the measures \( \mathcal{E}^{n}_{n} \) are regular, one can choose open sets \( G_{i} \) in such a manner that \( G_{i} \supset F_{i} \) and \( G_{i} \cap G_{j} = 0 \) for \( i = 1, 2, \ldots, N \) and \( \mathbf{v}(G_{i}) > \delta/2^{k+1} \) for \( i = 1, 2, \ldots, N \). Let

\[ A = \{ n > N: \mathbf{v}(G_{n}) > \delta/2^{k+1} \}. \]

Since the sets \( G_{i} \) are pairwise disjoint, we obtain

\[ \sum_{i=1}^{N} \mathbf{v}(G_{i}) \leq \sum_{i=1}^{N} \mathbf{v}(G_{i}) \leq c. \]

Hence every index \( n > N \) belongs to at least one of the sets \( A \). Thus there exists an index \( k_{n} \) such that the set \( A_{k_{n}} \) is infinite. Let us take \( G_{k} = O_{n}, \quad v_{k} = \mathbf{v}(G_{k}) \) of \( (\mathcal{E}^{n}_{n}) \) is a subsequence of the sequence \( (\mathcal{E}^{n}) \) made up of elements whose indices belong to the set \( A_{k_{n}} \). Also \( (\mathcal{E}^{n}_{n}) \) is a subsequence of the sequence \( (\mathcal{E}^{n}_{n} \cap (\bigcup_{k=1}^{\infty} \mathcal{G}_{k-1})) \) made up of elements whose indices belong to the set \( A_{k_{n}} \). Evidently, the sequences \( (\mathcal{E}^{n}_{n}), (\mathcal{E}^{n}_{n}) \) defined above,

where \( j(n) \) is an integer belonging to the set \( A_{k_{n}} \) and depending on \( n \). Hence condition (4b) is satisfied. Moreover,

\[ v_{k}(G_{k}) = \mathbf{v}(G_{k}) \geq \mathbf{v}(G_{k}) > \delta/2^{k+1} = \delta_{k+1}, \]

and

\[ G_{k} \subset G_{k} \subset O_{k} \cap O_{n} = 0 \quad \text{for} \quad i < k. \]

Thus condition (5b) is also satisfied.

Evidently, the sequences \( (\mathcal{E}^{n}) \) and \( (\mathcal{E}^{n}) \) satisfy Lemma 8.2.

**Proof of Theorem 8.3.** By Theorem 3.1, \( C(O) \) is the space \( C \). We show that if a set of functionals \( F \subset C(O) \) is not weakly compact, then there exists a weakly unconditionally convergent series

\[ \sum_{n=1}^{\infty} \mathbf{v}(G_{n}) \]

such that

\[ \lim_{n \to \infty} \mathbf{v}(G_{n}) > 0. \]

where \( \lim \) means the lower limit.

We now apply the theorem on the form of conditionally weakly compact sets in the space \( C(O) \) (Theorem 4.10, I). We find that there exists a sequence of measures \( (\mathcal{E}^{n}) \subset \mathcal{E} \) satisfying the assumptions of Lemma 8.3. We choose sequences \( (\mathcal{E}^{n}) \) and \( (\mathcal{E}^{n}) \) as in Lemma 8.3. Since the set \( G_{n} \) is open and \( v_{n}(G_{n}) > \delta/2 \), there exists a function \( f_{n} \in C(O) \) such that \( \|f_{n}\| = 1, f_{n}(x) > 0 \) for \( x \notin G_{n} \) and \( \int f_{n}(x) \, d\mu_{n} > \delta/2 \) (\( n = 1, 2, \ldots \)). Since the sets \( G_{n} \) are pairwise disjoint, the functions \( f_{n} \) vanish outside the set \( G_{n} \).

Here \[ \lim_{n \to \infty} \int f_{n}(x) \, d\mu_{n} > \delta/2 > 0. \]

**Theorem 8.4.** If \( X \) and \( Y \) are Banach spaces, then every weakly compact operator \( T \in B(X \to Y) \) is weakly unconditionally converging.
This Theorem can be formulated in a little stronger form:

**Theorem 8.4.** If $X$ and $Y$ are Banach spaces and an operator $T \in B(X \to Y)$ is not unconditionally converging, then there exists a subspace $X_0 \subset X$ isomorphic to the space $c_0$ and such that the operator $T$ is a one-to-one map of $X_0$ continuous in both directions.

**Proof.** By hypothesis, there exists a weakly unconditionally convergent series $\sum_{n=1}^{\infty} \alpha_n$ such that the series $\sum_{n=1}^{\infty} T\alpha_n$ is not unconditionally convergent. Hence one can choose sequences of indices $\{p_n\}$ and $\{q_n\}$ in such a manner that $\|T\beta\| > \delta$, where $\beta_k = \sum_{n=1}^{k} \alpha_n$. Evidently, the series $\sum_{n=1}^{\infty} \beta_n$ and $\sum_{n=1}^{\infty} T\beta_n$ are weakly unconditionally convergent. Hence the sequence $(T\beta_k)$ weakly tends to zero. Hence one can extract from the sequence $(T\beta_k)$ a subsequence which is a basis of the space spanned by itself. Thus, without loss of generality we may suppose at once that the sequence $(T\beta_k)$ has this property.

By Theorem 5.2, I, the basis $(T\beta_k)$ is equivalent to the standard basis in the space $c_0$. But the series $\sum_{n=1}^{\infty} T\beta_n$ is also weakly unconditionally convergent and $\|T\beta\| > \delta/\|T\|$. Hence one can extract a subsequence $(\beta_{p_k})$ in such a manner that this subsequence is a basis of the space $X_0$ spanned by itself, and this basis is equivalent to the standard basis of the space $c_0$, as follows from Theorem 5.2, I. This shows that the operator $T$ is a one-to-one map of the space $X_0$ onto the space $TX_0$, continuous in both directions.

**Theorem 8.5.** Let $X$ be a Banach space and let $T \in B(C(\Omega) \to Y)$. The following three conditions are equivalent:

(i) the operator $T$ is semicompact,

(ii) the restriction of the operator $T$ to a subspace of the space $C(\Omega)$ isomorphic to the space $c_0$ does not possess a continuous inverse,

(iii) the operator $T$ is weakly compact.

**Proof.** (i)$\Rightarrow$(ii). Evidently, spaces isomorphic to the space $c_0$ are infinite-dimensional. Hence the restriction of a semicompact operator to such a subspace cannot be invertible by definition.

(ii)$\Rightarrow$(iii). By Theorem 8.4 and condition (ii), the operator $T$ is unconditionally converging. According to Theorem 8.2 it is weakly compact.

(iii)$\Rightarrow$(i). This implication immediately follows from Theorem 7.5 and 7.7.

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**§ 8. Semicompact and co-semicoloncompact operators in $C(\Omega)$**

**Theorem 8.6.** Let $\Omega$ and $\Omega_1$ be compact Hausdorff spaces and let the space $\Omega_1$ be metrizable. If $T \in B(C(\Omega) \to C(\Omega_1))$, the following three conditions are equivalent:

(i) the operator $T$ is co-semicoloncompact,

(ii) there exists no continuous epimorphism $h_0 \in B(C(\Omega_1) \to c_0)$ such that the operator $h_0T \in B(C(\Omega) \to c_0)$ is a continuous epimorphism,

(iii) the operator $T$ is weakly compact.

**Proof.** The implication (i)$\Rightarrow$(ii) is obvious. We show that (ii)$\Rightarrow$(iii). Let us suppose that $T$ is not weakly compact. Then Theorem 8.5 implies the existence of a subspace of the space $E$ isomorphic to the space $c_0$ and such that the restriction of the operator $T$ to the space $E$ is an isomorphism of spaces $E$ and $TE$. Hence there exists an isomorphism $\iota$ between spaces $TE$ and $c_0$. Since the space $\Omega_1$ is metrizable, the space $C(\Omega_1)$ is separable. Applying Sobczyk's theorem 9.3, I, we find that there exists a continuous linear operator $p$ projecting the space $C(\Omega_1)$ onto its subspace $TE$ isomorphic with the space $c_0$. Let $h_0 = p \circ B(C(\Omega_1) \to c_0)$ and $h_0T \in B(C(\Omega) \to c_0)$. It is easily seen that $h_0$ and $h_0T$ are the required epimorphisms.

The implication (iii)$\Rightarrow$(i) is an immediate consequence of Theorem 7.7 and of the fact that the space $C(\Omega_1)$ possesses the Dunford-Pettis property.

As Pečenjak [1] has shown, the assumption of metrizability of the space $\Omega_1$ is essential.

**§ 9. Semicompact and co-semicoloncompact operators in the space $L(\Omega, \Sigma, \mu)$.**

Theorems on semicompact and co-semicoloncompact operators in spaces $L(\Omega, \Sigma, \mu)$ are in a certain sense dual to analogous theorems for spaces $C(\Omega)$. They are based on the following lemmas:

**Lemma 9.1.** If $X$ is a Banach space and the operator $T \in B(X \to l)$ is not compact, then there exists an operator $V \in B(l \to l)$ such that the operator $VT \in B(X \to l)$ is an epimorphism.

**Proof.** It follows from the assumption that there exists a number $\delta > 0$ and a sequence of elements $(\beta_n) \subset X$, $\|\beta_n\| < M$, such $\forall n$

$$\|T\beta_n - T\beta_n\| > \delta$$

$\rho \neq \gamma$ $(\rho, \gamma = 1, 2, ...)$.

By Theorem 1.3, B IV, on compact sets in spaces with a basis, one can find a subsequence $(\beta'_{n''})$ of the sequence $(\beta_n)$ such that the coefficients $T\beta'_{n''}$ of expansion of elements $T\beta'_{n''}$ with respect to the basis $(\alpha_m)$ in the space $l$ tend to zero:

$$\lim_{n \to \infty} |T\beta'_{n''}| = 0 \quad \text{for} \quad m = 1, 2, ...$$
§ 9. Semicompact and co-semicompact operators in \( L(\Omega, \Sigma, \mu) \)

It follows from Theorem 4.7, B II, that one can extract a subsequence \( (x'')_n \) of the sequence \( (x'_n) \) and an increasing sequence of indices \( (y'_k) \) in such a manner that

\[
\|Tx''_n - x'_y\| < 1/2^i, \quad \text{where} \quad y'_k = \max\{m \mid x''_n \in E_m\} \quad (i = 1, 2, \ldots).
\]

We may show in the same manner as in the proof of Theorem 8.3, I, that the space \( X \) spanned by the elements \( x'_n \) is a projection of the whole space \( I \). By Theorem 8.1, I, the space \( X \) spanned by the elements \( x''_n \) is a projection of the space \( I \). We denote this projection operator by \( P \).

On the other hand, the space \( F \) is block homogeneous (§ 1, I), whence the basis \( (x'_n) \) is equivalent to the standard basis in the space \( I \). Hence there exists an isomorphism \( R \) which maps the space \( X \) onto the space \( F \). The operator \( V = RP \) satisfies our lemma. \( \blacksquare \)

**Lemma 9.2.** If \( X \) and \( Y \) are Banach spaces, then the following conditions are equivalent for an arbitrary operator \( T \in B(\rightarrow Y) \):

(i) there exist epimorphisms \( h_X \in B(\rightarrow X) \) and \( h_Y \in B(\rightarrow Y) \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow{h_X} & & \downarrow{h_Y} \\
X & \xrightarrow{T} & Y
\end{array}
\]

(ii) there exist subspaces \( X_1 \) and \( Y_1 \) of spaces \( X \) and \( Y \), respectively, such that \( X \) and \( Y_1 \) are isomorphic to the space \( X \) and the following diagram is commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow{p_X} & & \downarrow{p_Y} \\
X_1 & \xrightarrow{T_1} & Y_1
\end{array}
\]

where \( p_X \) and \( p_Y \) are continuous projection operators onto the subspaces \( X_1 \) and \( Y_1 \), respectively, and the restriction \( T_1 \) of the operator \( T \) to the subspace \( X_1 \) has a continuous inverse.

(iii) there exists a subspace \( Y''_1 \) of the space \( Y^* \) isomorphic to the space \( X \) and such that the restriction of the operator \( T^* \) to the subspace \( Y''_1 \) has a continuous inverse.

**Proof.** (i)\(\rightarrow\)(ii). From the assumption that \( h_X \) is an epimorphism it follows that there exists a bounded sequence \( (x'_n) \subset X \) such that \( x'_n \) \(\rightarrow\) \( x_0 \) in the space \( X \). Let \( n \rightarrow \infty \) we have \( h(x'_n) \rightarrow h(x_0) \) in the space \( Y \). Since the diagram given in (i) is commutative, we have \( h_Y x'_n = y'_n \), where \( y'_n = T x'_n \). We obtain from the definition of the norm

\[
\|y'_n\| \leq \sum_{i=1}^n \|y_i\| \leq \sum_{i=1}^n \|T x'_n\| \leq \|T\| \sum_{i=1}^n \|x'_n\| \leq M \sum_{i=1}^n \|x'_n\|,
\]

where \( M = \sup \|x'_n\| \).

Hence the spaces \( X \) spanned by the elements \( x'_n \) and \( Y_1 \) spanned by the elements \( y'_n \) are isomorphic to the space \( I \). Moreover, the restriction \( T_1 \) of the operator \( T \) to the space \( X_1 \) maps \( X_1 \) onto the space \( Y_1 \), isomorphically.

We define the operators \( p_X \) and \( p_Y \) in the following manner:

\[
p_x(x) = \sum_{n=1}^\infty h(x(n) x'_n \quad \text{and} \quad p_y(y) = \sum_{n=1}^\infty h(y(n) y'_n),
\]

where \( x(n) \) means the \( n \)-th coefficient of expansion of the element \( x \in I \) with respect to the basis \( (x'_n) \). It is easily verified that \( p_X \) and \( p_Y \) are projection operators and the respective diagram is commutative.

(ii)\(\rightarrow\)(iii). Let us consider the diagram conjugate to the diagram (9.1):

\[
\begin{array}{ccc}
X & \xrightarrow{T^*} & Y^* \\
\downarrow{p_X^*} & & \downarrow{p_Y^*} \\
X_1 & \xrightarrow{T_1^*} & Y_1^*
\end{array}
\]

Evidently, projection operators are transformed into embeddings. Since the spaces \( X \) and \( Y \) are isomorphic to the space \( I \) and the subspaces \( X_1 \) and \( Y_1 \) are isomorphic to the space \( I \). Since the diagram is commutative, it follows that the space \( Y^* \) contains a subspace isomorphic to the space \( I \) and such that the restriction of the operator \( T^* \) to the space \( Y''_1 \) is invertible. But the space \( Y^* \) contains a subspace \( y_0 \). Hence there exists a subspace \( Y''_1 \) isomorphic to the space \( y_0 \) and such that the restriction of the operator \( T^* \) to the space \( Y''_1 \) is invertible.

(iii)\(\rightarrow\)(i). Let \( T_0 \) be the embedding of the space \( Y \) into the space \( Y^* \). Then \( T_0 \) is a projection operator onto the subspace \( Y''_1 \) of the space \( Y^* \). Let \( T_0 \in B(I \rightarrow Y^*) \). Then the restriction of the operator \( T_0 \) to the space \( Y \) is an isomorphism of the space \( Y \). But the restriction of the operator \( T_0 \) to the space \( Y''_1 \) is invertible. Hence the operator \( T^* \) is not compact. Consequently, the operator \( T_0 \) is not compact, either.

From Lemma 9.1 follows the existence of an operator \( V \in B(I \rightarrow Y) \).
such that the operator \( h_X = VU_1 \) is an epimorphism of the space \( X \) onto the space \( I \). Let \( h_Y = VU_1 \); then

\[
h_X = VU_1 = VU_1T = h_YT.
\]

This means that diagram \((9.1)\) is commutative. 

**Theorem 9.3.** (Pełczyński [3]) Let \( X \) be a Banach space and let the space \( L(Q, \Sigma, \mu) \) be arbitrary. The following conditions are equivalent for every operator \( T \in \{BX \to L(Q, \Sigma, \mu)\}:

(i) the operator \( T \) is co-semicompact,

(ii) for any two subspaces \( X_0 \) and \( Y_1 \) of spaces \( X \) and \( L(Q, \Sigma, \mu) \), respectively, isomorphic to the space \( I \), the following diagram is not commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{T} & L(Q, \Sigma, \mu) \\
p_X & & \downarrow p_Y \\
X_0 & \xrightarrow{Y_1} & Y_1
\end{array}
\]

(Here, \( p_X \) and \( p_Y \) denote projection operators on the spaces \( X_0 \) and \( Y_1 \), respectively, and the restriction \( T_0 \) of the operator \( T \) to the space \( X_0 \) has a continuous inverse),

(iii) the operator \( T \) is weakly compact.

Moreover, if the space \( X \) has the Dunford-Pettis property, then each of the above conditions is equivalent to the following one:

(iv) the operator \( T \) is semicompact.

Proof. The implication \((i)\rightarrow(ii)\) follows directly from the definition of a co-semicompact operator.

\((ii)\rightarrow(iii)\). Let us suppose that the operator \( T \) is not weakly compact. By Gantmacher's theorem (Corollary 4.6) the operator \( T^* \) is not weakly compact. Since the space conjugate to the space \( L(Q, \Sigma, \mu) \) is isomorphic to some space \( C(\Omega) \) (see Theorem 3.4, 4), Theorem 8.5 implies that the operator \( T^* \) satisfies condition (iii) of Lemma 9.2. Hence it follows from condition (ii) of Lemma 9.2 that condition (ii) is not satisfied.

\((iii)\rightarrow(i)\). The space \( L(Q, \Sigma, \mu) \) has the Dunford-Pettis property (Corollary 7.6). Thus, by Theorem 7.7, every weakly compact operator is co-semicompact.

Evidently, it follows from the definition of semicompactness that condition (iv) always implies (iii).

If the space \( X \) satisfies the Dunford-Pettis property, condition (iii) implies condition (iv) by Theorem 7.7. 

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**Chapter III**

\( \Phi \)-**Operators in Banach Spaces**

\section{Application of Neumann's series to the solution of equations.}

Suppose we are given a Banach space \( X \) and an operator \( B \in B(X \to Y) \) such that \( \|B\| < 1 \). Let us recall that the operator \( I - B \) is invertible and

\[
(I - B)^{-1} = \sum_{n=0}^{\infty} B^n
\]

(Theorem 1.2, 1). This immediately implies the following theorem:

**Theorem 1.1.** Let \( X \) be a Banach space and let an operator \( T \) be defined as \( T = B + K \), where \( B \in B(X) \), \( \|B\| < 1 \), and the operator \( K \) belongs to the ideal \( K(X) \) of finite-dimensional operators. Then the operator \( I + T \) has a simple regularizer \((I + B)^{-1}\) to the ideal \( K(X) \) and this regularizer is continuous.

**Proof.** Indeed,

\[
(I + T)(I + B)^{-1} = (I + B + K)(I + B)^{-1} = I + K(I + B)^{-1}
\]

where the operator \( K(I + B)^{-1} \) is obviously of a finite dimension. The same is obtained by means of a left-regularization.

Theorem 1.1 permits to solve effectively equations with operators of the above form. If there exists a basis in the space \( X \), then, of course, every compact operator can be written in this form.

**Example 1.1.** We shall solve the so-called Volterra integral equation of the second kind, i.e. the following equation:

\[
x(t) + \int_0^t K(t, s)x(s) ds = \phi(t).
\]

We shall suppose that the function \( K(t, s) \) is continuous in the square \( 0 \leq t, s \leq 1 \), that

\[
\sup_{0 \leq t, s \leq 1} |K(t, s)| = h < 1
\]

and that the function \( \phi(t) \) is continuous in the interval \( 0 \leq t \leq 1 \). Then the operator

\[
Bx = \int_0^t K(t, s)x(s) ds
\]
C. III. \( \varphi \)-operators in Banach spaces

satisfies the inequality
\[
|B_\varphi(t)| \leq \sup_{t \in cl \mathcal{C}} |\varphi(t)| \cdot \sup_{t \in cl \mathcal{C}} |K(t, s)| \cdot |s| \leq k \|\varphi\| \cdot |t|.
\]
Hence
\[
|B| = \sup_{t \in cl \mathcal{C}} |B_\varphi(t)| = \sup_{t \in cl \mathcal{C}} |B_\varphi(t)| \leq k < 1.
\]
Thus the operator \( I + B \) is invertible and
\[
x(t) = (I + B)^{-1}x_0(t) = \sum_{n=0}^{\infty} (-1)^n B^n x_0(t).
\]

We define a sequence \( \{x_n\} \) in the following manner:
\[
x_n = x_n(t), \quad x_n = x_n - B x_{n-1} \quad (n = 1, 2, \ldots).
\]
It is easily verified that
\[
x_n = \sum_{k=0}^{n} (-1)^k B^k x_0 \quad (k = 1, 2, \ldots).
\]
But the series \( \sum_{k=0}^{\infty} (-1)^k B^k \) is convergent in the norm. Consequently, the series with partial sums \( \{x_n\} \) is uniformly and absolutely convergent and
\[
x(t) = \lim_{n \to \infty} x_n(t).
\]
The sequence \( \{x_n\} \) is called a sequence of successive approximations. Evidently, the operators \( B^k \) are defined by the so-called \textit{iterated kernels} \( K_n(t, s) \), i.e.
\[
B_\varphi(t) = \int_0^t K_\varphi(t, s) \varphi(s) ds,
\]
where
\[
K_n(t, s) = \int_0^t K(t, \sigma) K_{n-1}(\sigma, s) d\sigma \quad (n = 2, 3, \ldots), \quad K_\varphi(t, s) = K(t, s).
\]

**EXAMPLE 1.2.** We shall find out for which values of the parameter \( \lambda \) the differential equation
\[
\frac{dx}{dt} = \lambda f(t) x(t)
\]
with the initial condition
\[
x(t_0) = x_0 \quad (a \leq t_0 \leq b)
\]
has a solution expressed by means of the von Neumann series. We assume the function \( f(t) \) to be continuous in the interval \( a \leq t \leq b \). We write
\[
m = \sup_{t \in cl \mathcal{C}} \|f(t)\|. \text{ As is known, the differential equation (1.2) with the initial condition (1.3) is equivalent to the Volterra integral equation }
\]
\[
(1.4) \quad x(t) = x_0 + \lambda \int_0^t f(s)x(s) ds.
\]
The integral operator in this equation \( B_\varphi(t) = \int_0^t f(s)x(s) ds \) satisfies the following inequality (obtained in the same manner as in the previous example):
\[
\|B_\varphi\| = \lambda \|f\| \|x\| \leq \lambda \|f\| \|x\| \|x\| \leq \lambda \|f\| \|x\|.
\]
Hence \( \|B\| < 1 \) if only \( |\lambda| \|f\| \|x\| < 1 \). Thus, if the parameter \( \lambda \) satisfies the inequality
\[
|\lambda| < \frac{1}{m(b-a)},
\]
equation (1.2) has a unique solution which is the limit of the sequence of successive approximations:
\[
x_0(t) = x_0,
\]
\[
x_n(t) = x_0 + \lambda \int_0^t f(s)x_{n-1}(s) ds \quad (n = 1, 2, \ldots).
\]

**EXAMPLE 1.3.** Let us consider the equation
\[
x(t) + \lambda x(t) = x(t_0) x_0(t),
\]
where \( \lambda \) is a parameter, and the given functions \( x_0 \) and \( x_2 \) are continuous and bounded on the whole straight line. The point \( t_0 \) is a fixed point on the straight line. We write
\[
B_\varphi(t) = \lambda x(t_0) x_0(t), \quad K_\varphi(t, s) = \lambda x(t_0) x_0(s).
\]
Thus the operator \( K \) is one-dimensional. The space \( X \) of all functions continuous and bounded on the whole straight line is a Banach space with the norm
\[
\|x\| = \sup_{-\infty < t < +\infty} |x(t)|.
\]
If \( |\lambda| < 1 \), we have \( \|B\| < 1 \) and
\[
(1.5) \quad x(t) = \sum_{n=0}^{\infty} (-1)^n \lambda^2 x_0(t) = \sum_{n=0}^{\infty} (-1)^n \lambda^{n+1} x_0(t - n) = \frac{1}{1 - \lambda^2} x(t) - \frac{\lambda}{1 - \lambda^2} x(t - \lambda^2).
\]
\[ (I + B)^{-1}Kx(t) = (I + B)^{-1}x(t_0) = \frac{x(t_0)}{1 - \lambda t_0} [x(t_0) - \lambda x(t_0)], \]

Thus, by Theorem 1.1, if \(|\lambda| < 1\), then equation (1.5) is equivalent to the equation
\[ x(t) + \frac{x(t_0)}{1 - \lambda t_0} [x(t_0) - \lambda x(t_0)] = \frac{x(t_0) - \lambda x(t_0)}{1 - \lambda t_0}. \]

Since the last equation is of the form \((I + K)x = x_1\), where the operator \(K_x\) is one-dimensional, arguing as in §3, A I, we finally obtain the following conclusions:

1. If \(|\lambda| < 1\) and \([x(t_0) - \lambda x(t_0)]/(1 - \lambda t_0) \neq 1\), then equation (1.5) has a unique solution given by the formula:
   \[ x(t) = \frac{x(t_0) - \lambda x(t_0)}{1 - \lambda t_0}, \]
   where \(t \neq t_0\).

2. If \(|\lambda| < 1\) and \([x(t_0) - \lambda x(t_0)]/(1 - \lambda t_0) = 1\), then equation (1.5) has a solution if and only if
   \[ \frac{x(t_0) - \lambda x(t_0)}{1 - \lambda t_0} = 0, \]
   and this solution is of the form
   \[ x(t) = \frac{x(t_0) - \lambda x(t_0)}{1 - \lambda t_0} - \mathcal{O}[x(t_0) - \lambda x(t_0)], \]
   where \(\mathcal{O}\) is an arbitrary constant. If \(|\lambda| > 1\), we obtain an analogous solution substituting \(t = -t_0, t_0 = -t_0, \mathcal{O} = \mathcal{O}(-t_0)\).

\section{Continuity of solutions}

If we are not able to solve the equation
\[ (I + A)x = x_0, \quad \text{i.e.} \quad x = x_0 - Ax \]
directly, and if we want to determine an approximation of the solution in such a manner that the error does not exceed a given number, we must know whether the solution of this equation is continuous with respect to the operator \(A\), i.e., whether "small" increments of the operator \(A\) cause "small" increments of the solution. This question will be answered by the following method, which, in many cases, is more convenient than von Neumann's method.

**Theorem 2.1.** If \(X\) is a Banach space and if an operator \(A \in B(X)\) is a limit (in the norm) of a sequence of uniformly bounded operators, \(\{A_n\} \subset B(X)\):
\[ \|A_n\| < q < 1, \]
then the equation
\[ (I + A)x = x_0, \quad x = x_0 - Ax \]
has a unique solution which is a limit of the sequence \(\{x_n\}\) of solutions of approximating equations:
\[ x_n = x_0 - A_n x_n \quad (n = 1, 2, \ldots). \]

**Proof.** Since \(q < 1\), each of the equations (2.2) has a unique solution. Hence equation (2.1) also has a unique solution. Indeed, let \(e\) be an arbitrary positive number. Then
\[ \|x - x_n\| = \|Ax - A_n x_n\|. \]
for sufficiently large \(n\). Since \(e\) is arbitrary, we obtain \(\|A\| < q < 1\). Subtracting equation (2.2) from equation (2.1) we get the following inequality:
\[ \|x - x_n\| = \|Ax - A_n x_n\| \]
\[ \leq \|A\| \|x - x_n\| + \|A - A_n\| \]
\[ \leq \|A\| \|x - x_n\| + \|A - A_n\| \|x - x_n\| \]
\[ < \|A - A_n\| \|x - x_n\| + q \|x - x_n\|, \]

i.e.
\[ \|x - x_n\| < \frac{\|x - x_n\|}{1 - q}. \]

Hence for an arbitrary number \(\varepsilon > 0\) there exists a number \(N_0\) such that if \(n > N_0\), then \(\|A - A_n\| < \frac{1 - q}{|x - x_n|}\) implies \(\|x - x_n\| < \varepsilon\).

Applying inequality (2.3) we remark that the error in the \(n\)th approximation is not greater than
\[ d_n = \frac{|x - x_n|}{1 - q} (n = 1, 2, \ldots). \]

**Example 2.1.** We consider the integral equation
\[ x(t) + \int_0^t \frac{s(x)}{1 - t_s} ds = 1 \]
in the space \(C[0, 1/2]\). Since
\[ \sum_{k=0}^n \frac{1}{1 - k} = \frac{1}{1 - n}, \]
for \(|u| < 1\), we take as \(A_n\) the following operators of finite dimension:
\[ A_n x = \sum_{k=0}^{n-1} \int_0^t k s(x) ds = \sum_{k=0}^{n-1} \int_0^t k s(x) ds. \]
Then
\[ \| A_n \| \leq \sum_{k=0}^{n-1} \left( \frac{1}{\sup_{x \in X} |t|^k} \right)^{3/2} \]
\[ = \frac{1}{2} \sum_{k=0}^{n-1} \frac{1}{2} \frac{1}{2} \frac{1}{1-1/4} \frac{1}{2} \frac{1}{4^k} = \frac{1}{3} \frac{1}{4^n}. \]

Hence the norms of all operators $K_n$ are uniformly bounded by the number $2/3$ and one may take $q = 2/3$. Moreover,
\[ \| A - A_n \| \leq \sum_{k=0}^{n-1} \left( \frac{1}{\sup_{x \in X} |t|^k} \right)^{3/2} \]
\[ = \frac{1}{2} \sum_{k=0}^{n-1} \frac{1}{2} \frac{1}{2} \frac{1}{1-1/4} \frac{1}{2} \frac{1}{4^k} = \frac{1}{3} \frac{1}{4^n}. \]

Hence the error of the $n$th approximation is not greater than the number
\[ \delta_n = \frac{\| a_n \|}{1-q} \| A - A_n \| \leq \frac{1}{2} \frac{1}{1-1/4} \frac{1}{3} \frac{1}{4^n}. \]

According to § 3, $A \sigma$, we expect the solution of the $n$th approximating equation
\[ (I + A_n) \sigma_n = 1, \quad i.e., \quad \sigma_n = 1 - A_n \sigma_n. \]
to have the form
\[ a_n(t) = 1 - \sum_{k=0}^{n-1} C_n t^k, \quad \text{where} \quad C_n = \int_0^1 s^k a_n(s) \, ds. \]

Since
\[ \int_0^1 s^k \, ds = \frac{1}{(k+1)^{k+1}} \]
the constants $C_n$ with a fixed $n$ satisfy the following system of equations:
\[ \sum_{m=0}^{n-1} \left( \frac{1}{(m+1)^{m+1}} \right)^{1/2} C_n = \frac{1}{(m+1)^{m+1}} \quad (m = 0, 1, \ldots, n-1). \]

The assumption $q < 1$ implies that this system has a determinant different from zero, and consequently, one solution only. We compute the first and the second approximation:

Let $n = 1$; we then have one equation: $(1 + \frac{1}{2}) C_0 = \frac{1}{2}$. Hence $C_0 = \frac{1}{2}$, and so the first approximation is
\[ a_1(t) = 1 - \frac{1}{2} t = \frac{1}{2}. \]

with an error $\delta_1 = 2 \frac{1}{2}/4 = 0.33...$

§ 2. Continuity of solutions

Let $n = 2$; we then have a system of equations
\[ \left( 1 + \frac{1}{2} \right) C_1 + \frac{1}{2} \frac{1}{2} C_0 = \frac{1}{2}; \quad \frac{1}{2} \frac{1}{2} C_0 + \left( 1 + \frac{1}{2} \right) C_1 = \frac{1}{2}. \]

Hence we obtain $C_0 = \frac{1}{2}, C_1 = \frac{1}{4}$. Thus the second approximation is
\[ a_2(t) = 1 - \frac{1}{2} \frac{1}{4} t = \frac{1}{2} \left( 36 - 33 \right) \]
with an error
\[ \delta_2 = \frac{2}{3} \sup_{x \in X} \| 36 - 33 \| = 26 \frac{13}{189} \approx 0.07. \]

§ 3. Normally resolvable operators.

Theorem 3.1. If $X$ and $Y$ are Banach spaces, if the operator $A \in B(X \rightarrow Y)$ and if the set $E_A$ is closed, then
\[ E_A = \{ \sigma \in X^+ : \text{if} \ Ax = 0, \ \text{then} \ \sigma \sigma(x) = 0 \}. \]

Proof. Let the functional $\sigma \sigma(x)$ satisfy the following condition: if $Ax = 0$, then $\sigma \sigma(x) = 0$. We define a linear functional $\gamma^s$ (not necessarily continuous) over the space $E_A$ by means of the formula:
\[ \gamma^s \sigma = \sigma \sigma(x). \]

The condition defining the functional $\sigma \sigma(x)$ implies that the functional $\gamma^s$ is defined uniquely. Since the spaces $X, Z_A$ and $E_A$ are isomorphic, we conclude from Banach's theorem (Theorem 3.2, B II) that there exists a constant $C$ such that for every $y \in E_A$ there exists an element $x$ satisfying the conditions $\| x \| \leq C \| y \|$ and $Ax = y$. Hence
\[ \gamma^s \sigma = \sigma \sigma(x) \leq C \| x \| \| y \|. \]

It follows from the Hahn-Banach theorem (Theorem 2.2, I) that the functional $\gamma^s$ can be extended to a linear functional $\gamma^s$ defined on the whole space $Y$ and such that $A^+ \gamma^s = \sigma \sigma$. We conclude from the definition of the operator $A^+$ that every element of the set $E_A$ of the values of this operator satisfies the imposed conditions. $\blacksquare$

Lemma 3.2. Let $X$ and $Y$ be Banach spaces and let $A \in B(X \rightarrow Y)$. If the operator $A^+$ is one-to-one and the set $E_A$ is closed, then $E_A = Y$.

Proof. Let $0 \neq y \in Y$ and let
\[ Y^+ = \{ y^* \in Y^+ : y^*(y) = 0 \}. \]

The set $Y^+$ is $Y$-closed in the space $X^+$.

First, let us suppose that the set $A^+ Y^+$ is closed in the $X$-topology and different from the set $A^+ Y^*$. Then there exists an element $x \in X$ satisfying the conditions
\[ \sigma(x) A^+ Y^* = 0, \quad \sigma(x) A^+ Y^+ = 0. \]
where $\sigma$ denotes the natural embedding of the space $X$ into the space $X^\ast$. This means that $\sigma x \neq 0$ and $y^\ast(\sigma x) = 0$ for every $y^\ast \in X^\ast$. Hence $\sigma x = \sigma y$, where $\sigma$ is a scalar. Thus $y \in E_\sigma$ and $E_\sigma = Y$, as was to be proved.

It remains to prove that the set $A^\ast Y^\ast$ is closed in the $X$-topology, but different from the set $A^\ast Y^\ast$. Since $\sigma \neq 0$, the set $Y^\ast$ is a proper subset of the space $X^\ast$; and since the operator $A$ is invertible, the set $A^\ast Y^\ast$ is a proper subset of the set $E_\sigma = A^\ast Y^\ast$. Finally, in order to prove that the set $A^\ast Y^\ast$ is closed in the $X$-topology it is sufficient to show (by Theorem 3.8, I) that the set $(A^\ast Y^\ast) \cap S(X^\ast)$ is closed, $S(X^\ast)$ denoting the closed unit ball in the space $X^\ast$. It follows from the continuity of the operator $(A^\ast)^{-1}$ that the set $(A^\ast)^{-1}S(X^\ast)$ is bounded. Hence $(A^\ast)^{-1}S(X^\ast) \subset \sigma S(Y^\ast)$ for some natural number $n$, where $\sigma S(Y^\ast)$ is the closed unit ball in the space $Y^\ast$. By Theorem 3.3, I, implies that the set $\sigma S(Y^\ast)$ is compact in the $Y$-topology of the space $Y^\ast$. But, by Theorem 10.4, B, I, the operator $A^\ast$ is a continuous transformation of the space $Y^\ast$ with its $Y$-topology into the space $X^\ast$ with its $X$-topology. Hence the image of the complete set $Y^\ast \cap n S(Y^\ast)$ by means of this transformation is closed. Consequently, the set

$$(A^\ast Y^\ast) \cap S(X^\ast) = S(X^\ast) \cap A^\ast Y^\ast \cap n S(X^\ast)$$

is $X$-closed.

Theorem 3.3. If $X$ and $Y$ are Banach spaces, $A \in B(X \rightarrow Y)$ and the set $E_\sigma$ is closed, then the set $E_\sigma$ is closed, and

$$E_\sigma = \{ y \in Y : (A^\ast y^\ast = 0, \text{then } y^\ast(y) = 0 \}.$$  

Proof. Let $A_1$ denote the map of the space $X$ into the space $Z = E_\sigma = A X$ defined by means of the equality $A_1 x = \sigma x$. Since the operator $A_1$ has a dense set of values, the operator $A_1^\ast$ is one-to-one. If $x^\ast \in X^\ast$ belongs to the closure of the set $A_1^\ast Z^\ast$, we have $x^\ast = \lim A_1^\ast z^\ast$, where $z^\ast \in Z^\ast$. If we denote by $y^\ast$ a continuous extension of the functional $x^\ast$ to the whole space $Y^\ast$, we get $x^\ast = \lim A_1^\ast y^\ast$, and since the set $E_\sigma$ is closed, we have $x^\ast = A^\ast y^\ast$ for some $y^\ast \in Y$. If $x^\ast$ denotes the restriction of the functional $y^\ast$ to the space $Z$, we have $x^\ast = A_1 x^\ast$. Hence the set $E_\sigma$ is also closed.

It follows from Lemma 3.2 that $E_\sigma = A_1 X = A X = E_\sigma$ is closed. Thus the set $E_\sigma$ is closed.

The following theorem, analogous to Theorem 7.1, Chapter I, Part A, holds for normally resolvable operators:

Theorem 3.4. If $X$ and $Y$ are Banach spaces, $A \in B(X \rightarrow Y)$ and $E_\sigma$ is a projection of the space $Y$, then

$$A \in D^\ast(X \rightarrow Y) \quad \text{and} \quad \sigma_a = \sigma_b$$

$$A \in D^\ast(X \rightarrow Y) \quad \text{and} \quad \beta_a = \sigma_a$$

if and only if $A = S + K$,

where $K$ is an operator of a finite dimension, and the operator $S$ has a left inverse (right inverse, respectively) $S_a = B(Y \rightarrow X)$.

The proof follows the same lines as that of Theorem 7.1, A I. It is sufficient to require that the functionals appearing in the definition of the operator $K$ be continuous. The continuity of the inverse operator is a consequence of Banach's theorem (Theorem 3.2, B II).

Corollary 3.5. If $X$ and $Y$ are Banach spaces, $A \in B(X \rightarrow Y)$ and $x_a = 0$, then $A = S + K$, where the operator $K$ is of a finite dimension and the operator $S$ is invertible.

Proof. By Theorem 3.4, we have $A = S + K$, where the operator $S$ is left-invertible, i.e., $x_a = 0$. But $x_a = x_a$. Hence we have also $\beta_a = 0$. Thus the operator $S$ is also right-invertible.

§ 4. Perturbations with a small norm. Theorem 2.1 can be formulated also in the following manner:

If $A = I + C$, where $C \in B(X)$ and $\|C\| < 1$, then for every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that if $B \in B(X)$ and $\|B\| < \delta$, then $\|A - B\| < \varepsilon$.

If the operator $A$ has finite $\delta$-characteristic, it is of course impossible to discuss the nearness of the solutions of the respective equations. There can be infinitely many solutions. However, some analogies of Theorem 2.1 can be proved.

Theorem 4.1. (Gohberg and Krein [1]) Let $X$ and $Y$ be Banach spaces, and let $A \in B(X \rightarrow Y)$ be any $\Phi$-operator. There exists a number $\varepsilon > 0$ such that for all operators $B \in B(X \rightarrow Y)$ satisfying the inequality $\|B\| < \varepsilon$, $A + B$ is a $\Phi$-operator and $x_{A + B} = x_A$.

In other words: for every $\Phi$-operator $A$ which maps a Banach space $X$ into a Banach space $Y$ there exists a positive number $\varepsilon$ such that all operators with a norm less than $\varepsilon$ are $\Phi$-perturbations of the operator $A$ which do not change the index.

Proof. We write the space $X$ as a direct sum: $X = E \oplus Z$, where $E$ is a closed subspace. To this decomposition there correspond two continuous projection operators

$$P_1 x = \begin{cases} x & \text{for } x \in E \\ 0 & \text{for } x \in Z \end{cases} \quad \text{and} \quad P_2 x = \begin{cases} 0 & \text{for } x \in E \\ x & \text{for } x \in Z. \end{cases}$$

We write $E = D \oplus Z$. Let $A_1$ be the restriction of the operator $A$ to $D$. Evidently, $A_1$ is also a $\Phi$-operator and its $\delta$-characteristic is $(0, \beta_a)$. The operator $A_1^{-1}$ is closed, and since it is defined on a closed set, it is also continuous (Theorem 1.4, B III). Let $A_1^{-1}$ be an arbitrary continuous extension of the operator $A_1^{-1}$ defined on the whole space $Y$. Such an
extension exists, because the \(d\)-characteristic is finite. Evidently, \(A_1^{-1}(A, \sigma) = x\) for \(x \in D_1\). Let us take

\[ \theta = \frac{1}{\|A_1^{-1}\|}. \]

Let \(B\) be an arbitrary operator with a norm less than \(\theta\) which maps the space \(X\) into the space \(Y\). Let \(B_1\) be the restriction of the operator \(B\) to the set \(D_1\). Then

\[ A_1 + B_1 = (I + B A_1^{-1}) A_1. \]

But \(\|B A_1^{-1}\| < \theta\|A_1^{-1}\| = 1\). Hence the operator \(C = I + B A_1^{-1}\) is invertible, and so it is a \(\Phi\)-operator and its index is equal to zero. Thus, by Theorem 2.2, III, the operator \(A_1 + B_1\) is a \(\Phi\)-operator and

\[ x_{A_1} + x_{B_1} = x_0 + x_{A_1} = x_{A_1} = \beta_{A_1}. \]

Hence it follows easily that \(x_{A_1} + x_{B_1} = \beta_{A_1} - a_{A_1} = x_{A_1}\). On the other hand,

\[ A_1 + B = A + B P_1 + B P_1 \]

and the operator \(B P_1\) is of a finite dimension, i.e. it does not change the index (Theorem 2.2, A I). Thus we have finally

\[ x_{A_1 + B} = x_{A_1 + B P_1} = x_{A_1}. \]

**Theorem 4.2.** (Gohberg and Krein [1]) Let \(X\) and \(Y\) be Banach spaces. If an operator \(A \in D^e(X \to Y)\) is normally resolvable, then there exists a number \(\varepsilon_1 > 0\) such that the conditions \(B \in B(X \to Y)\) and \(\|B\| < \varepsilon_1\) imply that the operator \(A + B\) is normally resolvable and \(a_{A_1 + B} = a_{A_1}\).

**Proof.** We write the space \(X\) as a direct sum \(X = Z_A \oplus \mathbb{C}\). Let \(A_1\) be the restriction of the operator \(A\) to the space \(Z_A\), i.e.

\[ A_1 x = Ax \quad \text{for} \quad x \in D_1 \cap Z_A. \]

Evidently, the operator \(A_1\) is normally resolvable and has a \(d\)-characteristic \((0, \beta_{A_1})\). Since \(a_{A_1} = 0\), there exists a number \(m > 0\) such that

\[ \|A_1 x\| > m \|x\| \quad \text{for} \quad x = D_1 - D_1 \cap Z_A. \]

Let \(B \in B(X \to Y)\) be an operator satisfying the inequality \(\|B\| < \varepsilon_1\), where \(\varepsilon_1 = m/5\). Then

\[ \|A_1 + B\| \geq (m - \|B\|) \|x\| \geq \frac{4}{5} m \|x\| \quad \text{for} \quad x \in D_1. \]

It follows from this inequality that the operator \(A_1 + B\) is normally resolvable and \(a_{A_1 + B} = 0\).

Hence the operator \(A_1 + B\) is also normally resolvable, as an extension of the operator \(A_1 + B\) to a space of a finite dimension (\(d\)-dimensional), and

\[ a_{A_1 + B} = a_{A_1} + a_B = a_{A_1}. \]

§ 4. Perturbations with a small norm

A theorem analogous to that on the nullity of an operator holds also for the deficiency:

**Theorem 4.3.** (Gohberg and Krein [1]) Let \(X\) and \(Y\) be Banach spaces. If an operator \(A \in D^e(X \to Y)\) is normally resolvable, then there exists a number \(\varepsilon_1 > 0\) such that the conditions \(B \in B(X \to Y)\) and \(\|B\| < \varepsilon_1\) imply that the operator \(A + B\) is normally resolvable and \(\beta_{A_1 + B} \leq \beta_{A_1}\).

**Proof.** First, let us suppose that the operator \(A\) is defined on the whole space \(X\): \(D_A = X\), and \(\beta_A = 0\), i.e. the operator \(A\) is a continuous epimorphism of the space \(X\) onto the space \(Y\). By Theorem 2.7, I, the conjugate operator \(A^* \in B(Y^* \to X^*)\) is an embedding. By Theorem 4.2, there exists a number \(\varepsilon' > 0\) such that if \(\|B\| = \|B^*\| < \varepsilon'\), then the operator \(A^* + B^*\) is an embedding. Hence, according to Theorem 2.6, the operator \(A + B\) is a continuous epimorphism. Consequently, it is a normally resolvable operator, and \(0 = \beta_{A_1 + B} \leq \beta_{A_1} = 0\).

We now proceed to the proof in the case of \(\beta_A \neq 0\). Let \(Y = E_A \oplus N\) and let \(M\) be a certain \(\beta_A\)-dimensional normed space. We denote by \(C\) a linear operator which maps the space \(M\) onto the space \(N\) (in particular, one can take \(M = N\) and \(C = I\)). Let \(X_1 = X \oplus M\) with a norm defined by the formula:

\[ \|x + y\| = \|x\| + \|y\| \quad (x \in X, \; y \in N). \]

We denote by \(\hat{A}\) the extension of the operator \(A\) to the space \(X\) defined by the formula:

\[ \hat{A} (x + y) = Ax + Cy \quad (x \in D_A, \; y \in M). \]

It is easily seen that the operator \(\hat{A}\) is normally resolvable and \(\beta_{\hat{A}} = 0\), \(a_{\hat{A}} = a_A\).

Hence we may apply the first part of the theorem which we have already proved to the operator \(\hat{A}\). Thus there exists a number \(\varepsilon_1 > 0\) such that for all operators \(B \in B(X \to Y)\), \(D_B = X_1\), satisfying the inequality \(\|B\| < \varepsilon_1\), the operator \(\hat{A} + B\) is normally resolvable and \(\beta_{\hat{A} + B} = 0\), \(a_{\hat{A} + B} = a_{\hat{A}}\),

where \(\hat{B}\) is the extension of the operator \(B\) to the space \(X_1\) defined by means of the formula \(\hat{B} M = 0\).

Let us remark that the operator \(\hat{A} + \hat{B}\) is an extension of the operator \(A + B\) to the space \(X_1\). Hence the operator \(A + B\) is normally resolvable and \(\beta_{A + B} \leq \beta_{\hat{A}}\).

Let us now consider the most general case. We define a new norm in the set \(D_A\): \(\|\cdot\|_a = \|\cdot\| + \|A\|\) (see § 1, B II). The set \(D_A\) with norm \(\|\cdot\|_a\)
is a Banach space which we shall denote by $X_d$. The operator $A$ induces in the space $X_d$ a bounded operator $A$ with the same set of values as the operator $A$. Hence the operator $A$ is normally resolvable and $\beta_d = \beta_d$.

We apply part of theorem which is already proved to the operator $A$. Hence for all operators $B, D = X_d$ for which $||B|| < \varepsilon_1$, the operator $A + B$ is normally resolvable and

$$\beta_{A + B} = \beta_d.$$  

In particular, the inequality $||B|| < \varepsilon_1$ holds for all operators $B \in B(X \to Y)$, $D = X_d$, satisfying the conditions $||B|| < \varepsilon_1$, $||B|| < ||B||$. Hence in this case the operator $A + B$ is normally resolvable and satisfies condition (4.1). Consequently, the operator $A + B$ is normally resolvable and

$$\beta_{A + B} = \beta_d.$$

It follows from Theorems 4.2 and 4.3 that if we consider the set $R(X \to Y)$ of continuous normally resolvable operators, then

$$\int R(X \to Y) \supset [D (X \to Y) \cup D (X \to Y)] \cap B(X \to Y).$$

As follows from a paper by M. A. Goldman [1], the sign of inclusion may be replaced by the sign of equality, i.e.

$$\int R(X \to Y) = [D (X \to Y) \cup D (X \to Y)] \cap B(X \to Y).$$

Let us remark that the positive constant $\varepsilon_1$ is the same in Theorems 4.2 and 4.3.

Theorems 4.2 and 4.3 show that $\Phi_{\varepsilon}$ and $\Phi_{\varepsilon}$-operators have hold theorems analogous to the first part of Theorem 4.1. In order to prove that $||B|| < \varepsilon_1$ implies $\lambda_{A + B} = \lambda_d$ we define the notion of the gap of two spaces.

We denote by $\delta(x, M)$ the distance between the point $x$ in a Banach space $X$ and a subspace $M \subset X$, i.e. the number

$$\delta(x, M) = \inf_{y \in M} \|x - y\|,$$

and by $\Theta(M, N)$, the gap of subspaces $M$ and $N$ of this space, i.e. the number

$$\Theta(M, N) = \sup_{x \in N} \sup_{y \in M} \delta(x, N) \delta(y, M).$$

Evidently, we always have

$$0 \leq \Theta(M, N) = \Theta(N, M) \leq 1,$$

and

$$\Theta(M, N) = \Theta(M, N).$$

Let us remark that the gap of spaces does not satisfy the triangle inequality. Hence sometime it is more convenient to use another notion, such called the distance of spaces introduced by Gohberg and Markus [1].

The distance of subspaces $M$ and $N$ is given by the formula:

$$\Theta(M, N) = \max \sup_{x \in B(M)} \sup_{y \in S(N)} \delta(x, N) \delta(y, N)$$

where $S(M)$ and $S(N)$ denote the unit spheres in the spaces $M$ and $N$ respectively.

Let us recall that a system of points $x_1, \ldots, x_n$ is called an $\varepsilon$-net of the set $E$ if $\sup_{x \in E} \delta(x, E) < \varepsilon$ for every point $x \in E$ (§ 1, B IV).

Let $M$ be an infinite-dimensional subspace. If any number $\varepsilon, 0 < \varepsilon < 1$, is given, the $\varepsilon$-net of the unit sphere $S(M)$ must be infinite. Indeed, let us suppose that this net is finite and consists of points $x_1, \ldots, x_n$. Let $L = \{ x \}$ of norm $\varepsilon < \|x\| < 1$ in the quotient space $M/L$. Hence there exists a point $x \in \{ x \}$ such that $\|x\| < 1$ but $\delta(x, L) > \varepsilon$. Thus the system $x_1, \ldots, x_n$ is not an $\varepsilon$-net.

If the space $N$ is of a finite dimension, the unit sphere $S(N)$ is precompact. Hence for every $\varepsilon > 0$ there exists a finite $\varepsilon$-net in this ball (see § 1, B IV).

Theorem 4.4. (Krein, Krasnosel'ski and Milman [1].) If $\Theta(M, N) < 1$ and if the subspace $M$ is infinite-dimensional, then the subspace $N$ is infinite-dimensional.

Proof. Let us suppose that the subspace $N$ is of a finite dimension. Let us form a finite $\varepsilon$-net $x_1, \ldots, x_n$ in $N$, where $\varepsilon < 1 - \Theta(M, N)$. It immediately follows from the definition of the number $\Theta(M, N)$ that the system $x_1, \ldots, x_n$ is an $\varepsilon$-net, where $\varepsilon = \varepsilon \Theta(M, N)$, in the sphere $S(M)$. But this is impossible because there can exist no finite $\varepsilon$-net in the sphere $S(M)$. Hence the subspace $N$ is infinite-dimensional.

Krein, Krasnosel'ski and Milman [1] show more in their paper: namely, that in the case of infinite-dimensional spaces the minimal power of an $\varepsilon$-net, $0 < \varepsilon < 1$, of the ball $S(M)$ in a Banach space $M$ is equal to the minimal power of a dense set. They formulate Theorem 4.4 as follows: if $\Theta(M, N) < 1$ (in the original paper, $\varepsilon < 1/2$), then the minimal powers of sets dense in subspaces $M$ and $N$ are equal.

We now give a few more facts in connection with the gap of spaces, which are necessary in our further considerations.

Let $M$ be a subspace of a Banach space $X$. We write

$$M^* = \{ x^* \in X^* : \delta(x^*, x) = 0 \text{ for } x^* \in M \}.$$  

Theorem 4.5. (Krein, Krasnosel'ski and Milman [1]) If $M$ and $N$ are subspaces of a Banach space $X$, then

$$\Theta(M^*, N^*) = \Theta(M, N).$$
§ 4. Perturbations with a small norm

**Theorem 4.7.** If the assumptions of Theorem 4.3 are satisfied and \( a_A = +\infty \) then \( a_{A+B} = +\infty \).

Proof. To begin with, let us suppose that the operator \( A \) is defined and continuous on the whole space \( X \) and that \( \beta_A = 0 \). Then the operator \( A \) is a continuous epimorphism. By Theorem 2.7, I, the operator \( A^+ \in B(X^+ \to X^+) \) is an embedding. Applying the estimation given in Theorem 4.2 we find that

\[
\Theta(E_{A^+}, E_A) < \frac{1}{2}.
\]

But Theorem 3.1 implies \( E_{A^+}, E_A \subseteq Z_{A+B} \) and \( E_A = Z_A \). Hence we conclude from Theorem 4.5 that

\[
\Theta(Z_{A+B}, Z_A) < \frac{1}{2}.
\]

By Theorem 4.4, we have \( a_{A+B} = +\infty \), because \( a_A = +\infty \).

Thus \( \beta_A > 0 \), we argue as in the proof of Theorem 4.3 considering operators \( A \) and \( B \) defined in the product \( X \times \mathbb{C} \), where \( \mathbb{C} \) is a \( \beta_A \)-dimensional space and the operator \( A \) is a continuous epimorphism. Since \( \beta_{A+B} = +\infty \), and consequently \( \beta_{A+B} = +\infty \).

The theorems on perturbations given so far involve one inconvenience: namely, the perturbation \( B \) of the operator \( A \) is required to be at least \( A \)-continuous, and this implies that the operator \( B \) must be defined in the domain of the operator \( A \). This inconvenience can be removed by applying the distance of the graphs of closed operators.

We call the set

\[
W_A = \{(x, y) : y = Ax, x \in D_A \} \subset X \times Y
\]

the graph of the operator \( A \subset L(X \to Y) \) (see § 1, A I). The distance \( q(A, B) \) of two closed operators \( A, B \subset L(X \to Y) \) is defined as the distance of their graphs, i.e.,

\[
q(A, B) = \|W_A - W_B\|.
\]

A similar metric was considered by J. D. Newburgh [1]. As has been proved by E. Berkson [1], it is equivalent to the metric \( q(A, B) \).

The following theorem holds for the metric \( q(A, B) \):

**Theorem 4.8.** (Paraskas [1]) Let \( X \) and \( Y \) be Banach spaces. For every \( \Phi \)-operator \( A \in L(X \to Y) \) there exists a number \( \delta > 0 \) such that every closed operator \( B \in L(X \to Y) \) satisfying the inequality

\[
q(A, B) < \delta
\]
is also a \( \Phi \)-operator. Moreover,

\[
x_B = x_A \quad \text{and} \quad a_B \leq a_A.
\]
4. Perturbations with a small norm

Hence it follows that the operator $B = B_1 - K$ is a $\Phi$-operator, and $x_0 = x_0$, and also $\beta_0 = \beta_A = \beta_{A-K} = x_0$. Since $a_0 = 0$, we obtain $x_0 = x_0$.

Finally, we show that $a_0 < a_0$. For an arbitrary $y \in Z_B$ we have $B_1 y = Ky$. Hence

$$B_1(Z_B) = K(Z_B),$$

and since $\beta_0 = 0$ and $\dim B_1 = a_0$, we get $a_0 = \dim B_1 = \dim B_1(Z_B) = \dim K(Z_B) = a_0$. ■

In an analogous manner we obtain

**Theorem 4.9.** (Paraske [1]) Let $X$ and $Y$ be Banach spaces and let $A \in L(X \rightarrow Y)$ be a $\Phi_-$-operator ($\Phi_-$-operator). There exists a number $\delta > 0$ such that every closed operator $B \in L(X \rightarrow Y)$ satisfying the inequality

$$\varrho(A, B) < \delta$$

is a $\Phi_-$-operator ($\Phi_-$-operator) and

$$a_0 \leq a_0, \quad (\beta_0 \leq \beta_0);$$

$$\beta_0 = \beta_0 \quad (\alpha_0 = \alpha_0).$$

§ 5. Improved estimations of the norms of small perturbations.

From Theorem 4.2 and 4.3 immediately follows the first part of Theorem 4.1; namely, that for operators $B \in L(X \rightarrow Y)$ such that $\|B\| < \varrho$ $A + B$ is a $\Phi$-operator. However, the constant $\varrho$, obtained in this manner is smaller than the constant $\varrho$ given in Theorem 4.1.

The following question arises: is it possible to prove theorems analogous to Theorems 4.2 and 4.3 with the same constant $\varrho$ which appears in Theorem 4.1. The answer is positive and is based upon the notion of the gap of two spaces. The proof of the fundamental property of the gap given in Theorem 5.1 makes use of a difficult topological theorem of Borsuk and therefore can be omitted at the same time.

**Theorem 5.1.** [Klein, Krasnoselski and Milman [1]] If $M$ and $N$ are subspaces of a Banach space $X$ and $\Theta(M, N) = \varrho < 1$, and if one of the numbers $\dim M$ or $\dim N$ is finite, then

$$\dim M = \dim N.$$
the unit sphere in the space \( X_1 \) does not contain any segment. Then for every \( x \in X_1 \), there exists only one element \( x \in M \) for which the number \( \varphi(x, M) \) will be assumed. It is easily seen that the mapping \( x \to Px \ ( x \in X_1 ) \) is continuous and satisfies the equality:

\[
P(x) = -Px.
\]

The "orthogonality" of the element \( x \) to the subspace \( M \) means that

\[
\varphi(x, M) = \|x - Px\| = \|x\|,
\]

i.e., \( Px = 0 \).

Let us now suppose

\[
P(y) \neq 0 \quad \text{for} \quad y \notin N, \quad \|y\| = 1.
\]

Since the space \( N \) is compact, the mapping \( P_y y = Py/\|Py\| \) is continuous in this space. \( P_z \) maps the \( n \)-dimensional sphere \( S_n = \{y \in N : \|y\| = 1\} \) in the \( (n-1) \)-dimensional sphere \( S_{n-1} = \{x \in M : \|x\| = 1\} \) in such a manner that symmetric points are associated with symmetric points:

\[
P_y (-y) = -P_y (y),
\]

but this is impossible, by Borsuk's theorem [1].

Thus the theorem is proved in the case where the unit sphere in the space \( M \) does not contain any segment, i.e., is strictly convex.

In the general case, choosing an arbitrary number \( \varepsilon > 0 \) one can construct a new norm \( \|\| \) in the space \( X_1 \) in such a manner that

\[
\|x\| \leq \|x\|_0 \leq (1 + \varepsilon)\|x\| \quad \text{for all} \quad x \in X_1,
\]

and that the new unit sphere \( \|x\|_0 = 1 \) be strictly convex, i.e., that, for arbitrary vectors \( x_1, x_2 \in X_1 \) in different directions,

\[
\|x_1 + x_2\|_0 < \|x_1\|_0 + \|x_2\|_0.
\]

Indeed, inequality (5.1) implies the following one:

\[
\theta(M, N) \leq (1 + \varepsilon)\theta(M, N),
\]

where \( \theta(M, N) \) is the gap between spaces \( M \) and \( N \) corresponding to the norm \( \|\|_0 \).

By condition (5.2), we have \( \theta(M, N) = 1 \). Since the number \( \varepsilon \) is arbitrary, we conclude that

\[
\theta(M, N) = 1.
\]

We now give a construction of the norm \( \|\|_0 \). Let \( \|\|_0 \) be an arbitrary norm in the space \( X_1 \) such that the unit sphere \( \|x\| = 1 \) is strictly convex. For example, one may take

\[
\|x\|_0 = \|x_1 + \ldots + x_k\|_0 = \sqrt{x_1^2 + \ldots + x_k^2},
\]

where \( \{x_1, \ldots, x_k\} \) is a basis of the space \( X_1 \).

\[\]
It follows from our previous considerations (Theorems 2.1, A I; 6.11, A I, and 4.1, I) that the index of an operator is a functional defined on the set $D_0(X)$ satisfying the following conditions:

1. the values of the functional $x_A$ are integers,
2. the functional $x_A$ is continuous over the set $D_0(X)$,
3. if $A, B \in D_0(X)$, then
   $$x_{AB} = x_A + x_B,$$
4. if the operator $A \in B(X)$ has an inverse $A^{-1} \in B(X)$, then
   $$x_A = 0.$$

We now show that every functional defined on the set $D_0(X)$ and satisfying conditions (1)-(4) is an index of operators if we disregard a constant integer coefficient. Namely, we have the following:

**Theorem 6.1.** (Gohberg [3].) If $X$ is a Banach space, then for every functional $v(A)$ defined on the set $D_0(X)$ and satisfying conditions (1)-(4) there exists an integer $p$ such that

$$v(A) = p x_A.$$

**Proof.** First, we show that properties (1) and (2) of the functional $v(A)$ imply the following condition: for an arbitrary operator $A \in D_0(X)$ and for an arbitrary operator $X$ of a finite dimension we have the equality

$$v(A + X) = v(A).$$

It follows from Theorem 3.2, A I, that we have $A + \lambda X \in D_0(X)$ for all complex numbers $\lambda$. Hence the function $\varphi(\lambda) = v(A + \lambda X)$ defined on the whole complex plane is continuous and integer-valued. Consequently, the function $v(A)$ is constant. This implies in particular that it assumes the same values at $\lambda = 0$ and $\lambda = 1$, i.e., $v(A + X) = v(A)$.

If $v(A) = 0$ for invertible operators, then $x_A = 0$ implies $v(A) = 0$. Indeed, if $x_A = 0$, then Corollary 3.5 implies $A = S + K$, where the operator $S$ is invertible and the operator $K$ is of finite dimension. Hence

$$v(A) = v(S + K) = v(S) = 0.$$

Thus, Theorem 6.11, A I, immediately implies our theorem.  

**Remark 6.2.** It follows from Remark 6.12, A I, that it is not necessary to define the functional $x$ on the whole set $D_0(X)$. It is sufficient that $x$ be defined on a set $W$ having the following properties:

1. if $A, B \in W$, then $AB \in W$,
2. if $A \in W$, then $A + T \in W$ for every compact operator $T$,
3. if $A \in W$, then there exists a compact operator $D_0$ of the operator $A$ to the ideal of compact operators.

In this case one cannot require $p$ to be an integer. The number $p$ may be a fraction of the form $k/p$, where $q = \inf\{w : w \geq 0, A_w \in W\}$.

Condition (4) can be replaced by a condition stating that the set of all invertible operators is connected.

If $X(Y)$ is the algebra of all continuous operators over a Hilbert space $Y$, then set $D_0(Y)$ is connected in the norm topology (see Gohberg, Markus, Feldman [1]; Krein [1]). For the case of spaces $F$ and $k$, this theorem was proved by G. Neuberger [2] (see also Arh [1]), who extended these results to some more general classes of spaces (Neuberger [3]). However, Donat [1] showed that there are Banach spaces for which this theorem is not true.

If we investigate the set of closed operators, and not the set of bounded operators, then the characterization of the index is the same as in case of bounded operators with the only difference that in place of the continuity of functionals with respect to the norm we require their continuity with respect to the graph norm $g(A, B)$ (see § 4 and Theorem 4.8).

If we consider the set of all closed operators over a separable Banach space $Y$, then the set of all invertible operators is connected in the graph metric (G. Neuberger [1]). This is a generalization of the results of H. O. Cordes and J. P. Labrousse [1] obtained for Hilbert spaces.

**§ 7. Operators preserving the conjugate space.** Let $X$ be a Banach space and let $A \in L_0(X)$. Let $S$ be a total family of linear functionals defined over the space $X$. As we know (§ 1, A III), the conjugate operator $A'$ is defined by means of the equality

$$A' \xi = \xi A \quad (\xi \in S).$$

does not always map the space $S$ into itself. However, if $SA \subset S$, we say that the operator $A$ preserves the conjugate space $S$. The set of all linear operators preserving the space $S$ has been denoted by $L_0(X, S)$.

We have denoted by $K_0(X, S)$ the ideal of operators of a finite dimension contained in the algebra $L_0(X, S)$. If $K \in K_0(X, S)$, then the operator $I + K$ is a $\Phi_2$-operator of index 0 (see § 2, A III).

Let $X$ and $S$ be Banach spaces. One can define the following new norm in the algebra $B_0(X, S) = B(X) \cap L_0(X, S)$:

$$\|A\|^* = \max \{\|A\|_S, \|A'\|_S\}.$$

If the topology in the space $S$ is equivalent to the topology determined by the norm of the functional, then of course the norms $\| \cdot \|$ and $\| \cdot \|_X$ are equivalent.
We denote by $K_0(X, \mathcal{S})$ the closure of the ideal $K_0(X, \mathcal{S})$ in the norm $\|\cdot\|$. Evidently, $K_0(X, \mathcal{S})$ is also a two-sided ideal.

**Theorem 7.1.** (On Simultaneous Approximation.) If $X$ is a Banach space and an operator $A \in B_0(X, \mathcal{S})$ has a left regularizer (right regularizer) to the ideal $K_0(X, \mathcal{S})$, then it has a left regularizer (right regularizer) to the ideal $K_0(X, \mathcal{S})$.

Proof. We perform the proof for a left regularizer; obviously, the proof for a right regularizer is the same. According to our assumption, the operator $A$ possesses a left $\mathcal{S}$-regularizer $R_0$ to the ideal $K_0(X, \mathcal{S})$, i.e.

$$R_0A = I + T, \quad \text{where} \quad T \in K_0(X, \mathcal{S}).$$

But the definition of the ideal $K_0(X, \mathcal{S})$ implies existence of an operator $K \in K_0(X, \mathcal{S})$ such that $\|T - K\| < 1$.

We write $B = T - K$. Since the spaces $X$ and $\mathcal{S}$ are complete, the operator $I + B$ is invertible and $(I + B)^{-1} \in B_0(X, \mathcal{S})$. Let

$$R_0^n = (I + B)^{-n}R_0.$$

Then

$$R_0^n A = (I + B)^{-n}R_0 A = (I + B)^{-n}(I + T) = (I + B)^{-n}(I + B + K) = I + (I + B)^{-n}K.$$

But we have $(I + B)^{-1}K \in K_0(X, \mathcal{S})$. Hence $R_0^n$ is a left regularizer of the operator $A$ to the ideal $K_0(X, \mathcal{S})$.

**Corollary 7.2.** If an operator $A \in B_0(X, \mathcal{S})$ has a simple regularizer to the ideal $K_0(X, \mathcal{S})$, in particular, if $A = I + T$, where $T \in K_0(X, \mathcal{S})$, then $A$ is a $\mathcal{S}$-operator.

**Corollary 7.3.** The ideal $K_0(X, \mathcal{S})$ is a $\mathcal{S}$-Fredholm ideal.

Proof. By Corollary 7.2, $K_0(X, \mathcal{S})$ is a $\mathcal{S}$-quasi-Fredholm ideal. It is sufficient to show that it is a Fredholm ideal. Let $T \in K_0(X, \mathcal{S})$, i.e. that the operator $I + T$ is of a finite $\mathcal{S}$-characteristic. By Theorem 4.1, there exists a number $\varepsilon > 0$ such that the inequality $\|T - \mathcal{T}\| < \varepsilon$ implies $\varepsilon + \varepsilon = \varepsilon + \mathcal{T} = \varepsilon + \mathcal{T}$. However, by hypothesis to every number $\varepsilon$ there exists an operator $K \in K_0(X, \mathcal{S})$ such that $\|K - \mathcal{T}\| < \varepsilon$. Hence

$$\varepsilon + \mathcal{T} = \varepsilon + \mathcal{T} = 0.$$

**Remark.** It is not known whether the assumption $T \in K_0(X, \mathcal{S})$ in Corollary 7.2 can be replaced by the assumption $T \in T(X, \mathcal{S})$, where $T(X, \mathcal{S})$ denotes the ideal of compact operators contained in the algebra $B_0(X, \mathcal{S})$. More generally, it is not known whether if $J$ is a Fredholm ideal contained in the algebra $L_0(X, \mathcal{S})$, then $\varepsilon + \mathcal{T} = 0$ for all operators $T \in J$.

**Theorem 7.4.** If $X$ is a Banach space and an operator $T \in B_0(X, \mathcal{S})$ is compact, then the conjugate operator $T^*$ is also compact.

The proof follows the same lines as that of Theorem 2.2, II; one must only replace the space $X^*$ by the space $\mathcal{S}$.


CHAPTER IV

Φ-POINTS AND THEOREM ON SPECTRAL DECOMPOSITION

In this chapter we shall consider linear spaces over the field of complex numbers only. The fundamental theorems of this chapter were given by Gohberg and Krein [1].

§ 1. Φ-points. Let us suppose that a linear closed operator \( A \) maps a Banach space \( X \) into itself. A point \( \lambda \) of the complex plane is called a Φ-point if the operator \( A - \lambda I \) is a Φ-operator. The set of all Φ-points of the operator \( A \) is called the Φ-set of the operator \( A \) and is denoted by \( \Phi_A \).

Let \( \lambda \in \Phi_A \). Since \( A - \lambda I = A - \lambda I - (\lambda - \lambda I)I \), according to Theorem 4.1, III, there exists a number \( \varepsilon > 0 \) such that all points of the disk \( |\lambda - \lambda| < \varepsilon \) are Φ-points; moreover,

\[
\kappa_{A-M} = \kappa_{A-M}.
\]

This immediately implies the following theorem:

Theorem 1.1. If \( X \) is a Banach space, then the Φ-set \( \Phi_A \) of a closed operator \( A \in L(X) \) is an open set, whence it is at most a countable union of connected components. In each connected component of the set \( \Phi_A \) the index \( \kappa_A \) of the operator \( A \) is constant.

Theorem 1.2. (Gohberg and Krein [1]). If \( X \) is a Banach space and every point \( \lambda \) of the complex plane is a Φ-point of an operator \( A \in B(X) \), then the space \( X \) is of a finite dimension.

Proof. If \( |\lambda| > \|A\| \), the operator \( A - \lambda I \) is of a finite dimension. Hence

\[
\kappa_{A-M} = 0 \quad (|\lambda| > \|A\|).
\]

On the other hand, \( \Phi_A \) consists of all points of the plane. Thus

\[
\kappa_{A-M} = 0 \quad \text{for all } \lambda.
\]

Let us consider the quotient algebra \( B(X)/L(X) \), where \( T(X) \) is the ideal of compact operators. We denote by \( |A| \) the cost determined by the operator \( A \). We define the norm of the cost \( |A| \) as follows:

\[
|A| = \inf_{x \notin T(X)} |A + T|.
\]

Let us suppose that the space \( X \) is not of a finite dimension. Then the algebra \( B(X)/L(X) \) cannot be of a finite dimension, for otherwise the space \( X \) would be locally compact, by Theorem 1.11, IV.

By Corollary 3.5, III, we have \( A - \lambda I = S + K \), where the operator \( S \) is invertible and the operator \( K \) is of a finite dimension. Hence the cost \( \{S \} \) generated by the operator \( S \) is inverse to the cost \( \{A - \lambda I\} \). Thus for every number \( \lambda \) the element \( A - \lambda I \) is invertible in the ring \( B(X) \). But this is impossible, because for every element \( a \) of a Banach algebra there must exist a number \( \mu \) such that the element \( a - \mu a \) is not invertible (see Theorem 1.6, II).

Theorem 1.2 can also be formulated in another way:

Theorem 1.2'. If a Banach space \( X \) is infinite-dimensional, then for every operator \( A \in B(X) \) there exists at least one point \( \lambda \) which is not a Φ-point of that operator.

§ 2. Properties of functions \( \sigma_{A-M} \) and \( \beta_{A-M} \). We shall now investigate the functions

\[
\sigma_{A}(\lambda) = \sigma_{A-M} \quad \text{and} \quad \beta_{A}(\lambda) = \beta_{A-M}
\]

inside each of the components of the set \( \Phi_A \). First, we prove the following lemma:

Lemma 2.1. Let \( X \) and \( Y \) be Banach spaces, let \( A \) and \( B \) belong to \( B(X \rightarrow Y) \) and let \( A \) be a Φ-operator. Then there exists a number \( \varepsilon > 0 \) such that for all \( \lambda \) satisfying the inequality \( 0 < |\lambda| < \varepsilon \) the equation

\[
(A - \lambda B)x = 0
\]

has the same number of linearly independent solutions.

Proof. First, we prove the lemma in the case of \( \kappa_A = 0 \). We denote by \( \{e_1, ..., e_n\} \) a basis of the space \( E \), and by \( \{g_1, ..., g_n\} \), a basis of the direct complement of the subspace \( E \) in the space \( Y \). Let \( \{f_1, ..., f_n\} \) be a system of functionals in the space \( Y^* \) such that

\[
f_j(e_k) = \delta_{jk} \quad (j, k = 1, 2, ..., n).
\]

The operator \( A \) defined by the equality

\[
A x = Ax + \sum_{j=1}^n f_j(x) g_j
\]

has a continuous inverse (compare the proof of Theorem 1.2). Hence the operator \( A - \lambda B \) is also continuously invertible for all \( \lambda \) belonging to the disc \( |\lambda| < \varepsilon = \|A^{-1}\|^{-1} \|B\| \) and

\[
R = (A - \lambda B)^{-1} = A^{-1} \left( I + \sum_{n=1}^\infty (\lambda B A^{-1})^n \right).
\]
However, the equation \((A + \lambda B)x = 0\) is clearly equivalent to the equation
\[
(A - \lambda B)x = \sum_{j=1}^{a_d} f_j(x) y_j
\]
or to the system of equations
\[
\begin{aligned}
\sum_{j=1}^{\alpha_d} \xi_j R_j y_j & = \sum_{j=1}^{\alpha_d} f_j(x) y_j \\
\xi_j & = f_j(x) \quad (k = 1, 2, \ldots, \alpha_d).
\end{aligned}
\]

Substituting in equation (2.2) the expression for \(x\) from equation (2.1) we obtain the following homogeneous system of \(\alpha_d\) linear equations determining the numbers \(\xi_k = (k = 1, 2, \ldots, \alpha_d): \)
\[
\sum_{j=1}^{\alpha_d} \left[ f_j(R_j y_j) \right] \xi_j = 0 \quad (k = 1, 2, \ldots, \alpha_d).
\]

Evidently, the number \(\alpha_d\) is equal to the number of linearly independent solutions of system (2.3). All elements of the determinant \(A(\lambda)\) of system (2.3) are analytic functions of the parameter \(\lambda\) inside the disc \(|\lambda| < \varrho\). If they are identically equal to zero, then the system has \(n = a_d\) linearly independent solutions. Therefore we have
\[
\alpha_d = n = a_d
\]
for all points \(\lambda\) of the disc \(|\lambda| < \varrho\).

Let us now suppose that at least one of the elements of the determinant \(A(\lambda)\) is different from zero at a certain point \(\lambda\) of the disc \(|\lambda| < \varrho\). We denote by \(A_\lambda(\lambda)\) an arbitrary minor of the highest rank among all the minors of the determinant \(A(\lambda)\) different from zero at least at one point \(\lambda\) of the disc \(|\lambda| < \varrho\), and by \(p\), the rank of that minor. Evidently, \(A_\lambda(\lambda) \neq 0\) at points of the disc \(|\lambda| < \varrho\), with the exception of some isolated points. At points \(\lambda\) such that \(A_\lambda(\lambda) = 0\) system (2.3) has \(n - p\) linearly independent solutions.

Let \(|\lambda| < \varepsilon\) be the largest disc such that \(A_\lambda(\lambda) = 0\) for all points inside this disc (with the possible exception of the point \(\lambda = 0\)). We have \(\alpha_d = n - p\) for all points \(\lambda\) satisfying the inequalities \(0 < |\lambda| < \varepsilon\). In this manner we have proved the lemma in the case of \(\kappa_\delta = 0\).

Let us now suppose that \(\kappa_\delta > 0\). Let \(N\) denote a certain \(|\kappa_\delta|\) dimensional space, and let \(\bar{Y} = \bar{Y} \oplus N\). Evidently, the space \(\bar{Y}\) can be considered as a Banach space if we define the norm in \(\bar{Y}\) as follows:
\[
|y + z| = |y| + |z| \quad (y \in \bar{Y}, z \in N).
\]

§ 2. Properties of functions \(a_{d - \varepsilon}\) and \(\beta_{d - \varepsilon}\)

In the following we shall consider the operators \(A\) and \(B\) as operators which map the space \(X\) into the space \(\bar{Y}\). The index of the operator \(A\) will become greater by \(\alpha_d\) hence it will be equal to zero. Applying the first part of the lemma to the operator \(A\) and taking into account the fact that changing the space \(\bar{Y}\) into the space \(\bar{Y}\) does not change the number \(\alpha_d\), we obtain our lemma also in this case.

It remains to consider the case \(\kappa_\delta > 0\). We denote by \(M\) a \(\kappa_\delta\) dimensional space, and by \(\bar{X}\) the direct sum of spaces \(X \oplus M\). We extend the operators \(A\) and \(B\) to the whole space \(\bar{X}\) taking
\[
\bar{A} = \bar{B} = \bar{0} \quad \text{for all } z \in M.
\]

Then \(\kappa_\delta = 0\) and we can apply the first part of the lemma to the operator \(\bar{A}\). Since we have
\[
a_{d - \varepsilon} = a_d - \varepsilon + \kappa_\delta
\]
for all \(\lambda\) satisfying the inequalities \(0 < |\lambda| < \varepsilon\), the lemma is proved also in the last case.

The lemma proved above makes it possible to investigate some properties of connected components of the set \(\Phi_d\).

**THEOREM 2.2.** If \(X\) is a Banach space and if a set \(G\) in a connected component of the \(\Phi\)-set \(\Phi_d\) of a closed operator \(A \in L(X)\), then the function \(a_d(\lambda)\) is constant for all points \(\lambda \in G\) with the exception of some isolated points:
\[
a_d(\lambda) = n.
\]

Moreover, we have \(a_d(\lambda) > n\) at the isolated points \(\lambda\).

Proof. Let \(n = \min_{\lambda \in G} a_d(\lambda)\) and let us suppose that \(a_d(\lambda)\) assumes this minimum at a point \(\lambda = \lambda_0\), i.e., \(a_d(\lambda_0) = n\).

We denote by \(\Lambda_0\) an arbitrary point of the component \(G\) at which \(a_d(\lambda) > n\). We show that the point \(\lambda_0\) is isolated, i.e., that there exists a number \(\varepsilon_0 > 0\) such that \(a_d(\lambda) = n\) for all points \(\lambda\) satisfying the inequalities \(0 < |\lambda - \lambda_0| < \varepsilon_0\). We join the points \(\lambda_0\) and \(\lambda_0\) by means of a curve \(\Gamma\) lying entirely in the component \(G\). Applying Lemma 2.1 to the operators \(A\) and \(B = I\), we conclude that to every point \(\lambda\) of the curve \(\Gamma\) there corresponds a number \(\varepsilon_0 > 0\) such that the function \(a_d(\mu)\) is constant for all \(\mu\) satisfying the inequalities \(0 < |\mu - \lambda| < \varepsilon_0\). In this manner with every point \(\lambda\) we have associated its neighbourhood \(U_\lambda\). Thus we have obtained a certain covering of the curve \(\Gamma\). From this covering we choose a finite subcovering \(U_1, U_2, \ldots, U_N\). Without loss of generality we may suppose that \(U_j \cap U_{j+1} = \emptyset\). Hence the function \(a_d(\lambda)\) assumes the same value at all points of neighbourhoods \(U_j\) \((j = 1, 2, \ldots, N)\) with the only possible exception of their centres. But \(a_d(\lambda) = n\) in the neighbourhood \(U_j\).
for all points $\lambda \in G$ with the exception of isolated points. At isolated points we have $\alpha(\lambda) > n$. Moreover, if $\alpha(\lambda) = 0$ at least at one point, then the operator $I - T_0$ has an inverse $(I - T_0)^{-1} \in \mathcal{B}(X)$ for all $\lambda \in G$ with the exception of isolated points. Let $\Gamma$ be a rectifiable curve contained in the domain $G$. By Theorem 11.5, B 1, the integral $\int_{\Gamma} A_\lambda \, d\lambda$ exists.

If $f(\lambda)$ is a continuous linear functional defined on the space $B(X \to Y)$, then

$$\int_{\Gamma} f(A_\lambda) \, d\lambda = \int_{\Gamma} f(A_\lambda) \, d\lambda = 0$$

for an arbitrary continuous linear functional defined on the space $B(X \to Y)$, because the scalar-valued function $P(\lambda) = f(A_\lambda)$ of the variable $\lambda$ is analytic. Since the functionals $f$ is arbitrary, it follows that

$$\int_{\Gamma} A_\lambda \, d\lambda = 0.$$

In a similar manner we verify Cauchy's formula:

$$A_\lambda = \frac{1}{2\pi i} \int_{\Gamma} \frac{A_{\mu}}{\mu - \lambda} \, d\mu,$$

where $\Gamma$ is a closed curve with the point $\lambda$ inside.

§ 4. Resolvent of an operator. A theorem on spectral decomposition. Let $X$ be a Banach space. A point $\lambda$ of the complex plane is called a regular point (§ 8, Chapter I) of an operator $A \in \mathcal{B}(X)$ if the operator $A - \lambda I$ has a continuous inverse, i.e. there exists an operator $R_\lambda$ bounded and defined on the whole space $X$ such that

$$R_\lambda (A - \lambda I) = (A - \lambda I) R_\lambda = I.$$

Such an operator is called a resolvent. If an operator $A$ has at least one regular point $\lambda_0$, then of course the operator $A - \lambda_0 I$ is closed, and so is the operator $A$. The set $O_\lambda$ of all regular points of an operator $A$ is open. Indeed, if $\lambda_0 \in O_\lambda$, then the equality

$$A - \lambda I = A - \lambda_0 I + (\lambda_0 - \lambda) I = (A - \lambda_0 I)(I + (\lambda_0 - \lambda) R_{\lambda_0})$$

implies the existence of a resolvent $R_{\lambda_0}$ in the disc $|\lambda - \lambda_0| < \|R_{\lambda_0}\|^{-1}$.
given by the formula
\[ B_0 = [(I - (\lambda - \mu))B_0]^{-1} B_0 = \sum_{k=0}^{\infty} (\lambda - \mu)^k B_0^k. \]

Hence follows

**Theorem 4.1.** If \( X \) is a Banach space, then in every connected component of the set \( \Omega_A \) of regular points of an operator \( A \in B(X) \) the resolvent \( R_\lambda \) of that operator is an analytic function (with values in \( B(X) \)).

Since the spectrum \( \sigma_A \) of an operator \( A \) (see § 8, Chapter I) constitutes the complement of the set \( \Omega_A \) of regular points to the whole complex plane, the spectrum \( \sigma_A \) is a closed set.

Let \( \Gamma \) be a rectifiable arc or a curve made up of such arcs. Let \( G_\mu \) be the domain closed by the curve \( \Gamma \). We suppose that the curve \( \Gamma \) consists of regular points of the operator \( A \), i.e., that \( R_\lambda = (A - \lambda I) \) is an analytic function on the curve \( \Gamma \) and that the curve is positively oriented with respect to the domain \( G_\mu \). Let us consider the integral
\[ P_\mu = -\frac{1}{2\pi i} \int_{\Gamma} R_\lambda d\lambda. \]

The existence of this integral follows from Theorem 11.5, B.I.

**Theorem 4.2.** If \( X \) is a Banach space and \( A \in B(X) \), then the operator \( P_\mu \) is a projector and
\[ X = X_\mu \oplus X_{\lambda}, \quad \text{where} \quad X_\mu = P_\mu X, \quad X_\lambda = (I - P_\mu)X. \]

Moreover, both components \( X_\mu \) and \( X_\lambda \) are invariant subspaces of the operator \( A \) having the following properties:

1. the restriction of the operator \( A \) to the space \( X_\mu \) is defined on the whole space \( X_\mu \) and its spectrum lies inside the domain \( G_\mu \);
2. the restriction of the operator \( A \) to the space \( X_\lambda \) is defined on the set \( D_A \cap X_\lambda \) and its spectrum lies outside the closure of the domain \( G_\lambda \).

Moreover, if \( \Gamma_\mu \) and \( \Gamma_\lambda \) are two curves with the above properties and if the domains \( G_\mu \) and \( G_\lambda \) are disjoint, then the respective projectors are orthogonal, i.e.,
\[ P_\mu P_\lambda = P_\lambda P_\mu = 0 \quad \text{if} \quad G_\mu \cap G_\lambda = 0. \]

**Proof.** First, we show that the operator \( P_\mu \) is a projector. If \( \lambda \) and \( \mu \) are two regular values of the operator \( A \), then
\[ R_\lambda - R_\mu = (A - \lambda I)^{-1} - (A - \mu)^{-1} = (A - \lambda I)^{-1} [(\lambda - \mu)^{-1} (A - \lambda I) - (A - \mu)^{-1}]. \]

Hence if \( \lambda \neq \mu \), then
\[ R_\lambda R_\mu = R_\lambda - R_\mu. \]

From the assumption that the curve \( \Gamma \) is made up of regular points of the operator \( A \) we conclude that the distance between the spectrum \( \sigma_A \) and the closed set which is the complement of the set \( G_\mu \) is positive. Hence there exists a curve \( \Gamma' \) contained inside the domain \( G_\mu \) such that \( \sigma_A \subset G_{\mu'}, \subset G_\mu \). But
\[ P_\mu = \frac{1}{2\pi i} \int_{\Gamma'} R_\lambda d\lambda = -\frac{1}{2\pi i} \int_{\Gamma'} R_\lambda d\mu \]
and
\[ \int_{\Gamma'} \frac{d\mu}{\lambda - \mu} = 0, \quad \int_{\Gamma'} \frac{d\lambda}{\lambda - \mu} = 2\pi i. \]

Hence the point \( \mu \) lies inside the domain \( G_\mu \) and the point \( \lambda \) outside the domain \( G_\mu \). Thus
\[ P_\mu = \left( \frac{1}{2\pi i} \right) \int_{\Gamma'} R_\lambda d\mu d\lambda = \frac{1}{2\pi i} \int_{\Gamma'} R_\lambda d\mu d\lambda \]
\[ = \frac{1}{2\pi i} \int_{\Gamma'} R_\lambda \left( \frac{1}{2\pi i} \right) \int_{\Gamma'} R_\lambda \int_{\Gamma'} \frac{d\lambda}{\lambda - \mu} d\mu \]
\[ = \frac{1}{2\pi i} \int_{\Gamma'} R_\lambda d\mu = P_\mu. \]

Consequently, \( P_\mu \) is a projector. We now show that if the domains \( G_\mu \) and \( G_\lambda \) are disjoint, then the respective projectors \( P_\mu \) and \( P_\lambda \) are orthogonal. Indeed, if \( \lambda \in \Gamma_\mu, \mu \in \Gamma_\lambda \), then
\[ \int_{\Gamma_\mu} \frac{d\lambda}{\lambda - \mu} = \int_{\Gamma_\lambda} \frac{d\mu}{\lambda - \mu} = 0. \]

Applying an analogous decomposition to that used in the previous case we obtain
\[ P_\mu P_\lambda = P_\lambda P_\mu = 0. \]

From the commutativity of the resolvent and the operator \( A \) follows the commutativity of the operator \( P_\mu \) and the operator \( A \). Hence
\[ A(D_A \cap X_\mu) = A(P_\mu X_\mu \cap D_A) = P_\mu A(X_\mu \cap D_A) \subset X_\mu \]
and analogously,
\[ A(D_A \cap X_\lambda) = A(P_\lambda X_\lambda \cap D_A) = P_\lambda A(X_\lambda \cap D_A) \subset X_\mu. \]

Moreover,
\[ (A - \lambda I) P_\mu = (A - \lambda I) R_\mu + (\lambda - \mu) P_\mu = I + (\lambda - \mu) P_\mu. \]
Hence
\[ A - \lambda I = \frac{1}{2\pi i} \int \frac{R_\mu}{\mu - \lambda} \, d\mu = \frac{1}{2\pi i} \int \frac{d\mu}{\mu - \lambda} I + \frac{1}{2\pi i} \int R_\mu d\mu \]
\[ = \begin{cases} 0, & I - P_\lambda = -P_\lambda \text{ if } \lambda \text{ lies outside the curve } \Gamma, \\ 1, & I - P_\lambda = I - P_\lambda \text{ if } \lambda \text{ lies inside the curve } \Gamma. \end{cases} \]

Hence it follows that if \( \lambda \) lies outside the curve \( \Gamma \), then the operator \( A - \lambda I \) is invertible on the set \( X_\Gamma \), and if \( \lambda \) lies inside the curve \( \Gamma \), then the operator \( A - \lambda I \) is invertible on the set \( X_\Gamma \).

Let us remark that according to Theorem 1.4, III, the operator \( A \) is bounded in the space \( X_\Gamma \), as a closed linear operator defined in a closed domain.

If there is a finite number of points \( \lambda_1, \ldots, \lambda_n \) of the spectrum \( S_A \) in the domain \( X_\Gamma \), then
\[ P_\lambda = P_{\lambda_1} + \cdots + P_{\lambda_n}, \quad P_{\lambda_i} P_{\lambda_j} = 0 \quad \text{for} \quad i \neq j, \]
where the operators \( P_{\lambda_i} (i = 1, 2, \ldots, n) \) are projectors and the projections \( P_{\lambda_i} X \subset C_0 Y \) of the space \( X \) are invariant spaces for the operator \( A \) such that in each of them the spectrum of the operator \( A \) consists of one number \( \lambda_i \) only.

Indeed, if \( \gamma_i \) are disjoint circles with centers at corresponding points \( \lambda_i \), which lie inside the domain \( X_\Gamma \), then
\[ P_\lambda = \sum_{i=1}^{n} \left( \frac{1}{2\pi i} \int R_{\lambda_i} d\lambda \right) = \sum_{i=1}^{n} P_{\lambda_i}. \]

§ 5. Decomposition of the operator \( P_{\lambda} \).

In the last section we defined the operator \( P_{\lambda} \). If the numbers \( \lambda_1, \ldots, \lambda_n \) are all values of the spectrum of an operator \( A \) contained inside a curve \( \Gamma \), then
\[ P_{\lambda} = P_{\lambda_1} + \cdots + P_{\lambda_n}. \]

In this section we shall deal with the question when the operators \( P_{\lambda} \) and \( P_{\lambda} \) are of finite dimensions. We recall the definition of a splittable space (§ 5, A.1).

Let
\[ G_{\lambda_i} = \{ x \in X : \text{there exists an exponent } n \text{ such that } (A - \lambda_i I)^n x = 0 \}. \]

If the space \( X \) can be written as a direct sum
\[ (5.1) \quad X = G_{\lambda_1} \oplus X_{\lambda_2}, \]
where the space \( X_{\lambda_2} \) is invariant and such that the operator \( (A - \lambda_1 I) \) is invertible on \( X_{\lambda_2} \), then \( G_{\lambda_2} \) is called a splittable space.

Let an operator \( A \in L(X) \) be closed. We say that the space \( G_{\lambda_2} \) (defined as above) is normally splittable if the subspace \( X_{\lambda_2} \) is closed and the operator
\[ (A - \lambda_1 I) \text{ is continuously invertible on the space } X_{\lambda_2}. \]
it is easily seen that the point \( \lambda_1 \) corresponding to a normally splittable principal space of a finite dimension of the operator \( A \), is a \( \phi \)-point of this operator. Moreover, all points \( \lambda \neq \lambda_1 \) in a certain neighborhood of the point \( \lambda_1 \) are regular points of the operator \( A \). Indeed, let us denote by \( A_1 \) and \( A_2 \) the operators induced by the operator \( A \) in subspaces \( G_{\lambda_1} \) and \( X_{\lambda_2} \). It follows from the definition that there is a number \( s \) such that \( (A_1 - \lambda_1 I)^s = 0 \).

Let \( n \) denote the least natural number satisfying the equality \( (A_1 - \lambda_1 I)^n = 0 \). Writing \( B_1 = A_1 - \lambda_1 I \) we obtain
\[ - (\lambda - \lambda_1)^n I = B_1^n = (\lambda - \lambda_1)^n I = (A_1 - \lambda_1 I) \left[ (\lambda - \lambda_1)^n I - (\lambda - \lambda_1)^{n-1} B_1 + \cdots + B_{n-1} \right]. \]

Hence
\[ - (A_1 - \lambda_1 I) = (\lambda - \lambda_1)^n I + \sum_{j=1}^{n-1} (\lambda - \lambda_1)^j B_1^j. \]

On the other hand, the operator \( A_1 - \lambda_1 I \) is continuously invertible in the subspace \( X_{\lambda_2} \). Hence for all numbers \( \lambda \) from the disc
\[ |\lambda - \lambda_1| < 1/||A_1 - \lambda_1 I||^{-1} \]
there exists a resolvent
\[ (A_1 - \lambda I)^{-1} = B_1 + (\lambda - \lambda_1) B_1^2 + \cdots + (\lambda - \lambda_1)^n B_1^n + \cdots, \]
where \( B_1 = (A_1 - \lambda_1 I)^{-1} \). Hence it follows that all points \( \lambda \) satisfying the inequalities \( 0 < |\lambda - \lambda_1| < ||B_1||^{-1} \) are regular points of the operator \( A \), and the resolvent \( R_{\lambda_1} \) for these points is defined by the formula
\[ (5.2) \quad R_{\lambda_1} = (\lambda - \lambda_1) B_1^{-1} + \cdots + (\lambda - \lambda_1)^n B_1^{-1} + \cdots + \sum_{j=1}^{n-1} (\lambda - \lambda_1)^j B_1^j + \cdots, \]
where the linear operators \( B_1 \) and \( B_2 \) are extended to the whole space \( X \) in such a manner that \( B_2 x = 0 \) for \( x \in G_{\lambda_1} \), \( y \in X_{\lambda_2} \), and \( T \) is a projection operator of the space \( X \) onto the subspace \( G_{\lambda_2} \).

Integrating both sides of equality (5.1) over the contour \( \Gamma \), we obtain
\[ (5.3) \quad P = P_{\lambda_1} \frac{1}{2\pi i} \int R_{\lambda_1} d\lambda. \]

Theorem 5.1. Let \( X \) be a Banach space. Let \( \Gamma \) be a rectifiable curve which is the boundary of a domain \( G \) and is made up of regular points of \( A \in L(X) \). The domain \( G \) contains a finite number of points of the spectrum of the operator \( A \) which are eigenvalues with normally splittable principal spaces of finite dimension if and only if the projection operator \( P_{\lambda} \) of a finite dimension.
Moreover, if the above condition is satisfied, the subspace $X_r = P_r X$ is the direct sum of all principal subspaces of the operator $A$ corresponding to eigenvalues $\lambda \in G_r$.

**Proof.** Let the spectrum of the operator $A$ contained in the interior of the domain $G_r$ consist of a finite number of eigenvalues $\lambda_1, \ldots, \lambda_n$ with corresponding normally split principal subspaces of finite dimensions. From (5.1) and (5.3) imply

$$P_r = P_{\lambda_1} + \cdots + P_{\lambda_n} \quad (P_{\lambda_j} P_{\lambda_k} = 0 \text{ for } j \neq k),$$

where the projection operator $P_{\lambda_j}$ ($j = 1, 2, ..., n$) projects the whole space $X$ on a principal subspace of a finite dimension of the operator $A$ corresponding to the eigenvalue $\lambda_j$. Hence the operator $P_r$ is of a finite dimension and

$$X_r = P_r X = \sum_{j=1}^{n} P_{\lambda_j} X = G_1 + \cdots + G_n.$$

Conversely, let us suppose that the projection operator $P_r$ is of a finite dimension. Then the space $X$ can be represented in the form of the direct sum of spaces $X_r$ and $D_r$ invariant with respect to the operator $A$:

$$X = X_r \oplus D_r.$$

We denote by $A_r$ and $A_s$ the restrictions of the operator $A$ to the subspaces $X_r$ and $D_r$ respectively. Since the subspace $X_r$ is of a finite dimension, the spectrum of the operator $A_r$ consists of a finite number of values $\lambda_j$ ($j = 1, 2, ..., n$; $\lambda_j \in G_r$). It follows from the well-known properties of the theory of finite matrices that the space $X$ can be decomposed into the direct sum of spaces $E_j$ ($j = 1, 2, ..., n$) invariant with respect to the operator $A$ and such that the operator $A - \lambda_j I$ is nilpotent in the space $E_j$. Hence it follows in particular that the operator $A_r - \lambda_j I$ is invertible on all subspaces $E_k (k \neq j)$.

The operator $A_r - \lambda_j I$ is invertible for all numbers $\lambda \in G_r$; hence the spectrum of the operator $A$ in the domain $G_r$ is the same as the spectrum of the operator $A_1$ in this domain. Hence the operator $A$ has a finite number of eigenvalues $\lambda_j$ ($j = 1, 2, ..., n$) with corresponding principal subspaces $E_j$ ($j = 1, 2, ..., n$) of a finite dimension in the interior of the domain $G_r$. These spaces are normally split because the space $X$ can be decomposed into the direct sum of spaces invariant with respect to the operator $A$:

$$X = E_1 \oplus M_1 \quad (j = 1, 2, ..., n),$$

where the operator $A - \lambda_j I$ has a continuous inverse in the space

$$N_j = X_r \oplus \sum_{m \neq j} M_m \quad (j = 1, 2, ..., n).$$

\section{6. Perturbations of the operator $P_r$.}

Let $\Gamma$ be an arbitrary rectifiable curve which is the boundary of a domain $G_{\Gamma}$ and has the following properties with respect to a closed operator $A \in L(X)$:

(a) The operator $A$ has a finite number of eigenvalues with corresponding normally split principal subspaces, inside the domain $G_{\Gamma}$.

(b) All the remaining points $\lambda \in \Gamma$ in the closure of the domain $G_{\Gamma}$ are regular points of the operator $A$.

The **root number of the operator $A$** corresponding to the curve is defined as the sum of all numbers $v_\lambda(\lambda_\Gamma)$ such that $\lambda_j (j = 1, 2, ..., n)$ is an $\lambda_\Gamma$-tuple eigenvalue of the operator $A$ inside the domain $G_{\Gamma}$, i.e. the number

$$v_\lambda(\Gamma) = v_\lambda(\lambda_j) + \cdots + v_\lambda(\lambda_n).$$

We infer from the formula $P_r = \sum_{\lambda \in \Gamma} P_{\lambda}$ that

$$v_\lambda(\Gamma) = \dim P_{\lambda} X,$$

where $P_{\lambda}$ is a projection operator defined by means of the formula

$$P_{\lambda} = \frac{1}{2\pi i} \int (A - \lambda I)^{-1} d\lambda.$$

**Theorem 6.1.** (Gohberg and Krein [1].) Let $X$ be a Banach space. Let $\Gamma$ be the boundary of a domain $G_{\Gamma}$, and $\Gamma$ a rectifiable curve having properties (a) and (b) with respect to a closed operator $A \in L(X)$.

There exists a number $\varrho > 0$ such that for all operators $B \in B(X)$, $D_B = X$, satisfying the inequality $\|B\| < \varrho$ the curve $\Gamma$ has properties (a) and (b) with respect to the operator $A + B$ and

$$v_\lambda(\Gamma) = v_\lambda(\Gamma) + v_\lambda(\Gamma).$$

**Proof.** As before, let $R_\lambda$ denote the resolvent of the operator $A$.

Let us set $d = 1/\max_{\lambda \in \Gamma} |R_\lambda|$. Then

$$\varrho = \frac{d^2}{d + |\Gamma|/2\pi} \quad (\text{evidently, } \varrho < d),$$

where $|\Gamma|$ denotes the length of the curve $\Gamma$. The number $d$ defined above satisfies the theorem. Indeed, let an operator $B \in B(X)$, $D_B = X$, satisfy the inequality $\|B\| < \varrho$. All points $\lambda \in \Gamma$ are regular points of the operator $A + B$ because, as can easily be seen, if $\lambda \in \Gamma$, then there exists an operator

$$\text{(6.1)} \quad (A + B - \lambda I)^{-1} = (I + BR_{\lambda}(A - \lambda I))^{-1} = R_{\lambda}(I + \sum_{j=1}^{n} (BR_{\lambda})^j)$$
where the inequality \( \|B\| < q \) implies that this series is convergent, because
\[
\|BR_i\| \leq \|B\| \|R_i\| < 1 \quad (i \in I).
\]
We now define a projection operator \( \tilde{P}_x \) by means of the equality
\[
\tilde{P}_x = \frac{1}{2\pi i} \int \frac{(A + B - \lambda I)^{-1}}{\lambda} \, d\lambda.
\]
Formula (6.1) implies
\[
\|\tilde{P}_x - P_x\| = \frac{1}{2\pi i} \int \left| \sum_{\lambda \in \Gamma} \frac{\|R_i\|}{\lambda} \right| \, |d\lambda| \leq 2\pi \max_{\lambda \in \Gamma} \frac{\|B\| \|R_i\|}{1 - \|B\| \|R_i\|}.
\]
Applying the inequality
\[
\|B\| < q = 2\pi d_d(2 \pi d_d + |\Gamma|), \quad \|R_i\| < d^{-1} \quad (\lambda \in \Gamma)
\]
we obtain
\[
\|\tilde{P}_x - P_x\| < 1.
\]
This and Theorem 5.1, III, imply
(6.2)
\[
\dim \tilde{P}_x X = \dim P_x X.
\]
Hence the operator \( \tilde{P}_x \) is of a finite dimension because so is the operator \( P_x \).

Thus, by Theorem 5.1, the curve \( \Gamma \) has properties (a) and (b) with respect to the operator \( A + B \). Moreover, equality (6.2) implies \( \psi_{A+B}(\Gamma) \) = \( \psi(\Gamma) \).

The above theorem is called the theorem on the continuity of the root number of an operator.

Remark. It follows from the proof that the theorem remains valid if the conditions imposed on the operator \( B \) are replaced by more general ones; namely: Let \( B \) be an \( A \)-bounded operator satisfying the inequality
\[
\|BR_i\| < 2\pi d_d(2 \pi d_d + |\Gamma|) \quad \text{for all } i \in \Gamma.
\]
It is easily seen that this inequality will be satisfied for all \( i \in \Gamma \) if the operator \( B \) is of a sufficiently small \( A \)-norm, i.e. if the inequality
\[
\|B\| < k(|\alpha| + \|A\|)
\]
is satisfied for a sufficiently small number \( k \), e. g. for
\[
0 < k < 2\pi d_d(2 \pi d_d + |\Gamma|)^{-1}(1 + 2/d)^{-1}.
\]
Moreover, if we define the number \( \psi(\Gamma) \) as \( \dim P_x X \) also in case when the space \( P_x X \) is infinite-dimensional, i. e. if the operator \( B \) satisfies the same conditions as before, the equality \( \psi_{A+B}(\Gamma) = \psi(\Gamma) \) holds also in this case.

CHAPTER V

PERTURBATIONS OF \( \Phi \), \( \Phi^- \) AND \( \Phi \)-OPERATORS

§ 1. \( \Phi \), \( \Phi^- \) AND \( \Phi \)-PERTURBATIONS. Let two Banach spaces \( X \) and \( Y \) be given. Let \( S(X \to Y) \subset B(X \to Y) \) be a space of linear operators. We denote by

\[
D_\Phi(X \to Y), \quad \text{the set of all } \Phi^- \text{-operators contained in the space } S(X \to Y),
\]
and by

\[
D_\Psi(X \to Y), \quad \text{the set of all } \Phi^- \text{-operators perturbations.}
\]

By Theorem 4.2, A 1, the sets \( D_\Phi(X \to Y), D_\Psi(X \to Y) \) are linear.

Theorem 4.1. (Gohberg, Markus, Feldman [1].) The set \( F_D(X \to Y) \) is closed in the space \( S(X \to Y) \).

Proof. Let us suppose that a sequence of operators \( T_n \in F_D(X \to Y) \) is convergent in the norm to an operator \( T \in S(X \to Y) \). Let \( \Lambda \in D_\Phi(X \to Y) \).

By Theorem 4.1, III, there exists an index \( n \) such that \( \Lambda + T_n \in D_\Psi(X \to Y) \). But \( T_n \in F_D(X \to Y) \). Hence \( \Lambda + T = \Lambda + (T - T_n) + T_n \in D_\Phi(X \to Y) \).

The proof for classes \( F_\Psi(X \to Y) \) is analogous, only in place of Theorem 4.1, III, one should apply Theorems 4.2 and 4.3 of that chapter, respectively.

Theorem 4.2. Let
\[
P(X \ri Y) = \left( \begin{array}{cc} A_0(X) & S_0(X \ri Y) \\ S_0(X \ri Y) & A_0(Y) \end{array} \right)
\]
be an arbitrary regularisable paralgebra. Moreover, let the space \( S_0(X \ri Y) \) contain at least one \( \Phi \)-operator \( F \). Then the set \( F_D(X \ri Y) \) of all perturbations of the class \( D_\Phi(X \ri Y) \) of \( \Phi \)-operators belonging to the paralgebra \( P(X \ri Y) \) is a maximal Fredholm ideal.
Proof. Let $A$ be an arbitrary operator from the paragleb $P(\mathcal{X} \rightarrow \mathcal{Y})$. The operator $A$ can be written as the sum of two $\Phi$-operators. Indeed, let $e.g. A = A_1(\mathcal{X}) + A_2(\mathcal{Y})$, then $A = aI + (A - aI)$. Let $a > |A|$, then the operator $A - aI$ is invertible by Theorem 1.2, I. Hence the operator $A$ is the sum of two invertible operators.

Let us now suppose that $A \in S((\mathcal{X} \rightarrow \mathcal{Y}))$. Then $A = aF + F(A/a - F)$. But $F$ is a $\Phi$-operator, Theorem 4.1, III, implies that $(A/a - F)$ is a $\Phi$-operator for sufficiently large values of $a$. Hence the operator $A$ is the sum of two $\Phi$-operators.

The arguments in the case of $A \in S_\Phi((\mathcal{X} \rightarrow \mathcal{Y}))$ are similar. Since the paragleb $P(\mathcal{X} \rightarrow \mathcal{Y})$ is regularizable, every $\Phi$-operator $P \in S_\Phi((\mathcal{X} \rightarrow \mathcal{Y}))$ has a simple regularizer $K_\Phi \in S_\Phi((\mathcal{X} \rightarrow \mathcal{Y}))$ and this regularizer is also a $\Phi$-operator. Hence we may apply Corollary 11.5, A, I, in order to show that the set $P(\mathcal{X} \rightarrow \mathcal{Y})$ is a maximal quasi-Fredholm ideal in the paragleb $P(\mathcal{X} \rightarrow \mathcal{Y})$. Thus in order to complete the proof it is sufficient to apply the following theorem:

**Theorem 1.3.** If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, then every quasi-Fredholm ideal contained in the paragleb $P(\mathcal{X} \rightarrow \mathcal{Y})$ is Fredholm.

**Proof.** Let $\mathcal{J}$ be a quasi-Fredholm ideal contained in the paragleb $P(\mathcal{X} \rightarrow \mathcal{Y})$. If $T \in \mathcal{J}$, then $aT \in \mathcal{J}$, where $a$ is a scalar. Since $\mathcal{J}$ is a quasi-Fredholm ideal, the index $f(a) = \kappa_{\mathcal{J}, a}$ is finite. By Theorem 1.1, IV, it is constant. But $f(0) = 0$. Hence $f(a) = \kappa_{\mathcal{J}, a} = 0$ and $\kappa_{\mathcal{J}, a} = 0$ for an arbitrary $T \in \mathcal{J}$. The set $T(\mathcal{X} \rightarrow \mathcal{Y})$ denote the ideal of compact operators in a paragleb $B(\mathcal{X} \rightarrow \mathcal{Y})$. We denote by $\mathcal{R}$ the radical of this paragleb and by $T_0(\mathcal{X} \rightarrow \mathcal{Y})$ the set of those operators which belong to cosets belonging to the radical $\mathcal{R}$:

$$T_0(\mathcal{X} \rightarrow \mathcal{Y}) = \{ T \in B(\mathcal{X} \rightarrow \mathcal{Y}) : [T] \in \mathcal{R} \}.$$

**Theorem 1.4.** The set $T_0(\mathcal{X} \rightarrow \mathcal{Y})$ is a maximal Fredholm ideal in the paragleb $B(\mathcal{X} \rightarrow \mathcal{Y})$.

**Proof.** By Corollary 5.3, B, II, the paragleb $B(\mathcal{X} \rightarrow \mathcal{Y})$ is regularizable. According to Remark 10.3, A, I, and Theorem 5.7, B, IV, the set $T_0(\mathcal{X} \rightarrow \mathcal{Y})$ is a Fredholm ideal. By Theorem 10.3 and Remark 10.3, A, I, it is a maximal Fredholm ideal.

By Theorem 11.1, A, I, every element of the ideal $T_0(\mathcal{X} \rightarrow \mathcal{Y})$ is a $\Phi_+$- and $\Phi_-$-perturbation. Hence we infer the following

**Corollary 1.5.** If a Banach paragleb $P(\mathcal{X} \rightarrow \mathcal{Y})$ is regularizable, then $F_\Phi(\mathcal{X} \rightarrow \mathcal{Y}) \subset F(\mathcal{X} \rightarrow \mathcal{Y})$, where $F_\Phi(\mathcal{X} \rightarrow \mathcal{Y})$ and $F(\mathcal{X} \rightarrow \mathcal{Y})$ denote the sets of $\Phi_-$, $\Phi_+$, and $\Phi_-$-perturbations, respectively, contained in the paragleb $P(\mathcal{X} \rightarrow \mathcal{Y})$.

It follows from Theorem 1.5, II (stating that a radical in a Banach paragleb is closed) that the ideal $T_0(\mathcal{X} \rightarrow \mathcal{Y})$ is closed. Hence one may consider the quotient paragleb $B(\mathcal{X} \rightarrow \mathcal{Y})/T_0(\mathcal{X} \rightarrow \mathcal{Y})$. The norm in this quotient paragleb induces the following norm in the paragleb $B(\mathcal{X} \rightarrow \mathcal{Y})$:

$$\|A\|_e = \inf_{T \in T_0(\mathcal{X} \rightarrow \mathcal{Y})} \|A + T\|.$$

**Theorem 4.1, III,** can be strengthened in the following manner:

**Theorem 1.6.** Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and let $A \in L(\mathcal{X} \rightarrow \mathcal{Y})$ be a $\Phi$-operator. There exists a number $q > 0$ such that for every operator $B \in B(\mathcal{X} \rightarrow \mathcal{Y})$, satisfying the inequality $\|B\| < q$, $A + B$ is also a $\Phi$-operator and $\kappa_{A+B} = \kappa_A$.

In other words: For every $\Phi$-operator $A$ which maps the space $\mathcal{X}$ into the space $\mathcal{Y}$ there exists a positive number $q$ such that all operators of a $C$-norm less than $q$ are $\Phi$-perturbations of the operator $A$ which do not change the index.

**Proof.** It immediately follows from Theorem 4.1 that if $\|B\| < q$, then $B$ is a $\Phi$-perturbation which does not change the index. But, by Theorem 3.2, I, every operator $T \in T_0(\mathcal{X})$ is a $\Phi$-perturbations of the operator $A + B$ not changing the index. Hence $B + T$ is a $\Phi$-perturbation of the operator $A$ not changing the index. This yields the theorem, because the operator $T$ has been arbitrary.

**§ 2. The form of the maximal Fredholm ideal in some concrete spaces.** We shall now give the form of the ideal $T_0(\mathcal{X}) = F_\Phi(\mathcal{X})$ for algebras $B(\mathcal{X})$ over some Banach spaces $\mathcal{X}$. Evidently, in spaces $P$ ($1 \leq p < +\infty$) and $c_0$ we have the equality $T_0 = T(\mathcal{X})$, because there cannot exist any two-sided ideal which would contain a non-compact operator (see Theorem 5.4, II). However, there exist spaces in which the ideal $T_0(\mathcal{X})$ is essentially wider than the ideal $T(\mathcal{X})$.

**Theorem 2.1.** If $\mathcal{X} = C[0, 1]$, then the ideal $W(\mathcal{X})$ of weakly compact operators is a Fredholm ideal wider than the ideal $T(\mathcal{X})$.

**Proof.** Let

$$T_0 = \int_0^1 x(s) t^{1-}ds.$$

The operator $T$ is well-defined since for each $t$ the integral (2.1) has only a weak singularity. Let $\mathcal{S}(\mathcal{X}) = \{ x \in \mathcal{X} : |x(s)| \leq 1 \}$. Let $\{x_n\} \subset \mathcal{S}(\mathcal{X})$ be an arbitrary sequence. The operators

$$T_0 x = \int_0^1 x(s) t^{1-}ds, \quad 0 < a < 1,$$
transforming the space \( C(0,1) \) into \( C[a,1] \) are compact (compare the Theorem 3.5, B III). Hence by the diagonal method we can find a subsequence \( \{n_k\} \) such that the sequence \( y_n = T_{n_k} \) is uniformly convergent on each interval \([a,1]\). Moreover, we are able to find a subsequence \( x_k(t) = y_{n_k}(t) \) which is convergent at the point \( t = 0 \). It is easy to check that the sequence \( \{x_n\} \) is weakly convergent. It implies that the operator \( T \) is weakly compact.

On the other hand, the operator \( T \) is not compact. Indeed, let

\[
\varepsilon_n(s) = \begin{cases} 
1 - ns & \text{for } s < 1/n, \\
0 & \text{for } s \geq 1/n.
\end{cases}
\]

By a simple calculation we obtain

\[
y_k(t) = \int_0^1 (1 - ns) e^{-(1/s)} ds = \frac{1}{t+2} \left( \frac{1}{n} \right)
\]

and it is easy to check that the sequence \( \{y_k(t)\} \) is not uniformly continuous, which is what was to be proved (see Theorem 2.5, B IV).

Therefore

\[
W(X) \neq T(X).
\]

On the other hand, by applying the Dunford-Pettis theorem (Theorem 7.5, II) we can see that squaring any weakly compact operator \( T \in W(X) \) we obtain a compact operator \( T^2 \in T(X) \). By Theorem 9.3, A I, the operator \( I - T \) is of a finite d-characteristic. By Theorem 1.3, \( W(X) \) is a Fredholm ideal.

It is possible to prove Theorem 2.1 for all spaces \( C(0,1) \) and for spaces \( L(p, \mathbb{Z}) \) if the measure \( \mu \) is not purely atomic.

Theorem 2.2. If \( X = C(0,1) \) then \( T_{n_k}(X) \rightarrow W(X) \).

Proof. Theorem 1.4 immediately implies \( W(X) \subset T_{n_k}(X) \). We shall prove the converse inclusion \( T_{n_k}(X) \subset W(X) \). Let us suppose that \( T \in T_{n_k}(X) \) but \( T \notin W(X) \). By Theorem 2.2, II, there exists a weakly unconditionally convergent series \( \sum \varepsilon_n \) such that the series \( \sum \varepsilon_n x_n \) is not unconditionally convergent. Hence there exists a permutation of the sequence \( \{x_n\} \), an increasing sequence of indices \( \{m_k\} \) and a constant \( \delta > 0 \) such that

\[
\|T_{m_k}y_k\|_p > 0,
\]

where \( y_k = \sum_{n=1}^{m_k} x_n \).

Evidently, this implies \( \|y_k\|_p > \delta \|T_{m_k}\| \).

On the other hand, the weak unconditionally convergent of the series implies that the sequence \( \{y_k\} \), and hence also the sequence \( \{T_{m_k}y_k\} \), are weakly convergent to zero.

§ 2. The form of the maximal Fredholm ideal

By Theorem 5.2, I, one can extract a subsequence \( \{y_{n_k}\} \) which is a basis of the space spanned over it; in fact: a basis equivalent to the standard basis in the space \( \mathbb{C} \).

We can deal similarly with the sequence \( \{T_{n_k}y_k\} \). Finally, we find that the operator \( T \) transforms the elements \( e_j = y_{n_k} \) into elements \( T_{n_k} \); moreover, the sequences \( \{e_j\} \) and \( \{T_{n_k}\} \) are both bases in spaces \( Y_k \) and \( Y_k \) spanned by those sequences, respectively, and equivalent to the standard basis in the space \( \mathbb{C} \). Hence the operator \( T^{-1} \) is well-defined and continuous on the space \( Y_k \). But the spaces \( Y_k \) and \( Y_k \) are isomorphic. By Sobczyk's theorem (Theorem 5.3, I), which states that at once the space \( Y_k \) and the space \( \mathbb{C} \), both bases are isomorphic, then \( Y_k \) is a projection of the space \( X \), we may extend the operator \( T^{-1} \) to an operator \( T^{-1} \) defined on the whole space \( X \). But

\[
(I - T^{-1}T)e_j = 0 \quad (j = 1, 2, \ldots).
\]

Hence the operator \( I - T^{-1}T \) does not possess a finite d-characteristic. Consequently, \( T \notin T_0 \), which is a contradiction.

§ 3. Semicompact and co-semicompact operators as \( \Phi_- \) and \( \Phi_- \)-perturbations.

Theorem 11.1. A I, shows that if an operator \( T \in B(X \rightarrow Y) \) belongs to the ideal \( T_{n_k}(X \rightarrow Y) \), then it is both a \( \Phi_- \)-perturbation and a \( \Phi_- \)-perturbation. However, there may exist \( \Phi_- \) and \( \Phi_- \)-perturbations with do not belong to this ideal.

Theorem 3.1. (Kato [1]). Let \( X \) and \( Y \) be Banach spaces and let \( A \in B(X \rightarrow Y) \) be a \( \Phi_- \)-operator. If \( T \) is an arbitrary semicompact operator, then \( I - T \) is a \( \Phi_- \)-operator.

Proof. Let \( X \) be a direct sum \( X = Z \oplus \mathbb{C} \), where \( \mathbb{C} \) is a closed subspace. Let \( A \) and \( T \) denote restrictions of operators \( A \) and \( T \) to the space \( \mathbb{C} \), respectively. Evidently, \( a_{n_k} = 0 \). Since \( E_{\varepsilon} = E_{\varepsilon} \) is a closed set by hypothesis, there exists a positive number \( \gamma \) such that

\[
\|A\varepsilon\| = \|A\varepsilon\| > \gamma \|\varepsilon\| \quad \text{for } \varepsilon \in \mathbb{C}.
\]

Let \( 0 < s < \gamma \), and let \( M \) denote an arbitrary subspace of the space \( \mathbb{C} \) such that

\[
\|A_{\varepsilon} + T_{\varepsilon}\| < s \|\varepsilon\| \quad \text{for } \varepsilon \in M.
\]

Then

\[
\|T_{\varepsilon}\| = \|T_{\varepsilon}\| > \|A_{\varepsilon}\| > \|A_{\varepsilon} + T_{\varepsilon}\| > (s - \varepsilon) \|\varepsilon\|.
\]

Since the operator \( T \) is semicompact, it follows that the space \( M \) is of a finite dimension. Hence

\[
a_{\varepsilon} \rightarrow +\infty \quad \text{as } \varepsilon \rightarrow +\infty.
\]

In order to complete the proof it is sufficient to show that the set \( E_{\varepsilon} \rightarrow T \) is closed. Since \( \dim Z \leq +\infty \), this is equivalent to the statement that the set \( E_{\varepsilon} \rightarrow T \) is closed. But this follows at once from formula (3.1) and from the following lemma:
LEMMA 3.2. (Kato [1]) Let $X$ and $Y$ be Banach spaces and let $A \in B(X \to Y)$. If $E_x$ is a closed set, then for every number $\epsilon > 0$ there exists an infinite-dimensional space $M_\epsilon$ such that

$$||A|| < \epsilon ||x||$$

for $x \in M_\epsilon$.

Proof. The operator $A$ induces an operator $[A]$ which is a one-to-one map of the quotient space $X/E_x$ onto the set $Z$. Let us suppose that there exists a number $\delta > 0$ such that $||A[x]|| > \delta ||x||$. Then the operator $A$ is invertible and, consequently, the set $E_x$ is closed. Thus for every number $\delta > 0$ there exists an element $[x]_k \in X/E_x$ such that

$$||A[x]_k|| < \delta ||x||.$$

It follows from the definition of the quotient space and of the operator on the cosets that there exists an element $x_0 \in X$ satisfying the inequality

$$||A x_0|| < \delta ||x_0||.$$

It is easily proved that the elements $x_0$ can be chosen from an arbitrary subspace of a finite codimension. Hence one can choose a sequence of elements $(x_n)$ and a sequence of functionals $(\lambda_n) \subset X^*$ such that $||x_n|| = 1$ and

$$f_i(x_j) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \quad \text{and} \quad ||A x_n|| < \frac{\epsilon}{\delta} ||x_n||.$$

Let $M_\epsilon$ be the subspace spanned by the elements $x_1, x_2, \ldots$. Evidently, the space $M_\epsilon$ is infinite-dimensional. Moreover, we have

$$||A|| \leq \sum_{n=1}^{\infty} |\xi_n| \frac{\epsilon}{\delta n} ||x_n|| \leq \max_{1 \leq n \leq \infty} |\lambda_n| \epsilon \max_{1 \leq i \leq \infty} |f_i(x)| \leq \epsilon ||x||$$

for every element $x = \sum_{n=1}^{\infty} \lambda_n x_n.$

Moreover, Lemma 3.2 implies

COROLLARY 3.3. If $X$ and $Y$ are Banach spaces and if $T \in B(X \to Y)$ is a compact operator, then for every number $\epsilon > 0$ there exists an infinite-dimensional subspace $M_\epsilon$ such that the restriction of the operator $T$ to the subspace $M_\epsilon$ is of a norm less than $\epsilon$.

A theorem analogous to Theorem 3.1 for $\Phi_-$-operators has been proved by J. N. Vladimirovskii.

THEOREM 3.4. (Vladimirovskii [1]) If $A \in B(X \to Y)$ is a $\Phi_-$-operator and $T \in B(X \to Y)$ is a co-semicompact operator, then $A + T$ is a $\Phi_-$-operator.

Proof. Let us suppose that $A + T$ is not a $\Phi_-$-operator. Lemma 6.3, III, implies that there is a subspace $M \subset Y$ with infinite codimension such that the operator $\Phi_M(A + T)$ is compact. Theorem 6.1, III, implies

that there is a subspace $N \subset M$ with infinite codimension such that $\Phi_N T$ is a compact operator. Obviously $\Phi_N(A + T)$ is a compact operator, whence $\Phi_M A$ is also a compact operator, which contradicts the assumptions. ■

Remark. The theorems of this section can easily be proved also for discontinuous $\Phi_-$-operators if we consider the space $X \Delta$ made up of the set $D_A$ provided with the norm $||x||_\Delta = ||x|| + ||A x||$ instead of the space $X$ (see § 1, B II). Evidently, the operator $A$ maps continuously the space $X \Delta$ into the space $Y$. Moreover, in place of the usual semicompactness (co-semicompactness) one may investigate $A$-semicompactness ($A$-co-semicompactness) defined in the following manner: An operator $T \in L(X \to Y)$ is called $A$-semicompact ($A$-co-semicompact) if the operator $T \in L(X \Delta \to Y)$ is semicompact (co-semicompact). Evidently, every semicompact (co-semicompact) operator is $A$-semicompact ($A$-co-semicompact).