CHAPTER I

LINEAR TOPOLOGICAL AND LINEAR METRIC SPACES

§ 1. Topological spaces and metric spaces. A non-void set $X$ is called a Hausdorff topological space if there exists a family $\mathcal{A}$ of sets $U \subseteq X$ called neighbourhoods satisfying the following axioms:

1. For every $x \in X$, if $x \in U$ and $x \in V$, $U, V \in \mathcal{A}$, then there exists a neighbourhood $W \subseteq U \cap V$, $W \in \mathcal{A}$, such that $x \in W$.

2. For every two points $x$ and $y$, $x, y \in X$, there exist neighbourhoods $U_x, U_y \in \mathcal{A}$ such that $x \in U_x, y \in U_y$ and $U_x \cap U_y = \emptyset$.

The family $\mathcal{A}$ of neighbourhoods determines a topology in the space $X$. We say that the topology determined by a family $\mathcal{A}$ is not finer (not stronger) than the topology determined by a family $\mathcal{B}$, or that the topology determined by the family $\mathcal{B}$ is not coarser (not weaker) than the topology determined by the family $\mathcal{A}$ if for every $x \in X$ and every $U \in \mathcal{A}$ such that $x \in U$ there exists a neighbourhood $V \in \mathcal{B}$ such that $x \in V$ and $V \subseteq U$.

Two topologies determined by families $\mathcal{A}$ and $\mathcal{B}$ are called equivalent if the first is not finer than the second and the second is not finer than the first one, simultaneously.

Suppose we are given a Hausdorff topological space, i.e. the following collection: set $X$ and topology determined by a family $\mathcal{A}$ of neighbourhoods in $X$. We say that a set $E \subseteq X$ is open if for every $x \in E$ there exists a neighbourhood $U \in \mathcal{A}$ such that $x \in U$ and $U \subseteq E$. A set $E \subseteq X$ is called closed if its complement, i.e. the set

$$CE = \{x \in X : x \notin E\},$$

is an open set.

It follows immediately that every neighbourhood is by definition an open set.

The union of an arbitrary number of open sets is an open set. Hence an intersection of an arbitrary number of closed sets is a closed set.

An intersection of a finite number of open sets is an open set. A union of a finite number of closed sets is a closed set.
If for an arbitrary family \( \mathcal{F} \) of disjoint closed sets there exists a family \( \mathcal{G} \) of disjoint open sets such that \( \mathcal{G} \supset \mathcal{F} \), the space is called normal.

If a set is a union of a countable number of closed sets, it is called a set of the class \( \mathcal{F} \). If a set is an intersection of a countable number of open sets, it is called a set of the class \( \mathcal{G} \).

The closure \( \overline{E} \) of a set \( E \) is the smallest closed set containing \( E \). It follows from the last remark that

\[
\overline{E} = \bigcap_{E \supseteq F, F \text{ closed set}} F.
\]

The interior \( \text{int} E \) of a set \( E \) is the greatest open set contained in \( E \). Evidently,

\[
\text{int} E = \bigcup_{E \supseteq G, G \text{ open set}} G = \{ x \in E : x \not\in \overline{E} \).
\]

The closure of a set \( E \) can be defined also as the set

\[
E_0 = \{ x \in X : U \cap E \neq \emptyset \text{ for every neighbourhood } U \text{ of the point } x \}.
\]

Indeed, every open set containing at least one point of the set \( E_0 \) has common points with the set \( E \). Hence the complement of the set \( E \) must be contained in the complement of the set \( E_0 \). But if \( y \not\in E_0 \), there exists a neighbourhood \( U_y \) of the point \( y \) having no common point with the set \( E \) and, consequently, no common point with the set \( E_0 \). Hence

\[
\text{cl} E_0 = \bigcup_{x \in E_0} U_y
\]

is an open set, as a union of open sets.

Points belonging to the set \( E_0 \) are called clusters points of the set \( E \).

The notion of a "cluster point" differs from the notion of an "accumulation point" essentially. Namely, we call a point \( p \) an accumulation point (limit point) of a set \( E \) if it is a cluster point of the set \( E(p) \).

A cluster point of a family of sets \( \mathcal{A} \) is a point which is a cluster point of all sets \( A \in \mathcal{A} \).

Evidently, the closure \( \overline{E} \) of a closed set \( E \) is equal to \( E : \overline{E} = E \). Hence \( \overline{E} = \overline{E} \) for an arbitrary set \( E \).

The set \( \overline{E} \cap \overline{E} \) is called the boundary of the set \( E \).

We say that a set \( E \) is dense in a set \( B \) if \( \overline{E} \supset B \). In particular, a set \( E \) is dense in the topological space \( X \) if \( \overline{E} = X \).

A space \( E \) is called separable if there exists a countable set dense in \( E \).

A set \( E \) is called nowhere dense (non-dense) if \( \overline{E} \) does not contain any open set.

A set \( E \) is called a set of the first category if it is the union of a countable number of nowhere dense sets. Evidently, a subset of a set of the first category is also a set of the first category. A set which is not of the first category is called a set of the second category. Since, by definition, a set of the second category is not nowhere dense, its closure must contain an open set.

A set \( X \) is called a metric space if there exists a real-valued, non-negative function \( \varrho(x, y) \) defined for all \( x, y \in X \) and called a metric, satisfying the conditions:

1. \( \varrho(x, y) = 0 \) if and only if \( x = y \);
2. \( \varrho(x, y) = \varrho(y, x) \);
3. \( \varrho(x, y) \leq \varrho(x, z) + \varrho(z, y) \) (triangle inequality).

Every metric induces a family of neighbourhoods \( \mathcal{N} \). Namely, a neighbourhood of the point \( x_0 \) is the set

\[
U_{x_0} = \{ x : \varrho(x, x_0) < \varepsilon \}.
\]

It is easily verified that the neighbourhoods defined above satisfy axioms (1) and (2) of a Hausdorff topological space. Hence every metric space is a topological space.

We say that two metrics are equivalent if the topologies induced by these metrics are equivalent.

In order to define a closed set in a metric space one can apply the notion of convergence of a sequence. A sequence \( \{ x_n \} \) is said to be convergent to an element \( x \), called the limit of the sequence, if

\[
\lim_{n \to \infty} \varrho(x_n, x) = 0;
\]

we shall denote this by \( x_n \to x \).

A set \( F \) is closed if and only if it contains limits of all convergent sequences \( \{ x_n \} \) of elements belonging to \( F \). Indeed, let us suppose that there exists a sequence \( \{ x_n \} \), \( x_n \in F \), such that \( x_n \to x \notin F \). Then every neighbourhood \( U \) of the point \( x \) contains points of the sequence \( \{ x_n \} \). Hence none of the neighbourhoods \( U \) of the point \( x \) contains in itself the complement of the set \( F \). Thus, the complement of the set \( F \) is not open. Consequently, the set \( F \) is not closed.

On the other hand, if \( x_n \to x \) for a certain sequence \( \{ x_n \} \subset F \) implies \( x \in F \), then for \( y \notin C \) of the points of the set \( C \).

Hence the set \( C \) is open and the set \( F \) is closed.

The product space \( X \times Y \) of two Hausdorff topological spaces \( X \) and \( Y \) is the set of ordered pairs \( (x, y) \) with the product topology, i.e. a neighbourhood of the point \( (x_0, y_0) \) is the set

\[
W(\{x_0, y_0\}, U, V) = \{ (x, y) : x \in U_{x_0}, y \in V_{y_0} \},
\]

where \( U_{x_0} \) and \( V_{y_0} \) are neighbourhoods of points \( x_0 \) and \( y_0 \) in spaces \( X \) and \( Y \), respectively.
A map \( f \) of a topological space \( X \) into a topological space \( Y \) is called a continuous transformation if the inverse image \( f^{-1}(G) \) of every open set \( G \), i.e. the set

\[
\{ x \in X : f(x) \in G \},
\]

is an open set or, equivalently, if the inverse image of every closed set is a closed set. One can give another definition of a continuous transformation:

A transformation \( f \) of a topological space \( X \) into a topological space \( Y \) is called continuous if for every point \( x \in X \) and for every neighbourhood \( V \) of the point \( f(x) \) there exists a neighbourhood \( U(V, x) \) of the point \( x \) such that \( f(U) \subseteq V \).

Both definitions are equivalent. Indeed, if we assume the first one, the set \( U = f^{-1}(V) \) satisfies the assumptions of the second definition.

If we assume the second definition, then for every open set \( G \),

\[
f^{-1}(G) = \bigcup_{x \in f^{-1}(G)} U(V, x),
\]

Hence the set \( f^{-1}(G) \) is open, as a union of open sets.

A superposition of two continuous transformations \( f \) and \( g \) is a continuous transformation. Indeed, the set \( f^{-1}(g^{-1}(G)) \) is open for every open set \( G \). Hence the set \( g^{-1}(f^{-1}(G)) \) is open. But \( (g^{-1}(f^{-1}(G))) = (g^{-1})^{-1}(G) \). Thus the set \( (g^{-1})^{-1}(G) \) is open for every open set \( G \), as we had to prove.

If \( X \) and \( Y \) are metric spaces, one can say that a transformation is continuous if for every sequence \( \{x_n\} \) convergent to a point \( x \) the sequence \( \{f(x_n)\} \) is convergent to the point \( f(x) \). Indeed, let \( F \) be a closed set. We prove \( f^{-1}(F) \) to be a closed set. Let some \( x_n \) be an arbitrary sequence convergent to a point \( x \), \( x_n \rightarrow x \). Then \( f(x_n) \rightarrow f(x) \) and since the set \( F \) is closed, also \( f(x) \in F \). Hence \( x \in f^{-1}(F) \), and the set \( f^{-1}(F) \) is closed.

On the other hand, let \( y_n \rightarrow x \). By hypothesis, the inverse image of every closed set is closed; hence \( f(x) \in \lim_n y_n \). Moreover, one can show in an analogous manner that \( f(x) = \lim_n y_n \). Hence there exists a convergent sequence \( \{y_n\} \) being arbitrary, we conclude that the sequence \( \{y_n\} \) is convergent to the point \( f(x) \).

A covering of a set \( E \) is a family of open sets \( \{P_n\} \) such that \( E \subseteq \bigcup P_n \).

A set \( E \) is called compact if from every covering of the set \( E \) by means of open sets \( \{P_n\} \) one can extract a finite system \( P_{n_i} \) such that \( E \subseteq \bigcup P_{n_i} \)

One can give the following dual definition of a compact set:

A set \( E \) is compact if for every family of closed subsets \( \{F_n\} \) of the set \( E \) such that the set \( \bigcap F_n \) is void there exists a finite system \( F_{n_i} \)

\[
(\cap_i F_{n_i} = 0).
\]

A linear space $X$ is called a linear topological space if it is a Hausdorff topological space and if the operations of addition of elements and of multiplication of an element by a scalar are continuous operations, i.e., if the operation of addition is a continuous transformation of the product $X \times X$ into the space $X$, and the operation of multiplication by a scalar is a continuous transformation of the product $X \times \mathbb{R}$ (or $\mathbb{R} \times X$) into the space $X$, where $\mathbb{R}$ denotes the field of complex numbers and $\mathbb{R}$ the field of real numbers.

Since addition is continuous, the set of neighbourhoods of the form $x + U$, where $U$ runs over the set of neighbourhoods of zero, determines a topology equivalent to the given one. Hence we can say that the topology in a linear topological space is determined by the set of neighbourhoods of zero.

In other words, a linear space $X$ is called a linear topological space if it possesses a topology having the following properties: For every open set $U$ the set $x + U$ is open, and for every neighbourhood of zero, $U$, there exists a neighbourhood of zero, $V$, such that $V + V \subseteq U$. Let us remark that the last fact implies $V \subseteq U$.

If a set $U$ is open, then the set $aU = \{ax : x \in U\}$ is open for every scalar $a \neq 0$.

A set $U$ is called symmetric, if $U = -U$.

A set $U$ is called balanced or circled if $aU \subseteq U$ for $|a| \leq 1$.

**Theorem 2.1.** If a set $V$ is a neighbourhood of zero in a linear topological space $X$, then there exists an open balanced set such that $V \subseteq U$.

**Proof.** It follows from the continuity of multiplication that there exist a neighbourhood of zero $V_0$ and a number $\varepsilon > 0$ such that $\varepsilon V_0 \subseteq V$ for $|a| \leq \varepsilon$. Let $V_1 = W_1$ and $aW_1 \subseteq V$ for $|a| \leq 1$. Let $U = \bigcup_{i=0}^{n} aW_i$. Evidently, $V \subseteq W$ and $aU \subseteq W$ for $|a| \leq 1$, and $U$ is an open set, as a union of open sets.

**Corollary 2.2.** If $X$ is a linear topological space, then there exists a topology determined by a family of balanced neighbourhoods of zero and equivalent to the given one.

**Proof.** Let a topology in the space $X$ be defined by a family $\mathfrak{B}$ of neighbourhoods. With every neighbourhood $V \in \mathfrak{B}$ one can associate an open balanced set $U$ contained in $V$. The family of these sets is denoted by $\mathfrak{B}$. Since sets from the family $\mathfrak{B}$ are open, each contains a neighbourhood of zero $V_0 \in \mathfrak{B}$. Hence the topologies determined by these families are equivalent.

A linear topological space is called a linear metric space if the topology given in the definition of a linear topological space is determined by a metric $g(x, y)$.

A metric $g'(x, y)$ is called an invariant metric if for every $\varepsilon \in X$

$$g'(x + \varepsilon, y + \varepsilon) = g'(x, y).$$

**Theorem 2.3.** (Kakutani [11]) If $X$ is a linear metric space with metric $g(x, y)$, then there exists an invariant metric $g'(x, y)$ equivalent to the metric $g(x, y)$.

**Proof.** It follows from the continuity of multiplication by a scalar that for every neighbourhood of zero $V$ there exists a neighbourhood of zero $U$ such that $V \subseteq U \subseteq V$. By Theorem 2.1, we may assume without loss of generality that the neighbourhood $U$ is balanced.

Let us fix one balanced neighbourhood $U$ and let us denote it by $U(1/2)$. By induction, a sequence of neighbourhoods $U(1/2^n), n = 1, 2, 3, 4, \ldots$, can be constructed satisfying the conditions

1. $aU(1/2^n) = U(1/2^n)$ for $|a| = 1$,
2. $U(1/2^{n+1}) \subseteq U(1/2^n) \subseteq U(1/2^n)$,
3. $U(1/2^n) \subseteq U(1/2^n)$.

By $U(1)$ we denote the whole space $X$.

Let $r$ be a dyadic number: $r = \sum_{i=0}^{\infty} r_i (1/2^i)$, where $0 < r < 1$ and $r_i$ is equal to $0$ or to $1$. We write

$$U(r) = \sum_{i=0}^{\infty} r_i U(1/2^i).$$

(\sum_{i=0}^{\infty} r_i U(1/2^i) is an algebraic sum of sets).

From formulae (1) and (2) we obtain

1. $aU(r) = U(r)$ if $|a| = 1$,
2. $U(r_1 + r_2) \supseteq U(r_1) + U(r_2)$.

Let us take

$$g'(x, y) = \inf \{r : x - y \in U(r)\}.$$

Condition (1) implies $g'(x, y) = g'(y, x)$, and from (2) we obtain $g'(x, y) \leq g'(x, z) + g'(z, y)$. The invariance of $g'$ is proved immediately, since

$$g'(x + z, y + z) = \inf \{r : (x + z) - (y + z) \in U(r)\}
= \inf \{r : x - y \in U(r)\} = g'(x, y).$$

If $g'(x, y) \to 0$, then the continuity of addition gives $x_n \to x$. Since $U(1/2^n)$ are neighbourhoods, given an arbitrary $n$ there exists a number $h_n$ such that $x_n - x \in U(1/2^n)$ for $h > h_n$. Hence $g'(x, x) \leq 1/2^n$ for $h > h_n$. Thus $g'(x, x) \to 0$ as $h \to 0$. So $g'(x, x) = 0$.
and, consequently, \( g'(x, x) \to 0 \). On the other hand, if \( g'(x, x) \to 0 \), condition (3) implies \( g(x - x, 0) \to 0 \), and continuity of addition gives \( g(x, x) \to 0 \).

Hence it follows that \( g'(x, y) = 0 \) if and only if \( x = y \), and that metrics \( g'(x, y) \) and \( g(x, y) \) are equivalent. ■

In the proof of Theorem 2.3 we did not apply the existence of a metric in an essential way. We made use only of the fact that there exists a countable family of neighbourhoods of zero determining the topology. In our case it was the family of neighbourhoods \( V_n \), and the norm in the space \( X \) induces a norm in the quotient space \( X / X_0 \).

Theorem 2.3. If a topology in a linear topological space is determined by a countable family of neighbourhoods of zero, then there exists an invariant metric \( g(x, y) \) determining a topology equivalent to the given one.

Let us remark that in the proof of Theorem 2.3 we did not apply multiplication by a scalar. Hence the theorem on the existence of an invariant metric can be transferred to the case of Abelian metric groups, i.e. groups which are metric spaces with the continuous operation of addition.

Let \( X \) be a linear metric space with an invariant metric \( g(x, y) \). Let us write \( g(x, 0) = ||x|| \). Then

(a) \( ||x|| = 0 \) if and only if \( x = 0 \),

(b) \( ||x + y|| \leq ||x|| + ||y|| \) (subadditivity, the so-called triangle condition).

A non-negative function satisfying conditions (a), (b), (c) is called a norm. Every invariant metric induces a norm uniquely. On the other hand, every norm induces the invariant metric \( g(x, y) = ||x - y|| \).

Two norms are equivalent if the metrics induced by these norms are equivalent.

Let us remark that condition (c) immediately implies continuity of addition. Indeed, if \( x_n \to x \), \( y_n \to y \), then

\[ ||x + y - x_n - y_n|| \leq ||x - x_n|| + ||y - y_n|| \to 0 \, . \]

Let \( X \) be a linear topological space, and let \( X_0 \) be a closed subspace of \( X \). As before we denote the quotient space by \( X / X_0 \). The topology in the space \( X \) induces the following topology in the space \( X / X_0 \):

With every neighbourhood \( U \subseteq X \) we associate a neighbourhood \( [U] \) made of all cosets \( [x] \) having common points with the neighbourhood \( U \).

It is easily verified that the family of all sets \([U]\) satisfies all axioms of a family of neighbourhoods. In order to prove that these neighbourhoods distinguish between points one has to apply in an essential way the fact that the space \( X_0 \) is closed.

\[ \|x\| = \inf_{a \in X} ||x - a|| \, . \]

This norm determines a topology in the quotient space corresponding to the previously defined topology of quotient spaces.

The map \( \Phi_{X_0} \) of a space \( X \) into the space \( X / X_0 \) which associates with every element \( x \) of the space \( X \) the corresponding coset \([x]\) (defined in §1, A1) is a continuous map. Indeed, let \( A \) be an open set in the space \( X \).

The inverse image of the set \( A \), i.e. the set \( A_0 = \Phi_{X_0}^{-1}(A) = \{x: x \in A\} \) is also an open set. For if a point \( x \in A_0 \) belonged to the closure of the complement of the set \( A_0 \), the corresponding coset \([x]\) would belong to the closure of the complement of the set \( A \), contradicting the assumption that \( A \) is an open set.

§ 3. Examples of linear metric spaces.

Example 3.1. Let a set \( A \) and a countably additive algebra \( \Sigma \) of subsets of \( A \) be given. Let \( \mu \) be a measure defined on \( \Sigma \). We consider the set of all \( \mu \)-measurable functions \( x(t) \) such that

\[ \|x\| = \int_0^\infty \frac{|x(t)|}{1 + |x(t)|} \, dt < +\infty . \]

We identify all functions which differ only on a set measure \( \mu \) zero.

The set of these cosets will be denoted by \( \mathcal{B}(A, \Sigma, \mu) \).

It is easily seen that the function \( \|x\| \) is a norm. Indeed,

(a) The identification of elements of the same coset implies that \( \|x\| = 0 \) if and only if \( x(t) = 0 \).

(b) \( \|x\| = \int_0^\infty \frac{|x(t)|}{1 + |x(t)|} \, dt = \int_0^\infty \frac{|x(t)|}{1 + |x(t)|} \, d\mu = |x| \) if \( |a| = 1 \).

(c) Let us observe that the following inequality holds:

\[ \frac{|a + b|}{1 + |a + b|} \leq \frac{|a|}{1 + |a|} + \frac{|b|}{1 + |b|} . \]

Indeed, if \( |a + b| > \max(|a|, |b|) \), then the inequality \( |a + b| < |a| + |b| \) implies

\[ \frac{|a + b|}{1 + |a + b|} \leq \frac{|a|}{1 + |a|} + \frac{|b|}{1 + |b|} \leq \frac{|a|}{1 + |a|} + \frac{|b|}{1 + |b|} . \]
Let $|a + b| \leq \max(|a|, |b|)$, and let us suppose $|a| \geq |a + b|$. Then 
$$
\frac{|a + b|}{1 + |a + b|} \leq \frac{|a|}{1 + |a|} \leq \frac{|a|}{1 + |a| + |b|}.
$$
for $\frac{|b|}{1 + |b|}$ is an increasing function of the variable $|b|$. Condition (c) implies continuity of addition. We shall show that multiplication by a scalar is continuous. Indeed, let $a_n \to a$, $x_n \to x$. We have
$$
(a_n x_n - ax) = a_n (x_n - x) + (a_n - a)x.
$$
Let us observe that if $|a| < 1$, then the monotonicity of the function $|t|/(1 + |t|)$ implies $|b_n x_n| < |a_n|$. Let $k$ be a natural number such that $|a_k| < 1$. Then
$$
|b_n x_n - b_n x| = \left| \frac{b_n}{k} \right| \left| b_n (x_n - x) \right| < k \left| \frac{x}{x} \right| (x_n - x) \to 0.
$$
Let $x$ be a fixed element, and let $x$ be an arbitrary positive number. There exists a set $K$ of finite measure $\mu$ such that
$$
\int_{K} \frac{|x|}{1 + |x|} \, d\mu < e^3.
$$
We consider the function $\sigma(t)$ on the set $K$. Since this function is defined almost everywhere, we have $\lim_{n \to \infty} \mu(K_n) = 0$, where $K_n = \{ t \in K : |\sigma(t)| > n \}$. Let $s_n$ in such a manner that $\mu(K_n) > e^3$. Since $a_n \to a$, there exists an index $N$ such that $|a_n - a| < \mu(K_n) < e^3$ for $n > N$. Hence
$$
\int_{K_{n+1}} |(a_n - a)x| \, d\mu + \int_{K_n} |(a_n - a)x| \, d\mu + \int_{K_n} |(a_n - a)x| \, d\mu = \int_{K_n} |(a_n - a)x| \, d\mu < e^3 + e^3 + e^3 = e.
$$
Consequently, $(a_n - a)x \to 0$, and we can conclude the continuity of multiplication by a scalar. Hence the space $S(Q, \Sigma, \mu)$ is a linear metric space.

**Example 3.1.a.** Let $Q$ be the closed interval $[0, 1]$, $\mu$ the Lebesgue measure, $\Sigma$ the field of measurable sets. Then we denote $S(Q, \Sigma, \mu)$ by $S[0, 1]$.

**Example 3.1.b.** Let $Q$ be the set of natural numbers, $\Sigma$ the field of all its subsets, and $\mu(\{n\}) = 2^{-n}$. Then $S(Q, \Sigma, \mu)$ is the space of all sequences $x = \{x_n\}$ with the norm
$$
|x| = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n|}{1 + |x_n|}.
$$
We denote this space by $(S)$. 

$\|x\| = \int_{\Omega} |x(t)|^p \, d\mu < +\infty$. 

**Example 3.2.** Suppose we are given a set $\Omega$, a countably additive algebra $\Sigma$ of its subsets, and a measure $\mu$ defined on $\Sigma$. Moreover, let $p$ be a number satisfying the inequalities $0 < p \leq 1$. We consider the set of all $\mu$-measurable functions $x(t)$ such that 
$$
\|x\| = \int_{\Omega} |x(t)|^p \, d\mu < +\infty.
$$
We identify all functions which differ on a set of measure $\mu$ zero only. The set of all cosets obtained in this manner is denoted by $P(\Omega, \Sigma, \mu)$. This is a linear metric space. Indeed, 
(a) the identification defined above implies that $\|x\| = 0$ if and only if $x(t) = 0$; 
(b) we have 
$$
\|ax\| = \int_{\Omega} |ax(t)|^p \, d\mu = \int_{\Omega} |x(t)|^p \, d\mu = \|x\| \quad \text{if} \quad |a| = 1;
$$
$$
\|x + y\| = \int_{\Omega} |x(t) + y(t)|^p \, d\mu < +\infty
$$
and this was to be proved.

**Example 3.3.** Let $Q$ be a set, $\Sigma$ a countably additive algebra of its subsets, and $\mu$ a measure defined on $\Sigma$. We consider all $\mu$-measurable functions $x(t)$ such that 
$$
\|x\| = \left( \int_{\Omega} |x(t)|^p \, d\mu \right)^{\frac{1}{p}} < +\infty,
$$
where $p \geq 1$.
We identify all functions $x(t)$ and $y(t)$ such that $x(t) \neq y(t)$ only on sets of measure $\mu$ equal to zero. We denote the set of all such cosets by $P(\Omega, \Sigma, \mu)$. Let us remark that if $x, y \in P(\Omega, \Sigma, \mu)$, then 
(a) the identification defined above implies that $\|x\| = 0$ if and only if $x = 0$; 
(b) we have 
$$
\|ax\| = \left( \int_{\Omega} |ax(t)|^p \, d\mu \right)^{\frac{1}{p}} = \left( \int_{\Omega} |x(t)|^p \, d\mu \right)^{\frac{1}{p}} \quad \text{if} \quad |a| = 1;
Example 3.3. Let $\Omega$ be the interval $[0, 1]$, $\mu$ the Lebesgue measure, and $\Sigma$ the field of sets measurable in Lebesgue sense. Then $L^p(\Omega, \Sigma, \mu)$ is the space of functions integrable with power $p$ on the interval $[0, 1]$. We shall denote this space by $L^p$.

Example 3.3. Let $\Sigma$ be the family of all subsets of a countable set $\Omega$ and let the measure $\mu$ be equal to one at each point of $\Omega$. Then $L^p(\Omega, \Sigma, \mu)$ is the space of all sequences summable with power $p$. We shall denote this space by $P(\Omega)$. If $\Omega$ is the set of all natural numbers, we denote $P(\Omega)$ briefly by $P$.

Example 3.4. Let $\Omega$ be a set, $\Sigma$ a countably additive algebra of subsets of the set $\Omega$, and $\mu$ a measure defined on $\Sigma$. We consider the set of $\mu$-measurable, essentially bounded functions $\pi(t)$ on the set $\Omega$, i.e., functions for which

$$||\pi|| = \text{ess sup}_{t \in \Omega} |\pi(t)| = \inf_{t \in \Omega} \sup_{E \in \Sigma, \mu(E) \leq \epsilon} |\pi(t)| < +\infty.$$  

As in Example 3.3, we identify all functions which differ at most on a set of measure $\mu$ equal to zero. We denote the set of cosets obtained in this manner by $M(\Omega, \Sigma, \mu)$. Then

(a) identification of functions which differ on a set of measure zero implies that $||\pi|| = 0$ if and only if $\pi = 0$;

(b) $||\pi|| = \text{ess sup}_{t \in \Omega} |\pi(t)| = \text{ess sup}_{t \in \Omega} |\pi(t)| = ||\pi||$, if $||\pi|| = 1$;

(c) we have

$$||\pi|| + ||\eta|| = \inf_{E \subseteq \Omega \cap \pi(E) \leq \epsilon} \sup_{E \subseteq \Omega \cap \pi(E) \leq \epsilon} |\pi(t)| + \sup_{E \subseteq \Omega \cap \eta(E) \leq \epsilon} |\eta(t)| = \inf_{E \subseteq \Omega \cap \eta(E) \leq \epsilon} \sup_{E \subseteq \Omega \cap \eta(E) \leq \epsilon} |\pi(t)| + \sup_{E \subseteq \Omega \cap \eta(E) \leq \epsilon} |\eta(t)| = ||\pi|| + ||\eta||.$$

Hence the space $M(\Omega, \Sigma, \mu)$ is a linear metric space.

Example 3.4. Let $\Omega$ be the interval $[0, 1]$, $\mu$ the Lebesgue measure, and $\Sigma$ the field of Lebesgue measurable sets. Then $M(\Omega, \Sigma, \mu)$ is the space of all measurable, essentially bounded functions defined on the interval $[0, 1]$. We denote this space by $M$.

Example 3.5. Let $\Omega$ be the set of natural numbers, $\Sigma$ the algebra of all subsets of the set $\Omega$, and $\mu$ a measure equal to one at each point of $\Omega$. Then $M(\Omega, \Sigma, \mu)$ is the space of all bounded sequences. We denote this space by $m$.

Example 3.5. Let $\Omega$ be a compact set. We denote by $C(\Omega)$ the set of functions $\pi(t)$ defined and continuous on the set $\Omega$ with the norm

$$||\pi|| = \sup_{t \in \Omega} |\pi(t)|.$$

Evidently, $C(\Omega)$ is a linear space, since a linear combination of continuous functions is a continuous function. Moreover,

(a) $||\pi|| = 0$ if and only if $\pi = 0$;

(b) $||\pi|| = \sup_{t \in \Omega} |\pi(t)|$ if $||\pi|| = 1$;

(c) $||\pi + \eta|| = \sup_{t \in \Omega} |\pi(t) + \eta(t)|$.

Hence the space $C(\Omega)$ is a linear metric space.

Example 3.5. Let $\Omega$ be the closed interval $[0, 1]$. $C(\Omega)$ is the space of continuous functions on the interval $[0, 1]$.

Example 3.5. Let $\Omega$ be the sequence of points $1, 1/2, 1/3, \ldots$, together with the point 0. Then $C(\Omega)$ is the space of convergent sequences. We denote this space by $c$.

Example 3.5. Let $\Omega$ be a compact set, and let $\Omega_0$ be a closed subset of $\Omega$. We denote by $C(\Omega_0)$ the subset of those functions belonging to $C(\Omega)$ which are equal to zero on the set $\Omega_0$ with the same norm as in the space $C(\Omega)$. Then $C(\Omega) \cap C(\Omega_0)$ is a linear metric space.

Example 3.5. Let $\Omega$ be the sequence $(1/n)$ together with the point 0. Let $\Omega_0 = \{0\}$. Then $C(\Omega) \cap C(\Omega_0)$ is the space of sequences convergent to zero. We denote this space by $c_{00}$.

Example 3.6. Let a set $\Omega$ be the union of an increasing sequence of a countable number of compact sets $\Omega_i$:

$$\Omega = \bigcup_{i=1}^{\infty} \Omega_i.$$

We denote by $C_0(\Omega)$ the space of all continuous functions on the set $\Omega$. We define $C_0(\Omega)$

$$||\pi|| = \sum_{i=1}^{\infty} \frac{1}{2^i} \sup_{t \in \Omega_i} |\pi(t)|,$$

where $||\pi|| = \sup_{t \in \Omega} |\pi(t)|$.

Considering the fact that $||\pi||$ is a norm in the space $C(\Omega)$ (Example 3.6), arguments similar to those used in Example 3.1 show that $||\pi||$ is a norm in the space $C_0(\Omega)$. Hence $C_0(\Omega)$ is a linear metric space.

Example 3.7. If $\Omega_i = \{1, 2, \ldots, i\}$, then $C_0(\Omega)$ is the space of all sequences and is denoted by $(1)$. 

§ 3. Examples of linear metric spaces

Example 3.3. Let $\Omega$ be a compact set. We denote by $C(\Omega)$ the set of functions $\pi(t)$ defined and continuous on the set $\Omega$ with the norm

$$||\pi|| = \sup_{t \in \Omega} |\pi(t)|.$$

Evidently, $C(\Omega)$ is a linear space, since a linear combination of continuous functions is a continuous function. Moreover,

(a) $||\pi|| = 0$ if and only if $\pi(t) = 0$;

(b) $||\pi|| = \sup_{t \in \Omega} |\pi(t)| = ||\pi||$ if $||\pi|| = 1$;

(c) $||\pi + \eta|| = \sup_{t \in \Omega} |\pi(t) + \eta(t)|$.

Hence the space $C(\Omega)$ is a linear metric space.

Example 3.5. Let $\Omega$ be the closed interval $[0, 1]$. $C(\Omega)$ is the space of continuous functions on the interval $[0, 1]$.
EXAMPLE 3.8. Let \( \Omega \) be a closed bounded domain in an \( n \)-dimensional Euclidean space. We denote by \( C^m(\Omega) \) the set of all functions infinitely differentiable on the set \( \Omega \). If \( k_1, \ldots, k_n \) are positive integers, we write
\[
k = (k_1, k_2, \ldots, k_n), \quad |k| = k_1 + k_2 + \ldots + k_n.
\]
The vector \( k \) is called a multi-index. We define
\[
|a| = \sum_{k_1, \ldots, k_n = 0}^{\infty} \frac{1}{1 + |a|^k},
\]
where
\[
|a| = \sup_{t \in \Omega} \sum_{k_1, \ldots, k_n = 0}^{\infty} \frac{1}{1 + |a|^k} |a(t)|.
\]
Applying the subadditivity of \( |a| \) and arguing as in Example 3.1 we easily verify that \( \|
\| \) is a norm. Hence \( C(\Omega) \) is a linear metric space.

EXAMPLE 3.9. We denote by \( \mathcal{B}(E^n) \) the space of all functions infinitely differentiable on the \( n \)-dimensional Euclidean space \( E^n \) and such that
\[
|a|_{m,b} = \sup_{t \in E^n} \sum_{m_1 + \ldots + m_n = m} \frac{1}{1 + |a|^b} < +\infty
\]
for arbitrary multiindices \( m = (m_1, \ldots, m_n) \) and \( k = (k_1, \ldots, k_n) \).

It is easily verified that \( |a|_{m,b} \) is symmetric and subadditive function. Moreover, \( |a|_{m,b} = 0 \) implies \( \|a(t)\| = 0 \). Hence arguments analogous to those applied in Example 3.1 show that
\[
|a| = \sum_{m_1 + \ldots + m_n + b = 0}^{\infty} \frac{1}{1 + |a|^b} |a(t)| < +\infty
\]
is a norm. Consequently, \( \mathcal{B}(E^n) \) is a linear metric space.

EXAMPLE 3.10. Let a topological space \( \Omega \) be given, and let \( \mathcal{B} \) denote the set of all Borel subsets of \( \Omega \). Evidently, \( \mathcal{B} \) is a countably additive algebra. A countably additive measure \( \mu \) (complex-valued or real-valued) is called regular if for every set \( E \in \mathcal{B} \) and for every number \( \varepsilon > 0 \) there exist a set \( F \) whose closure is contained in \( E \) and open set \( G \) such that \( E \) is contained in \( G \), satisfying the inequality
\[
\mu(F) < \varepsilon
\]
for every set \( G \subset F \), \( G \in \mathcal{B} \).

We denote by \( \text{rc} \mathcal{B} \mathcal{O} \) the set of all regular measures \( \mu \) such that
\[
|\mu| = \operatorname{var} \mu = \sup_{G \in \mathcal{O}} \left\{ \sum_{i=1}^{n} |\mu(G_i)| : G_1, \ldots, G_n \in \mathcal{O}; G_i \cap G_j = 0, \quad i \neq j \right\} < +\infty,
\]
with the norm \( |\mu| \). Evidently,
\[
|\mu| = 0 \quad \text{if and only if} \quad \mu(E) = 0 \quad \text{for all sets} \quad E \in \mathcal{B};
\]
\[
|\mu| = \operatorname{var} \mu = \operatorname{var} \mu = |\mu|, \quad \text{if} \quad |\mu| = 1;
\]
\[
|\mu + \nu| = \operatorname{var} \mu + \operatorname{var} \nu = \operatorname{var} \mu + \operatorname{var} \nu = |\mu + \nu|,
\]
and we have
\[
|\mu + \nu| = \operatorname{var} \mu + \operatorname{var} \nu = \operatorname{var} \mu + \operatorname{var} \nu = |\mu + \nu|,
\]
\[
\begin{align*}
&|\mu + \nu| = \operatorname{var} \mu + \operatorname{var} \nu = \operatorname{var} \mu + \operatorname{var} \nu = |\mu + \nu|,
&\leq \sup_{G \in \mathcal{O}} \sum_{i=1}^{n} |\mu(G_i)| + \sum_{i=1}^{n} |\nu(G_i)|,
&\leq \sup_{G \in \mathcal{O}} \sum_{i=1}^{n} |\mu(A_i)| + \sup_{G \in \mathcal{O}} \sum_{i=1}^{n} |\nu(B_i)| = |\mu| + |\nu|,
\end{align*}
\]
where \( G_i, A_i, B_i \in \mathcal{O} \) and
\[
A_i \cap A_j = 0 \quad \text{for} \quad i \neq j, \quad G_i \cap G_j = 0.
\]
Hence \( \text{rc} \mathcal{B} \mathcal{O} \) is a linear metric space.

§ 3. Examples of linear metric spaces
(c) We have
\[
\|x + y\| = \sup_{t \in A} |x(t) + y(t)| = \sup_{t_1, t_2 \in \Omega} \left| \frac{[x(t_1) + y(t_2)] - [x(t_1) + y(t_1)]}{t(t_1, t_2)} \right|
\]
\[
\leq \sup_{t \in A} |x(t)| + \sup_{t \in A} |y(t)| + \sup_{t_1, t_2 \in \Omega} \left| \frac{[x(t_1) - x(t)]}{t'} \right| + \sup_{t_1, t_2 \in \Omega} \left| \frac{y(t') - y(t'')}{t'} \right|
\]
\[
= |x| + |y|.
\]
Hence the space \( \mathbb{H}(\mathcal{G}) \) is a linear metric space.

**Example 3.12.** We say that a scalar product (inner product) is defined in a linear space \( \mathcal{X} \) if there exists a function defined for all pairs \((x, y)\), where \( x, y \in \mathcal{X} \), with values in a field of scalars, such that

1. \( (x_1 + x_2, y) = (x_1, y) + (x_2, y) \),
2. \( (\alpha x, y) = \alpha (x, y) \) (where \( \alpha \) is the complex number conjugate to \( \alpha \)),
3. \( (x, y) = \overline{(y, x)} \),
4. \( |(x, x)| > 0 \) for \( x \neq 0 \).

A linear space with a scalar product is called a *pre-Hilbert space*. A pre-Hilbert space is a linear metric space if we define the norm in the following manner:

\[
|x| = \sqrt{(x, x)}. \]

Condition (1) implies \( |0| = \sqrt{(0, 0)} = 0 \). Condition (4) implies \( |x| > 0 \) for \( x \neq 0 \).

In order to prove the triangle inequality, we first prove the following

**Schwarz inequality:**

\[
|(x, y)| \leq |x| \cdot |y|.
\]

Indeed, we have for an arbitrary number \( \alpha \)

\[
0 \leq (x + \alpha y, x + \alpha y) = (x, x) + \alpha (x, y) + \overline{\alpha} (y, x) + \alpha \overline{\alpha} (y, y) = |x|^2 + \alpha (x, y) + \alpha \overline{\alpha} |y|^2.
\]

Hence the discriminant of the last trinomial satisfies the inequality

\[
\frac{(x, y)^2}{4} - |x|^2 \cdot |y|^2 \leq 0.
\]

Thus

\[
|\frac{x}{x} + \frac{y}{y}|^2 \leq |\frac{x}{x}| \cdot |\frac{y}{y}|.
\]

But there exists a number \( \beta, |\beta| = 1 \) such that the product \((x, y)\) is a real number. Let \( y = \beta y \); then

\[
|(x, y)| = \beta \cdot |(x, y)| = \frac{(x, y) + (\beta y, x)}{2} \leq |x| \cdot |y| = |x| \cdot |y|.
\]

Now we prove the triangle inequality. We obtain

\[
|x + y|^2 = |(x + y, x + y)| = |(x, x) + (y, y) + (x, y) + (y, x)|
\]

\[
\leq |x|^2 + |y|^2 + 2 |x| \cdot |y| = |(x, y)|^2,
\]

which was to be proved.

The space \( \mathcal{D}(\mathcal{G}, \Sigma, \mu) \) can be considered as a pre-Hilbert space if we define the scalar product by the formula

\[
(x, y) = \int_0^1 x(t) y(t) d\mu(t).
\]

**§ 4. Complete linear topological spaces.** Let a linear topological space \( \mathcal{X} \) be given. A fundamental family is a non-void family \( \mathcal{F} \) of sets such that for any two sets \( M, N \in \mathcal{F} \) there exists a set \( E \in \mathcal{F} \), \( E \subset M \cap N \), and for every neighbourhood of zero \( U \) there exists a set \( M \in \mathcal{F} \), \( M \subset U \cap U \).

A fundamental family may have at most one cluster point. Indeed, let us suppose that \( x \) and \( y \) are cluster points of a fundamental family \( \mathcal{F} \). Let \( U \) be an arbitrary neighbourhood of \( x \). From the assumption that the family \( \mathcal{F} \) is fundamental we infer the existence of a set \( M \in \mathcal{F} \) such that \( M \subset U \). On the other hand, since \( x \) and \( y \) are cluster points of the family \( \mathcal{F} \), there exist points \( x_1, y_1 \in M \) such that \( x - x_1, y - y_1 \in U \).

Hence

\[
x - y = (x - x_1) + (y_1 - y) + (y_1 - y) \in U + U + U.
\]

Since the neighbourhood \( U \) is arbitrary, it follows that \( x = y \).

A subset \( E \) of a linear topological space \( \mathcal{X} \) (in particular the space \( \mathcal{X} \) itself) is called a complete set if every fundamental family \( \mathcal{F} \) of subsets of the set \( E \) possesses a cluster point belonging to the set \( E \).

**Theorem 4.1.** A subset \( E \) of a complete linear topological space \( \mathcal{X} \) is complete if and only if it is closed.

**Proof.** Let \( \mathcal{F} \) be an arbitrary fundamental family of subsets of the set \( E \). Since the space \( \mathcal{X} \) is complete, the family \( \mathcal{F} \) possesses a cluster point \( x \), i.e. for every neighbourhood \( U_x \) of the point \( x \) and for every \( \forall \in \mathcal{F} \) such that \( \forall \cap E \neq \emptyset \) we have \( \forall \cap U_x \neq \emptyset \). Hence \( U_x \cap E \neq \emptyset \), and this proves that \( x \in E \).

On the other hand, if \( x \) is a point belonging to the closure of the set \( E \), and if \( \mathcal{F} \) is a fundamental family with a cluster point \( x \), then the
family \( \mathcal{B} = \{ U + V : U \in \mathcal{A}, V \in \mathcal{B} \} \).

Evidently, this is a fundamental family of subsets of the set \( E \) with cluster point \( x \). The completeness of the set \( E \) implies \( x \in E \). Hence the set \( E \) is closed. 

Not every linear topological space is complete. But

**Theorem 4.2.** If \( X \) is a linear topological space, there exists a complete linear topological space \( \tilde{X} \) such that \( \tilde{X} \) is a dense subset of \( X \) and the topology induced in \( X \) by the space \( \tilde{X} \) is equivalent to the topology given in \( X \).

Proof: We define points of the space \( \tilde{X} \) as fundamental families in the space \( X \). Addition of fundamental families is defined as follows:

\[
\{ \mathcal{A} + \mathcal{B} = \{ U + V : U \in \mathcal{A}, V \in \mathcal{B} \} \}.
\]

It follows at once from the continuity of addition that the family \( \mathcal{A} + \mathcal{B} \) is a fundamental family. Multiplication by a scalar is defined similarly.

We say that two fundamental families \( \mathcal{A} \) and \( \mathcal{B} \) belong to the same class if \( 0 \) is the cluster point of the family \( \mathcal{A} \subset \mathcal{B} \). We denote by \( s \) the class of fundamental families with cluster point \( x \). Evidently, the set

\[
\tilde{X} = \{ \tilde{x} : \tilde{x} \in X \}
\]

is a linear space. With each point \( x \in X \) we associate the class \( \tilde{x} \); in this sense, \( X \subset \tilde{X} \). Topology in the space \( \tilde{X} \) can be introduced by means of closed sets. We call a set \( A \subset \tilde{X} \) closed if

(i) the set \( A \subset \tilde{X} \) is closed in the space \( \tilde{X} \).

(ii) every fundamental family \( \mathcal{A} \) made of subsets of a set \( A \subset \tilde{X} \) determines a point belonging to the set \( A \).

It is easily verified that the space \( \tilde{X} \) with topology determined by means of the closed sets defined above satisfies the theorem.

The space \( \tilde{X} \) satisfying Theorem 4.2 is called the completion of the space \( X \).

§ 5. Complete linear metric spaces. We say that a sequence \( (a_n) \) of elements of a metric space \( X \) is a fundamental sequence or a Cauchy sequence if for every \( \varepsilon > 0 \) there exists a number \( N \) such that \( d(a_n, a_m) < \varepsilon \) for \( n, m > N \).

**Theorem 5.1.** If a subsequence \( (a_{n_k}) \) of a fundamental sequence \( (a_n) \) is convergent to a point \( x \), then the sequence \( (a_n) \) is convergent to \( x \).

Proof. Let \( a_{n_k} \to x \), and let \( \varepsilon \) be an arbitrary positive number. There exists an index \( k \) such that \( d(a_k, a') \leq \varepsilon /2 \) for \( k > k \). On the other hand, since the sequence \( (a_n) \) is fundamental, there exists a number \( N \) such that \( d(a_n, a_m) < \varepsilon /2 \) for \( m > N \). Let \( m = n \) for \( k > k \). Then

\[
d(a_n, x) \leq d(a_n, a_m) + d(a_m, x) < \varepsilon /2 + \varepsilon /2 = \varepsilon.
\]

A metric space \( X \) is called complete if every fundamental sequence has a limit.

**Theorem 5.2.** (Baire) A complete metric space is of the second category.

Proof. Let us suppose that \( X \) is of the first category. Then \( X = \bigcup_{n=1}^{\infty} F_n \), where the sets \( F_n \) are nowhere dense. One can suppose without loss of generality that the sets \( F_n \) are closed. Since the set \( F_1 \) is nowhere dense, there exists a ball \( K_1 \) of radius not greater than 1 such that \( K_1 \cap F_1 = 0 \). Again the set \( F_2 \) is nowhere dense and hence there exists a ball \( K_2 \) of radius not greater than \( 1/2 \), \( K_2 \subset K_1 \), such that \( K_2 \cap F_2 = 0 \). In this manner we define by induction a sequence \( (K_n) \) of balls such that \( K_1 \cap K_n = 0 \), the radius of the ball \( K_n \), \( r(K_n) < 1/n \), and \( K_1 \cap F_n = 0 \).

Let us consider the intersection \( \bigcap_{n=1}^{\infty} K_n \). It is non-empty. Indeed, taking any sequence \( (a_n) \) such that \( a_n \in K_n \), we have \( d(a_n, a_m) < 1/n \) for \( m > n \). Hence the sequence \( (a_n) \) is fundamental. But \( K_1 \cap K_n \subset K_1 \). Thus the limit \( x \) of this sequence belongs to \( K_n \) for \( n = 1, 2, ... \). Consequently \( \bigcap_{n=1}^{\infty} K_n \neq 0 \).

Now, we have \( K_1 \cap F_n = 0 \), and so \( \bigcap_{n=1}^{\infty} F_n = 0 \) for \( m = 1, 2, ... \), contradicting the assumption \( X = \bigcup_{n=1}^{\infty} F_n \).

**Corollary 5.3.** The complement \( X = \bigcup_{n=1}^{\infty} F_n \) of a set \( B \) in one of the first category in a complete metric space is a set of the second category.

Proof. \( X = E \cup \Omega \). The set \( E \) is of the first category. If we assumed the set \( \Omega \) to be also of the first category, the space \( X \) would be of the first category, as the union of two sets of the first category.

A linear metric space is called complete if it is complete as a metric space.

If a sequence \( (a_n) \) is fundamental, then the family \( \mathcal{A} \) of sets \( U_n = \{ a_n, a_{n+1}, ... \} \) is fundamental. On the other hand, if a family \( \mathcal{A} \) is fundamental, then there exists a sequence of neighborhoods \( \{ U_n \} \subset \mathcal{A} \) such that

\[
\sup_{a_n, a_m \in U_n} d(a_n, a_m) > 1/n.
\]

If \( (a_n) \) is an arbitrary sequence satisfying the condition \( a_n \in \bigcap_{n=1}^{k} U_n \), then \( (a_n) \) is a fundamental sequence. Hence the definition of the completeness
of a linear metric space given above is the same as the definition of the completeness of linear topological spaces given in the preceding section.

**Theorem 5.4.** (Klee [1]). If \( X \) is a complete linear metric space with metric \( \rho(x, y) \), and if \( \rho'(x, y) \) is an invariant metric equivalent to the metric \( \rho(x, y) \), then the space \( X \) with metric \( \rho'(x, y) \) is also complete.

The proof of Theorem 5.4 is based on the following lemmas:

**Lemma 5.5.** (Sierpiński [1]). Let \( E \) be a complete linear metric space with metric \( \rho'(x, y) \). Suppose that the space \( E \) is embedded in a complete metric space \( E' \) with a metric \( \rho'(x, y) \) in such a manner that the embedding is continuous in both directions, i.e., \( \rho'(x_0, x) \to 0 \), if and only if \( \rho'(x_0, x) \to 0 \) (here the element \( x \in E \) is identified with its image in the space \( E' \)). Under these assumptions the space \( E \), considered as a subset of the space \( E' \), is a \( G_2 \)-set (see § 1).

**Proof.** By hypothesis, if \( x \in E \) is any given element, there exists a positive number \( r_0(x) < 1/n \) such that \( y \in E \) and \( \rho'(x, y) < r_0(x) \) imply \( \rho(x, y) < 1/n \). Let

\[
U_a(x) = \{ y \in E' : \rho'(x, y) < r_0(x) \} \quad \text{and} \quad G_n = \bigcup_{a \in E} U_a(x),
\]

\[
G_n = \bigcup_{n=1}^{\infty} G_n.
\]

From the definition of the sets \( U_a(x) \) it follows that they are open. Hence the sets \( G_n \) are open. Thus, \( G_n \) is an intersection of a countable number of open sets. Evidently, \( E \subseteq G_n \). It remains to show that \( E \subseteq G_n \). Let \( a_n \in E \); then \( a_n \in G_k \) for \( n = 1, 2, \ldots \). By definition, there exist elements \( a_n \in E \) such that \( \rho'(a_n, a_0) < r_0(a_n) \). It follows from the definition of the number \( r_0(a_n) \) that \( \rho'(a_n, a_0) < 1/n \). Hence the sequence \( \{a_n\} \) is convergent to the element \( a_0 \) in the sense of the metric \( \rho' \).

Let \( e \) be an arbitrary positive number, and let \( n \) be a natural number satisfying the inequality \( 2/n < e \). Finally, let \( k_0 \) be a natural number such that

\[
1/k_0 < r_0(a_0) \quad \text{and} \quad \rho'(a_k, a_0) = \rho'(a_k, a_0) \quad \text{for} \quad k > k_0.
\]

Hence it follows that \( \rho(a_k, a_0) < 1/n \), by the definition of the number \( r_0(a_n) \). This proves the sequence \( \{a_n\} \) is fundamental in the metric \( \rho' \). Thus, the completeness of the space \( E \) implies that \( a_0 \in E \). □

**Lemma 5.6.** (Mazur, Sternbach [1]). If \( X \) is a complete linear metric space with an invariant metric, and if \( X_0 \) is a linear subspace of the space \( X \), dense in \( X \) and such that \( X_0 \) is a \( G_2 \)-set, then \( X_0 = X \).

**Proof.** By hypothesis, \( X_0 = \bigcap_{n=1}^{\infty} G_n \), where each of the sets \( G_n \) is open and dense in \( X \). Hence the set \( X \setminus X_0 \) is nowhere dense and the set \( X \setminus X_0 \) is of the first category. Thus, \( X_0 \) is a set of the second category. Let us suppose that the set \( X \setminus X_0 \) is non-empty, i.e., there exists an element \( x \in X \setminus X_0 \). Since the metric is invariant, the cost \( y + X_0 \) is of the second category. But \( y + X_0 \subseteq X \setminus X_0 \), and the last set is of the first category, which gives a contradiction. Hence the set \( X \setminus X_0 \) is void. □

**Proof of Theorem 5.4.** Let us denote by \( Y \) the completion of the space \( X \) in the metric \( \rho'(x, y) \). By Lemma 5.5, the set \( X \) is the union of a countable number of open sets in the metric \( \rho'(x, y) \). Hence \( X = Y \), by Lemma 5.6. □

A consequence of Theorem 5.4 is the following useful test for the completeness of the space \( X \). We say that a series \( \sum_{n=1}^{\infty} a_n \) convergent to a point \( x \) if the sequence \( \{a_n\} = \{n x_n\} \) is convergent to the point \( x \).

**Theorem 5.7.** A linear metric space \( X \) is complete if, for every convergent series of positive numbers \( \sum_{n=1}^{\infty} a_n \), any series \( \sum_{n=1}^{\infty} e_n \) satisfying the inequalities \( |e_n| < c \) is convergent.

**Proof.** Let \( \{g_n\} \) be an arbitrary fundamental sequence. One can extract a subsequence \( \{g_{k_n}\} \) such that

\[
||g_{k_{n+1}} - g_{k_n}|| < c_n \quad (k = 1, 2, \ldots).
\]

Hence the series \( \sum_{k=1}^{\infty} a_k \), where \( a_k = g_{k_{n+1}} - g_{k_n} \), is convergent. Let us denote its sum by \( a \). In other words, the sequence \( \{g_n\} \) is convergent to the point \( a \). We show that \( g_n \to a \). Let \( \epsilon \) be an arbitrary positive number. There exists a number \( N \) such that \( ||g_n - g_m|| < \epsilon/2 \) for \( n, m > N \). Then

\[
||g_n - a|| \leq ||g_n - g_m|| + ||g_m - a|| < c \quad \text{for} \quad n > N.
\]

**Theorem 5.8.** If \( X \) is a complete linear metric space and if \( X_0 \) is a closed subspace of \( X \), then the quotient space \( X/X_0 \) is complete.

**Proof.** By Theorem 5.4, one can assume the space \( X \) to be metrizable in a complete manner by means of an invariant metric \( \rho(x, y) \) defined by a norm \( ||.|| \).

Let \( \{a_n\} \subset X/X_0 \) be an arbitrary sequence satisfying the inequalities \( ||a_n|| < 1/2^n \). By the definition of the norm in the quotient space, there exist elements \( x_n \in [a_n] \) such that \( ||x_n|| < 1/2^{n-1} \). But the space \( X \) is complete.
According to Theorem 5.7, the series \( \sum_{n=1}^{\infty} a_n \) is convergent and has a sum \( a \). The definition of the norm in the quotient space gives
\[
\left| \sum_{k=1}^{\infty} [a]_k - [a] \right| < \left| \sum_{n=1}^{\infty} a_n - a \right| \quad (k = 1, 2, \ldots).
\]
Hence the series \( \sum_{n=1}^{\infty} [a]_n \) is convergent to the element \([a]\). By Theorem 5.7, the completeness of the space \( X/X_2 \) follows. ■


**Example 6.1.** Spaces \( S(\Omega, \Sigma, \mu) \) and \( LP(\Omega, \Sigma, \mu) \) are complete.

Let us take a sequence \( (a_n) \subset S(\Omega, \Sigma, \mu) \) (resp. \( (a_n) \subset LP(\Omega, \Sigma, \mu) \)) such that \( \|a_n\| < 1/4^n \). Let
\[
A_n = \{ t : |a_n(t)| > 1/2^{n-1} \}. \quad (\text{resp. } A_n = \{ t : |a_n(t)| > (1/2^n)^{(n)} \}).
\]
Evidently, \( \|a_n\| < 1/4^n \) implies \( \mu(A_n) < 1/2^n \).

Let \( B_n = \bigcap_{k=1}^{n} A_k \). We have \( |a_n(t)| < 1/2^{k-1} \) (resp. \( |a_n(t)| < (1/2^n)^{(n)} \)) in the complement of the set \( B_n \). Hence the sum of the series \( \sum_{n=1}^{\infty} a_n(t) \) exists in the complement of the set \( B = \bigcap_{n=1}^{\infty} B_n \). Moreover, this series is uniformly convergent on each of the sets \( \Omega \setminus B_n \). Let us denote the sum of the series \( \sum_{n=1}^{\infty} a_n(t) \) by \( a(t) \). The function \( a(t) \) is measurable on the set \( \Omega \setminus B_n \). Moreover, let us remark that
\[
\mu(B_n) \leq \sum_{n=1}^{\infty} \mu(A_n) \leq 1/2^{k-1}.
\]
Hence \( \mu(B) = 0 \), and the function \( a(t) \) is measurable on the whole set \( \Omega \) and determined uniquely with the exception of a set of measure \( \mu \) equal to 0.

Since the series \( \sum_{n=1}^{\infty} a_n(t) \) is uniformly convergent on sets \( \Omega \setminus B_n \), the function
\[
\sum_{n=1}^{\infty} a_n(t) = a(t) \quad \text{for } t \notin B_n.
\]
belongs to the space \( S(\Omega \setminus B_n, \Sigma, \mu) \) (resp. \( LP(\Omega \setminus B_n, \Sigma, \mu) \)), and the sequence \( \{ \sum_{n=1}^{\infty} a_n - a(t) \} \) tends to zero in the respective norm, where \( k \) is arbitrary.

Hence it follows that \( a \in S(\Omega, \Sigma, \mu) \) (resp. \( a \in LP(\Omega, \Sigma, \mu) \)), and the series \( \sum_{n=1}^{\infty} a_n \) is convergent to the function \( a \).

Thus, by Theorem 5.7, the space \( S(\Omega, \Sigma, \mu) \) (resp. \( LP(\Omega, \Sigma, \mu) \)) is complete.

A complete pre-Hilbert space is called a **Hilbert space**. Hence spaces \( LP(\Omega, \Sigma, \mu) \) are Hilbert spaces.

**Example 6.2.** The space \( M(\Omega, \Sigma, \mu) \) is complete.

Indeed, let \( \sum a_n(t) \) be a series satisfying the condition \( \sum |a_n(t)| < \infty \).

Given any natural number \( n \), there exists a set \( A_n \) such that \( \mu(A_n) = 0 \) \( \mu(A_n) \geq |a_n(t)| \) for \( t \notin A_n \). Let us consider the series \( \sum a_n \) on the set \( \Omega \setminus A \), where \( A = \bigcup_{n=1}^{\infty} A_n \). This series is uniformly convergent. Hence it has a bounded measurable function \( a(t) \) as the sum. Moreover,
\[
\lim_{n \to \infty} \sup_{t \in \Omega \setminus A} |a_n(t) - a(t)| = 0.
\]
Let
\[
\alpha = \begin{cases} a_n(t) & \text{for } t \notin A, \\ 0 & \text{for } t \in A. \end{cases}
\]
Since \( \mu(A) = 0 \), the series \( \sum a_n(t) \) is convergent to the function \( a(t) \) in the norm.

Hence, by Theorem 5.7, the space \( M(\Omega, \Sigma, \mu) \) is complete.

**Example 6.3.** \( C(\Omega) \) is a complete space.

Indeed, let \( (a_n(t)) \) be a fundamental sequence. This sequence is convergent at every point. Hence it converges to a function \( a(t) \). The function \( a(t) \) is continuous as the limit of a uniformly convergent sequence of continuous functions.

Let \( \varepsilon \) be an arbitrary positive number. Since \( (a_n(t)) \) is a fundamental sequence, there exists an index \( k \) such that \( |a_k(t) - a(t)| \leq \varepsilon \) for \( k' > k \).

This means that \( |a_k(t) - a(t)| \leq \varepsilon \) for every \( t \). Taking \( k' \to \infty \) we obtain \( |a_k(t) - a(t)| \leq \varepsilon \) for an arbitrary \( t \). Hence \( |a_n - a| \leq \varepsilon \), which was to be proved.

**Example 6.4.** The space \( C(\Omega | \Omega_2) \) is complete.

Indeed, \( C(\Omega | \Omega_2) \) is a closed subspace of the space \( C(\Omega) \), since if \( a_n(t) \to a(t) \) in \( C(\Omega) \), then \( a(t) = 0 \) for \( t \in \Omega_2 \).

**Example 6.5.** The space \( C_0(\Omega) \) is complete.

Indeed, let \( (a_n(t)) \subset C_0(\Omega) \) be a fundamental sequence, i.e.
\[
\lim_{n \to \infty} |a_n - a| = 0.
\]
§ 6. Completeness of some linear metric spaces

By the definition of the norm in the space $C_0(\Omega)$, this implies

$$\lim_{n \to \infty} ||s_n - s_m|| = 0 \quad (i = 1, 2, \ldots),$$

where $||s||$ is the norm in the space $C(\Omega)$. Thus, according to Example 6.4, the sequence $\{s_n(t)\}$ is uniformly convergent on each set $\Omega_i$ to a function $x(t)$ continuous on the set $\Omega_i$. But $\Omega = \bigcup_{i=1}^{\omega} \Omega_i$; hence the function $x(t)$ is continuous on the set $\Omega$ and belongs to the space $C_0(\Omega)$. It is easily verified that the definition of the norm $||s||$ implies

$$\lim_{n \to \infty} ||s_n - s_m|| = 0.$$ 

This proves the completeness of the space $C_0(\Omega)$.

**Example 6.6.** The space $C^\infty(\Omega)$ is complete.

Indeed, let $\{s_n\}$ be a fundamental sequence in the space $C^\infty(\Omega)$, i.e.

$$\lim_{n \to \infty} ||s_n - s_m|| = 0.$$ 

By the definition of the norm, this implies

$$\lim_{n \to \infty} ||s_n - s_m||_k = 0 \quad (k = 0, 1, 2, \ldots).$$

Applying the fact that this equality holds for $k = 0$, we conclude that the sequence $\{s_n(t)\}$ is uniformly convergent to a continuous function $x(t)$ (Example 6.4). In a similar manner we verify that for an arbitrary multindex $k = (k_1, \ldots, k_\lambda)$ the sequence of derivatives $\frac{\partial^k s_n}{\partial t_1^{k_1} \cdots \partial t_\lambda^{k_\lambda}}$ is uniformly convergent on the set $\Omega$ and its limit is equal to the respective derivative of the function $x(t)$ by a well-known theorem of the calculus.

Hence it follows at once from the definition of the norm that the space $C^\infty(\Omega)$ is complete.

**Example 6.7.** The space $S(E^n)$ is complete.

Indeed, let $\{s_n\}$ be a fundamental sequence in the space $S(E^n)$, i.e.

$$\lim_{n \to \infty} ||s_n - s_m|| = 0.$$ 

Thus, according to the definition of the norm,

$$\lim_{n \to \infty} ||s_n - s_m||_{k,m} = 0$$

for arbitrary two multindices $k, m$. This equality holds also for $m = (0, 0, \ldots, 0)$. Hence the sequence $\{s_n(t)\}$ is uniformly convergent together with all its derivatives to an infinitely differentiable function $x(t)$ (see Example 6.6). However, ac-
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Thus \( \sigma(t) \) belongs to the space \( H^p(U) \) as the sum of functions \( a_n(t) \) and \( \pi(t) - a_n(t) \). On the other hand, \( |a_n(t)| - \sigma(t) |t| \ll \epsilon \) (see Example 6.3). Hence

\[
||a_n - \sigma|| < 2\epsilon
\]

and the space \( H^p(U) \) is complete.

§ 7. Bounded sets and locally bounded spaces. Let a linear topological space \( X \) be given. We say that a set \( E \subset X \) is bounded if for every neighborhood \( U \) there exists a scalar \( a \neq 0 \) such that \( aE \subset U \). It follows from the continuity of addition that if the sets \( E_1 \) and \( E_2 \) are bounded, then the set \( E_1 + E_2 \) is bounded. Indeed, let \( U \) be an arbitrary neighborhood of \( 0 \). There exists a balanced neighborhood \( V \) such that \( V + V \subset U \).

Since the sets \( E_1 \), \( E_2 \) are bounded, there exist numbers \( a_1 \) and \( a_2 \) satisfying the conditions \( |a_1| \leq 1 \), \( |a_2| \leq 1 \), \( a_1E_1 \subset V \), \( a_2E_2 \subset V \). Hence

\[
a_1a_2(E_1 + E_2) \subset V + V \subset U.
\]

The closure \( \overline{E} \) of a bounded set \( E \) is a bounded set, since \( a\overline{E} \subset \overline{U} \).

If \( X \) is a linear metric space, then a set \( E \) is bounded if and only if \( \Delta_{\overline{a}a} \to 0 \) for every sequence \( \{a_n\} \subset E \) and an arbitrary sequence \( \{a_n\} \to 0 \).

Evidently, it follows from the continuity of multiplication by a scalar that every convergent sequence \( \{a_n\} \) in a linear metric space is a bounded set.

A space \( X \) is called locally bounded if there exists a bounded neighborhood \( V \) of \( 0 \) in \( X \). By the definition of a bounded set, the sequence \( \{\frac{1}{n}V\} \) determines a topology equivalent to the given one. Thus, according to Theorem 2.1, one may construct in \( X \) an invariant metric determining a topology equivalent to the given one.

We say that a norm \( |||| \) (see § 2) is \( p \)-homogeneous, \( 0 < p < 1 \), if

\[
||a|| = ||a^p||. 
\]

A 1-homogeneous norm is called briefly homogeneous. If there exists a \( p \)-homogeneous norm in a space \( X \), a set \( E \subset X \) is bounded if and only if

\[
\sup_{a \in E} ||a|| < M < \infty.
\]

Indeed, let \( \{a_n\} \) be a bounded sequence: \( ||a_n|| \leq M \), and let \( \{a_n\} \) be a sequence of numbers convergent to zero. Then

\[
||a_n a_n|| = ||a_n^p|| |a_n| \to 0,
\]

and so the set \( E \) is bounded. On the other hand, if \( \sup ||a|| = +\infty \), one can choose a sequence \( \{a_n\} \subset E \) such that \( ||a_n|| > n \). Let \( a_n = (1||a_n||)^{1/p} \); then \( a_n \to 0 \), but \( ||a_n a_n|| = 1 \). Hence the set \( E \) is not bounded.

Hence it follows that if there exists a \( p \)-homogeneous norm determining the topology in a linear metric space \( X \), then the space \( X \) is locally bounded. On the other hand, we show that if a space \( X \) is locally bounded, then there exists a \( p \)-homogeneous norm in \( X \) determining a topology equivalent to the given one.

Let us suppose that \( V \) is a bounded neighborhood of \( 0 \), and let \( U = \bigcup_{|s| < \alpha} V \). Evidently, \( U \) is a neighborhood of \( 0 \). We show that \( U \) is a bounded set. Indeed, let \( \{a_n\} \subset U \); then \( a_n = a_n s_n \), where \( s_n \in V \) and \( |s_n| \leq 1 \). If \( a_n \to 0 \), then \( a_n a_n \to 0 \), since \( s_n \to 0 \) and the set \( V \) is bounded. Evidently, \( aU \subset U \) for \( |a| \leq 1 \).

We denote by \( \mathfrak{M} \) the class of bounded open sets such that \( aV \subset V \) for \( |a| \leq 1 \).

Let \( V \in \mathfrak{M} \). We call the number

\[
\zeta(V) = \inf\{\alpha > 0: V + V \subset aV, V \in \mathfrak{M}\}
\]

the modulus of concavity of the set \( V \). \( \zeta(V) \) is a finite number, since \( V + V \) is a bounded set, and hence there exists a number \( \alpha = 1/\zeta \) such that the set \( a(V + V) \) is contained in the open set \( V \). The modulus of concavity of the space \( X \) is the number

\[
\zeta(X) = \inf\{\zeta(V): V \in \mathfrak{M}\}.
\]

THEOREM 7.1. (Aoki [1]). Rolewicz [11]. If \( X \) is a locally bounded space, then for every \( p \) satisfying the inequalities \( 0 < p < p_0 = \log_{1+\alpha} 2 \) there exists a \( p \)-homogeneous norm determining a topology equivalent to the given one.

Proof. Let \( s = 2^{1/p} \). By the definition of the number \( \zeta(X) \) there exists a set \( V \in \mathfrak{M} \) such that

\[
V + V \subset aV.
\]

Let us write \( U(2s) = aV \), where \( s \) is an integer. For every dyadic number \( r = \sum_{i=0}^{\infty} n_i 2^i \), where \( n_i, m_i \) are integers and \( \epsilon_i = 0 \) or \( 1 \), we define (as in Theorem 2.1) a neighborhood \( U(2s) = \bigcup_{i=0}^{\infty} U(t_i) \). Condition (7.1) implies

\[
U(r + 1) \subset U(r) + U(\epsilon t) \cdot
\]

The construction of the neighborhood \( U(\epsilon t) \) implies \( U(r) \in \mathfrak{M} \). Hence

\[
aU \subset U \quad \text{for} \quad |a| \leq 1,
\]

and

\[
U(2r) = U(r).
\]

Let us write \( ||a|| = \inf\{\epsilon: a \in U(\epsilon t)\} \). Considerations analogous to those used in Theorem 2.1 show that this is a norm and that this norm determines a topology equivalent to the given one. Moreover,

\[
||a|| = ||a|| \quad \text{for} \quad |a| = 1
\]
The intersection of an arbitrary number of convex sets \( W = \bigcap W_e \) is a convex set. Indeed, let \( x, y \in W \). Then, \( x, y \in W_e \) for all \( e \). Hence \( ax + by \in W_e \) for \( a, b \geq 0 \), \( a + b = 1 \) and consequently \( ax + by \in W \).

The closure of a convex set \( W \) is a convex set. Indeed, if \( x \in \overline{W} \) and \( y \in \overline{W} \), then arbitrary neighbourhoods \( U_x \) and \( U_y \) of points \( x \) and \( y \), respectively, have common points with the set \( W \). Hence the neighbourhood \( aU_x + bU_y \) of the point \( ax + by \) has common points with the set \( W \). Thus, by the continuity of addition and multiplication by a scalar, \( ax + by \in \overline{W} \).

An algebraic sum \( E + F \) of two convex sets \( E \) and \( F \) is a convex set.

The smallest convex set containing a set \( E \subset X \) is called the convex hull of the set \( E \) and is denoted by \( \text{conv} E \). It is easily verified that

\[
\text{conv} E = \left\{ \sum_{i=1}^{n} a_i x_i : a_i \geq 0, \sum_{i=1}^{n} a_i = 1, x_i \in E \right\}.
\]

If \( E \) is an open set, then the set \( \text{conv} E \) is also open. This follows from the continuity of addition and multiplication by a scalar and from the form of the set \( \text{conv} E \).

The set \( E \) is balanced, then the set \( \text{conv} E \) is also balanced. Indeed, let \( p \in \text{conv} E \). We infer from the form of the set \( \text{conv} E \) that the element \( p \) can be written as

\[
p = \sum_{i=1}^{n} a_i x_i, \quad x_i \in E, \quad a_i \geq 0, \quad \sum_{i=1}^{n} a_i = 1.
\]

Let \( |a| < 1 \); then

\[
ap = \sum_{i=1}^{n} a_i a x_i = \sum_{i=1}^{n} a_i a_i x_i.
\]

But the set \( E \) is balanced. Hence \( ax + by \) and consequently \( ap \in \text{conv} E \).

If a continuous linear functional \( f \) exists in a space \( X \), then there exist convex open sets, for instance the set \( U = \{x : |f(x)| < 1\} \). The set \( U \) is open, as an inverse image of the interval \((-1,1)\) by means of a continuous transformation. Moreover, the set \( U \) is convex, since if \( x, y \in U \), \( a, b \geq 0 \), \( a + b = 1 \), then

\[
|f(ax + by)| < a |f(x)| + b |f(y)| < 1.
\]

On the other hand, let \( X \) be a linear topological space. If there exist convex open sets in \( X \), different from the whole space \( X \), then (as we show below) there exist continuous linear functionals.

Let us suppose \( X \) to be a linear topological space. Let \( U \) be a convex open set different from the whole space \( X \). Since a translation of sets
maps open sets onto open sets and convex sets onto convex sets, we can assume without loss of generality that \( 0 \in U \). Let
\[
\|x\|_U = \inf \left\{ t > 0 : \frac{x}{t} \in U \right\} = \inf \left\{ t > 0 : \frac{x}{t} \in U \right\}.
\]
Evidently,
\[
U = \{ x : \|x\|_U < 1 \} , \quad \overline{U} = \{ x : \|x\|_U \leq 1 \}.
\]
Since the set \( U \) is open, the function \( \|x\|_U \) is continuous at 0. Moreover, for \( t > 0 \) (positive homogeneity) and
\[
\|x + y\|_U \leq \|x\|_U + \|y\|_U \quad (\text{subadditivity}) .
\]
Indeed, \( \frac{x}{\|x\|_U} , \frac{y}{\|y\|_U} \in U \). Hence, by the convexity of the set \( U \),
\[
\frac{\|x + y\|_U}{\|x\|_U + \|y\|_U} \leq \frac{\|x\|_U}{\|x\|_U + \|y\|_U} \frac{x}{\|x\|_U} + \frac{\|y\|_U}{\|x\|_U + \|y\|_U} \frac{y}{\|y\|_U} = \frac{x}{\|x\|_U} + \frac{y}{\|y\|_U} \in U.
\]
Thus
\[
\|x + y\|_U \leq \|x\|_U + \|y\|_U \quad \|x\|_U , \|y\|_U \leq 1 ,
\]
which was to be proved.

If the set \( U \) is balanced, condition (8.1) can be replaced by the following condition:
\[
\|tx\|_U = |t| \cdot \|x\|_U \quad \text{for all scalars } t \quad (\text{homogeneity}).
\]
A non-negative function satisfying conditions (8.2) and (8.3) is called a pseudonorm.

Evidently, if \( f(x) \leq \|x\|_U \), then the functional \( f \) is continuous. Indeed, let \( O \) be an arbitrary neighbourhood of zero in the field of scalars. There exists a positive number \( s \) such that \( O \subset K_s = \{ s : |s| < s \} \). It is easily seen that \( f(U) \subset K_s \subset O \).

**Theorem 8.1.** (Hahn, Banach.) Let \( p \) be a functional defined on a linear space \( X \) over the field of real numbers satisfying the conditions
\[
\begin{align*}
(1) \quad p(x+y) & \leq p(x) + p(y) \quad (\text{subadditivity}), \\
(2) \quad p(tx) & = tp(x) \quad \text{for } t > 0 \quad (\text{positive homogeneity}).
\end{align*}
\]
If \( f_0 \) is a linear functional defined on a subspace \( X_0 \subset X \) and satisfying the inequality
\[
f_0(x) \leq p(x),
\]
then there exists a linear functional \( f \) defined on the whole space \( X \), identical with \( f_0 \) on the subspace \( X_0 \), and such that
\[
f(x) \leq p(x)
\]
on the whole space \( X \).

\[\text{§ 8. Convex sets and continuous linear functionals}\]

Proof. Let \( x_0 \) be an arbitrary element of the space \( X \) not belonging to \( X_0 \). Suppose that \( X_1 = \text{lin}(x_0) + X_0 \), i.e., that every element of \( X_1 \) can be written in the form
\[
x = x_0 + x' \quad (x' \in X_0). \tag{8.6}
\]
If \( x' , x'' \in X_0 \), inequality (8.4) gives
\[
f_0(x') + f_0(x'') = f_0(x' + x'') \leq p([x_0 + x'] + [x_0 + x''])
\]
\[
\leq p(x_0 + x') + p(-x_0 + x'').
\]
Hence
\[
f_0(x') - p(-x_0 + x') = -f(x' + p(x_0 + x')).
\]
Since this inequality holds for arbitrary \( x' , x'' \in X_0 \), we infer
\[
A = \sup \{ f_0(x') - p(-x_0 + x') \} \leq \inf \{ -f(x) + p(x_0 + x') \} = B.
\]
Let \( A \leq t_0 \leq B \). We define a functional \( f \) on the space \( X_1 \) by means of the formula
\[
f(x) = f_0 + f_0(x') \quad (x = x_0 + x' , \quad x' \in X_0).
\]
Evidently, the functional \( f \) is linear, and it is identical with \( f_0 \) on the subspace \( X_0 \). We show that inequality (8.5) is satisfied for all \( x \in X_1 \). Let us suppose that \( f(x) > 0 \) in formula (8.6). Then
\[
f(x) = f_0 + f_0(x') \leq \lambda B + f_0(x') \leq \lambda [-f_0(x'')] + p(x_0) + f_0(x') = f_0(x') + p(x_0 + x'') + f_0(x') = p(x_0).
\]
If \( \lambda < 0 \), the proof follows the same lines, but the inequality \( t_0 \geq A \) must be applied in place of \( t_0 \leq B \).

In the same manner as in the proof of Theorem 0.3, Part B, we represent \( X \) as a direct sum \( X = X_0 \oplus X_1 \), where \( X = \text{lin}(y_0) \) and the elements \( y_0 \) are linearly independent. Let \( X = \text{lin}(y_1 , \ldots , y_n \geq \beta) \). We prove the theorem by applying transfinite induction. If the set of all \( a \) such that \( a \leq \beta \) contains a greatest element, the arguments are the same as those described above. In other cases we have \( X_1 = X_0 \cup \beta \), but the sets \( X_1 = \{ x : \beta \} \) satisfy the theorem, by the induction hypothesis. Hence the theorem is satisfied for the sum \( X_2 \) also. \( \blacksquare \)

**Corollary 8.2.** Let \( X \) be a linear topological space over the field of real numbers, and let \( U \subset X \) be a convex open set. If \( \alpha \in U \), then there exists a continuous linear functional \( f \) such that
\[
f(x) = 1 \quad \text{and} \quad f(x) < 1 \quad \text{for } x \notin U.
\]

Proof. Let \( x' \in U \). The set \( U_x = U - y \) is convex. Since \( x_0 - y \notin U_x \), we have \( [x_0 - y] \geq 1 > 1 \). We define a functional \( f_0 \) on the one-dimensional Equations in linear spaces
space $X_0$ spanned by the element $x_0 - y$ in the following manner:

$$ f(x_0 - y) = f(x_0 - y) \cdot \|x_0 - y\|_{X_0}. $$

Evidently,

$$ f(x) \leq \|x\|_{X_0}, \quad x \in X_0. $$

We can extend this functional to the whole space $X$, leaving the last inequality unchanged. Let $\tilde{f}$ be such an extension of the functional $f$. Then

$$ \tilde{f}(x) < 1 \text{ for } x \in U_0 \quad \text{and} \quad \tilde{f}(x_0 - y) = \|x_0 - y\|_{X_0} > 1. $$

Let $c = 1 + \tilde{f}(y)$. Then

$$ \tilde{f}(x_0) > 1 + \tilde{f}(y) = c, \quad \text{and} \quad \tilde{f}(x - y) < 1 \text{ for } x \in U. $$

Hence

$$ \tilde{f}(x) < 1 + \tilde{f}(y) = c. $$

The functional $f = \frac{1}{c} \tilde{f}$ possesses the required properties.

We shall now consider linear topological spaces over the field of complex numbers.

**Theorem 8.3.** Let $X$ be a linear topological space over the field of complex numbers. If there exists a convex open set $U$ different from the whole space $X$, then there exists a continuous linear functional (with multiplication by complex numbers) different from zero defined on the space $X$.

**Proof.** The space $X$ may also be treated as a linear space over the field of real numbers. By the Hahn–Banach theorem, there exists a real continuous linear functional $f$, i.e. such that $f(x + y) = f(x) + f(y)$ and $f(tx) = tf(x)$ for real scalars $t$.

Let $g(z) = f(z) - f(iz)$. Evidently, the functional $g$ is continuous and additive: $g(x + y) = g(x) + g(y)$. Moreover, $g$ is homogeneous as regards multiplication by real numbers. In order to show $g$ to be homogeneous as regards multiplication by complex numbers it is sufficient to remark that

$$ g(iz) = f(iz) - if(-iz) = if'(z) = ig(z). $$

**Corollary 8.4.** Let $U$ be a convex open set in a linear topological space $X$ over the field of complex numbers. If $x \in U$, then there exists a continuous linear functional $g(x)$ such that

$$ \text{reg}(x) > 1 \quad \text{and} \quad \text{reg}(x) < 1 \quad \text{for } x \in U. $$

**Proof.** It is sufficient to repeat the construction of the functional $f$ from Corollary 8.2. Then we take $g(x) = f(x) - if'(x)$ and we remark that $\text{reg}(x) = f(x)$.

**§ 8. Convex sets and continuous linear functionals**

Remark. If the convex set $U$ in the assumptions of Theorem 8.3 is balanced, then the condition $f(x) \leq \|x\|_U$ implies $|g(x)| < \|x\|_U$.

**§ 9. Locally convex spaces.** A linear topological space $X$ is called locally convex if there exists a family $\mathcal{W}$ of convex sets in $X$ determining a topology in $X$ equivalent to the given topology. In other words, a linear topological space is locally convex if every neighbourhood of zero in the given topology contains a convex neighbourhood of zero.

If $X$ is locally convex, one can introduce a topology in $X$ not only by means of convex neighbourhoods of zero, but also by means of balanced convex neighbourhoods of zero, i.e. such that $aU \subseteq U$ for $|a| \leq 1$.

Indeed, let $W$ be a convex neighbourhood of zero. By Theorem 5.1, there exists a balanced open set $V \subseteq W$. Let $U = \text{conv}V$. $V$ is a convex set, as a convex hull. Moreover, since $W$ is convex, we have $U \subseteq W$.

A subspace of a locally convex space is locally convex.

**Theorem 9.1.** If $X$ is a locally convex space and if $f$ is a continuous linear functional on a subspace $X_0 \subseteq X$, then $f$ can be extended to a continuous linear functional on the whole space $X$.

**Proof.** Since the space $X$ is locally convex, there exists a balanced convex open set $U$ containing zero and such that

$$ U \cap X_0 \subseteq \{x \in X_0 : |f(x)| < 1\}. $$

Hence we have $|f(x)| \leq \|x\|_U$ on the subspace $X_0$. By the Hahn–Banach theorem, $f(x)$ can be extended to a functional $f(x)$ such that $f(x) \leq \|x\|_U$. Evidently, $f(x)$ is a continuous functional.

**Corollary 9.2.** If $X$ is a locally convex space, then for every $x \in X$, $x \neq 0$, there exists a continuous linear functional $f$ such that $f(x) \neq 0$.

**Proof.** Let $x \neq 0$ be a one-dimensional space spanned by the element $x$. Let $f(x) = 1$. The extension $f$ of the functional $f$ satisfies the statement of the Corollary.

**Theorem 9.3.** Let $X$ be a locally convex space, and let $W \subseteq X$ be a convex set. If $x_0 \in X$ and $x_0 \notin W$, then there exists a continuous linear functional $g(x)$ such that

$$ \text{reg}(x_0) > 1 \quad \text{and} \quad \text{reg}(x) < 1 \quad \text{for } x \in W. $$

**Proof.** Since the space $X$ is locally convex, there exists a convex neighbourhood $U$ of zero in $X$, such that $x_0 \notin \text{conv}U$. The set $W + U$ is convex and open. By Corollary 9.2, there exists a functional $g(x)$ satisfying the inequalities $\text{reg}(x_0) > 1$ and $\text{reg}(x) < 1$, for $x \in W + U$, and hence for $x \in W$.

**Corollary 9.4.** A linear space $X$ has two convex topologies $\tau_1$ and $\tau_2$. If the spaces $(X, \tau_1)$ and $(X, \tau_2)$ have the same set of continuous functions, then $\tau_1 = \tau_2$. 

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linear functionals, then a convex set in $X$ is closed in $(X, \tau_2)$ if and only if it is closed in $(X, \tau_1)$.

Proof. Let a convex set $W$ be closed in the space $(X, \tau_2)$ and let $x_0 \notin W$. By Theorem 7.3, there exist a continuous linear functional $g$ on the space $(X, \tau_2)$ and a number $\varepsilon > 0$ such that

$$\sup_{x \in W} \{ g(x) - g(x_0) \} < \varepsilon.$$ 

Since the functional $g$ is continuous also on the space $(X, \tau_1)$, the neighbourhood $[x_0 : g(x_0) - g(x_0) < \varepsilon]$ of the point $x_0$ in the space $(X, \tau_1)$ does not intersect the set $W$. Hence $W$ is closed in the space $(X, \tau_2)$.

Evidently, this Corollary does not imply the equivalence of the topologies $\tau_1$ and $\tau_2$.

If a locally convex space $X$ is a linear metric space, then the topology in $X$ can be determined by means of a countable sequence of pseudonorms (see § 8). Namely, as a family of neighbourhoods of zero we may take, say, the countable family of sets $[x: g(x, 0) < 1/m]$ from each of these sets we may choose a convex and balanced neighbourhood $U_m$.

Next, we may construct a pseudonorm $\|x\| = \|x\|_m$ for each of these neighbourhoods. It is easily verified that $x_m \to x$ if and only if $\lim_{m \to \infty} \|x_m - x\|_m = 0$ for $m = 1, 2, ...$.

Conversely, let us suppose a sequence of pseudonorms $\{\|\cdot\|_m\}$ is given in a linear space $X$ and determines a topology in $X$. In other words, there exists a topology in $X$ such that $\lim_{m \to \infty} \|x_m - x\|_m = 0$ for $m = 1, 2, ...$ if and only if $x_m \to x$. Under this assumption the space $X$ is metrizable. Then a norm can be defined in $X$ by means of the formula

$$\|x\| = \left( \sum_{m=1}^{\infty} \frac{\|x\|_m^2}{2^m} \right)^{1/2}.$$ 

Locally convex linear metric spaces are called briefly $B_2^*$-spaces. If $B_2^*$-space is complete, it is called a $B_2$-space. It follows from this definition that the spaces

$$C_c(\Omega), \quad C_0(\Omega), \quad C_b(\Omega), \quad C_0^\infty(\Omega), \quad S(E^p), \quad H^p(\Omega), \quad rca(\Omega),$$

$$E^p(\Omega, \Sigma, \mu) \quad \text{for} \quad p \geq 1, \quad M(\Omega, \Sigma, \mu)$$

are $B_2$-spaces.

Arguing as in the proof of Theorem 9.1 one can prove the following

**Theorem 9.5.** (Mazur, Orlicz [11].) Let $X$ be a $B_2$-space with a topology determined by a sequence of pseudonorms $\{\|\cdot\|_m\}$. A linear functional $f$ on $X$ is continuous if and only if there exist a pseudonorm $\|\cdot\|_m$ and a positive constant $K_f$ such that

$$|f(x)| \leq K_f \|x\|_m.$$
If a space \( X \) is simultaneously locally convex and locally bounded, then for any natural number \( n \) there exists a positive number \( \delta_n \) such that
\[
\|x\|_n \leq \delta_n \|x\|_i \quad \text{for all } x \in X.
\]
Hence convergence with respect to the pseudonorm \( \| \cdot \|_i \) implies convergence with respect to all pseudonorms. Consequently, \( \| \cdot \|_i \) is a norm determining a topology equivalent to the given one. Let us remark that this pseudonorm is homogeneous, i.e., \( \|ax\|_i = |a| \|x\|_i \) for an arbitrary scalar \( a \).

Spaces with a homogeneous norm are called normed spaces. Normed spaces will be discussed in the next part.

§ 10. \( \mathcal{E} \)-topology and \( \mathcal{E} \)-convergence. Let there be given a linear space \( X \) and a total linear space \( \mathcal{E} \) of linear functionals defined on \( X \). We consider neighbourhoods of the form
\[
U = \{x \in X : \|\xi(x) - \xi(a_i)\| < \varepsilon_i, \xi \in \mathcal{E} \ (i = 1, 2, ..., n)\}.
\]

Neighbourhoods of this type determine a topology. Indeed, let
\[
W = \{x : \|\xi(x) - \xi(a_i)\| < \varepsilon_i, \varepsilon = n + 1, ..., n + m\}.
\]
Let us suppose that \( x_0 \in U \cap W \). This means that \( a_i = \|\xi(x_0) - \xi(a_i)\| < \varepsilon_i \) for \( i = 1, 2, ..., n \) and \( a_i = \|\xi(x_0) - \xi(a_i)\| < \varepsilon_i \) for \( i = n + 1, ..., n + m \).

Let
\[
V = \{x : \|\xi(x) - \xi(a_i)\| < \varepsilon_i, \xi \in \mathcal{E} \ (i = 1, 2, ..., n + m)\}.
\]
It is easily seen that \( V \) is a neighbourhood of the point \( x_0 \) and \( V \subseteq U \cap W \).

A topology defined in this manner is called the \( \mathcal{E} \)-topology. The \( \mathcal{E} \)-topology is a locally convex topology. Indeed, if
\[
\|\xi(x) - \xi(a_i)\| < \varepsilon \quad \text{and} \quad \|\xi(x') - \xi(a_i)\| < \varepsilon,
\]
then
\[
\|\xi(x + (1 - t)x') - \xi(a_i)\| = \|\xi(x) - \xi(a_i) + (1 - t)(\xi(x') - \xi(a_i))\|
\leq t\varepsilon + (1 - t)\varepsilon = \varepsilon.
\]
Hence \( U \) is a convex set, as an intersection of sets of the form \( \{x : \|\xi(x) - \xi(a_i)\| < \varepsilon\} \).

\textbf{Theorem 10.1.} A linear functional \( f \) is continuous in the \( \mathcal{E} \)-topology if and only if \( f \in \mathcal{E} \).

\textit{Proof.} If \( f \in \mathcal{E} \), then the inverse image of the set \( \{x : \|\xi(x) - a_i\| < \varepsilon\} \) is the set \( \{x : \|f(x) - a_i\| < \varepsilon\} \), i.e., a neighbourhood in the \( \mathcal{E} \)-topology.

On the other hand, let \( f \) be a \( \mathcal{E} \)-continuous functional. There exists a neighbourhood of zero \( U = \{x : \|\xi(x) - a_i\| < \varepsilon_i, \xi \in \mathcal{E} \ (i = 1, 2, ..., n)\} \) such that \( |f(x)| < 1 \) for \( x \in U \). Let \( H_i = \{x : \xi(x) = 0\} \) and \( H = \bigcap_{i=1}^n H_i \).

Evidently, \( a_i \in H \) implies \( m a_i \in H \). Since \( H \subseteq U \), we conclude that \( |f(ma_i)| < 1 \) and, consequently, \( f(ma_i) = 0 \). Hence \( \xi(x) = 0 \) for \( i = 1, 2, ..., n \) implies \( f(x) = 0 \). By Theorem 1.3, \( \Lambda(I) \) is a linear combination of the functionals \( I_i \). Hence \( f \) is in \( \mathcal{E} \). \( \blacksquare \)

\textbf{Theorem 10.3.} The space \( X' \) is a complete linear topological space in the \( \mathcal{E} \)-topology.

\textit{Proof.} Let \( \mathfrak{F} \) be a family of subsets of the space \( X' \), fundamental in the \( \mathcal{E} \)-topology. It follows from the definition of fundamentalness that the family of sets of numbers
\[
P(\mathfrak{F}) = \{f(a) : f \in \mathfrak{F}\}
\]
is fundamental for every \( a \in X' \). Hence \( P(\mathfrak{F}) \) has one cluster point \( f(a) \).

If \( A \in P(\mathfrak{F}) \), then \( A \subset A_1 + A_3 \), where \( A_1 \subset P(\mathfrak{F}) \) and \( A_3 \subset P(\mathfrak{F}) \). Hence it follows immediately that the functional \( f \) is additive. In a similar manner one can prove the homogeneity of \( f \). Hence \( f \in X' \). By the definition of the \( \mathcal{E} \)-topology, the functional \( f \) is a cluster point of the family \( \mathfrak{F} \). \( \blacksquare \)

\textbf{Theorem 10.3.} If \( f(a) \) is a real-valued positive function, then the set
\[
K = \{x : f(x) \leq c(a)\}
\]
is compact in the \( \mathcal{E} \)-topology.

\textit{Proof.} By definition, the \( \mathcal{E} \)-topology is given by neighbourhoods of zero, \( U \), of the form
\[
U = \{x : |f(x)| < \varepsilon, \varepsilon > 0, \xi(x', i = 1, 2, ..., n)\}.
\]
Let us take an arbitrary neighbourhood of this form and let us consider a sequence \( \{f_m\} \) of functionals satisfying the inequalities
\[
\text{(1.1)} \quad \sup_{m, n} |f_m(x') - f_n(x')| > \varepsilon \quad \text{for} \quad m \neq n.
\]
According to the condition \( |f(x)| < c(a) \) the set of functionals satisfying (1.1) is finite. Hence
\[
K = \bigcap_{m=1}^n (f_m + U).
\]
Since the neighbourhood \( U \) is arbitrary, this condition shows that the set \( K \) is compact.

But the set \( K \) is closed and the space \( X' \) is complete in the \( \mathcal{E} \)-topology. Hence the set \( K \) is compact. \( \blacksquare \)

Together with the \( \mathcal{E} \)-topology one can consider also \( \mathcal{E} \)-convergence. We say that a sequence \( \{a_n\} \) is \( \mathcal{E} \)-convergent to an element \( x \) if \( \lim_{n \to \infty} \xi(a_n - x) = 0 \) for every \( \xi \in \mathcal{E} \).

\((\dagger)\) I.e. Theorem 1.3 of Chapter I, Part A.
A sequence \((\{x_n\})\) is called \(E\)-fundamental if the sequence \((\xi(x_n))\) is fundamental for every \(\xi \in E\).

If a space \(X\) is metrizable in the \(E\)-topology, a sequence \((x_n)\) is convergent in the \(E\)-topology if and only if it is \(E\)-convergent.

**Theorem 10.4.** Let \(A\) be an operator which maps a linear space \(X\) with a \(E\)-topology into a linear space \(Y\) with an \(H\)-topology, and let us suppose that the conjugate operator maps the space \(H\) into the space \(E\), i.e., that \(A \in L(H, E)\). Then the operator \(A\) is continuous and maps \(E\)-convergent sequences in \(H\)-convergent sequences.

**Proof.** Let \(U\) be an arbitrary neighborhood of the point \(y_0 = Ax_0\) in the space \(Y\). Then

\[
U = \{y : |\eta(y) - \eta(y_0)| < \epsilon, \ i = 1, 2, \ldots, n\}.
\]

Let

\[
V = \{x : |\xi(x) - \xi(x_0)| < \epsilon, \ i = 1, 2, \ldots, n\}.
\]

Evidently, \(V\) is a neighborhood and \(AV \subset U\), which was to be proved.

Let \((x_n)\) be a sequence \(E\)-convergent to \(x\). We consider the sequence \((Ax_n)\). We obtain

\[
\lim_{n \to \infty} |Ax_n - Ax| = \lim_{n \to \infty} |\xi(x_n) - \xi(x)| = 0,
\]

where \(\xi = A\eta\).

We say that a subset \(E\) of the space \(X\) is \(E\)-closed if it is closed in the \(E\)-topology.

A linear subspace \(X_0 \subset X\) is \(E\)-closed if and only if it is \(E\)-describable.

In order to prove it we need only to remark that the notions of a \(E\)-closed subspace and of a \(E\)-describable subspace are both equivalent to the following condition:

For every element \(x \in X\) there exists a functional \(\xi \in E\) such that

\[
\xi(x) \neq 0
\]

and

\[
\xi(x) = 0
\]

for \(x \in X_0\).

**Section 11. Riemann integral in complete linear metric spaces.** Let \(X\) be a linear metric space over the field of complex numbers (or real numbers).

Let \(L\) be a rectifiable curve (i.e., of finite length) on the complex plane, \(L = \{c(t) : a \leq t \leq b\}\). Finally, let \(x(t)\) be a function defined on the curve \(L\) with values in the space \(X\). The Riemann integral of the function \(x(t)\) is defined in the same manner as the Riemann integral of a complex-valued (or real-valued) function.

A subdivision \(D^t\) of the curve \(L\) is a system of \(n_t\) points

\[
a = t_0^0 < t_1^0 < \ldots < t_{n_t}^0 = b.
\]

A sequence \((D^t)\) of subdivisions is called normal if

\[
\lim_{t \to \infty} \left(\sup_{1 \leq k \leq n_t} |t_k^0 - t_{k+1}^0|\right) = 0.
\]

Let

\[
\tilde{S}(x(t), D, \tau) = \sum_{k=1}^{n_t} x(t_k^0) \left[\int_{t_{k-1}^0}^{t_k^0} |\dot{x}(s)| ds - \int_{t_k^0}^{t_{k+1}^0} |\dot{x}(s)| ds\right],
\]

where \(\tau_k\) is an arbitrary point satisfying the inequalities \(\tau_{k-1} \leq \tau_k \leq \tau_k^0\).

If the limit

\[
\lim_{t \to \infty} \tilde{S}(x(t), D^t, \tau)\]

exists for an arbitrary normal sequence of subdivisions and for an arbitrary choice of points \(\tau_k\), then this limit is called the Riemann integral of the function \(x(t)\) on the curve \(L\) and is denoted by

\[
\int_L x(t) dt.
\]

In the same manner as for complex-valued (real-valued) functions it is proved that this limit does not depend on the choice of the normal sequence of subdivisions or on the choice of the points \(\tau_k\).

Functions which possess integrals are called integrable. Other functions are called non-integrable.

If \(L = L_1 \cup L_2\) and if a function \(x(t)\) is integrable on each of the curves \(L_1, L_2\), then it is integrable on the curve \(L\). Hence, moreover, the curves \(L_1, L_2\) intersect at a finite number of points, then it is easily proved that

\[
\int_{L_1 \cup L_2} x(t) dt = \int_{L_1} x(t) dt + \int_{L_2} x(t) dt.
\]

Just as for complex-valued (real-valued) functions, it is proved that

\[
\int_L (ax(t) + by(t)) dt = a \int_L x(t) dt + b \int_L y(t) dt.
\]

Evidently, if \(x(t) = \varphi(t) - \chi\), where \(\varphi(t)\) is a complex-valued (real-valued) function integrable in the sense of Riemann, then the integral

\[
\int_L \varphi(t) dt = \int_L \varphi(t) dt x(t) \text{ exists.}
\]

In particular, if \(L = \bigcup_{i=1}^{n} L_i\) and \(x = \sum_{i=1}^{n} \chi_i a_i\),

where \(L_i = \{a_i \leq t \leq b_i\}, a_i \in X, \chi_i\) is the characteristic function of the arc \(L_i,\) the integral on the arc \(L_i\) exists and

\[
\int_{L_i} x(t) dt = \sum_{i=1}^{n} \int_{L_i} \chi_i a_i x(t) dt.
\]

**Theorem 11.1.** Let \(x(t)\) be a function with values in a linear metric space \(X\). If for an arbitrary neighborhood of zero \(U \subset X\) there exists an integrable function \(x(t)\) such that

\[
\tilde{S}(x(t) - x(t) \cdot \varphi(t), D^t, \tau) \in U
\]

for any subdivision \(D^t\), then the function \(x(t)\) is integrable.
B. I. Linear topological and linear metric spaces

Proof. Let \((d^j)\) be a normal sequence of subdivisions of the arc \(L\). Since the function \(σ_τ(τ)\) is integrable, there exists a positive integer \(i_0\) such that for \(i, j > i_0\)
\[
S(σ_τ(τ), d^i, \tau) - S(σ_τ(τ), d^j, \tau) ∈ U.
\]
Hence
\[
S(σ_τ(τ), d^j, \tau) - S(σ_τ(τ), d^j, \tau) = \frac{1}{i_0} ∑_{i=1}^{∞} |σ_τ(τ) - σ_{τ}(τ)| + \frac{1}{i_0} ∑_{i=1}^{∞} |σ_{τ}(τ) - σ_{τ}(τ)| + \frac{1}{i_0} ∑_{i=1}^{∞} |σ_τ(τ) - σ_{τ}(τ)| ∈ U + U + U.
\]
Since the neighbourhood \(U\) is arbitrary, this proves the existence of the integral. □

Corollary 11.2. If \(σ_τ(τ) = ∑_{i=1}^{∞} |σ_τ(τ)\|\), where \(σ_τ(τ)\) are uniformly bounded \(n\)-valued functions: \(|σ_τ(τ)| < M\), and if the series \(\sum_{i=1}^{∞} |σ_τ(τ)|\) is convergent, then the function \(σ_τ(τ)\) is integrable.

Proof. Let \(U\) be an arbitrary neighbourhood of zero. The continuity of multiplication by a scalar implies the existence of a positive number \(M\) such that
\[
\sum_{i=1}^{∞} |σ_τ(τ)| < M.
\]
Hence, if \(σ_τ(τ) = ∑_{i=1}^{∞} |σ_τ(τ)\|\), then \(S(σ_τ(τ) - σ_{τ}(τ), d^j, \tau) ∈ U\).

Corollary 11.3. (Mazur, Olechk [1]) If \(X\) is a complete, locally convex space and \(σ_τ(τ)\) a continuous function, then the integral \(∫σ_τ(τ) dτ\) exists.

Proof. A continuous function \(σ_τ(τ)\) can be approximated by means of step functions. Let \(U\) be an arbitrary neighbourhood of zero, convex and balanced. We find a simple function \(σ_τ = ∑_{i=1}^{∞} a_i x_i\) satisfying the condition \(σ_τ - σ_{τ}(τ) ≤ \frac{1}{i_0} U\), where \(a_i\) are scalars, \(x_i\) are characteristic functions of measurable sets \(E_i\), and \(|E|\) is the length of the arc \(L\). Let us remark that the local convexity of the space \(X\) implies
\[
S(σ_τ - σ_{τ}(τ), d^j, \tau) ∈ U
\]
for an arbitrary subdivision \(d^j\). □

The following theorem can be treated as, to a certain extent, converse to Corollary 11.3:

Theorem 11.4. (Mazur, Olechk [1]) If a complete linear metric space is not locally convex, then there exists a non-integrable continuous function.

Proof. Let \(L\) be the interval \([0, 1]\) and let \(∥∥\) be the norm in the space \(X\). If the space \(X\) is not locally convex, there exists a number \(ε > 0\) with the following property: For every \(ε > 0\) there exists a system of points \(x^1, ..., x^n\) such that \(∥x^i - x^j∥ < ε\) for \(i = 1, ..., n\), and
\[
\frac{1}{n} ∑_{i=1}^{n} x^i < 0.
\]
Let a sequence \(x_n \to 0\) be given. We write briefly \(x^i = x^i_n\) and \(n = n_x\).
We define a function \(σ_τ(τ)\) in the following manner:
\[
σ_τ(τ) = \begin{cases} 0 & \text{for } τ = \frac{1}{2^k} + \frac{1}{n_x 2^k}, \\ x^i & \text{for } τ = \frac{1}{2^k} + \frac{2i-1}{n_x 2^k}, \quad (i = 0, 1, ..., n_x) \end{cases}
\]
and elsewhere as a linear function.

Geometrically, the function \(σ_τ(τ)\) looks like a sequence of decreasing "spikes" converging to zero (Fig. 8).

\[\text{Fig. 8. Graph of non-integrable continuous function}\]

Evidently, the function \(σ_τ(τ)\) is continuous.
Let us take a normal sequence of subdivisions
\[
(d^j): 0 = t^0 < t^1 < ... < t^{n_x+1} = 1,
\]
CHAPTER II
CONTINUOUS LINEAR OPERATORS
IN LINEAR TOPOLOGICAL SPACES

§ 1. Continuous linear operators. Let \( X \) and \( Y \) be linear topological spaces. If an operator \( A \in L(X \to Y) \) is continuous, we call \( A \) a continuous linear operator. If \( X \) and \( Y \) are linear metric spaces, this means that the conditions \( x_n \to x, [x_n] \subseteq D_A, \|x\| \in D_A \) imply \( A x_n \to A x \).

Let us remark that if \( X \) and \( Y \) are linear spaces over the field of real numbers, and if an operator \( A \in L(X \to Y) \) is additive and continuous, then \( A \) is linear. Indeed, the additivity of \( A \) implies

\[
A(ax) = n(Ax)
\]

for every integer \( n \). But

\[
Ax = A(\frac{1}{m} + \ldots + \frac{1}{m}) = nA(\frac{1}{m}x).
\]

Hence

\[
A(\frac{1}{m}x) = \frac{1}{m}Ax
\]

Consequently,

\[
A(ax) = aAx
\]

for an arbitrary rational number \( a \).

We prove that equality (1.1) is true also in the case where \( \omega \) is an arbitrary real number. Let \( U \) be an arbitrary neighbourhood of zero. There exists a rational number \( c_n \) such that \( A(\omega - c_n)x \in U \) and \( (\omega - c_n)Ax \in U \). Hence

\[
A(\omega x) - A(ax) = A(\omega - c_n)x + [A(\omega - c_n)x - A((\omega - c_n)x)] + A((\omega - c_n)x) \in U.
\]

Since the neighbourhood \( U \) is arbitrary, this implies \( \omega Ax = A(ax) \).

We say that two linear topological spaces \( X \) and \( Y \) are isomorphic if there exists a one-to-one linear operator \( A \) mapping the whole space \( X \) onto the whole space \( Y \) and such that both \( A \) and the inverse operator \( A^{-1} \) are continuous operators (compare § 1, A I (1)).

\( 1 \) I.e., § 1 of Part A, Chapter I.
The isomorphism of two spaces considered as linear spaces does not imply their isomorphism as linear topological spaces.

An operator \( A \in L(X \to Y) \) is called bounded if it maps bounded sets onto bounded sets.

**Theorem 1.1.** A continuous linear operator is bounded.

Proof. Let \( A \in L(X \to Y) \), and let us suppose that the operator \( A \) is continuous but not bounded. There exists a bounded set \( E \) such that the set \( AE \) is not bounded. This means that there exists a neighbourhood \( V \subseteq Y \) for which \( A(aE) = a(\text{im} A) \cap V \) for every scalar \( a \neq 0 \). But the set \( E \) is bounded. Hence for every neighborhood of zero \( U \subseteq X \), there exists a positive number \( a \) such that \( aE \subseteq U \). Thus \( AU \subseteq V \) for every neighborhood \( U \subseteq X \), contradicting the assumption of continuity of the operator \( A \).

**Corollary 1.2.** Let \( X \) and \( Y \) be locally bounded spaces, and let \( \| x \| \) and \( \| y \| \) be \( p_x \)- and \( p_y \)-homogeneous norms in \( X \) and \( Y \), respectively. A linear operator \( A \) from \( X \) into \( Y \) is continuous if and only if

\[
\| A \| = \sup_{\| x \| < +\infty} \| Ax \| < +\infty.
\]

Proof. Since the ball \( K = \{ x \in X : \| x \| < 1 \} \) is a bounded set, the image of \( K \) is also bounded. Hence

\[
\sup_{\| x \| < c} \| Ax \| = \sup_{\| x \| < c} \| y \| < +\infty.
\]

On the other hand, if \( \| A \| = +\infty \), then the unit ball in \( X \) is transformed into a bounded set in \( Y \). Hence for an arbitrary neighborhood of zero \( U \subseteq Y \) there exists a neighborhood of zero \( V \subseteq X \) such that \( A(V) \subseteq U \). Thus the operator \( A \) is continuous.

If \( p_x = p_y \), then \( \| Ax \| \leq \| A \| \| x \| \), the number \( \| A \| \) is called the norm of the operator \( A \).

Let \( X \) and \( Y \) be arbitrary locally bounded spaces. There always exists a number \( \rho \) for which a \( p_{x \rho} \)-homogeneous norm exists in both \( X \) and \( Y \). Indeed, by Theorem 7.1, there exists a \( p_x \)-homogeneous norm in the space \( X \) and a \( p_y \)-homogeneous norm in the space \( Y \). Without loss of generality we may suppose that \( \rho \leq p_y \). Let us remark that

\[
\| y \|_p = (\| y \|_p)^{p_y/p_x}
\]

is a \( p_y \)-homogeneous norm in the space \( Y \).

Hence a norm of the operator can be defined for all continuous operators which transform a locally bounded space \( X \) into a locally bounded space \( Y \). Such norms may be different according to the choice of the norms \( \| x \| \) and \( \| y \| \), but they determine the same topology.

**§ 1. Continuous linear operators**

Let \( X \) and \( Y \) be locally bounded spaces. A continuous operator \( A \in L(X \to Y) \) is called an isometry if \( \| Ax \| = \| x \| \) for all \( x \in X \). It follows from this definition that if an isomorphism \( A \) is an isometry, we have

\[
\| A \| = \| A^{-1} \| = 1.
\]

If \( X \) and \( Y \) are linear metric spaces, then the following theorem, converse to Theorem 1.1, is true:

**Theorem 1.3.** If \( X \) and \( Y \) are linear metric spaces and if an operator \( A \in L(X \to Y) \) is bounded, then \( A \) is continuous.

Proof. Let us suppose that the operator \( A \) is not continuous. There exists a sequence \( (a_n) \) convergent to zero such that \( \| a_n \| > \delta > 0 \), where \( y_n = Ax_n \). Let us write

\[
a_n' = y_n/\| y_n \|, \quad a_n = \text{entier} \left( \frac{1}{\| y_n \|} \right).
\]

By the subadditivity of the norm,

\[
\| a_n \| \leq \| a_n \|_X + \sup_{\| x \| \leq 1} \| x_n \| + \sup_{\| x \| \leq 1} \| x_n \|
\]

\[
\leq \| y_n \|/\| y_n \| + \sup_{\| x \| \leq 1} \| y_n \| \to 0 \quad \text{as} \quad n \to \infty.
\]

Let \( t_n = y_n/\| y_n \| \). Evidently, \( t_n \to 0 \) and \( A(t_n) = y_n \to 0 \). Hence the bounded set made of elements of the sequence \( (a_n) \) (see § 7, I) is transformed onto an unbounded set.

Let us now suppose that \( X \) and \( Y \) are locally convex linear metric spaces. In each of these spaces there exists a countable set of homogeneous pseudonorms \( \| x \|_i \) and \( \| y \|_i \), respectively.

The following theorem is a consequence of Theorems 1.1 and 1.3.

**Theorem 1.4.** If \( X \) and \( Y \) are locally convex linear metric spaces, then an operator \( A \in L(X \to Y) \) is continuous if and only if it satisfies the following condition:

\[
(1.1) \quad \text{for every index } k \text{ there exists an index } n_k \text{ and a non-negative number } a_k \text{ such that}
\]

\[
\| Ax_k \|_Y \leq a_k \sup_{i \leq k} \| x_i \|_X \quad \text{for all } x \in X.
\]

Proof. Sufficiency. Let \( E \subseteq X \) be an arbitrary bounded set. There exists a sequence \( (M_k) \) of positive constants for which

\[
\| x \|_i \leq M_k \quad \text{for all } x \in E \quad (n = 1, 2, \ldots)
\]

(see § 9, I). Hence, supposing (1.1) to be satisfied, we obtain

\[
\| Ax \|_Y \leq a_k \sup_{i \leq k} M_i \quad \text{for all } x \in E \quad (k = 1, 2, \ldots).
\]

Thus the set \( A \) is also bounded. By Theorem 1.1, the operator \( A \) is continuous.
Necessity. If condition (1.1) is not satisfied, there exists an index $k_4$ such that for every positive integer $n$ there corresponds an element $a_n$ with the property
\[ \|A a_n\|_\infty > n \sup_{x \in S} \|x\|^p.\]
Let $a'_n = a'_n \sup_{x \in S} \|x\|^p$. The sequence $(a'_n)$ is bounded, and the sequence $(A a'_n)$ is unbounded, since $\|A a'_n\|_\infty > n$. By Theorem 1.3, the operator $A$ is not continuous. \[ \square \]

**Theorem 1.5.** If a continuous linear operator $A$ maps a linear topological space $X$ into a complete linear topological space $Y$, then there exists one and only one extension of the operator $A$ to a continuous linear operator $\hat{A}$ which maps the completion $\bar{X}$ of the space $X$ into the space $Y$.

**Proof.** By the definition of completion (see § 4 of the previous chapter), elements of the space $\bar{X}$ are fundamental families. Let $\hat{x} = \{x\}$ be a fundamental family in the space $X$. Then $\hat{A}(\hat{x}) = \{A(U): U \subset \mathbb{R}\}$ is a fundamental family in the space $Y$. Since $Y$ is complete, each fundamental family determines an element $y \in Y$. The operator $\hat{A}$ defined by means of the equality $\hat{A} \hat{x} = y$, where $y$ is the element determined by the fundamental family $A(\{x\})$, is the required extension. \[ \square \]

Let $X$ and $Y$ be complete linear topological spaces. Theorem 1.5 shows that every continuous operator $A \in L(X, Y)$ defined on a dense linear subset $D \subset X$ has one and only one extension $\hat{A} \in L(\bar{X}, Y)$. Hence we limit ourselves to the consideration of continuous operators defined in closed domains. This is justified also by the fact that the essential properties of operators $A$ and $A$ are the same.

If $X$ and $Y$ are linear topological spaces, we denote by $B_d(X, Y)$ the set of all continuous operators belonging to the space $L_d(X, Y)$. We write briefly $B_d(X) = B_d(X, X)$. The set $B_d(X, Y)$ is a linear space.

Indeed, let $A, B \in B_d(X, Y)$, and let $V$ be an arbitrary neighborhood of zero in the space $Y$. There exists a neighborhood of zero $W \subset Y$ such that $W + W \subset V$. Hence, the operators $A$ and $B$ are continuous, there exist neighborhoods of zero $U_1$ and $U_2$ in the space $X$ satisfying the conditions $A U_1 \subset W$ and $B U_2 \subset W$. Then, $U = U_1 \cap U_2$ and $A + B \subset U \subset A U_1 + B U_2 \subset W + W \subset V$.

In a similar manner we verify that the product of a continuous operator by a number is a continuous operator. Since the superposition of two continuous operators is a continuous operator, we regard the parasmooth of continuous linear operators
\[ B_d(X \Rightarrow Y) = \left( \begin{array}{c} B_d(X) \\ B_d(X \rightarrow Y) \\ B_d(Y \rightarrow X) \\ B_d(Y) \end{array} \right). \]

Let $\sigma$ be a family of bounded sets in a linear topological space $X$. We denote by $B_d(X \Rightarrow Y)$ the space $B_d(X \rightarrow Y)$ with the topology determined by neighborhoods of the following form:

A neighborhood of an operator $A_0$ is the set $U(A_0, B, Y)$ of all operators $A$ such that $(A - A_0) B \subset V$, where $B$ is an arbitrary set belonging to $\sigma$, and $Y$ is a neighborhood of zero in the space $Y$. The space $B_d(X \rightarrow Y)$ is a linear topological space with this topology.

If $\sigma$ is the family of all bounded sets, this topology is called the topology of bounded convergence. The space $B_d(X \rightarrow Y)$ with this topology will be denoted by $B(X \rightarrow Y)$. The space $B(X \rightarrow X)$ will be denoted briefly by $B(X)$.

If the spaces $X$ and $Y$ are locally bounded with $p$-homogeneous norms, then the topology in the space $B(X \rightarrow Y)$ is equivalent to the topology determined by the norm of the operator, i.e., the set $U = \{A: \|A - A_0\| < \epsilon\}$ is a neighborhood of the operator $A_0$.

We say that a subspace $Y$ of a linear topological space $X$ is a projection of the space $X$ if there exists a continuous projection operator $P$ such that $Y = \{x \in X: P x = x\}$. Evidently, the set $Y_0 = \{x \in X: P x = 0\}$ is also a projection of the space $X$, and $Y = Y_0 \oplus Y_1$. The subspace $Y_0$ will be called a complementary subspace of the subspace $Y$ to the space $X$.

Let us remember that every projection operator defines a decomposition of the space into a direct sum of two subspaces.

If a projection operator $P$ is continuous, then the subspaces $Y_1$ and $Y_2$, induced by $P$ are closed. Indeed, the inverse image of a one-point set $\{0\}$, which is obviously closed, is the space $Y_1$. Since $I - P$ is also a continuous projection operator and the space $Y$ is the inverse image of $\{0\}$, the spaces $Y_1$ and $Y_2$ are both closed.

By Theorem 0.3, if $X_0$ is a subspace of a linear space $X$, the space $X$ can be written as the direct sum of $X_0$ and some space $Z = X_0 \oplus Z$.

If $X_0$ is a closed subspace of a linear topological space $X$, we cannot always find a closed subspace $Z$ such that $X = X_0 \oplus Z$. Hence it is not for every subspace $X_0$ that there exists a continuous projection operator. In the general case it is not sufficient even if $X_0$ is a finite-dimensional space. This follows from

**Theorem 1.6.** Let $X_0$ be an $n$-dimensional subspace of a linear topological space $X$. The subspace $X_0$ is a projection of the space $X$ if and only if there exists a system of continuous functionals $f_1, \ldots, f_n$ such that the condition
\[ \pi \times X_0 \text{ and } f_i(x) = 0 \text{ for } i = 1, 2, \ldots, n \]
implies $x = 0$.

Equations in linear spaces
Proof. Necessity. Since $X_\sigma$ is finite-dimensional, there exists a system of continuous functionals $\{f_i\}$ on $X_\sigma$ such that $f_i(a) = 0$ for $i = 1, 2, \ldots, n$ implies $a = 0$. Let $f(x) = R(x)a_i$, where $R$ is a projection operator on the subspace $X_\sigma$. The functionals $f_i$ are defined on the whole space $X$ and are continuous as superpositions of continuous operators. Evidently, if $a \in X_\sigma$ and $f_i(a) = R(x)a_i = 0$ for $i = 1, 2, \ldots, n$, then $a = 0$.

Sufficiency. It is easily shown that there exist elements $e_i \in X_\sigma$ such that

$$f_i(e_j) = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

Indeed, let $\{e_1, \ldots, e_n\}$ be a basis of the space $X_\sigma$ and let

$$P = \sum_{i=1}^{n} f_i(a_i).$$

Since $P$ is a sum of continuous operators, $P$ is continuous. Moreover,

$$P^2 = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} f_i(a_j) e_j \right) e_i = \sum_{i=1}^{n} f_i(a) e_i = P a.$$ 

Hence $P$ is a projection operator.

Corollary 1.7. If there exists a total family of linear functionals on a linear topological space $X$, or $f_i$, in particular, $X$ is a locally convex space, then every finite-dimensional subspace $X_\sigma$ is a projection of the space $X$.

The following notion of continuity with respect to an operator (B. Sz. Nagy [1], [2]) is of importance in the theory of perturbations of unbounded operators.

Let an operator $A \in L(X \to Y)$ be given. We define a new topology in the set $D_A$ of all sets of the following form as a family of neighbourhoods of zero:

$$U \cap A^*(V),$$

where $U$ and $V$ are neighbourhoods in spaces $X$ and $Y$, respectively, and $A^*(V)$ is the inverse image of the set $V$.

The set $D_A$ with this topology will be denoted by $X_A$. It is easily seen that the operator $A$ transforms the space $X_\sigma$ into the space $Y$ continuously. An operator $B \in L(X \to Y)$ is called $A$-continuous if $D_B \subseteq D_A$ and the restriction of $B$ to the set $D_A$ transforms $X_\sigma$ into $Y$ continuously.

Evidently, every continuous operator is $A$-continuous.

If $X$ and $Y$ are linear metric spaces and if $B \in L(X \to Y)$ is an $A$-continuous operator, then the topology in the space $X_A$ can be defined by means of the norm

$$\|a\| = \|a\|_X + \|A\| \|a\|_Y,$$

where $\|\cdot\|_X$ and $\|\cdot\|_Y$ are norms defining topologies in spaces $X$ and $Y$, respectively.

§ 2. Equicontinuous operators. Let $X$ and $Y$ be linear topological spaces. Let $\mathcal{F}$ be a subset of the set $B(X \to Y)$, not necessarily linear.

We say that the operators belonging to the set $\mathcal{F}$ are equicontinuous if for every neighbourhood of zero $V \subseteq Y$ there exists a neighbourhood of zero $U \subseteq X$ such that $A U \subseteq V$ for all $A \in \mathcal{F}$.

Let $\mathcal{F}$ be a family of operators from $B_0(X \to Y)$. The family $\mathcal{F}$ is a family of equicontinuous operators if there exists an operator $A_0$ such that $A \in \mathcal{F}$ for every operator $A \in A_0$. If $X$ and $Y$ are linear metric spaces, the condition $A(U) \subseteq A_0(U)$ can be expressed by means of the norms as follows: $\|A\| \leq \|A_0\|$ for all $A \in \mathcal{F}$.

A closed subset $V$ of a linear topological space $X$ is called a barrel if for every element $x \in X$ there exists a positive number $a_x$ such that $bx \in V$ for $|b| < a_x$.

A linear topological space $X$ is called a barrel space if every barrel $V \subseteq X$ contains an open set (Vilansky [1], p. 224).

Theorem 3.1. If a linear topological space $X$ is a barrel space, then every convex barrel $V \subseteq X$ contains a neighbourhood of zero.

Proof. Since $X$ is barrel space, every barrel $V \subseteq X$ contains an open subset $U$. Let $x \in U$. Since $V$ is a barrel, there exists a number $a > 0$ such that $-ax \in V$. It follows from the properties of convex sets that the set

$$\text{conv}(U \cup (-ax)) \setminus (-ax)$$

is open. Obviously, this set contains zero. $\blacksquare$

Theorem 3.2. Every linear topological space $X$ of the second category is a barrel space.

Proof. Let $V \subseteq X$ be an arbitrary barrel. Let us write

$$nV = \{nx : x \in V\},$$

where $n = 1, 2, \ldots$. Since $V$ is a barrel, we have $C \subseteq nV$. But the space $X$ is of the second category. Hence there exists an index $n_0$ such that the set $n_0V$ is of the second category. Therefore the set $V$ is of the second category. Since $V$ is closed, $U$ contains an open set. $\blacksquare$

Theorem 3.3. (Banach, Steinhaus.) Let $X$ be a barrel space and let $Y$ be a linear topological space. If a family $\mathcal{F} \subseteq B_0(X \to Y)$ of operators is such that the set $\{A : A \in \mathcal{F}\}$ is bounded for every $x \in X$, then the family $\mathcal{F}$ is equicontinuous.

Proof. Let $U$ be an arbitrary neighbourhood of zero in the space $X$ and let $V$ be a balanced neighbourhood of zero such that $V_1 + V_1 \subseteq V$. 

\(\blacksquare\)
§ 2. Equicontinuous operators

A set $B \subset X$ is called total if the set of linear combinations of elements of $B$ is dense in $X$.

**Theorem 2.7.** If $X$ and $Y$ are complete linear metric spaces and if a sequence $(A_n) \subset B_{B}(X \to Y)$ of equicontinuous operators is convergent to an operator $A$ on a total set $E$, then $A \ast B_{B}(X \to Y)$ and $A_n \to A$ for all $x \in X$.

**Proof.** If $A_n x \to Ax$ on a set $E$, then this convergence holds also for any linear combination of elements of $E$, i.e. on a certain dense set $D$. Let $x \in D$ be an arbitrary positive number. By the assumption of equicontinuity, there exists a $\delta > 0$ such that the inequality $|x - x'| < \delta$ implies $|A_n x - A_n x'| < \epsilon$ for all $n$. Hence

$$|A_n x - A_n x'| \leq |A_n x - A_n x'| + |A_n x - A_n x'| + |A_x x - A_n x'| < 3\epsilon.$$  

Since the space $X$ is complete, $A x = \lim A_n x$ exists. By Theorem 2.3, the operator $A$ is linear and continuous.

§ 3. Continuity of the inverse of a continuous operator in complete linear metric spaces.

**Theorem 3.1.** (Banach [2].) If $X$ and $Y$ are complete linear metric spaces and if $A \subset B_{B}(X \to Y)$ maps $X$ onto $Y$, then the image $AY$ of any open set $U \subset X$ is open.

**Proof.** Let $A \subset B_{B}(X \to Y)$. We prove that the closure $\overline{A U}$ of the image of an arbitrary neighbourhood of zero $U$ in the space $X$ contains a neighbourhood of zero in the space $Y$. Since $a - b$ is a continuous function of arguments $a$ and $b$, there exists a neighbourhood of zero $M$ in the space $X$ such that $M \cdot M \subseteq U$. The sequence $(a/n)$ tends to zero for every $x \in X$. Hence $a + n M$ for sufficiently large $n$. Thus

$$X = \bigcup_{n=1}^{\infty} n M, \quad Y = AX = \bigcup_{n=1}^{\infty} n A M.$$

By the Baire theorem (Theorem 5.2, I) on categories, at least one of the sets $n A M$ contains a non-void open set. Since the map $y \to y$ is a homeomorphism of the space $Y$ onto itself, the set $A M$ contains also a non-void open set $V$. Hence

$$\overline{A U} \supset A M \cdot A M \supset A M \cdot A M \supset V \cdot V.$$

The set $(a - V)$ is open because the map $y \to a - y$ is a homeomorphism. The set $V - V = \bigcup_{n=1}^{\infty}(a - V)$ is open as union of open sets. Moreover, $V - V$ contains $0$. Hence it is a neighbourhood of zero. Thus the closure of the image of a neighbourhood of zero contains a neighbourhood of zero.
Given any \( \varepsilon > 0 \), we denote by \( X_\varepsilon \) and \( Y_\varepsilon \) balls with centre at the point zero and radii \( \varepsilon \) in spaces \( X \) and \( Y \), respectively. Let \( \varepsilon_q > 0 \) be arbitrary and let \( \varepsilon_i > 0 \), where \( \sum \varepsilon_i < \varepsilon_q \). As we have already shown, there exists a sequence \( \{ \varepsilon_i \} \) of positive numbers convergent to zero such that

\[
\left( X_{\varepsilon_i} \right) \supset Y_\varepsilon \quad (i = 0, 1, \ldots).
\]

Let \( y \in Y_\varepsilon \). We show that there is an element \( x \in X_{\varepsilon_\varepsilon} \) such that \( Ax = y \). Formula (3.1) implies the existence of an element \( x_\varepsilon \in X_{\varepsilon_\varepsilon} \) satisfying the inequality \( \| y - Ax_\varepsilon \| < \varepsilon_\varepsilon \). Since \( y - Ax_\varepsilon \in X_\varepsilon \), taking \( i = 1 \) in formula (3.1) we conclude that there exists an element \( x_1 \in X_{\varepsilon_1} \) such that \( \| y - Ax_1 - Ax_\varepsilon \| < \varepsilon_\varepsilon \). In this manner we may define a sequence of points \( \{ x_n \} \),

\[
x_n \in X_{\varepsilon_n},
\]

such that

\[
\left| y - \left( \sum_{k=0}^{n} x_k \right) \right| < \varepsilon_{n+1} \quad (n = 0, 1, \ldots).
\]

We take \( x_n = x_0 + \ldots + x_n \). Then \( \| x_m - x_n \| = \| x_{n+1} + \ldots + x_m \| < \varepsilon_{n+1} + \ldots + \varepsilon_m \) for \( m > n \). Hence the sequence \( \{ x_n \} \) is fundamental. Consequently, the series \( x_0 + x_1 + \ldots \) is convergent to a point \( x \) for which

\[
\| x \| = \lim_{n \to \infty} \| x_n \| < \lim_{n \to \infty} (\varepsilon_{0} + \varepsilon_{1} + \ldots + \varepsilon_{n}) = \varepsilon_\varepsilon.
\]

Since the operator \( A \) is continuous, formula (3.2) implies \( y = Ax \). This proves that an arbitrary ball \( X_{\varepsilon_\varepsilon} \) with centre 0 in the space \( X \) contains a certain ball \( Y_{\varepsilon_\varepsilon} \) in the space \( Y \). Hence the image of a neighbourhood of zero in the space \( X \) by means of the operator \( A \) contains a certain neighbourhood of zero in the space \( Y \).

Now, let \( U \subseteq X \) be a non-void open set, let \( x \in U \), and let \( N \) be a neighbourhood of zero in \( X \) such that \( x + N \subseteq U \). We denote by \( M \) a neighbourhood of zero in the space \( Y \) satisfying the condition \( Ax \subseteq M \).

Then

\[
A \left( x + N \right) = Ax + AN \subseteq M.
\]

Hence \( AU \) contains a neighbourhood of each of its points. \( \blacksquare \)

**Theorem 3.2.** If \( X \) and \( Y \) are complete linear metric spaces and the operator \( A \in B_d(X \to Y) \) is an isomorphism, then the inverse operator \( A^{-1} \in B_d(X \to Y) \).

**Proof.** Let \( AX = Y \). The map \( (A^{-1})^{-1} = A \) transforms open sets onto open sets (Theorem 3.1). Hence the operator \( A^{-1} \) is continuous. \( \blacksquare \)

**Corollary 3.3.** If \( X \) and \( Y \) are complete linear metric spaces and the operator \( A \in B_d(X \to Y) \) is of finite deficiency: \( \beta_A < +\infty \), then the set \( E_A \) is closed.

\[ \text{§ 3. Continuity of the inverse of a continuous operator} \]

**Proof.** Let \( A \in B_d(X \to Y) \). Let \( \mathcal{C} \) be the quotient space \( X / E_A \). By hypothesis, \( Y = E_A / \mathcal{G} \), where \( \dim \mathcal{G} < +\infty \).

Let \( X_\beta = \mathcal{C} \times \mathcal{G} \) with the natural topology of a product. Evidently, \( X_\beta \) is a complete space.

\[ A_\beta x = \begin{cases} A^\beta x & \text{for } x \in \mathcal{C}, \\ x & \text{for } x \in \mathcal{G}, \end{cases} \]

where \( A^\beta \) is an operator induced by \( A \) in the quotient space \( \mathcal{C} \).

The operator \( A_\beta \) is a continuous and one-in-one map of the space \( X_\beta \) onto the space \( Y \). Hence \( A_\beta \) has an inverse \( A_\beta^{-1} \). By Theorem 3.2, \( A_\beta^{-1} \) is continuous.

Since the subspace \( \mathcal{C} \) is closed in the space \( X_\beta \), the subspace

\[ E_A = (A_\beta^{-1})^{-1}(\mathcal{C}) \]

is closed as the inverse image of a closed set by means of a continuous operator.

**Corollary 3.4.** If \( X \) and \( Y \) are complete linear metric spaces with total families of functionals and \( A \in B_d(X \to Y) \) is a continuous operator, then there exists an operator \( R_A \in B_d(X \to Y) \) such that \( R_A A - I \) and \( A R_A - I \) are finite dimensional operators.

**Proof.** We write the spaces \( X \) and \( Y \) as direct sums:

\[ X = Z_\beta \oplus \mathcal{C}, \quad Y = E_A \oplus \mathcal{G}, \]

where \( \mathcal{C} \) is a closed space. (See Corollary 1.7.)

Let \( A_\beta \) be the restriction of the operator \( A \) to the space \( \mathcal{C} \). By Corollary 3.3, the set \( E_A \) is closed. Hence, by Theorem 3.2, the operator \( A_\beta^{-1} \), which maps the subspace \( E_A \) onto the subspace \( \mathcal{C} \), is continuous. Let \( R_A \) be an arbitrary extension of the operator \( A_\beta^{-1} \) to the space \( Y \). Since the subspace \( \mathcal{G} \) is finite-dimensional, the operator \( R_A \) is continuous. It is easily verified that the operator \( R_A \) possesses the required properties.

**Corollary 3.5.** If \( X \) and \( Y \) are complete linear metric spaces with total families of functionals then the paralelgebra

\[ B_d(X \to Y) = \left( B_d(X), B_d(X \to Y), B_d(Y) \right) \]

is regularizable.

\[ \text{§ 4. Locally algebraic operators.} \]

An operator \( A \in B_d(X) \) is called **locally algebraic** if for every \( x \in X \) there exists a (non-zero) polynomial \( P_a(x) \) such that \( P_a(A)x = 0 \).

**Theorem 4.1.** (Kaplansky [3]) If \( X \) is a complete linear metric space, then every locally algebraic operator \( A \in B_d(X) \) is algebraic.
Proof. We apply the method of categories. Let \( X_n = \{ x \in X : \text{there exists a polynomial } P \text{ of degree } \leq n \text{ such that } P(A)x = 0 \} \).

We show that \( X_n \) is a closed set. Let us suppose that \( \{ x_n \} \subseteq X_n \), i.e., that there exist polynomials \( P_i \) of degree \( \leq n \) such that \( P_i(A)x_n = 0 \). Moreover, let the sequence \( \{ x_n \} \) be convergent to an element \( x \in X \). One can normalize all coefficients of the polynomials \( P_i \) so as to make them absolutely \( \leq 1 \). Moreover, one of those coefficients can be assumed to be equal to 1. There exists a subsequence \( \{ P_{n_k} \} \) of the sequence of polynomials \( \{ P_n \} \) convergent \( \to 0 \) to a non-zero polynomial \( P \) such that \( P(A)x = 0 \) and the degree of \( P \) is \( \leq n \). Hence \( x \in X_n \), and the set \( X_n \) is closed.

Since the space \( X \) is equal to \( \bigcup_{n=1}^\infty X_n \), Baire's theorem on categories (Theorem 5.2, I) shows that at least one of the sets \( X_n \) has a non-empty interior \( U \). Let \( y \) be an arbitrary element of that interior. The set \( U - y \) is a neighbourhood of zero and each of its elements is annihilated by a certain polynomial of degree \( \leq 2n \). Multiplying the neighbourhood \( U \) by scalars we find that this property holds for an arbitrary element of the space \( Y \). By the Knozisky Theorem 5.2, A II, the operator \( A \) is algebraic.

§ 5. Basis of a linear metric space and its properties. Let a complete linear metric space \( X \) be given. A sequence of elements \( \{ x_n \} \subseteq X \) is called a Schauder basis (J. Schauder [1]) or simply a basis of the space \( X \) if every element \( x \in X \) can be represented uniquely as the sum of the series

\[
x = \sum_{n=1}^\infty t_n x_n,
\]

where the coefficients \( t_n \) are scalars.

Evidently, if a space has a basis, then it is separable. Let us write

\[
x_n = \sum_{i=1}^\infty t_i x_i.
\]

Theorem 5.1. If \( X \) is a complete linear metric space with a basis \( \{ x_n \} \), then all operators \( P_n x = x_n \) are equicontinuous.

Proof. Let us denote by \( X_n \) the linear space of all sequences of numbers \( \{ y_i \} \), such that the series \( \sum_{i=1}^\infty y_i t_i \) is convergent. We define a norm in \( X_n \) in the following manner:

\[
\|y\| = \sup_n \left\| \sum_{i=1}^\infty y_i t_i \right\|.
\]

By the convergence of a sequence of polynomials we understand the convergence of all sequences of coefficients of these polynomials.

It is easily shown that \( X_n \) is a linear metric space with this norm. We will show that \( X_n \) is complete. Let a sequence \( \{ y_n \} \) be given, where

\[
y_n = (y_{n,i}) \in X_n \quad (i = 1, 2, \ldots),
\]

and let \( \{ y_n \} \) satisfy the Cauchy condition. For an arbitrary \( \varepsilon > 0 \) there exists a natural number \( m_\varepsilon \) such that

\[
\|y_n - y_k\| < \varepsilon \quad \text{if} \quad m_\varepsilon < m_\varepsilon < m.
\]

Consequently, the inequality

\[
\left\| \sum_{i=1}^n [y_{n,i} - y_{k,i}] e_i \right\| < \varepsilon
\]

holds for \( k, m \geq m_\varepsilon \) and for an arbitrary \( n \). Hence it follows that

\[
\left\| \sum_{i=1}^n [y_{n,i} - y_{k,i}] e_i \right\| = \left\| \sum_{i=1}^n [y_{n,i} - y_{k,i}] e_i \right\| < 2\varepsilon.
\]

Consequently,

\[
\lim_{m_\varepsilon \to \infty} \left\| \sum_{i=1}^n [y_{n,i} - y_{k,i}] e_i \right\| = 0
\]

for an arbitrary \( n \). Thus, the sequence of numbers \( \{ p_{m,i}^{(n)} \} \) is convergent for every fixed \( n \). We denote its limit by \( p_n \).

If we take \( k \to \infty \) in inequality (5.2), we obtain

\[
\left\| \sum_{i=1}^n [y_{i} - y_{k,i}] e_i \right\| < \varepsilon
\]

for an arbitrary \( m \geq m_\varepsilon \) and for an arbitrary \( n \). Now, let us write

\[
\begin{align*}
    s_n^{(m)} &= \sum_{i=1}^m y_{n,i} e_i, \\
    s_n &= \sum_{i=1}^\infty y_{n,i} e_i.
\end{align*}
\]

Taking into account inequality (5.3) we obtain

\[
\left\| s_n - s_n^{(m)} \right\| < \| s_n - s_n^{(m)} \| + \varepsilon
\]

for \( m \geq m_\varepsilon \) and for arbitrary indices \( m \) and \( n \). Let an arbitrary number \( \varepsilon > 0 \) be given. We choose a number \( \varepsilon > 0 \) in such a manner that \( 2\varepsilon < \frac{1}{2} \varepsilon \). Now, let us fix an index \( m \geq m_\varepsilon \) and let us choose a number \( n \) such that the inequality

\[
\left\| s_n^{(m)} - s_n^{(m)} \right\| < \frac{1}{2} \varepsilon
\]

holds for \( n \geq n_\varepsilon \) and for an arbitrary \( p \). This is always possible, because the series \( \sum_{i=1}^\infty y_{n,i} e_i \) is convergent. Hence the inequality

\[
\left\| s_n - s_n^{(m)} \right\| < \varepsilon
\]

is satisfied.
holds for \( n > n_0 \) and for an arbitrary \( p > 0 \). Thus the series
\[
\sum_{i=1}^{\infty} \eta_i \epsilon_i
\]
is convergent and \( y = \{ \eta_i \} \in X_1 \). Since inequality (5.3) gives the estimation
\[
\sup_n \left| \sum_{i=1}^{n} [\eta_i^{(m)} - \eta_i^{(m)}] \eta_i \right| \leq \epsilon \quad \text{for} \quad m > m_0,
\]
i.e. the inequality
\[
\| y - y_n \| \leq \epsilon \quad \text{for} \quad m > m_0,
\]
the space \( X_1 \) is complete.

Evidently, to every element \( x = \sum_{i=1}^{m} \xi_i \epsilon_i \in X_1 \) there corresponds exactly one element \( y = \{ \eta_i \} \in X_1 \). Conversely, to every element \( y = \{ \eta_i \} \in X_1 \) there corresponds exactly one element \( x = \sum_{i=1}^{m} \eta_i \epsilon_i \). Thus an operator \( A: X \rightarrow Y \) is defined and is a one-to-one map of the space \( X_1 \) onto the space \( X \). It is easily seen that \( A \) is a linear operator. It is also continuous, because
\[
\| A y \| = \| x \| = \left\| \sum_{i=1}^{\infty} \xi_i \epsilon_i \right\| \leq \sup_n \left\| \sum_{i=1}^{n} \xi_i \epsilon_i \right\| = \| y \|.
\]
Hence \( A \) is a continuous linear operator which maps the complete linear metric space \( X_1 \) onto the complete linear metric space \( X \) one-to-one.

By Theorem 3.2, there exists the inverse operator \( A^{-1} \), which is also linear and continuous. Consequently, \( A^{-1} \) is bounded. Hence it follows that
\[
\| x_n \| = \left\| \sum_{i=1}^{\infty} \xi_i \epsilon_i \right\| < \sup_n \left\| \sum_{i=1}^{n} \xi_i \epsilon_i \right\| = \| y \|.
\]
Thus the operators \( P_n x = [x_n] \) are equicontinuous. \( \square \)

Hence, if \( X \) is a locally bounded complete space with a \( p \)-homogeneous norm \( \| \cdot \| \) and with a basis \( \{ \epsilon_i \} \), then there exists a positive number \( K \) such that
\[
\| x_n \| < K \| x \| \quad \text{for all} \quad n.
\]
The least number \( K \) satisfying condition (5.4) is called the norm of the basis.

Theorem 1.1 implies that \( \epsilon_i \) from the equality \( x = \sum_{i=1}^{\infty} \xi_i \epsilon_i \) are continuous linear functionals. These functionals will be called basis functionals and will often be denoted by \( \epsilon_i = \varphi_i(x) \) (i = 1, 2, ...).
Proof. It follows from the continuity of the operators \( P_n \) that \( P_n \) can be extended to the space \( X_1 \) uniquely. Moreover, the extensions \( P_n \) are also equicontinuous.

Let \( X_1 = \{ x : x = \sum_{n=1}^{\infty} a_n e_n \} \). Evidently, \( X_1 \subset X \). Since the operators \( P_n \) are equicontinuous, the sequence \( \{e_n\} \) is a basis of the space \( X_1 \). We show that \( X_1 \) is a complete space.

As in the previous theorem, we show that the space \( X_2 \) of all sequences of numbers \( \{t_i\} \) such that
\[
\|\{t_i\}\| = \sup_{n} \sum_{i=1}^{n} |t_i e_i| < +\infty
\]
is complete in the norm \( \|\{t_i\}\| \). Evidently, \( \|s\| \leq \|\{t_i\}\| \), where \( s = \sum_{i=1}^{\infty} t_i e_i \).

On the other hand, \( s \to 0 \) implies \( \|\{t_i\}\| \to 0 \), by the equicontinuity of the operators \( P_n \). Hence the map associating the element \( s \) with the sequence \( \{t_i\} \) is an isomorphism continuous in both directions. By Theorem 3.4. 1, the space \( X_1 \) is complete. Since the space \( X_2 \) is dense in the space \( X_1 \), we have \( X_2 = X_1 \).

**COROLLARY 5.4.** Let \( X \) be a complete linear metric space with a basis \( \{e_n\} \). Let \( t_1, t_2, \ldots \) be an arbitrary sequence of numbers, and let \( P_0, P_1, \ldots \) be an increasing sequence of indices. Then the sequence \( \{t'_n\} \), where
\[
t'_n = \sum_{i=1}^{P_n} t_i e_i,
\]
is a basis of the space spanned by \( \{e_i\} \).

**COROLLARY 5.5.** A sequence \( \{e_n\} \) of linearly independent elements of a linear metric space \( X \) is a basis of this space if and only if the following two conditions are satisfied:

1. Linear combinations of elements \( e_n \) are dense in the space \( X \),

2. Operators \( P_n x = x_{|P_n|} \) are equicontinuous in the space \( \mathbb{E} \) of \( \{e_n\} \).

**COROLLARY 5.6.** A sequence \( \{e_n\} \) of linearly independent elements of a locally bounded complete space \( X \) with a \( p \)-homogeneous norm \( \|\cdot\| \) is a basis of \( X \) if and only if the following two conditions are satisfied:

1. Linear combinations of elements \( e_n \) are dense in the space \( X \),

2. There exists a number \( X \) such that
\[
\left| \sum_{i=1}^{n} t_i e_i \right| \leq X \left| \sum_{i=1}^{n} t_i e_i \right|
\]

for an arbitrary \( n \).

\[\text{§ 5. Basis of a linear metric space}\]

**THEOREM 5.7.** Let \( X \) be a complete linear metric space with a basis \( \{e_n\} \). Let \( \{e_n\} \), \( \|e_n\| = 1 \), be a sequence of elements of the form
\[
x_n = \sum_{i=1}^{\infty} t_i^n e_i, \quad \text{where} \quad \lim_{n \to \infty} t^n_i = 0.
\]

If \( \{e_n\} \) is an arbitrary sequence of positive numbers, then there exist an increasing sequence of indices \( \{p_n\} \) and a subsequence \( \{x_{p_n}\} \) of the sequence \( \{x_n\} \) such that
\[
\left\| x_{p_n} - \sum_{i=1}^{p_n} t_i^n e_i \right\| < \varepsilon.
\]

Proof — by induction. Let \( p_1 = 0 \), \( x_{p_1} = x_1 \). We denote by \( p_k \) an index satisfying the inequality
\[
\left| x_{p_k} - \sum_{i=1}^{p_k} t_i^n e_i \right| < \varepsilon.
\]

Let us suppose that the element \( x_{p_{k+1}} \) and the number \( p_{k+1} \) are already chosen. The assumption \( \lim_{n \to \infty} t^n_i = 0 \) implies the existence of an element \( x_{p_{k+1}} \) such that
\[
\left| \sum_{i=p_{k+1}}^{p_{k+1}} t_i^n e_i \right| < \frac{\varepsilon}{2} \varepsilon.
\]

Let \( p_{k+1} \) be a number satisfying the inequality
\[
\left| x_{p_{k+1}} - \sum_{i=1}^{p_{k+1}} t_i^n e_i \right| < \frac{\varepsilon}{2} \varepsilon.
\]

Then
\[
\left| x_{p_{k+1}} - \sum_{i=1}^{p_{k+1}} t_i^n e_i \right| < \varepsilon.
\]

**THEOREM 5.8.** If a locally bounded space \( X \) has a basis \( \{e_n\} \), then every infinitely dimensional subspace \( X \times C \subset X \) contains a subsequence \( \{e_n\} \)
\[
= \left( \sum_{i=1}^{\infty} t_i^n e_i, \|e_n\| = 1, \text{such that} \lim_{n \to \infty} t^n_i = 0 \text{ for } i = 1, 2, \ldots \right).
\]

Proof. Let us suppose that the theorem is false. There exists a positive integer \( k \) such that the conditions \( x \in X, \|x\| = 1, x = \sum_{i=1}^{k} t_i e_i \) imply \( \|x\| = \max \{t_i\} > c \). Hence there exists a one-to-one transformation of the space \( X_b \) onto the space \( X_b \) of all systems of numbers \( \{t_1, \ldots, t_k\} \) which is continuous in both directions. Consequently, \( X_b \) is a finite-dimensional space, which contradicts the assumption.

**Example 6.1.** The sequence

\[ e_n = \{ \delta_{nk} \}, \quad n = 1, 2, \ldots \]

(\( \delta_{nk} \) being the Kronecker symbol) is a basis in spaces \( e_n \) and \( l^p \), \( p > 0 \).

This basis is called a *standard basis*.

**Example 6.2.** There exists also a Schauder basis in the space \( C[0, 1] \) (Schauder [7]). It is constructed in the following manner:

![Schauder basis](image)

We define a function \( u_n(t) \) (\( 0 \leq t < 2^k; \ k = 0, 1, \ldots \)):

- if \( t \in [\frac{k}{2^k}, \frac{k+1}{2^k}] \), then \( u_n(t) = 0 \);
- if \( t \in [\frac{k}{2^k}, \frac{k+1}{2^k}] \), then the graph of \( u_n(t) \) is an isosceles triangle with altitude \( 1 \).

Every continuous function \( x(t) \) in the interval \([0, 1]\) can be uniquely written in the form of a series

\[ x(t) = a_0 + a_1(t - t_1) + \sum_{k=0}^{\infty} \sum_{n=1}^{2^k} a_{nk} u_n(t) , \]

where \( a_0 = x(0) \), \( a_1 = x(0) \), and the coefficients \( a_{nk} \) can be uniquely determined by a certain geometric construction. Namely, we draw the chord \( t(t) \) of the area \( x = x(t) \) through the points \( i/2^k \) and \( (i+1)/2^k \). The number \( a_{nk} \) is given by means of the formula

\[ a_{nk} = \left( \frac{2^k+1}{2^{k+1}} \right) - \left( \frac{2^k+1}{2^{k+1}} \right) . \]

Evidently, the graph of the partial sum

\[ a_0 t + a_1(1 - t) + \sum_{k=0}^{n} \sum_{n=1}^{2^k} a_{nk} u_n(t) \]

is a polygon with \( 2^n+1 \) vertices lying on the curve \( x = x(t) \) at points with equidistant abscissae. It is proved that the sequence of functions

\[ t, 1-t; \ u_n(t); \ u_{n+1}(t); \ u_{n+2}(t); \ u_{n+3}(t); \ u_{n+4}(t); \ u_{n+5}(t); \ldots \]

is a basis of the space \( C[0, 1] \).

**Example 6.3.** Let \( H \) be a Hilbert space with a scalar product \((x, y)\). A sequence of elements \( \{e_n\} \), \( e_n \neq 0 \), is called *orthogonal* if \((e_m, e_n) = 0\) for \( m \neq n \). If, moreover, \([e_n] = V(e_n, e_n) = 1\), the sequence \( \{e_n\} \) is called *orthonormal*.

Every orthogonal sequence \( \{e_n\} \) is a basis of the space \( H = \text{lin} \{e_n\} \).

Indeed, if \( x = \sum a_n e_n \), then

\[ ||x||^2 = (\sum a_n e_n, \sum a_n e_n) = \sum a_n ||e_n||^2 = ||a||^2 . \]

Hence

\[ ||x||^2 = \sum a_n ||e_n||^2 \leq ||a||^2 . \]

By Theorem 4.3, the sequence \( \{e_n\} \) is a basis.

Let us remark that if \( \{e_n\} \) is an orthonormal sequence, then the coefficients of expansions of elements \( x \in H \) constitute the space \( P \), and the map \( x \mapsto \{a_n\} \) is an isometry, i.e., the norm \( ||x|| \) in the space \( H \) is equal to the norm \( ||{a_n}|| \) in the space \( P \).

If linear combinations of elements \( e_n \) are dense in the space \( H \), then \( H = H \) and the sequence \( \{e_n\} \) is a basis of the space \( H \). A basis made of elements of an orthogonal (orthonormal) sequence is called an *orthogonal* (orthonormal) *basis*.

Since the space \( H \) and \( P \) are isometric, a necessary and sufficient condition for \( H = H \) is that \( a_n = (x, e_n) = 0 \) for \( n = 1, 2, \ldots \), should imply \( x = 0 \).

In every separable Hilbert space there exists an orthonormal basis. Indeed, let \( \{a_n\} \) be a sequence of elements such that linear combinations of \( a_n \) are dense in the space \( H \). Without loss of generality we may suppose that all elements \( a_n \) are linearly independent. We construct an orthonormal sequence \( \{e_n\} \) by induction. We require the subspace spanned by elements \( a_1, \ldots, a_n \) and by elements \( e_1, \ldots, e_n \) to be equal. Let us take
§ 6. Examples of bases

Because taking

\[ w(t) = \begin{cases} +1 & \text{for } \frac{1}{2} < t \leq 1, \\ -1 & \text{for } \frac{1}{4} < t \leq \frac{1}{2}, \end{cases} \]

we have \((x, r_j) = 0\) for all \(j\), but \(x \neq 0\).

We now give the construction of the Haar orthonormal system. Let

\[ h_{1,0}(t) = 1 \quad \text{for } t \in [0, 1], \]
\[ h_{n,0}(0) = 1 \quad \text{for } j > 0, \]
\[ h_{n,j}(t) = \begin{cases} 1 & \text{for } (j-1)2^n < t \leq (j-1)2^n + 1, \\ -1 & \text{for } (2j-1)2^n + 1 < t \leq (2j-1)2^n, \\ 0 & \text{for remaining } t \in [0, 1] \quad (j = 0, 1, \ldots, 2^n; \; n = 1, 2, \ldots). \end{cases} \]

The sequence

\(1, h_{0,1}, h_{1,1}, h_{2,1}, h_{3,1}, h_{4,1}, h_{5,1}, \ldots\)

is an orthogonal system. Dividing functions of this system by their norms in the space \(L^2[0, 1]\), we obtain an orthonormal system \(\{h_m\}\), where

\[ h_m = h_{m,0} ||h_{m,0}||, \quad m = 2^n + j. \]

This system is called the Haar system.

--

Fig. 11. The Haar system

We show that every simple function

\[ g = \sum_{i=1}^{\infty} h_{n_i}(t) \begin{cases} 1 & \text{if } \frac{1}{2^n} \leq t < \frac{1}{2^{n+1}}, \\ 0 & \text{otherwise}, \end{cases} \]

can be written as a linear combination of the functions

\[ h_{1,0}, h_{1,1}, \ldots, h_{n,m}, \ldots. \]

Proof. — By induction with respect to \(n\). If \(n = -1\) the theorem is obvious. Let us suppose that the theorem is true for \( n < k \). Equations in linear spaces
Let a function
\[ g(t) = \sum_{k=1}^{2^n} b_k X^{t - \frac{k}{2^n}} \]
be given, and let
\[ g_d(t) = \sum_{k=1}^{2^n} b_{k-1} + \frac{b_k}{2} X^{t - \frac{k}{2^n}} \]
\[ h_k = \frac{b_k}{2} X^{t - \frac{k}{2^n}} \]

It is easily verified that
\[ g(t) = g_d(t) + \sum_{k=1}^{2^n} b_{k-1} + b_k h_k \]

By the induction hypothesis, \( g_d(t) \) is a linear combination of functions
\( h_{k-1}, \ldots, h_{k-2^n+1} \). Hence \( g(t) \) is a linear combination of functions
\( h_{k-1}, \ldots, h_{k+2^n} \). \[ \square \]

Since the functions of the form
\[ g(t) = \sum_{k=1}^{2^n} b_k X^{t - \frac{k}{2^n}} \]
are dense in the space \( L^2(0,1) \), the Haar system is a basis of this space.

**Example 6.4.** The Haar system is a basis of the space \( L^2[0,1] \), \( p \geq 1 \).

Indeed, let us write
\[ g_{n,i} = X^{t - \frac{i}{2^n}} \]

In the same manner as in the last example one can prove that each of the functions
\[ g_{n,i+1}, \ldots, g_{n,2^n}, g_{n+1,1}, \ldots, g_{n+1,2^n} \]
can be written as a linear combination of functions \( h_{k}, \ldots, h_{k+2^n} \). The functions \( g_{n,i+1}, \ldots, g_{n+1,2^n} \) form an orthogonal system. Hence the projection operator \( P_n \) on the space \( H_n \) spanned by the elements \( h_k, \ldots, h_m \) can be written in two ways:
\[ [x]_m = P_n x \]

where \( P_n = \sum_{j=1}^{m} (x, h_j) h_j \) and
\[ P_n = \frac{1}{\|g_{n+1,1}\|} (x, g_{n+1,1}) g_{n+1,1} + \frac{1}{\|g_{n+1,2^n}\|} (x, g_{n+1,2^n}) g_{n+1,2^n} \]

Since
\[ [x(t)]_m = \int_{I_i} x(t) dt \quad \text{for} \quad \tau \in I_i \]

\[ I_i = \left\{ \frac{j}{2^n}, \frac{j+1}{2^n} \right\} \quad \text{for} \quad i \leq \frac{2^n - 1}{2^n} \]

we get
\[ \|P_n x\| = \|x\| = \left( \sum_{i=1}^{m} \|f_i\| \right)^{1/p} \]

for Hölder's inequality (see the Appendix) implies
\[ \int_{I_i} |x(t)| dt = \frac{1}{|I_i|} \int_{I_i} |x(t)| dt \leq \left( \sum_{i=1}^{m} \int_{I_i} |x(t)|^p dt \right)^{1/p} |I_i|^{-1/p} \]

Hence it follows that the operators \( P_n \) are linear and continuous, with the norms \( \|P_n\| \leq 1 \) (substituting \( \alpha = 1 \) we find that \( \|P_n\| = 1 \)). Moreover, the operators \( P_n x = [x]_m \) are equicontinuous. Hence the Haar system is a basis of the space \( L^p([0,1]) \), by Theorem 4.3.

**§ 7. Continuous operators in spaces with a basis.** Let us suppose that \( \{e_n\} \) and \( \{f_n\} \) are bases in the linear metric spaces \( X \) and \( Y \), respectively. Elements of the spaces \( X \) and \( Y \) can be represented by means of sequences of coefficients of expansions with respect to the bases. Those sequences will be denoted by \( \{x_n\} \) for \( x \in X \) and by \( \{y_n\} \) for \( y \in Y \). To any map \( y = A x \), \( A \in B_r(X \to Y) \), the corresponding map is denoted by the same letter \( A \).

Thus, with every operator \( A \) one can associate the matrix of transformation of the coefficients
\[ A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \]

(7.1)

Let
\[ x = \sum_{k=1}^{n} \xi_k e_k \in X, \quad y = \sum_{k=1}^{n} \eta_k f_k \in Y \]

We consider the operators

\[ A_n x = [A_n x]_n \quad \text{and} \quad A_m x = [A_m x]_m. \]

We write

\[
[a_{ik}]_n = \begin{cases} a_{ik} & (i \leq n, \ k \leq m), \\ 0 & (i > n \text{ or } k > m). \end{cases}
\]

Applying this notation we introduce the following matrices:

\[
\begin{bmatrix}
[a_{11}]_n & [a_{12}]_n & \cdots & [a_{1k}]_n \\
[a_{21}]_n & [a_{22}]_n & \cdots & [a_{2k}]_n \\
\cdots & \cdots & \cdots & \cdots \\
[a_{n1}]_n & [a_{n2}]_n & \cdots & [a_{nk}]_n
\end{bmatrix}
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\
0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
[a_{11}]_m & [a_{12}]_m & \cdots & [a_{1k}]_m \\
[a_{21}]_m & [a_{22}]_m & \cdots & [a_{2k}]_m \\
\cdots & \cdots & \cdots & \cdots \\
[a_{n1}]_m & [a_{n2}]_m & \cdots & [a_{nk}]_m
\end{bmatrix}
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\
0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0
\end{bmatrix}
\]

It is easily seen that these matrices correspond to the operators \( A_n \) and \( A_m \) introduced before. Indeed, if we put \( y = Ax \) and \( A_n x = ([a_{nk}]_n), \)

\[
A_{nm} x = ([a_{nm}]_m),
\]

we obtain

\[
[a_{nk}]_n = \sum_{i=1}^{\infty} a_{ik} \xi_i \quad (i \leq n),
\]

\[
0 \quad (i > n),
\]

i.e.

\[
[y]_n = \sum_{i=1}^{\infty} [a_{ik}]_n \xi_i \quad (i = 1, 2, \ldots).
\]

Hence it follows that

\[
[y]_m = \sum_{i=1}^{\infty} [a_{ik}]_n \xi_i = \sum_{i=1}^{\infty} [a_{ik}]_{nm} \xi_i \quad (i = 1, 2, \ldots).
\]

**Theorem 7.1.** (Cohen–Dunford [1].) If linear metric spaces \( X \) and \( Y \) possess bases and \( A \in B_d(X \rightarrow Y) \), then for every \( x \in X \)

\[
\lim_{n \to \infty} A_n x = A x \quad \text{and} \quad \lim_{m \to \infty} A_{nm} x = A x.
\]

**Proof.** Since \( \lim_{n \to \infty} [y]_n = y \), we have

\[
\lim_{n \to \infty} A_n x = \lim_{n \to \infty} [Ax]_n = Ax \quad (x \in X).
\]
§ 1. Closed operators. Let $X$ and $Y$ be linear topological spaces. We have called the set

$$W_A = \{(x, y): x \in D_A, y = Ax\} \subset X \times Y$$

the graph of the operator $A \in L(X \to Y)$ (compare § 1, A I).

We say that an operator $A \in L(X \to Y)$ is closed if its graph is closed. If $X$ and $Y$ are linear metric spaces, then this condition can be formulated as follows: an operator $A \in L(X \to Y)$ is closed if the conditions $x_n \to x$ and $Ax = y$ imply $x \in D_A$ and $y = Ax$.

**Theorem 1.1.** Let $X$ and $Y$ be linear topological spaces. If $A \in B(X \to Y)$ and if the domain $D_A \subset X$ is closed, then the operator $A$ is closed.

**Proof.** It is sufficient to show that the complement of the graph $W_A$ is open. Let $(a_0, y_0) \in W_A$. If $a_0 \in D_A$, then there exists a neighbourhood of zero $U$ such that

$$(a_0 + U) \cap D_A = 0.$$

By the definition of the graph, it follows that for every neighbourhood of zero $V$ in the space $X$ we have

$$W[(a_0, y_0), U, V] \cap W_A = 0,$$

where $W[(a_0, y_0), U, V]$ is the neighbourhood of the point $(a_0, y_0)$ in the product $X \times Y$, determined by the neighbourhoods $U$ and $V$. If $a_0 \notin D_A$ and $(a_0, y_0) \notin W_A$, there exists a neighbourhood of zero $V$ in the space $Y$ such that $y_0 \notin Ax + V$. Let $V_1$ be a neighbourhood of zero in $X$ satisfying the condition $V_1 + V_1 \subset V$. It follows from the continuity of the operator $A$ that there exists a neighbourhood of zero $U$ in the space $X$ such that $Ax + V \subset Ax + V_1$. It is easily verified that

$$W[(a_0, y_0), U, V] \cap W_A = 0.$$

If the domain $D_A$ of a continuous operator $A$ is not closed, it is evident that $A$ is not closed.

**Theorem 1.2.** If $X$ and $Y$ are linear topological spaces and if a closed operator $A \in L(X \to Y)$ is one-to-one, then the inverse operator $A^{-1}$ is closed.

---

**Proof.** The graph of the inverse operator $A^{-1}$ is a subset of the product of spaces $Y \times X$ of the form

$$W_{A^{-1}} = \{(y, A^{-1}y): y \in Y\}.$$

The transformation of the product $Y \times X$ onto the product $X \times Y$ associating the pair $(x, y)$ with the pair $(y, x)$ is an isomorphism, continuous in both directions, which maps the graph $W_{A^{-1}}$ onto the graph $W_A$. The graph $W_A$ is closed by hypothesis. Hence the graph $W_{A^{-1}}$ is also closed.

There exist closed discontinuous operators. Indeed, it is sufficient to consider a continuous one-to-one operator which does not possess a continuous inverse, for example the integral operator

$$\int x(t) \, dt$$

which maps the space $C[0, 1]$ of continuous functions into itself. The inverse operator is the differential operator $d/dt$, which is discontinuous and closed. It is defined in the set of all differentiable functions in the interval $[0, 1]$.

**Theorem 1.3.** (Banach [23]) If $X$ and $Y$ are complete linear metric spaces and $A \in L(X \to Y)$ is a closed operator, then $A$ is continuous.

**Proof.** By hypothesis, the graph $W_A$ of the operator $A$ is a closed linear subspace of the complete metric space $X \times Y$. Hence $W_A$ is also a complete metric space. But the projection operator $P$ of the space $W_A$ onto the space $X$ is continuous, one-to-one and linear; hence it is an isomorphism. Since the inverse of $P$ is the operator associating the pair $(x, Ax)$ with the element $x \in X$, $A$ is a continuous operator.

**Corollary 1.4.** Let $X$ and $Y$ be complete linear metric spaces. If $A, B : L(X \to Y)$ are closed operators and $D_B \supset D_A$, then the operator $B$ is $A$-continuous.

**Proof.** By hypothesis, $B : L(X \to Y)$. Since the topology in the space $X_A$ is finer than the topology, $B$ is a closed operator which maps the space $X_A$ into $Y$. Let us remark that the space $X_A$ is isomorphic with the graph $W_A$ of the operator $A$. Since the graph $W_A$ is closed, the space $X_A$ is complete. By Theorem 1.3, the operator $B$ maps the space $X_A$ into the space $Y$ continuously.

If an operator $A : B(L(X \to Y)$ is given and $Y$ is a complete space, then the operator $A$ can be uniquely extended to an operator $A : B(L(X \to Y)$, where $X$ is the completion of the space $X$ (Theorem 1.4, II). This theorem does not hold for closed operators, as the following example shows:
EXAMPLE 1.1. Let $Y = \mathcal{C}$. We define in $Y$ a continuous operator $A^{-1}$ in the following manner: $A^{-1}(y_a) = (y_{a+1} + y_{a}/n)$. The operator $A^{-1}$ maps the space $\mathcal{C}$ into itself, but it is not one-to-one. Indeed, $A^{-1}(1, 0, 0, 0, \ldots) = (1/2) = A^{-1}(0, 1, 1/2, \ldots)$. However, if we limit ourselves to the space $Y_0$ of sequences of a finite number of elements $y_a \neq 0$, $A^{-1}$ is a one-to-one map. Let $X = A^{-1}(Y_0)$. The set $X$ is dense in the space $\mathcal{C}$.

The operator $A^{-1}$ which maps the space $Y_0$ into the space $X$ is closed. Indeed, if $y_{a} \to y$, $A^{-1}y_{a} \to z$, then $z \in X$. Hence $y \in Y_0 = D_{A^{-1}}$ and $Ay = x$.

By Theorem 1.2, the operator $A = (A^{-1})^{-1}$ is closed in the space $X$; hence its graph $W_A$ is closed. However, the closure of $W_A$ in the space $X \times Y$ is a closed set, but it is not the graph of the operator, for $A^{-1}$ is not one-to-one on the whole space $\mathcal{C}$.

THEOREM 1.5. Let $X$ and $Y$ be linear topological spaces, and let $A \in L(X \to Y)$ be closed and $B \in L(Y \to X)$ be continuous. Then $A + B$ is a closed operator.

Proof. We prove that the map of the graph $W_A$ of the operator $A$ onto the graph $W_{A+B}$ of the operator $A + B$ associating the point $(x, (A+B)x) \in W_{A+B}$ with the point $(x, Ax) \in W_A$ is a continuous operator. If $U \subseteq W_{A+B}$ is a neighbourhood, we take $U_0 = \{(x, 0) : (x, y) \in U\}$. Evidently, every neighbourhood $V \subset X$ there exists a neighbourhood $U$ such that $U_0 \subset V$. Let $W$ an arbitrary neighbourhood of zero in the graph $W_{A+B}$. Since $B$ is a continuous operator, there exists a neighbourhood $U_0$ such that $(U_1 - B(U_0)) \cap W_A \subset W$. Hence the map defined above is continuous. Thus the graph $W_{A+B}$ is closed as an inverse image of the closed set $W_A$.

Hence it follows that continuous operators are perturbations of closed operators. A sum of two closed operators is not necessarily a closed operator. Indeed, let $A$ be an arbitrary closed operator whose domain $D_A$ is not closed, and let $B$ be a continuous operator. Let $A_1 = A + B$, $A_2 = -A$. Evidently, the operators $A_1$ and $A_2$ are closed. Their sum is a continuous operator $B$ defined in a domain $D_B$ which is not closed. Hence this sum is not a closed operator.

§ 2. $\Phi$-operators. Let $X$ and $Y$ be linear topological spaces. A closed operator $A \in L(X \to Y)$ is called normally resolvable if the set $E_A$ of its values is closed.

A normally resolvable operator

with finite $d$-characteristic will be called $\Phi$-operator,

with nullity will be called $\Phi_+^-$-operator,

with deficiency will be called $\Phi_-$-operator.

(Gohberg and Krein [1]).

We denote by $Y^+$ the set of all continuous linear functionals defined on the space $Y$ and having values in a field of scalars. Obviously this is a linear space. The corresponding operator conjugate to an operator $A \in L(X \to Y)$ will be denoted by $A^+$. (See also § 1, A III.) This operator is well defined only if the spaces $X^-$ and $Y^+$ are total.

In § 5, A III, we have defined $\Phi_+^-$-operators as operators whose $d_\ast$-characteristics are equal to their $d$-characteristics. According to these definitions a normally resolvable $\Phi_+^-$-operator is a $\Phi$-operator.

THEOREM 2.1. If $X$ and $Y$ are linear topological spaces and $Y^-$, $Y^+$ are total spaces, then every normally resolvable operator $A \in L(X \to Y)$ with a finite $d$-characteristic is a $\Phi$-operator.

Proof. By hypothesis, the set $E_A$ is a closed subspace with a finite defect, i.e., there exists a system of elements $y_1, \ldots, y_k$ such that every element $y \in Y$ can be written in the form

$$y = y_0 + \sum_{i=1}^k a_i y_i,$$

in a unique manner.

The functionals $\eta(y) = a_i$ are linear. Since $E_A$ is a closed set, they are continuous. Hence $\beta_A = a_{\ast},$ where $A^\ast$ is the conjugate of the operator $A$.

THEOREM 2.2. (Atkinson [1].) Let $X$, $X$, $Z$ be linear topological spaces, and let $A \in L(X \to Y)$ and $B \in L(Y \to X)$ be $\Phi$-operators. If the set $D_A$ is dense in the space $X$, then the superposition $AB$ is a $\Phi$-operator and

$$\eta_{AB} = \eta_A + \eta_B.$$

The proof is based on the following lemma:

LEMMA 2.3. Let a linear topological space $X$ be a direct sum of the form

$$X = R \oplus F,$$

where $F$ is a finite-dimensional space. If a linear set $D$ is dense in the space $X$, then the set $D_A = D \cap R$ is dense in $R$, and $D$ can be written as a direct sum:

$$D = D_A \oplus F',$$

where $F' \subset D$.

Proof. We denote a basis of the space $F$ by $(e_1, \ldots, e_m)$. We define linear functionals $f_i$ in the space $F^+$ as follows:

$$f_i(e_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

We now define extensions $f_i$ of functionals $f_i$ on the whole space $Y$ in the following manner: if $y \in X$, then $y = s + z$, where $s \in R$, $z \in F$; we assume that $f_i(y) = f_i(z)$. By definition, the functionals $f_i$ are continuous.
Moreover, \( f(y) = 0 \) for \( y \in E \) and, conversely, \( f_i(y) = 0 \) for \( i = 1, 2, \ldots, n \) implies \( y \in E \).

Since the set \( D \) is dense, one can choose a set of points \( z_1, \ldots, z_n \in D \) such that \( \text{det}(f(z)) \neq 0 \).

For every \( y \in E \) and for every neighbourhood of zero \( U \) we have \( (y + U) \cap D \neq \emptyset \). Let \( z_m \in (y + U) \cap D \) and let

\[
\tilde{z}_m = z_m + \sum_{k=1}^n a_k y_k,
\]

where the numbers \( a_k \) are chosen in such a manner that \( \tilde{z}_m \in D \). Such numbers \( a_k \) exist. Indeed, \( \tilde{z}_m \in D \) if and only if

\[
f_k(\tilde{z}_m) = 0 \quad \text{for } i = 1, 2, \ldots, n.
\]

Thus we obtain a system of linear equations with the coefficients \( a_k \) as unknowns:

\[
\sum_{k=1}^n a_k y_k + f(\tilde{z}_m) = 0 \quad (i = 1, 2, \ldots, n).
\]

By hypothesis, \( \text{det}(f(z)) \neq 0 \). Hence this system of equations has a solution.

Let \( \delta \) be an arbitrary positive number. Since \( f_i \) are continuous functions, there exists a neighbourhood of zero \( U \) such that if \( z \in U \), then

\[
f_i(z) < \delta \quad (i = 1, 2, \ldots, n).
\]

On the other hand, if \( \varepsilon \) is an arbitrary positive number, then there exists a \( \delta > 0 \) such that the condition \( |f(z)| < \delta \) implies that the solutions \( a_k \) of the system of equations (2.1) are absolutely less than \( \varepsilon \). Moreover, if \( U \) is an arbitrary neighbourhood of zero, there exists an \( \varepsilon > 0 \) such that the inequalities \( |a_k| < \varepsilon \) imply \( \sum_{k=1}^n a_k y_k \in U \). Generally, we may conclude that for an arbitrary neighbourhood of zero \( U \), there exists a neighbourhood of zero \( U \) such that \( \sum_{k=1}^n a_k y_k \in U \).

Let \( U \) be an arbitrary neighbourhood of zero. Let \( U \) be a neighbourhood of zero satisfying the condition \( \text{cond} \), \( \text{cond} : U_+ \cup U_0 \subset U \), and let \( U \) be a neighbourhood of zero constructed in the manner described above. Writing \( U = U_+ \cup U_0 \), we get \( y + U \cap D \neq \emptyset \), by hypothesis. Let \( z_m \in (y + U) \cap D \). Then \( \tilde{z}_m \in z_m + U_0 \). Hence \( \tilde{z}_m \in (y + U) \cap D \). By hypothesis, we have \( \tilde{z}_m \in D \). Therefore, \( D \cap D \) is dense in \( U \).

Let \( P = \text{lin}(e_1, \ldots, e_n) \). It is easily verified that \( D = D_0 \cap P \).

Proof of Theorem 2.2. There is only one difference between the proof of this theorem and the proof of Theorem 1.1, A. \( \Xi \). Namely, defining decomposition (1.4) one has to require additionally that \( G \in D_0 \). This can be obtained by applying the above lemma. Moreover, one has to remark that \( AD_0 = E_0 \) and \( AD_0 = E_{AD_0} \), where \( D_0 = D_0 \cap D_0 \).

§ 3. Operators conjugate to \( \Phi \)-operators. Let a linear topological space \( X \) be given. It may happen that \( X \) contains only a trivial functional (i.e., a functional equal to zero everywhere), for example as in the case of the space \( S(0, 1) \).

By Theorem 1.3, the set \( X \) is linear. It may be considered as a linear topological space with the topology of bounded convergence. This topology will be called the strong topology.

Theorem 3.1. If \( X \) and \( Y \) are linear topological spaces, \( X^+ \) and \( Y^+ \) are total spaces and \( X \subset B_0 \), then the conjugate operator \( A^* \) is \( (X^+ \to Y^+) \) is continuous in the strong topology.

Proof. Evidently, the general properties of conjugate operators imply that the operator \( A^* \) is linear. Let \( U \) be a neighbourhood of zero in the space \( X^+ \). This neighbourhood contains a neighbourhood \( U_0 \) of the form

\[
U_0 = \{ \xi : |\xi(x)| < \varepsilon : x \in B \},
\]

where \( B \) is a certain bounded set. Let \( B_1 = A \subset B_1 \). Evidently, the set \( B_1 \subset Y \) is bounded. Let \( V \) be a neighbourhood of zero in the space \( Y^+ \) of the form

\[
V = \{ \eta : |\eta(y)| < \varepsilon, y \in B_1 \}.
\]

Let us consider the set

\[
A^*V = \{ \xi : \xi = A^*\eta, \eta \in V \} = \{ \xi : A^* \in \eta(x) < \varepsilon : x \in B_1 \}
\]

Thus proves the continuity of the operator \( A^* \).

Corollary 3.2. Let \( X \) and \( Y \) be complete linear metric spaces. Let \( X^+ \) and \( Y^+ \) be total spaces. If a \( \Phi \)-operator \( A \) belongs to \( L(X^+ \to Y^+) \), then the conjugate operator \( A^* : (X^+ \to Y^+) \) is a \( \Phi \)-operator.

Proof. As in the proof of Theorem 4.1. A. \( \Xi \), using in addition the fact that \( E_0 \) is a closed set we conclude that \( A^* \) is a direct sum \( E_0 \). On the other hand, applying Lemma 2.3 we write the set \( D_0 \) as a direct sum \( D_0 = E_0 \oplus D_1 \), and the space \( Y \) as a direct sum \( Y = E_0 \oplus G_0 \). The operator \( A \) considered as a map of the set \( D_0 \) onto the set \( E_0 \) is one-to-one and closed; by Theorem 1.2, the inverse operator \( A^{-1} \) is closed. By hypothesis, the set \( E_0 \) is a complete space, as a closed subset of a complete space. Hence the operator \( A^{-1} \) is continuous, by Theorem 1.3. Hence it follows that every
continuous linear functional \( \xi \) defined on the set \( D \) is the image of some continuous linear functional defined on \( D \) by means of the conjugate operator \( A^+ \). Indeed, \( \xi = A^+ \eta \), where \( \eta(y) = \xi(A^{-1}(y)) \). Since \( X = Z \oplus D \), and since every continuous linear functional defined on \( D \) can be extended to a continuous linear functional defined on \( D \), in one way only, we have

\[
E_{A^*} = \{ \xi : \xi(x) = 0 \text{ for } x \in Z \}.
\]

Hence the operator \( A^+ \) is normally resolvable and \( \beta_{A^*} = \alpha_A \). Thus we have proved that \( A^+ \) is a \( \Phi \)-operator. \( \blacksquare \)

CHAPTER IV

COMPACT OPERATORS IN LINEAR TOPOLOGICAL SPACES

§ 1. Compact and precompact sets. In § 1, I, a subset \( K \) of a linear topological space \( X \) was called compact if every covering of this set contains a finite subcovering. A subset \( K \) of a linear topological space \( X \) is called relatively compact if its closure is a compact set.

![Fig. 12. \( \varepsilon \)-net of the set \( E \)](image)

We say that a subset \( K \) of a linear topological space \( X \) is \emph{precompact} if for every neighbourhood \( V \subseteq X \) there exists a finite system of points \( x_1, \ldots, x_n \in X \) such that \( K \subseteq \bigcup_{i=1}^n (x_i + V) \).

If \( X \) is a linear metric space, a set \( K \subseteq X \) is precompact if and only if for every positive number \( \varepsilon \) there exists a \emph{finite} \( \varepsilon \)-net, i.e. a system of points \( x_1, \ldots, x_n \) such that for every point \( x \in K \) there exists an index \( i \) satisfying the inequality \( \varepsilon(x, x_i) < \varepsilon \).
A subset of a precompact set is precompact. If the sets $E_1$ and $E_2$ are precompact, then the set $E_1 \cup E_2$ is precompact.

**Theorem 1.1.** If the sets $E_1$ and $E_2$ are precompact, then the set $E_1 \cup E_2$ is precompact.

**Proof.** Let $U$ be an arbitrary neighbourhood of zero, and let $V$ be a neighbourhood of zero such that $V + V \subset U$. By hypothesis, there exist finite systems of points $x_1, x_2, \ldots, x_m \in X$ and $x_1', x_2', \ldots, x_m' \in X$ satisfying the conditions

$E_1 \subset \bigcup_{i=1}^{m} (x_i + V)$ and $E_2 \subset \bigcup_{i=1}^{m} (x_i' + V)$.

Let $y_{ij} = x_i + x_j$; then

$E_1 + E_2 \subset \bigcup_{i=1}^{m} \bigcup_{j=1}^{m} (y_{ij} + V) \subset \bigcup_{i=1}^{m} \bigcup_{j=1}^{m} (y_{ij} + U)$.

Hence the set $E_1 + E_2$ is compact. ■

From the definition of compactness it follows immediately that every relatively compact set in a linear topological space is precompact. The converse theorem is not true in general. However, it holds for complete spaces, as follows from the next theorem:

**Theorem 1.2.** (Bourbaki [1]). If a linear topological space $X$ is complete, then every precompact closed set $K \subset X$ is compact.

**Proof.** Let $F_s$ be an arbitrary filter. We refine this filter by an ultrafilter $F$ made of subsets of the set $K$. We show that the ultrafilter $F$ is a fundamental family. Indeed, if $V$ is a precompact neighbourhood of zero, there exists a system of points $x_1, x_2, \ldots, x_m$ such that $K \subset \bigcup_{i=1}^{m} (x_i + V)$.

However, the properties of ultrafilters (see § 1, I) imply the existence of a point $x_1$ such that $A_{x_1} = (x_1 + V) \cap K \in F$. But $A_{x_1} = A_{x_1} \cap Y - V$. Since the neighbourhood $V$ is arbitrary, the ultrafilter $F$ must be a fundamental family.

Since the space $X$ is complete, the family $F$ has a cluster point $x$. But the set $K$ is closed. Hence $x \in K$, and since $x$ is a cluster point of the ultrafilter $F$, it is a cluster point of the filter $F_s$. Thus it follows from Theorem 1.1, I, that $K$ is a compact set. ■

**Theorem 1.3.** Let $K$ be a precompact set. If $V$ is a neighbourhood of zero and $\{x_n\} \subset K$ is a directed family of points such that $x_n + V \subset Y$, then the family $\{x_n\}$ is finite.

**Proof.** Suppose that the family $\{x_n\}$ is infinite. Let $U$ be a balanced neighbourhood of zero satisfying the condition $U + U \subset V$. The condition $x_n + V$ implies $(x_n + U) \cap (x_n + U) = \emptyset$. Hence if we take any point $y$ and suppose that $\alpha \neq \gamma$, the points $x_\alpha$ and $x_\gamma$ cannot both belong to the set $y + U$. Consequently, there is no finite system of points $y_1, \ldots, y_n$ such that $K \subset \bigcup_{i=1}^{n} (y_i + U)$. Thus the set $K$ is not compact, which contradicts the assumption. ■

In our further considerations we shall need the following theorem of a purely topological character:

**Theorem 1.4.** (Lebesgue [1]). If $Y$ is a topological space, $K$ a compact space, $f(t, k)$ a continuous transformation of the product $Y \times K$ into a topological space $X$, and $F$ a closed set in the space $X$ such that $F$ does not intersect the set $(f(t, K))$, then there exists a neighbourhood $V$ of the point $t$ such that $F$ does not intersect the set $f(V, K)$.

**Proof.** Let $k \in K$. There exists a neighbourhood $V(k)$ of the point $t$ and a neighbourhood $W(k)$ of the point $k$ such that $F$ does not intersect the set $f(V(k), W(k))$. If we cover the set $K$ by a finite number of neighbourhoods $W(k)$, then the desired neighbourhood $V$ is the intersection of the neighbourhoods $V(k)$ corresponding to $W(k)$. ■

**Theorem 1.5.** Every compact set $K$ in a linear topological space $X$ is bounded.

**Proof.** Let $V$ be an arbitrary neighbourhood of zero in the space $X$. We denote by $F$ the complement of the set $V$: $F = CV$. Let us take the field of scalars in place of $Y$, let $K$ be a compact set contained in $X$, and let $f(t, k) = tk \in X$, where $k \in K$, $t$ is a scalar.

Then there exists a neighbourhood of zero $A = \{x: \|x\| < \delta\}$ in the space $Y$ such that

$\Delta K = \{x = k \in A, k \in K\} \subset V$.

Since the neighbourhood $V$ is arbitrary, the set $K$ is bounded. ■

**Corollary 1.6.** Every precompact set $K$ in a linear topological space $X$ is bounded.

**Proof.** Let $\bar{X}$ be the completion of the space $X$. Let $\bar{K}$ denote the closure of the set $K$ in the space $X$. By Theorem 1.2, the set $\bar{K}$ is compact. Hence it is bounded, by Theorem 1.5. Thus the set $K$ is bounded, as a subset of a bounded set. ■

**Theorem 1.7.** If $F$ is a closed set in a linear topological space $X$ and $K$ is a compact set in $X$, then the set $F + K$ is closed.

**Proof.** If $x \in F + K$, then $F$ has no common points with the set $x - K$. By Theorem 1.4, there exists a neighbourhood $V$ of the point $x$ such that $F$ does not intersect the set $V - K$. Hence $V$ does not intersect the set $F + K$. ■

**Theorem 1.8.** Let $B$ be a closed set of scalars different from zero, and let $F$ be a closed set of points $\neq 0$ of a linear topological space $X$. Then the set $BF$ is closed.
Proof. Let \( \theta \) be a compact subset of the field of scalars, made up of the number 0 and of numbers \( k^{-1} \), where \( k \in B \). If \( x \in BF \), then \( F \) has no common points with the set \( Gx \). By Theorem 1.4, there exists a neighbourhood \( V \) of the point \( x \) such that \( F \) does not intersect the set \( GV \). Hence \( V \) does not intersect the set \( BF \).

**Theorem 1.9.** If \( W \) is a neighbourhood of the point 0 and \( \Delta \) is a neighbourhood of the number 0, then the intersection \( V \cap W \) of \( V \) and \( W \), where \( b \in \Delta \), is a neighbourhood of the point 0.

Proof. We apply the previous theorem to the complements \( R \) and \( F \) of \( \Delta \) and \( W \), respectively, taking into account the fact that \( \Delta \) and \( W \) are open.

**Theorem 1.10.** If there exists an open neighbourhood \( V \) of the point 0 in a linear topological space \( X \) such that \( V \) compact, then for an arbitrary closed subspace \( Y \) of the space \( X \) (\( Y \neq X \)) there is a point \( x \in V \) such that \( x \notin Y + V \).

Proof. Let \( x \in X \), but \( x \notin Y \). Since the subspace \( Y \) is closed, there exists a neighbourhood \( W \) of the point 0 such that \( x \notin Y + W \). By Theorem 1.5, there exists a number \( \varepsilon > 0 \) such that \( V \subseteq W \). Hence \( x \notin Y + V \) and \( \varepsilon \leq d(Y + V, X) \), i.e., \( X \neq Y + V \).

Let us suppose that the theorem is false, i.e., that \( Y \subseteq X + V \). Then \( Y + V = Y + V \). However, by Theorem 1.7, the set \( Y + V \) is closed. On the other hand, the set \( Y + V = \bigcup_{y \in Y} (y + V) \) is open as a union of open sets. Since \( X \) is a connected space, it follows that \( Y + V = X \), contradicting the condition \( X \neq Y + V \).

**Theorem 1.11.** If \( Y \) is a finite-dimensional subspace of a linear topological space \( X \), then

1. the space \( Y \) is an Euclidean space,
2. the subspace \( Y \) is closed in \( X \).

Proof. First, we prove that condition (a) is satisfied. We denote by \( \{y_1, y_2, \ldots, y_n\} \) the basis of the space \( Y \). With every point \( \{x_1, x_2, \ldots, x_m\} \) of the Euclidean space \( \mathbb{E}^m \) one can associate the point \( x_1 y_1 \) + \( x_2 y_2 \) + \( \ldots \) + \( x_m y_m \) of the space \( Y \). This correspondence \( f \) is a one-to-one linear and continuous map of the space \( \mathbb{E}^m \) onto the space \( Y \). One has to prove that the inverse operator \( f^{-1} \) is continuous. Let a ball \( U = \{(x_1, \ldots, x_m) : |x_1|^2 + \ldots + |x_m|^2 < 1 \} \) be given in the space \( \mathbb{E}^m \). It is sufficient to show that the set \( f(U) \) is a neighbourhood of 0 in the space \( Y \). But the set \( U'' = U - U \) is compact and \( 0 \notin U'' \). Hence the set \( f(U'') \) is compact (\( f(U') \)) and 0 \notin f(U''). It follows that the point 0 in the space \( Y \) has a convex neighbourhood \( V \) which does not intersect the set \( f(U') \). However, the set \( f^{-1}(V) \) is convex, contains the point 0 and does not intersect the boundary \( U' \) of the ball \( U \) with centre in 0. Thus \( f^{-1}(V) \subseteq U \), and so \( V \subseteq f(U) \). Consequently, \( f(U) \) is a neighbourhood of the point 0 in the space \( Y \).

We now proceed to the proof of (b). Let \( x \in Y \), but \( x \notin Y \). Let us denote by \( Z \) the subspace of \( X \) spanned by the basis \( \{x, y_1, \ldots, y_m\} \). It follows from condition (a) that the space \( Z \) is Euclidean. Hence \( Y \) is a subspace of an Euclidean subspace \( Z \), and \( Y \) is not closed in \( Z \), which is impossible.

A linear topological space \( X \) is called locally compact if there exists a precompact neighborhood of zero in \( X \).

**Theorem 1.12.** Every locally compact subspace \( X \) of a linear topological space \( X \) is closed and Euclidean.

Proof. Without loss of generality, we can assume that the space \( X \) is complete. Let \( V \) be a neighbourhood of zero in the space \( Y \) such that the set \( V \) is compact (see Theorem 1.3). Applying induction we define a sequence of closed Euclidean subspaces \( Y_n \) of the space \( Y \) in the following manner. We take \( Y_0 = 0 \). If \( Y_{n+1} \neq Y_n \), we have already defined \( Y_{n+1} \), and \( Y_n \neq Y_n \), we apply Theorem 1.10: there exists a point \( y_n \notin V \) such that \( y_n \notin Y_n \). Then we take \( Y_{n+1} = \text{lin}(y_n, Y_n) \). Evidently, \( Y_n = n \). By Theorem 1.11, the space \( Y_n \) is a closed Euclidean subspace of the space \( Y \). Hence \( y_n \in V \) and \( y_n \notin V_n \) for \( m < n \). By Theorem 1.3, the sequence of subspaces \( (Y_n) \) is finite and its last element is \( Y \). Thus the space \( Y \) is an Euclidean subspace of the space \( X \). Applying the previous theorem we find that \( Y \) is closed.

**§ 2. Characterization of precompact sets in concrete spaces.**

**Theorem 2.1.** (Cohn and Dunford) If a linear metric space \( X \) has a basis \( (e_n) \), then a set \( E \subseteq X \) is precompact if and only if

1. \( q(e) < M \) for all \( x \in K \), where \( q(e) \) are basis functionals, i.e., \( x = \sum_{i=1}^{n} q(e_i)e_i \),
2. the series \( x = \sum_{i=1}^{n} q(e_i)e_i \) is uniformly convergent for all \( x \in K \).

Proof. Let us suppose that assumptions (1) and (2) are satisfied. To an arbitrary number \( \varepsilon > 0 \) one can choose a natural number \( n \) satisfying the inequality

\[
\left| \sum_{i=1}^{n} q(e_i)e_i \right| < \frac{\varepsilon}{4} \quad \text{for all } x \in K.
\]

Let us consider the set

\[
K_n = \{x \in K : \text{dist}(x, K_n) < \varepsilon/4\},
\]

where \( [x]_n = \sum_{i=1}^{n} q(e_i)e_i \).

By assumption (1), the set \( K_n \) is precompact. Hence there exists a finite system of points \( x_1, \ldots, x_m \) such that to every \( x \in K_n \) there is an equilibrium in linear spaces
index $i$ satisfying the inequality $|z_i - z_i| < \frac{1}{4} \epsilon$. Thus

$$|x - z_i| < |x - [x]| + |[x] - z_i| < \epsilon.$$  

Since $x$ is an arbitrary element of $K$, the set $K$ is precompact.

Conversely, let us suppose that $K$ is a precompact set. Since $(a_n)$ is a basis, we conclude from Theorem 5.1, $\Pi$, that the transformations

$$f_n(x) = \sum_{i=1}^{\infty} \varphi_i(x) a_i$$

are equicontinuous, i.e., for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $n$ and for $|\varepsilon| < \delta$ we have

$$|f_n(x) - f_n(y)| < \frac{1}{2} \epsilon.$$ 

The set $K$ is precompact. Hence there exists a finite system of points $x^1, \ldots, x^m$ such that for every $x \in K$ there is an index $i$ satisfying the inequality

$$|x - x^i| < \min(\delta, \frac{1}{4} \epsilon).$$

It follows from the convergence of the sequence $(f_n(x))$ to the element $x$ that there exists a number $N_x$ such that

$$||f_n(x^i) - x^i|| < \frac{1}{2} \epsilon$$

for $n > N_x$, $i = 1, 2, \ldots, m$.

Hence

$$||f_n(x) - x|| = ||f_n(x) - f_n(x^i)|| + ||f_n(x^i) - x^i|| + ||x^i - x|| < \epsilon$$

for $x \in K$ and $n > N_x$. This proves the uniform convergence on $K$ of the expansion of $x$ with respect to the basis $\Pi$.

**Corollary 2.2.** A set $K$ is precompact in the space $P$ ($0 < p < +\infty$) if and only if $|x_i| < M$ and the series $\sum_{i=1}^{\infty} |x_i|^p$ is uniformly convergent for all sequences $(x_i) \subset K$.

**Corollary 2.3.** A set $K$ is precompact in the space $c_0$ if and only if $|x_i| < M$ and $\lim \sup \|f_n(x)\| = 0$.

Let $X$ be a locally convex linear metric space. It follows from the definition that a set $K$ is precompact in the space $X$ if and only if a finite $\varepsilon$-net can be defined in $K$ with respect to every pseudonorm.

**Theorem 2.4.** Let $M(a_{nm})$ be the space of all sequences of complex numbers $x = (x_n)$ such that

$$\|x\|_m = \sup_{n \geq 0} a_{nm} |x_n| < +\infty,$$

where $(a_{nm})$ is an infinite matrix with positive elements, and $a_{nm} \leq a_{nm+1}$ ($n, m = 1, 2, \ldots$). If we define the topology in $M(a_{nm})$ by means of pseudonorms $\|x\|_m$, then $M(a_{nm})$ becomes a linear metric locally convex complete space.

\[ \text{§ 2. Characterization of precompact sets} \]

A set $K \subset M(a_{nm})$ is precompact if and only if

(a) $\sup_{m=1}^{\infty} |a_{nm}| M_m < M_m$,  
(b) $\lim_{n \to \infty} \sup_{x \in K} |x_n| = 0$.

If for every number $m$ there exists a number $k$ such that $\lim_{n \to \infty} (a_{nm}/a_{nm+k}) = 0$, then the condition (a) implies the condition (b).

**Theorem 2.5.** (Arzelà [1]) A set $K \subset C(I)$ is precompact if and only if it consists of uniformly bounded and equicontinuous functions.

**Proof.** Necessity. Let the set $K$ be precompact and let $\varepsilon$ be an arbitrary positive number. There exists a system of functions $x_1, \ldots, x_m \in C(I)$ such that for every function $x \in K$ one can choose a function $x_i$ satisfying the inequality

$$|x - x_i| = \sup_{t \in I} |x(t) - x(t)| < \frac{1}{4} \varepsilon.$$ 

Hence it follows

$$|x| < |x_i| + \frac{1}{8} \varepsilon \leq \sup_{t \in I} |x(t)| + \frac{1}{8} \varepsilon,$$

i.e., the functions $x \in K$ are uniformly bounded. Moreover, since the functions $x(t)$ are continuous on a compact set $I$, they are uniformly continuous on $K$. Hence for every $x \in K$ there exists a number $\delta_i > 0$ such that the condition $g(t, t') < \delta_i$ implies

$$|x(t) - x(t')| < \frac{1}{4} \varepsilon.$$ 

Let $\delta = \min\{\delta_i\}$ and let $g(t, t') < \delta$. Then

$$|x(t) - x(t')| < \frac{1}{4} \varepsilon.$$ 

Hence all functions $x(t)$ in $K$ are equicontinuous.

Sufficiency. Let us suppose that a family $K$ of functions $x(t)$ is equicontinuous. This means that to any number $\varepsilon > 0$ and to an arbitrary point $t_i \in \Omega$ there exists a neighborhood $V_{t_i}$ of the point $t_i$ such that

$$|x(t) - x(t_i)| < \varepsilon$$

for all functions $x \in K$.

Sets $V_{t_i}$ form a covering of the compact set $\Omega$. Let us choose a finite subcovering $V_{t_1}, \ldots, V_{t_m}$. Let us now choose an arbitrary system $S$ of functions $(x_i(t))$ in such a manner that

$$\sup_{i \in S} |x_i(t_0) - x_i(t_m)| > \frac{1}{4} \varepsilon$$

for $i \neq j$.

Since the functions $x_i(t)$ are uniformly bounded, the system $S$ is finite, by Theorem 1.3. Evidently, to every function $x_i(t) \in K$ one can choose an index $i$ satisfying the inequality

$$\sup_{i \in S} |x_i(t_0) - x_i(t)| < \frac{1}{4} \varepsilon,$$
since otherwise the function \( s(t) \) could be added to the system \( S \). Hence
\[
|x(t) - x(t_0)| < |x(t) - x(t_0)| + |x(t_0) - x(t_0)| + |x(t_0) - x(t_0)| < \varepsilon + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]
and \( |x - x| < \varepsilon \). This shows that the finite system \( S \) is an \( \varepsilon \)-net.

**Corollary 2.8.** A set \( K \subset C^0([0,1]) \) is precompact if and only if
\[
\sup_{x \in K} \| x \|_K < M_K, \quad \text{where} \quad \| x \|_K = \sup_{t \in [0,1]} \| x(t) \|.
\]

**Proof.** If a set \( K \) is precompact, then it is precompact in each pseudonorm. By the Arzelà-Ascoli Theorem 2.5, the functions \( \frac{d^n x}{dt^n} \) are uniformly bounded in each pseudonorm. On the other hand, if (2.1) holds, then these functions satisfy the Lipschitz condition with the constant \( M_K \). Hence they are equicontinuous and uniformly bounded in each pseudonorm. Thus the set \( K \) is precompact in each pseudonorm, i.e., precompact.

**§ 3. Compact operators.** Let \( X \) and \( Y \) be linear topological spaces. An operator \( T \in L(X \to Y) \) is called compact (or completely continuous) if there exists a neighbourhood of zero \( U \subset X \) such that the set \( T(U) \) is precompact.

Every compact operator is continuous. Indeed, let \( V \) be an arbitrary open set in the space \( Y \). By Corollary 1.6, the set \( T(U) \) is bounded. Hence there exists a number \( \lambda \) such that \( y + \lambda \cdot T(U) \subset V \). Thus
\[
T^{-1}(y + \lambda \cdot T(U)) \subset T^{-1}(V) \subset \bigcup_{x \in T^{-1}(V)} y + \lambda \cdot T(U).
\]
But
\[
T^{-1}(y + \lambda \cdot T(U)) \subset \bigcup_{x \in T^{-1}(V)} x + \lambda \cdot T(U) = \bigcup_{x \in T^{-1}(V)} x + \lambda \cdot T(U)
\]
is an open union of open sets. Hence the set
\[
T^{-1}(V) \subset \bigcup_{y \in T^{-1}(V)} T^{-1}(y + \lambda \cdot T(U))
\]
is open, which is what was to be proved.

The sum of two compact operators is a compact operator. Indeed, if \( T_1, T_2 \in L(X \to Y) \) are compact operators, then there exist neighbourhoods of zero \( U \subset X \) such that the sets \( T_1(U) \) and \( T_2(U) \) are precompact. The neighbourhood of zero \( U \subset X \) satisfies the condition
\[
(T_1 + T_2)(U) \subset T_1(U) + T_2(U) = U.
\]
By Theorem 1.1, the set \( (T_1 + T_2)(U) \) is precompact.

In a similar manner we show that the product of a compact operator by a number is a compact operator.

Evidently, the restriction \( T_1 \) of a compact operator \( T \in L(X \to Y) \) to a subspace \( X' \subset X \) is a compact operator.

Let three linear topological spaces \( X, Y, Z \) be given. Let \( T_1 \in L(X \to Y) \) and \( T_2 \in L(Y \to Z) \). If one of the operators \( T_1, T_2 \) is continuous and the other one is compact, then the superposition \( T_1 T_2 \) is a compact operator.

In order to prove this fact, we first show that the image of a precompact set \( K \subset X \) by means of a continuous operator \( T \in L(X \to Y) \) is precompact. Indeed, let \( U \) be an arbitrary neighbourhood of zero in the space \( Y \). There exists a neighbourhood of zero \( V \) in the space \( X \) such that \( TV \subset U \). Since the set \( K \) is precompact, there exists a finite system of points \( x_1, \ldots, x_n \in X \) satisfying the condition \( K \subset \bigcup_{i=1}^n (x_i + V) \). Hence
\[
TK \subset \bigcup_{i=1}^n (T(x_i + V)) \subset \bigcup_{i=1}^n (y_i + U), \quad \text{where} \quad y_i = T(x_i).
\]
If the operator \( T_1 \) is compact, there exists a neighbourhood of zero \( V \) in the space \( X \) such that the set \( T_1(U) \) is precompact. Hence the set \( T_1(U) \) is also precompact. If \( T_1 \) is a compact operator and \( U \) is a neighbourhood of zero in the space \( X \) such that the set \( T_1(U) \) is precompact, then the continuity of the operator \( T_1 \) implies the existence of a neighbourhood of zero \( U \subset X \) for which the inclusion \( T_1(U) \subset U \) holds. Consequently, the set \( T_1 T_2(U) \subset U \) is precompact.

Hence the set \( T(X \to Y) \) of all compact operators forms a two-sided ideal in the algebra \( B(X \to Y) \). If at least one of the spaces \( X, Y \) is of infinite dimension, this ideal is a proper one, since one of the identities \( I_X \) and \( I_Y \) is not contained in it. This follows from Theorem 1.12, which states that a space of infinite dimension is not locally compact. We shall denote by \( T(X) \) the ideal of compact operators in the algebra \( B(X) \).

Let us suppose that there are two topologies, \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \), in a space \( X \). We say that the topology \( \mathcal{T}_1 \) is compact with respect to the topology \( \mathcal{T}_2 \) if there exists a neighbourhood \( U \subset \mathcal{T}_2 \) precompact in the topology \( \mathcal{T}_1 \). We denote by \( X_{\mathcal{T}_1}(i = 1, 2) \) the space \( X \) with the topology \( \mathcal{T}_i \). If a linear operator \( A \) maps the space \( X_{\mathcal{T}_1} \) in the space \( X_{\mathcal{T}_2} \) continuously, then \( A \) is considered as an operator which maps the space \( X_{\mathcal{T}_1} \) into itself compactly, since the topology \( \mathcal{T}_1 \) is compact with respect to \( \mathcal{T}_2 \). Moreover, we have the following:

**Theorem 3.1.** If \( \mathcal{T}_1 \) is a topology compact with respect to the topology \( \mathcal{T}_2 \), and if \( A \in B(X_{\mathcal{T}_1} \to X_{\mathcal{T}_2}) \), then \( A \) considered as an operator from the algebra \( B(X_{\mathcal{T}_1}) \) is compact.

**Proof.** Let \( U \subset \mathcal{T}_2 \) be a neighbourhood of zero precompact in the topology \( \mathcal{T}_2 \). We show the set \( AU \) to be precompact in the topology \( \mathcal{T}_2 \).

\[
\mathcal{T}_2 \subset \bigcup_{i=1}^n (y_i + U), \quad \text{where} \quad y_i = T(x_i).
\]
Let \( U \) be an arbitrary neighbourhood of zero in the topology \( \mathcal{C}_V \). Let \( V \) be a neighbourhood of zero in the topology \( \mathcal{C}_U \) such that \( AV \subseteq U \).

Since the neighbourhood \( U_n \) is precompact in the topology \( \mathcal{C}_V \), there exists a system of points \( x_1, \ldots, x_n \) such that \( U_n \subseteq \bigcup_{i=1}^n (x_i + V) \). Hence

\[
A U_n \subseteq \bigcup_{i=1}^n (Ax_i + U).
\]

Thus the set \( AU_n \) is precompact in the topology \( \mathcal{C}_V \).

In investigating perturbations of discontinuous operators the notion of \( A \)-compactness is very useful. It is defined in the same manner as the notion of \( A \)-continuity (see § 1, II). We say that an operator \( B \in L(X,Y) \) is \( A \)-compact if \( D_B \supseteq D_A \) and \( B \) is a compact operator which maps the space \( X_A \), the space \( Y \). As in § 1, II, we denote by \( X_A \), the set \( D_A \) provided with the topology determined by neighbourhoods of the form \( U \cap A^{-1}(V) \), where \( U \) and \( V \) are neigbourhoods of zero in spaces \( X \) and \( Y \), respectively. Evidently, every compact operator is \( A \)-compact.

The set of compact operators is not necessarily closed in the algebra \( B_0(X) \).

**Example 3.1.** Let \( X = (s) \) be the space of all sequences (see Example 3.1, b, § 1). It is easily verified that the closure of the ideal of finite-dimensional operators in this space contains the identity.

However, if there exists a bounded neighbourhood of zero in the space \( X \), then the following theorem holds:

**Theorem 3.2.** If the spaces \( X \) and \( Y \) are locally bounded, the ideal \( T(X,Y) \) of compact operators is closed in the paraproduct \( B_0(X,Y) \).

**Proof.** Let \( U \) be a bounded neighbourhood of zero in \( X \). An operator \( T \) is compact if and only if the set \( TU \) is precompact. Let an operator \( T \) belong to the closure of the set \( T(X,Y) \). Let \( V \) be an arbitrary neighbourhood of zero in the space \( Y \). Evidently, there exists a neighborhood of zero \( V \), satisfying the condition \( V_1 \cup V \subseteq V \). The definition of topology and the condition \( T \in T(X,Y) \) imply the existence of an operator \( T \in T(X,Y) \) such that \( T(x) = T(V) \) for \( x \in U \). But the operator \( T \) is precompact. Hence there exists a finite system of points \( x_1, \ldots, x_n \) of the space \( X \) such that \( T U \subseteq \bigcup_{i=1}^n (x_i + V) \). Thus

\[
T U \subseteq \bigcup_{i=1}^n (x_i + V) \subseteq \bigcup_{i=1}^n (x_i + V).
\]

It follows that \( \lim_{n \to \infty} \sum_{i=1}^n \|X_n\|^p = 0 \).

**Theorem 3.3.** Let \( X \) and \( Y \) be linear metric spaces and let \( Y \) have a basis \( (e_i) \). If \( T \in B_0(X,Y) \) is a compact operator, then the sequence of operators \( T_n \), where \( T_n x = T(e_i) \), is convergent to the operator \( T \) in the sense of bounded convergence.

**Proof.** Let \( B \) be an arbitrary bounded set. If \( T \) is a compact operator, there exists a neighborhood of zero \( U \) such that the set \( TU \) is precompact. Since \( B \) is a bounded set, there exists a number \( \lambda \) for which the inclusion \( \lambda U \subseteq U \) holds. By the Cohen–Dunford Theorem 1.1, the sequence \( \{T_n x - T x\} \) tends to zero uniformly for \( x \in TU \). Hence the sequence \( \{T_n x - T x\} \) tends to zero uniformly for \( x \in B \). Thus the sequence \( \{T_n x - T x\} \) tends to zero uniformly for \( x \in B \).

**Example 3.1.** shows that the converse of this theorem is not true. However, if the space \( X \) is locally bounded, Theorem 3.3 implies that the condition given in Theorem 3.3 is also sufficient in order that \( T \) be a compact operator.

**Corollary 3.4.** If a matrix \( (a_n) \) satisfies the condition

\[
\sum_{n=1}^\infty \left( \sum_{k=1}^n |a_{nk}|^p \right)^{1/p} < +\infty,
\]

then the operator \( A \) corresponding to this matrix is compact in the space \( X \), where \( 1/p + 1/q = 1 \).

Indeed,

\[
\|A - A_n\| \leq \sum_{n=1}^\infty \left( \sum_{k=1}^n |a_{nk}|^p \right)^{1/p}.
\]

Hence it follows that \( \lim_{n \to \infty} \|A - A_n\| = 0 \). Thus \( A \) is a compact operator.

Let us remark that if \( p = r = 2 \), then condition (3.1) assumes a simpler form:

\[
\sum_{n=1}^\infty \sum_{k=1}^n |a_{nk}|^2 < +\infty,
\]

and if \( p = 1 \), then (3.1) is to be written in the form

\[
\sum_{n=1}^\infty \sup_{k} |a_{nk}| < +\infty.
\]

**Theorem 3.5.** If \( T(s,t) \) is a continuous function in the square \( a \leq s, t \leq b \), then the integral operator \( y = T x \), where

\[
y(s) = \int_a^b T(s,t)x(t)dt,
\]

which maps the space \( C[a,b] \) into itself, is compact in \( C[a,b] \).
Proof. Let $E$ be a bounded set in the topological space $C[a, b]$; then $||x|| \leq M$ for all $x \in E$. The set $TE$ is also bounded, since

$$||y|| \leq M ||T|| \quad \text{for} \quad y \in TE.$$ 

If $x \in E$, then writing $y = Ax$ we have

$$||y(t) - y(t')|| \leq \int_a^b \int (T(s, t) - T(s, t')) \alpha(s) \, ds \, dt \leq M \int_a^b \int (T(s, t) - T(s, t')) \, dt \, ds.$$

Since $T(s, t)$ is a continuous function, if the difference $t - t'$ is sufficiently small, then the right-hand side of the last inequality is arbitrarily small, independently of $x \in E$. Hence the functions from the set $TE$ are equicontinuous and uniformly bounded. Thus the set $TE$ is precompact (see Theorem 2.5). This proves that the operator $T$ is compact. \[\square\]

**Theorem 3.6.** Let a function $T(s, t)$ be integrable with power $r'$ in a domain $Q \times Q'$, where $r = \min(p, q')$ (given $a$, we denote by $a'$ the number satisfying the equality $1/a + 1/a' = 1$):

$$\left( \int_a^b \int (T(s, t))^{r'} \, dt \, ds \right)^{1/r'} \leq C < \infty.$$ 

Then the integral operator $T$

$$Tz = y(s) = \int_a^b T(s, t)z(t) \, dt$$

is a compact operator which maps the space $I^r$ in the space $I^a$.

**Proof.** It follows from Hölder's inequality that

$$\|y\| \leq \left( \int_a^b \int (T(s, t))^{r'} \, dt \, ds \right)^{1/r'} \leq \left( \int_a^b \int \alpha(s) \, ds \, dt \right)^{1/r'} \|z\| \leq C \|z\|.$$

Hence we obtain

$$\|y\| = \left( \int_a^b \int (y(t))^{r'} \, dt \, ds \right)^{1/r'} \leq C \|z\| \left( \int_a^b \int (T(s, t))^{r'} \, dt \, ds \right)^{1/r'}.$$

Since $r' \leq q'$, we have $r \geq q$. Applying Hölder's inequality with the exponents $r'/q'$ and $r'/q'$ to the first integral on the right-hand side of the last inequality we obtain

$$\|y\| \leq \|z\| \left( \int_a^b \int (T(s, t))^{r'} \, dt \, ds \right)^{1/r'} \left( \int_a^b \int \alpha(s) \, ds \, dt \right)^{1/r} \leq C \|z\|.$$

Since $C$ is the measure of $Q_1$ in the power $1/q(r'/q')$.

---

§ 3. Compact operators

Since the function $T$ is an element of the space $L(Q \times Q')$, one can find a sequence of continuous functions $(T_n(s, t))$ such that

$$\left( \int_a^b \int (T_n(s, t) - T_n(s, t'))^2 \, ds \, dt \right)^{1/2} \leq c_n \quad (n = 1, 2, ...),$$

where $c_n \to 0$. Denoting by $T_n$ the integral operator determined by the function $T_n(s, t)$ and taking into account the fact that $T_n$ is a compact operator (the proof being similar to the proof of Theorem 3.5), we conclude that $T$ is a compact operator as a limit of a sequence of compact operators (Theorem 3.2). \[\square\]

The function $T(s, t)$ determining the integral operator $Tz = \int_a^b T(s, t)x(t) \, dt$ is called the integral kernel of the operator $T$.

**Theorem 3.7.** Let $T(s, t)$ be a function infinitely differentiable defined in the square $[0, 1] \times [0, 1]$. The operator

$$Tz = y(s) = \int_a^b T(s, t)z(t) \, dt$$

maps the space $C^m[0, 1]$ of functions infinitely differentiable in the interval $[0, 1]$ into itself. Moreover, $T$ is a compact operator.

**Proof.** Since

$$y^{(m)}(s) = \int_a^b T^{(m)}(s, t)z(t) \, dt,$$

Theorem 3.5 implies that the set $TU_n$ is precompact, where $U_n = \{x: ||x|| \leq 1\}$. \[\square\]

§ 4. Properties of compact operators which map a space into itself.

We begin with three lemmas. The first one is of a purely algebraic character and the other two are topological lemmas.

Let $A \in L(X)$ and let $y \notin E_A$ be given. We write

$$X_n = \text{lin} \{y, Ay, ..., A^{n-1}y\} \quad (n = 1, 2, ...).$$

**Lemma 4.1.** If $A \in L(X)$ is a monomorphism (i.e. $X_n = \{0\}$) and $n$ is an arbitrary positive integer, then

1. $\dim X_n = n$;
2. $X_n \cap E_A = \{0\}$.

**Proof.** (1). Let us suppose that (1) does not hold. Let $m$ be the least number for which (1) is not true, i.e.

$$a_0y + a_1Ay + ... + a_{m-1}A^{m-1}y = 0,$$

where $a_{m-1} \neq 0$. Since $y \notin E_A$, we have $a_0 = 0$. Hence

$$A(a_1Ay + ... + a_{m-1}A^{m-2}y) = 0,$$
and the assumptions on $A$ imply

$$a_1 y + \cdots + a_{n-1} A^{n-1} y = 0.$$  

This means that $\dim Y_{n-1} < m-1$, contradicting the definition of the number $m$.

(2) Let us suppose that $0 \neq x \in Y_{n-1} \cap A^n X$. Then

$$x = b_1 y + b_1 A y + \cdots + b_{n-1} A^{n-1} y = A^n x \neq 0$$

for some $y \in X$. Since $y \notin E_A$, this implies $b_0 = 0$. However, $A^n = A^e$ implies $u = v$ because $A$ is a monomorphism. Hence

$$b_1 y + \cdots + b_{n-1} A^{n-1} y = A^n x \neq 0,$$

i.e., $Y_{n-1} \cap A^n X \neq \{0\}$. Repeating these arguments we finally obtain $Y_1 \cap A^n X \neq \{0\}$, contradicting our assumption.

**Lemma 4.2.** Let $X$ be a linear topological space and let an operator $A \in B(X)$ with a closed set of values $E_A$ have a left inverse $A \in B(X)$. If $y \notin E_A$ and $X = \lim(y)$, then for every neighbourhood of zero $U$ there exists a neighbourhood of zero $U'$ such that $AU \supset (Y + U') \cap E_A$.

**Proof.** Since $A$ has a continuous left inverse $A_1$, there exists a neighbourhood $U_0$ satisfying the condition $AU \supset E_A \cap U_0$. Let $U = B(x, k)$ be a balanced neighbourhood such that $U_0 \supset U \supset U_0 \supset U$. Since $E_A$ is a linear space, the operator $A_1$ is a linear space, we have $(a_1 U) \cap E_A = 0$ for $|a_1| < k$. The neighbourhood we are looking for is $U' = U + U_0$. Indeed, we have

$$AU \supset (U + U_0) \cap E_A = (a_1 U + U_0) \cap E_A = 0 = (a_1 U_0 \cap E_A).$$

**Theorem 4.5.** Let $X$ be a finite-dimensional subspace of a linear topological space $Y$, and let $U_0$ be a neighbourhood transformed by the operator $T \in T(X)$ in a precompact set. If $Ty = 0$ for $y \in X$, then $y = 0$. Therefore, $A_0 \cap Y = 0$ is compact.

**Proof.** The restriction $T_0$ of the operator $T$ to the subspace $X$ maps the set $Y \cap Y_0$ onto a compact set, and the inverse of the operator $T$ is continuous because the space $Y$ is finite-dimensional.

**Theorem 4.6.** Let $X$ be a linear topological space, and let $T \in T(X)$, $A = I - T$. We denote by $U_0$ a neighbourhood of zero transformed by the operator $T$ in a precompact set. Then the inverse image $A^{-1}(0)$ is a closed Euclidean subspace of the space $X$ and the set $A^{-1}(0) \cap U_0$ is a precompact neighbourhood of zero in this space.

**Proof.** The set $Y = A^{-1}(0)$ is closed because the operator $A$ is continuous. Moreover, $x \in U$ implies $x = T x$. Hence $Y \cap X = Y \cap U_0$. Thus $X \cap U_0$ is a precompact neighbourhood of zero in the subspace $X$. This and Theorem 1.12 imply $Y$ to be an Euclidean space.

**Theorem 4.7.** Under the assumptions of the previous theorem the inverse image $A^{-1}(0)$ is a closed Euclidean space and $A^{-1}(0) \cap U_0$ is a precompact neighbourhood of zero in this space.

**Proof.** We replace the operator $A$ in the previous theorem by the operator $A^e$. Then $I - A^e$ is a compact operator as a polynomial in $T$ without a free term.

§ 5. The Riesz theory. In this section we show that if $T \in L_d(X \rightarrow Y)$ is a compact operator, then the operator $T - I$ has a finite $d$-characteristic and its index is equal to zero. The first theorems of this type were given by F. Riesz [1] and therefore this theory is named the Riesz theory.

**Theorem 5.1.** If $X$ is a complete linear topological space and if $T \in T(X)$, $A = I - T$, then the subspace $E_A$ is closed in the space $X$.

**Proof.** Let $U_0$ be a neighbourhood of zero transformed by the operator $T$ in a precompact set, and let $F_0 = A^{-1}(AU_0) = Y_0 + A^{-1}(0)$. By Theorem 4.4, the set $F_0$ is closed. However, the set

$$0 = A^{-1}(AU_0) = U_0 + A^{-1}(0) = Y_0 + A^{-1}(0)$$

is open as a union of open sets. Hence the set $F_0 = F_0 \cap X$ is closed and $F_1 = F_0 \cap X \subset F_0$. Let $x \notin F_1$. Since $0$ is an interior point of the set $F_1$,
Let us remark that if $x_0$ is a cluster point of the family $\mathcal{TB}_1$, then it is also a cluster point of the family $\mathfrak{B}_1$. Indeed, let us take an arbitrary set $B \in \mathfrak{B}_1$. Let $U$ be an arbitrary neighbourhood of zero. Moreover, let a set $B_1 \in \mathfrak{B}_1$ satisfying the condition $AB_1 \subset U$ be given, and let $B_2 = B_1 \cap B$. Since $x_0$ is a cluster point of the family $\mathcal{TB}_1$, we have $(x_0 + U) \cap \mathcal{TB}_1 \neq \emptyset$. This means that there exists a point $x \in B_2$ such that $Ax \in x_0 + U$. Hence $x = Tx + Ax \in x_0 + U + U$.

Thus $B \cap (x_0 + U + U) \neq \emptyset$. The sets $U$ and $B$ being arbitrary, it follows that $x_0 \in \mathcal{TB}_1$. Hence $x_0 \neq 0$. But the continuity of the operator $A$ implies that $A\mathfrak{B}_1$ is a cluster point of the family $\mathcal{TB}_1$, hence $A\mathfrak{B}_1 = 0$, in contradiction to our assumption. Hence case (b) is excluded from our considerations.

Now, let us suppose that condition (c) is satisfied. Let $y \notin A\mathfrak{B}_1$. We write

$$Y_n = \text{lin}(y, A_0 y, ... , A_{n-1} y), \quad n = 0, 1, 2, ...$$

By Lemma 4.1, $\dim Y_n = n$ and $Y_n \cap E_{e_1} = \{0\}$. Hence Theorem 4.3 implies the set $Y_n \cap U_1$ to be compact for any positive integer $n$. Thus, if $U \subset U_1$, then the set $Y_n \cap U$ is also compact.

We suppose that the operator $A$ satisfies condition (c). By Theorem 5.1, the set $E_1$ is closed. Applying Lemma 4.2 we conclude that there exists a neighbourhood $U'$ such that $U_1 \cap (Y_n + U') \subset E_1$. Let $U$ be an open balanced neighbourhood satisfying the condition $U \subset U_1 \cap U'$. By Theorem 1.10, if $n$ is an arbitrary natural number, then there exists a point $y'_n \in Y_n \cap U$ such that $y'_n \notin Y_{n-1} + U$. Hence $y'_n = Ax_{n-1} + ay_b$, where $x_{n-1} \in Y_{n-1}$. This implies $Ax_{n-1} \in Y_{n-1} + U$ for any natural number $n$, and since the operator $A$ is one-to-one, we obtain $x_{n-1} \in U$ for an arbitrary $n$.

On the other hand,

$$T_{x_n} - T_{x_{n-1}} = -y_{n+1} + (x_n + a_{n+1} y + y_{n+1} - z_n - a_{n-1} y) .$$

But if $m > n$, the expression in brackets on the right-hand side of this equality is a point of the space $Y_m$. Hence $T_{x_m} - T_{x_n} + U$ for $m > n$. But the sequence $(T_{x_n})$ is a subset of the precompact set $T(U)$ by Theorem 1.3, the sequence $(z_n)$ is finite, in contradiction to our assumption. Hence condition (c) cannot be satisfied, either. The only possible case is (a), which proves the theorem.
Proof. Since $A^n x = 0$ implies $A^{n+1} x = 0$, we have $A^{-n+1}(0) \subset A^{-n}(0)$ for $n > 0$. Let us suppose that

$$A^{-n+1}(0) \neq A^{-n}(0).$$

(5.1)

By Theorem 1.10, there exists a point $x_n$ such that

$$x_n \in A^{-n}(0) \cap U_g, \quad x_n \notin A^{-n+1}(0) + U_g.$$

(5.2)

According to assumption (5.1), $A^n x_n = 0$. Hence $A^n x_n \notin A^{-n+1}(0)$. The condition $A^n x_n = 0$ implies for $m < n$

$$x_m \in A^{-n+1}(0), \quad A x_m \in A^{-n+1}(0).$$

(5.3)

Hence we obtain from (5.1), (5.2) and (5.3)

$$x_n \notin (A x_n + x_n - A^n x_n + U_g),$$

i.e. $T x_n \notin \overline{T(U_g + U_n)}$. On the other hand, we obviously have $T x_n \in \overline{T(U_g)}$. Applying Theorem 1.5, we conclude that the sequence of indices satisfying condition (5.1) is finite. We denote by $s$ the last term of this sequence. ■

Theorem 5.4. Keeping the notation of the last theorem unchanged, we have $A^{-s}(0) \subset \overline{A x} = 0$ for an arbitrary $n > 0$. ■

Proof. Let $x \in A^{-s}(0) \cap \overline{A x}$. There exists an element $y \in X$ such that $x = A^s y$ and $A x = 0$. Hence $A^{s+1} y = 0$, and the previous lemma implies

$$y \in A^{-s}(0).$$

This implies $A y = 0$, that is $x = 0$. ■

Theorem 5.5. If $X$ is a complete linear topological space, $T \in T(X)$ and $A = I - T$, then the subspaces $E_n$ are closed,

$$X = \overline{A x} \supset \overline{A^2 x} \supset \overline{A^3 x} \supset \cdots \supset \overline{A^k x} = \cdots (k > s),$$

and the operator $A^{-1}$ maps the space $E_n$ onto itself continuously.

Proof. By Theorem 5.1, the set $E_n = A x$ is closed. Let $n > 0$ and let us suppose that we have already proved the set $E_n = A^n x$ to be a closed set. The restriction of the operator $T$ to the subspace $A^n x$ maps the set $A^n x$ into itself, since $T A^n x = A^n x$ and $T A^n$ is a compact operator. Hence $A^{n+1} x$ is a closed subspace. Moreover, the above restriction is one-to-one on the space $A^n x$ (see Theorem 5.4). Therefore (Theorem 5.2) the restriction of the operator $A$ to the subspace $A^n x$ has a continuous inverse which maps the subspace $A^n x$ onto itself. In particular, $A^{n+1} x = A^n x$. ■

Theorem 5.6. In the notation from Theorem 5.5, the space $X$ is a direct sum:

$$X = A^{-n}(0) \oplus A^s x.$$
Moreover, let
\[ \lambda \varphi = T \varphi, \]
where \( \varphi \neq 0 \). Let us suppose that the element \( \varphi \) is linearly dependent on the elements \( \varphi_1, \ldots, \varphi_n \), i.e., \( \varphi = \alpha_1 \varphi_1 + \cdots + \alpha_n \varphi_n \). Applying the operator \( \lambda I - T \) to both sides of this equality, we obtain by (6.1),
\[ \alpha_1 (\lambda - \lambda_1) \varphi_1 + \cdots + \alpha_n (\lambda - \lambda_n) \varphi_n = 0. \]
Hence there exists an element \( \varphi_0 \), \( \mu < \nu \), linearly dependent on the elements \( \varphi_1, \ldots, \varphi_n \). Repeating these arguments we finally obtain \( \varphi \neq 0 \), contradicting the assumption \( \varphi \neq 0 \). Hence the elements \( \varphi_1, \ldots, \varphi_n \) are linearly independent. We denote by \( X \), the linear space spanned by these elements. By Theorem 1.11, the spaces \( X \), are closed and Euclidean. Since \( X \neq X_{++} \), \( X \subset X_{++} \subset \cdots \subset X \), we conclude from Theorem 1.10 that there exists a \( y \), such that
\[ y \in X \cap U_0, \quad y \notin X_{++} \cup U_0, \]
(6.2) Here \( U_0 \) is a neighbourhood of zero transformed by the operator \( T \) in a precompact set. Since \( y \in X \), formula (6.1) and the definition of the space \( X \), imply
\[ \lambda y \notin (\lambda y - T y + U_0), \]
i.e., \( T y \notin (T y + U_0) \). By Theorem 1.9, there exists a neighbourhood of zero \( V \) such that \( V \subset \lambda U_0 \). Hence \( T y \notin (T y + V) \). On the other hand, formula (6.2) implies \( T y \notin T U_0 \). Applying Theorem 1.3 we conclude that the sequence \( \{ \lambda \} \) is finite. 

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\text{Equations in linear spaces}
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