§ 0. Auxiliary notions. We assume the reader to be acquainted with notions of a set, a subset and a family of sets, and with the operations on sets. We denote by \( A \cup B \) the union (sum), by \( A \setminus B \) the difference, and by \( A \cap B \) the intersection (product) of sets \( A \) and \( B \). Let a family \( \mathcal{F} \) of sets be given. We denote by \( \bigcup_{A \in \mathcal{F}} A \) the union, and by \( \bigcap_{A \in \mathcal{F}} A \) the intersection of sets of the family \( \mathcal{F} \). In the sequel we shall often speak about points instead of elements of a set. A set containing only one element \( x \) will be denoted by \( \{x\} \).

We also assume that the reader is acquainted with the notions of a relation, a function and a one-to-one function. In the sequel, the terms "map" and "transformation" will also mean a function.

Sets \( A \) and \( B \) are said to be of the same power if there exists a one-to-one map of \( A \) onto \( B \). Sets of the same power as the set of all positive integers are called countable.

A set \( A \) is called ordered by the relation \( \preceq \) if this relation satisfies the following axioms for any two elements \( x, y \in A \):

1. \( x \preceq x \) (reflexivity);
2. if \( x \neq y \), then \( x \preceq y \) or \( y \preceq x \) (connectivity);
3. if \( x \preceq y \) and \( y \preceq x \), then \( x = y \) (weak asymmetry);
4. if \( x \preceq y \) and \( y \preceq z \), then \( x \preceq z \) (transitivity).

A relation satisfying axioms 1-4 is called an order relation. A set \( A \) is called well-ordered with respect to the order relation \( \preceq \) if each non-void subset \( B \) of \( A \) contains a first element, i.e. an element \( x_0 \) satisfying the condition \( x_0 \preceq y \) for all \( y \in B \). It follows from the axiom of choice that every set can be well-ordered. Let \( A \) be a well-ordered set. We write

\[ O_A(x) = \{ y \in A : y \preceq x , y \neq x \} .\]

If \( B \) is a non-void subset of \( A \) and if, for every \( x \), \( O_A(x) \cap B \) implies \( x \in B \), then \( B = A \).

This theorem is called the principle of transfinite induction. The method of proving theorems by transfinite induction is the same as that
of complete induction; the only difference is that one has to suppose the theorem to be true for all \( y \in O(a)(x) \) instead of all positive integers less than \( n \).

We say that a set \( Z \) is partially ordered if there exists an order relation in \( Z \), but this order relation need not be defined for all pairs of elements \( x, y \in Z \). The principle of transfinite induction is equivalent to Kuratowski-Zorn's lemma:

If a set \( Z \) has the following property: every ordered subset \( Z, \subseteq Z \) has an upper bound (i.e., an element \( z_0 \) such that \( x \leq z_0 \) for all \( x \in Z \), and \( y \leq z_0 \) and \( x \leq y \) for \( x \in Z \), imply \( y = z_0 \)), then the set \( Z \) also has an upper bound.

Let a set \( X \) be given. Let us suppose that to every two elements \( x \) and \( y \) of the set \( X \) there corresponds in a unique manner a third element \( x \odot y \) of that set. If the operation \( \odot \) is associative, i.e.

\[(x \odot y) \odot z = x \odot (y \odot z),\]

and if to any two elements \( x, y \in X \), there exist elements \( z_1, z_2 \in X \) such that \( x \odot z_1 = y \) and \( z_2 \odot x = y \), then the set \( X \) is called a group.

If \( X \) is a group and the equality

\[x \odot y = y \odot x\]

holds for any two elements \( x, y \in X \), then \( X \) is called a commutative group or an Abelian group. The operation \( \odot \) in Abelian groups is denoted traditionally by \(^+\) and called addition. The element \( a \) satisfying the equation \( x + x = a \) is called the difference of elements \( y \) and \( x \) and is denoted by \( x = y + x \); the operation \(^-\) is called subtraction. The element \( x - x \) is denoted by 0 and is called the zero element or neutral element. Let us remark that \( y + 0 = y \) for every \( y \in X \). Evidently, the zero element 0 is unique. Indeed, let us suppose that 0 and 0\(_1\) are two zeros in a group \( X \); then \( 0 + 0 = 0_1 + 0 = 0_1 \). A subset \( M \) of a group \( X \) is called a subgroup of the group \( X \) if it has the same operation as \( X \). If \( M, N \) are subsets of a group \( X \), we write

\[M \odot N = \{x \odot y : x \in M, y \in N\}.\]

If \( X \) is an Abelian group, the set \( M + N \) is called the algebraic sum of sets \( M \) and \( N \). The sum \( (X) + M \) is denoted briefly by \( x + M \).

A commutative group \( X \) with another associative operation \( a \odot b \) besides \( a + b \) is called a ring if the following distributivity conditions are satisfied:

\[(0.1)\]

\[x(y + z) = xy + xz; \quad (x + y)z = xz + yz.\]

The operation \( a \odot b \) is called multiplication.
Elements $x, y \in X$ such that $xy = yx$ are called a commutative pair of elements. If every pair of elements of a ring $X$ is commutative, $X$ is called a commutative ring.

A ring $X$ such that every non-zero element of $X$ has an inverse is called a field. For instance, the sets $\mathbb{R}$ of real numbers and $\mathbb{C}$ of complex numbers are fields with the usual definitions of addition and multiplication as ring-operations.

A right ideal (left ideal) in a ring $X$ is a set $M \subseteq X$ such that

1. if $x, y \in M$, then $x-y \in M$;
2. if $x \in X$, then $xM \subseteq M$ (resp. $xM \subseteq M$).

If $M$ is both a right ideal and a left ideal, $M$ is called a two-sided ideal.

Obviously, the whole ring $X$ and also the set $\{0\}$ are ideals. These two ideals will be called trivial ideals. All other ideals will be called non-trivial or proper ones.

It is easily verified that an intersection of an arbitrary number of left ideals (right ideals) is a left ideal (right ideal), not necessarily a proper one.

A proper ideal cannot contain the unit of the ring. A proper right ideal (left ideal) cannot contain a right-invertible (left-invertible) element. Conversely, if an element $a$ is not right-invertible (left-invertible), there exists a proper right ideal (left ideal) containing $a$. Indeed, such an ideal is the set $aX$ (resp. $a\mathbb{C}$).

A proper right ideal (left ideal, two-sided ideal) $M$ is called a maximal right ideal (left ideal, two-sided ideal) if every proper right ideal (left ideal, two-sided ideal) $M'$ containing $M$ is equal to $M$, i.e. $M' \supset M$ implies $M' = M$.

Every proper right ideal (left ideal, two-sided ideal) is contained in a maximal right ideal (left ideal, two-sided ideal). Evidently, if $M$ is a two-sided ideal, the maximal right ideal and left ideal may be different from each other.

A radical of a ring $X$ is a set $R(X)$ of elements $s$ such that the element $s+xy$ is invertible for all $x, y \in X$. We shall prove that a radical $R(X)$ is a two-sided ideal. Since $x$ and $y$ are arbitrary elements of the ring, $s \in R(X)$ implies $s,x \in R(X)$ and $xy \in R(X)$ for arbitrary $s \in X$ and $y \in X$. Let us suppose that $s \in R(X)$. Let $s \in X$. Then the element

$$e+x(s-x)y = x+xy-xy$$

is invertible.

A radical may also be defined as the intersection of all maximal right ideals (left ideals) (see Jacobson [1]).

Let a group $X$ and a subgroup $M$ of $X$ be given, and let $y \circ M = M \circ y$ for every $y \in X$. Every element $x \in X$ has its corresponding coset, i.e. the set of elements of the form $[x] = \{x \circ s : s \in M\}$. This correspondence is easily proved to be unique. We define

$$[x \circ y] = [x] \circ [y],$$

where $[x] \circ [y] = \{x \circ y : x \in [x], y \in [y]\}$.

The set of such cosets with the group operation defined above is denoted by $X/M$ and is called the quotient group or the factor group of $X$ by $M$.

Let $X$ be a ring, and let $M \subseteq X$ be a two-sided ideal. The operation of multiplication in the quotient-group $X/M$ is defined as follows:

$$[x \cdot y] = [x] \cdot [y] = \{x \cdot y : x \in [x], y \in [y]\}.$$  

This operation is associative and distributive. Hence the set $X/M$ may be considered as a ring. This ring is called a quotient ring. Obviously, one can write $[x] = x+M$.

If $X$ is a commutative ring, then $X/M$ is a field if and only if $M$ is a maximal ideal.

A linear space over the field of complex (real) numbers is a commutative group $X$ such that the multiplication of elements of $X$ by complex (real) numbers is defined and satisfies the following conditions:

$$(s \cdot x) = \{s \cdot x : s \in \mathbb{C}, x \in X\};$$

$$(t,s)x = t(sx);$$

$$(t+s)x = (t+s)x;$$

$$(t \cdot s)x = t(sx);$$

$$(0, x) = 0 \cdot x = 0.$$}

It follows from the above conditions that if $t \cdot s = 0$ and $s \neq 0$, then $t = 0$.

Since most results for linear spaces over the field of real numbers and over the field of complex numbers are the same, in contents as well as in formulation, we shall understand by "linear space" both kinds of linear spaces. Obviously, the words, "number" or "scalar" will mean a number from the field under consideration. The same is valid with regard to functions assuming numerical values; considering real-valued and complex-valued functions, we obtain examples of linear spaces over the field of real numbers and the field of complex numbers, respectively.

Let a linear space $X$ and a subset $Y$ of $X$ be given, and let a sum of two elements of $X$ and a product of an element of $Y$ by a scalar again belong to $X$. Such a subset $Y$ of the space $X$ is called a linear subset, a linear manifold, or a subspace of the space $X$.

Let $B$ be an arbitrary subset of a space $X$. The smallest linear subset containing the set $B$ is called the space spanned by the set $B$, or the linear span of $B$; it is denoted by $\text{lin}B$.  

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It is easily shown that

\[ \text{lin} E = \{ \alpha \in X : \alpha = \sum_{i=1}^{n} t_i a_i, \ t_i \text{ scalars}, \ a_i \in E \} . \]

We say that an element \( \alpha \in X \) is linearly dependent on a set \( E \) (or on the elements of the set \( E \)) if \( \alpha \in \text{lin} E \). If \( \alpha \) is not linearly dependent on the set \( E \), we say that \( \alpha \) is linearly independent on the set \( E \). A set \( E \) is called linearly independent if there is no element \( \alpha \in E \) linearly dependent on the set of the remaining elements of \( E \), i.e. if \( \alpha \notin \text{lin} E \setminus \{ \alpha \} \).

It follows from the form of the set \( \text{lin} E \) that elements \( a_i \) are linearly dependent if the equality

\[ t_1 a_1 + \cdots + t_n a_n = 0 \quad (a_i \neq a_j) \]

implies

\[ t_1 = t_2 = \cdots = t_n = 0 . \]

If the maximal number of linearly independent elements belonging to a linear space \( X \) is finite, we call this number the dimension of the space \( X \) and we denote it by \( \dim X \). Otherwise we say that the dimension of the space \( X \) is infinite and we write \( \dim X = +\infty \). If \( \dim X < +\infty \), we call \( X \) finite-dimensional; if \( \dim X = +\infty \), \( X \) is called infinite-dimensional.

Let us remark that if the same finite-dimensional space \( X \) is considered as a space over the field of complex numbers and as a space over the field of real numbers, its dimension in the second case is twice as great as in the first one.

In further considerations, speaking about linear spaces over the field of complex numbers we shall always have in mind the dimension over this field.

A basis of a linear space \( X \) is a set of linearly independent elements \( \{a_i\} \) such that \( \text{lin} \{a_i\} = X \).

The power of the set of indices \( \{a_i\} \) is called the power of the basis \( \{a_i\} \).

It can be shown that this power depends only on the space \( X \) itself, and does not depend on the choice of the basis. A special case of this general theorem is the following

**Theorem 0.2.** If \( X \) is an \( n \)-dimensional space: \( \dim X = n \), then each basis of \( X \) contains exactly \( n \) elements.

**Proof.** It follows immediately from the linear independence of elements of a basis \( \{a_i\} \) that the number of elements of the basis is not greater than \( \dim X \). Let us suppose that a sequence \( a_1, \ldots, a_m \) is a basis and \( m < n = \dim X \). From the definition of \( \dim X \) we infer the existence of a sequence \( y_1, \ldots, y_n \) of linearly independent elements. But the set

\[ \{y_1, \ldots, y_n\} \]

is a basis. Consequently,

\[ y_1 = \sum_{j=1}^{n} a_j y_j . \]

Let us consider a linear combination of elements \( y_j \):

\[ \sum_{i=1}^{m} t_i y_i = \sum_{i=1}^{m} \sum_{j=1}^{n} t_i a_j y_j = \sum_{j=1}^{n} \left( \sum_{i=1}^{m} t_i a_j \right) y_j . \]

Since \( m < n \), we may find a number \( t_i \neq 0 \) such that \( \sum_{i=1}^{m} t_i a_j = 0 \) for \( j = 1, 2, \ldots, m \).

Hence the elements \( y_j \) are not linearly independent, which is a contradiction.

**Corollary 0.2.** The linear span of \( n \) linearly independent elements \( a_i \) is of dimension \( n "): \( \dim \text{lin} \{a_1, \ldots, a_n\} = n \).

The product \( X \times Y \) of two linear spaces \( X \) and \( Y \) is the space of all ordered pairs \((x,y)\) with the addition of elements and the multiplication of an element by a scalar defined by the formula:

\[ (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2); \quad \lambda(x, y) = (\lambda x, \lambda y) . \]

If \( Y \) and \( Z \) are subspaces of a linear space \( X \) and if \( Y \cap Z = \{0\} \), i.e. if the intersection of subspaces \( Y \) and \( Z \) consists of the element 0 only, then the set \( Y + Z \) is called the direct sum of subspaces \( Y \) and \( Z \) and is denoted by \( Y \oplus Z \). Let us remark that the condition \( Y \cap Z = \{0\} \) implies that every element \( v \in Y \oplus Z \) can be written in the form \( v = y + z \), where \( y \in Y \), \( z \in Z \), in a unique way.

If \( X = Y \oplus Z \), we say that \( X \) can be decomposed into a direct sum \( Y \oplus Z \).

**Theorem 0.3.** If \( X \) is a linear space and \( Y \) is a subspace of the space \( X \), then there exists a subspace \( Z \) such that \( X \) can be decomposed into a direct sum of subspaces \( Y \) and \( Z \): \( X = Y \oplus Z \).

**Proof.** (by transfinite induction). Let us take a relation \( \prec \) well-ordering the set \( X \). As we have already seen, such a relation exists by the axiom of choice. If \( Y = X \), we obtain the theorem by taking \( Z = \{0\} \).

Let us suppose that \( Y \neq X \). There exists an element \( y \in X \), first with respect to the relation \( \prec \). We write

\[ X = \{ y + \lambda y : y \in Y, \lambda \text{ being an arbitrary scalar} \} . \]
Evidently, this is a linear set. Let us denote by \( y_i \) the first element of \( X \) not belonging to \( X \). Let

\[ X_0 = \{ y + y_i; \ y \in X, \ \lambda \text{ being an arbitrary scalar} \} . \]

We denote by \( y_0 \) the first element of \( X \) not belonging to \( X_0 \). Repeating this argument we obtain a set \( X_0 = \{ y_1, y_2, \ldots \} \) well-ordered with respect to the relation \( \preceq \), and consisting of elements \( y_0 \) such that

\[ y_0 \not\in \text{lin}(Y, y_i; y_i < y_0) . \]

Let \( Z = \text{lin} Y_0 \). Obviously, \( X + Z = X \). Let us suppose \( z \in Y \) and \( z \not\in Z \), \( z \neq 0 \). It follows from the general form of the elements of the set \( Y \) that

\[ z = \sum_{i=1}^{n} \lambda_i y_i , \quad \text{where} \quad y_i \in Z, \quad \lambda_i \neq 0; \quad y_i \not\in Y_0 . \]

Hence

\[ y_i = \frac{1}{\lambda_i} \sum_{i=1}^{n} \lambda_i y_i , \]

which contradicts the definition of the element \( y_i \). Thus \( z = 0 \) and \( Y \subseteq Z \) is a direct sum.

Let \( X \) be a linear space over the field of real numbers. We show that \( X \) can be embedded in a linear space over the field of complex numbers in a natural way. We consider the space of all ordered pairs \( x, y \) with operations defined as follows:

\[ (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) ; \]
\[ (a + b)(x, y) = (ax + by, ay + bx) . \]

We denote this space by \( X + iX \). The rules of distributivity for the operations defined above are easily verified. We only prove that the multiplication is associative. Indeed,

\[ [(a + b)(c + id)](x, y) = (ac - bd, bc + ad)(x, y) = [(ac - bd)x - (bc + ad)y, (ac - bd)y + (bc + ad)x] , \]
\[ (a + b)(c + id)(x, y) = (a + b)(cx - dy, cy + dx) = [(ac - bd)x - (bc + ad)y, (ac - bd)y + (bc + ad)x] . \]

On the other hand, if \( X \) is a linear space over the field of complex numbers, there always exists a linear space \( Y \) over the field of real numbers such that \( Y + iY = X \).

Indeed, let us write all elements of the space \( X \) in a transfinite sequence \( \{x_i\} \). In the same way as in the proof of Theorem 0.2, we choose a subsequence \( \{y_i\} = \{x_{\beta}\} \) such that

\[ y_\beta \not\in \text{lin}(y_\beta; y < y_\beta) \quad \text{and} \quad \text{lin}(y_\beta) = X . \]

Obviously, the linear span means a linear subset with multiplication by complex numbers. If \( X \) is the smallest linear space with real multiplication spanned by the sequence \( \{y_i\} \), we get \( X + iX = X \).

Let \( X \) be a subspace of a linear space \( X \). Obviously, the quotient group \( [X] = X/X_0 \) is also a linear space. This space will be called the quotient space.

The defect (or codimension) of a subspace \( X \) of a linear space \( X \) is the dimension of the quotient space \( X/X_0 \):

\[ \text{codim} X_0 = \dim X/X_0 . \]

**Theorem 0.4.** If \( X_0 \) is a subspace of a linear space \( X \) and \( \text{codim} X_0 < +\infty \), then there exists a space \( X_0 \) such that \( X = X_0 \otimes X_0 \) and \( \text{dim} X_0 = \text{codim} X_0 \).

**Proof.** Let us suppose that \( \text{codim} X_0 = n \). We write \( [X] = X/X_0 \). There exist exactly \( n \) linearly independent elements \( \{a_0, \ldots, a_n\} \subseteq [X] \), and every element \( [x] \subseteq [X] \) can be written in the form \( [x] = \sum_{i=0}^{n} a_i [x_i] \), where \( a_i \) are scalars, in a unique manner. Let \( a_0, \ldots, a_n \) be arbitrary fixed elements such that \( a_0 \not\in [x]. \) Elements \( a_i \) are linearly independent. Hence every element \( x \in X \) can be written in the form

\[ x = x_0 + \sum_{i=0}^{n} a_i [x_i] , \quad \text{where} \quad x_0 \in X_0 , \]

in a unique manner. Thus \( X = X_0 \otimes X_0 \), where \( X_0 = \text{lin} (a_0, \ldots, a_n) \), and \( \text{dim} X_0 = n = \text{codim} X_0 \), by definition.

If a linear space \( X \) is a ring (with the same definition of addition), then \( X \) is a linear space over the field of complex numbers, there always exists a linear space \( Y \) over the field of real numbers such that \( Y + iY = X \). If a subalgebra \( X_0 \) is a two-sided ideal, we call it a subalgebra of \( X \). By a subalgebra of an algebra \( X \) we mean a linear space \( X \) with a ring structure that is a ring with respect to the same operations as \( X \). If a subalgebra \( X_0 \) is a two-sided ideal, we call it a subalgebra of \( X \). By a subalgebra of an algebra \( X \) we mean a linear space \( X \) with a ring structure that is a ring with respect to the same operations as \( X \).

**Example 0.1.** The \( n \)-dimensional vector space with addition and multiplication by scalars defined as usual operations on vectors is a linear space.

**Example 0.2.** The space \( C(0, 1) \) of continuous functions in the closed interval \( [0, 1] \) is a linear space, for the sum of two continuous functions and the product of a continuous function by a scalar are continuous.
EXAMPLE 0.3. The space $C^0[0, 1]$ of functions defined in the interval $[0, 1]$ and possessing the $\mu$th continuous derivative is a linear space, for the derivative of a linear combination of two such functions is a linear combination of their derivatives.

EXAMPLE 0.4. The space $C^\infty[0, 1]$ of functions infinitely differentiable in the interval $[0, 1]$ is a linear space.

EXAMPLE 0.5. The space $C^0[0, 1]$ of all measurable functions defined in the interval $[0, 1]$ is a linear space, for a linear combination of measurable functions is a measurable function.

EXAMPLE 0.6. The space $C^1[0, 1]$ of functions defined in the interval $[0, 1]$ and satisfying Hölder’s condition with the exponent $\mu$, i.e. the condition

\[(\#) \quad \|f(t) - f(t')\| < c|t - t'|^\mu \quad \text{for arbitrary } t, t' \in [0, 1] \quad (0 < \mu < 1),\]

is a linear space, for a linear combination of two functions satisfying Hölder’s condition satisfies Hölder’s condition (if $\mu = 1$, condition (\#) is called the Lipshitz condition).

CHAPTER I

OPERATORS WITH A FINITE AND SEMIFINITE DIMENSIONAL CHARACTERISTIC

§ 1. Linear operators. A \textit{linear operator} is a map $A$ of a linear subset $D_A$ of a linear space $X$ into a linear space $Y$ (both over the same field of numbers) such that

\[ A(x + y) = Ax + Ay; \quad A(tx) = t(Ax); \quad x, y \in D_A, \quad t \text{ is a scalar} . \]

The set $D_A$ is called the \textit{domain} of the operator $A$. Indeed, a linear operator is a pair $(D_A, A)$, for it is defined both by the domain $D_A$ and by the form of the map $A$. However, to be brief, we shall use the traditional notation $A$ instead of the pair $(D_A, A)$.

Let $G$ be a subset of the set $D_A$. We write

\[ AG = \{ y \in Y : y = Ax, \quad x \in G \} . \]

The set $E_A = AD_A$ is called the \textit{range} of the operator $A$ or the \textit{set of its values}. The \textit{graph of an operator} $A$ is the set

\[ W_A = \{ (x, y) : x \in X, y \in Y, \quad y = Ax \} . \]

If $D_A = X$ and $E_A = Y$, the operator $A$ is called an \textit{epimorphism}.

A linear operator $A$ which maps the space $X$ into the space $Y$ is called a \textit{monomorphism} if $D_A = X$ and if the operator $A$ is one-to-one. If an operator $A$ is a monomorphism and an epimorphism simultaneously, then $A$ is called an \textit{isomorphism}.

The set of all linear operators defined in the space $X$ with values in the space $Y$ will be denoted by $L(X \to Y)$.

If an operator $A : L(X \to Y)$ is a monomorphism, we can define the \textit{inverse operator} $A\textsuperscript{-1}$ in such a manner that every element $x \in D_A$ corresponds to an element $y \in E_A$ satisfying the condition $y = Ax$. It is easily verified that the operator $A\textsuperscript{-1}$ is linear and $D_{A\textsuperscript{-1}} = E_A$, $E_{A\textsuperscript{-1}} = D_A$. Hence it follows that if the operator $A : L(X \to Y)$ is an isomorphism, then $A\textsuperscript{-1}$ exists and is also an isomorphism.

Two linear spaces $X$ and $Y$ are called \textit{isomorphic} if there is an isomorphism mapping the space $X$ onto the space $Y$. If spaces $X$ and $Y$ have bases $\{e_1, \ldots, e_n\}$, $\{f_1, \ldots, f_m\}$, of the same power, then $X$ and $Y$ are isomorphic.
Indeed, the operator $A$ defined by means of the formula
\[ A \left( \sum_{i=1}^{n} t_i x_i \right) = \sum_{i=1}^{n} t_i y_i \]
is an isomorphism.

We define the addition of operators $A$, $B \in L(X \rightarrow Y)$ and the multiplication of an operator by a scalar as follows:
\[(A+B)x = Ax + Bx; \quad (\lambda A)x = \lambda (Ax), \quad x \in D_A \triangleq D_A \cap D_B.
\]

Evidently, an operator $C$ such that $A + C = B$ does not exist for any two operators $A$, $B \in L(X \rightarrow Y)$. This follows from the fact that the domains $D_A$ and $D_B$ may be different. If the operator $C$ exists, we write $C = B - A$ and then $C$ is called the difference of operators $B$ and $A$; the operation "-" is called a subtraction.

We denote by $L_0(X \rightarrow Y)$ the set of all operators $A \in L(X \rightarrow Y)$ such that $D_A = X$. The operations of addition, subtraction and multiplication by a scalar are well-defined in the set $L_0(X \rightarrow Y)$. Hence $L_0(X \rightarrow Y)$ is a linear space. The zero element is the zero operator $0: \theta(x) = 0$ for all $x \in X$. In the sequel we denote this operator by $0$, in the same manner as the zero element of the space; this will not lead to any misunderstanding.

Let $A \in L(Y \rightarrow X)$ and $B \in L(X \rightarrow Y)$, and let $D_A \supset E_B$. The superposition or the product of operators $A$ and $B$ is defined as an operator $AB$ defined by means of the equality
\[(AB)x = A(Bx).
\]

We have $AB \in L(X \rightarrow Z)$. If all the operators written below exist, the following rules of distributivity hold:
\[(1) \quad (A_1 + A_2)B = A_1 B + A_2 B; \quad A \in L(Y \rightarrow Z), \quad B \in L(X \rightarrow Y),
\]
\[(2) \quad (A_1 B + A_2 B) = A_1 B + A_2 B; \quad A_1, A_2 \in L(Y \rightarrow Z), \quad B \in L(X \rightarrow Y).
\]

We write briefly $L(Y \rightarrow X) = L(Y)$ and $L_0(X \rightarrow Y) = L_0(X)$. Formulas (1.1) show that $L_0(X)$ is not only a linear space but also an algebra.

The algebra $L_0(X)$ contains a unit, namely the identical operator (identity) $I: IX = x$ for every $x \in X$.

An operator $P \in L_0(X)$ is called a projector (projection operator), if $P^2 = P$ (where $P^2 = P \circ P = P$). Each projector defines a decomposition of the space into a direct sum $X = Y \oplus Z$, where
\[Y = \{ x \in X: Px = 0\}, \quad Z = \{ x \in X: Pg \neq 0\}.
\]

Indeed, if $x \in Y \oplus Z$, then $x = 0$, since $Pg = 0$. If $x \in X$, then $x = x - Px \in Z$, because $P(x - Px) = P - Px = P - P = 0$, and so $x = y + z$, where $y = Px \in Y$, $z = x - Px = (I - P)x \in Z$. The set $Y$ is called the projection space, and the set $Z$ is called the direction of the projection.

On the other hand, if $X = Y \oplus Z$ is a decomposition of the space $X$ into a direct sum, then the operator $P = y$, where $z = y + z$, $y \in Y$, $z \in Z$, is a projector on the projection space $Y$ in the direction $Z$.

Let us remark that the operator $I - P$ is also a projection operator with projection space $Z$ in the direction $Y$.

Let $X_0$ be a subspace of a linear space $X$. Every operator $A \in L_0(X \rightarrow Y)$ induces an operator $[A] \in L_0([X] \rightarrow [Y])$, where $[X] = X/X_0 \supset [Y] = Y/Y_0$, defined as follows:
\[[A]x = [Ax] \quad \text{for} \quad x \in [X],
\]
where $[x]$ is the coset defined by the element $x$.

Let $X_0$ be a subspace of a linear space $X$, and let $A \in L(X \rightarrow Y)$. The operator $A_0 \in L_0(X_0 \rightarrow Y)$ defined by means of the formula
\[A_0x = Ax \quad \text{for} \quad x \in D_A \cap X_0,
\]
is called the restriction of the operator $A$ to the subspace $X_0$.

If $A \in L_0(X_0 \rightarrow Y)$, then an operator $A_1 \in L_0(X \rightarrow Y)$ satisfying the condition
\[A_1x = Ax \quad \text{for} \quad x \in X_0
\]
is called an extension of the operator $A$ to the space $X$.

The following theorem is a consequence of Theorem 0.3:

Theorem 1.1. Every operator $A \in L(X \rightarrow Y)$ defined on a linear subspace $X_0$ of a linear space $X$ may be extended to an operator $\hat{A}$ defined on the whole space $X$, and $\hat{A}_{|X_0} = A$.

Proof. Let us write $X$ as a direct sum $X = X_0 \oplus Z$. As has been proved before, there exists a projection operator $P \in L_0(X \rightarrow X)$, projecting the space $X$ onto the subspace $X_0$. The operator $\hat{A} = AP$ possesses all the required properties. 

A linear operator whose domain is $D_A = X$ and whose values belong to the field of scalars is called a linear functional defined on the space $X$. We denote by $X'$ the set of all linear functionals defined on the space $X$.

Let $\{a_n\}$ be a basis of the linear space $X$. Since every element $x \in X$ can be written in the form
\[x = \sum_{i=1}^{n} t_i a_i \quad \text{where} \quad t_i \text{ are scalars},
\]
in one and only one manner, every linear functional $f$ can be written in the form
\[f(x) = \sum_{i=1}^{n} a_i a_n \quad \text{where} \quad a_n = f(a_n).
In particular, if \( X \) is an \( n \)-dimensional space spanned by the elements \((e_1, \ldots, e_n)\), every linear functional \( f \) is of the form
\[
f(x) = \sum_{i=1}^n a_i e_i,
\]
where \( x = \sum_{i=1}^n a_i e_i \).

Evidently, linear functionals defined on the whole space \( X \) form a linear space \( L_0(X \to \mathbb{C}) \) or \( L_0(X \to \mathbb{R}) \), depending on the field of scalars under consideration. Hence one can speak about the linear independence of elements of these spaces, i.e., functionals. It follows from previous considerations regarding the form of functionals that \( \dim X = \dim \mathbb{C} \).

**Theorem 1.2.** If \( f_1, \ldots, f_n \) are linear functionals defined on a linear space \( X \), and if \( f_i(x) = 0 \) for \( i = 1, 2, \ldots, n \) implies \( g(x) = 0 \), then the functional \( g \) is linearly dependent on functionals \( f_i \).

**Proof.** Let us consider the linear operator \( T \in L_0(X \to \mathbb{C}^n) \) defined by the equality
\[
Tz = [f_1(x), f_2(x), \ldots, f_n(x)].
\]
We define a transformation \( h \) on the linear subspace \( TX \) of the space \( \mathbb{C}^n \) as follows:
\[
h[T(x)] = [f_1(x), \ldots, f_n(x)] = g(x).
\]
The transformation \( h \) is defined uniquely, for \( Tz = Ty \) implies \( (x - y) = 0 \) and consequently \( g(x) = g(y) \). Obviously, \( h \) is a linear functional defined on the subspace \( TX \) of the space \( \mathbb{C}^n \); hence
\[
h[y_1, \ldots, y_n] = \sum_{i=1}^n a_i y_i,
\]
and so
\[
g(x) = \sum_{i=1}^n a_i f_i(x) \quad \text{for} \quad x \in X.
\]

Let an operator \( A \in L_0(X) \) be given. Since \( E_n \subseteq X \), \( E_n \subseteq A(E_n) \subseteq E_{n+1} \), easy induction shows that
\[
X \supseteq E_1 \supseteq E_2 \supseteq \cdots \supseteq E_n \supseteq E_{n+1} \quad (n = 1, 2, \ldots).
\]

**Theorem 1.3.** Let \( A \in L_0(X) \). If
\[
E_{m+n} = E_{m+n+1}, \quad \text{for a fixed positive integer } m,
\]
then
\[
E_{m+n} = E_{m+n} \quad \text{for all } n > m.
\]

**Proof by induction.** It is easily seen that we only need to prove the theorem for \( n = m + 2 \). Let us suppose that \( E_{m+n} \cap E_{m+n+1} \neq 0 \), i.e., that there exists an element \( y \in E_{m+n} \), such that \( y \notin E_{m+n+1} \). From the definition of sets \( E_{m+n} \) follows the existence of an element \( z \in E_{m+n} \) satisfying the equality \( Ax = y \). But \( E_{m+n} = E_{m+n+1} \), and so \( x \notin E_{m+n+1} \). Hence \( y = Ax \notin E_{m+n+1} \), which contradicts the definition of the element \( y \). Consequently, \( E_{m+n} = E_{m+n+1} \).

**§ 2. Dimensional characteristic of linear operators.** Let \( X \) and \( Y \) be linear spaces and let \( A \in L(X \to Y) \). We denote by \( Z_A \) the set of zeros of the operator \( A \):
\[
Z_A = \{ x \in X : Ax = 0 \}.
\]

Evidently, sets \( Z_A \) and \( E_A = \text{ker } A \) are linear spaces. The space \( Z_A \) is called the kernel of the operator \( A \), and the quotient space \( X/E_A \) is called the cokernel of the operator \( A \). The number \( \text{dim } Z_A \) is called the nullity of the operator \( A \), and the number \( \beta_A = \dim Y/E_A = \text{codim } E_A \) is called the deficiency of the operator \( A \). Hence, by definition, the deficiency of an operator \( A \) is equal to the defect of its range.

![Fig. 1](image)

The ordered pair \( (\alpha_A, \beta_A) \) is called the *dimensional characteristic* of the operator \( A \) or, more briefly, its *d-characteristic*. We say that the *d-characteristic* of an operator \( A \) is *finite* if numbers \( \alpha_A, \beta_A \) are both finite. If at least one of the numbers \( \alpha_A, \beta_A \) is finite, we say that the operator \( A \) has a *semifinite d-characteristic*.

We write
\[
\begin{align*}
D(X \to Y) & \quad \text{the set of operators } A \in L(X \to Y) \\
D^+(X \to Y) & \quad \text{such that } \\
D^-(X \to Y) & \quad \text{such that }
\end{align*}
\]

and
\[
\begin{align*}
\alpha_A & < +\infty, \quad \beta_A < +\infty \\
\beta_A & < +\infty, \quad \alpha_A = +\infty \\
\alpha_A & < +\infty, \quad \beta_A = +\infty
\end{align*}
\]
Moreover, we write
\[
D_A^*(X \to Y) = D^*(X \to Y) \cap L_D(X \to Y), \\
D_A^v(X \to Y) = D^v(X \to Y) \cap L_D(X \to Y), \\
D_A(X \to Y) = D(X \to Y) \cap L_D(X \to Y).
\]
We shall write briefly:
\[
D^+(X) = D^+(X \to X); \\
D^-(X) = D^-(X \to X); \\
D(X) = D(X \to X); \\
D_A^+(X) = D_A^+(X \to X); \\
D_A^-(X) = D_A^-(X \to X); \\
D_A(X) = D_A(X \to X).
\]

The above notions are of use in solving linear equations. For, if \( A \) is a linear operator, and if we consider the equation
\[
Ax = y,
\]
a solution of this equation exists if and only if \( y \in E_A \). On the other hand, if \( Ax = y \) for a certain \( x \), the general solution of equation (2.1) is of the form
\[
x = x_0 + x_1,
\]
where \( x_0 \) is an arbitrary element of the set \( Z_A \). Thus, when solving equation (2.1), it is essential to know sets \( Z_A \) and \( E_A \). Numbers \( a_A \) and \( \alpha_A \) characterize these sets in a certain way (but they do not describe them). As we shall see later, the knowledge of deficiency and nullity may be very useful in investigating the solvability of equation (2.1).

Very often the given equation can be reduced to another one in such a manner that the nullity and deficiency of the respective operator are easy to determine. Therefore, the following theorems play a fundamental role in our further considerations.

The \textit{index} \( \kappa_A \) of an operator \( A \in L(X \to Y) \) is defined in the following manner:
\[
\kappa_A = \begin{cases} 
\beta_A - a_A & \text{if } A \in D(X \to Y), \\
+\infty & \text{if } A \in D^+(X \to Y), \\
-\infty & \text{if } A \in D^-(X \to Y).
\end{cases}
\]

The following important theorem holds for the index of a superposition of two operators:

**Theorem 2.1.** If \( D_A \supset E_B \) and if
\[
B \in D^+(X \to Y), \quad D^+(X \to Y), \quad \text{and} \quad A \in D^+(X \to Y), \quad D^+(Y \to Z), \quad D^-(Y \to Z), \quad D(-X \to Z),
\]

then
\[
AB \in D^+(X \to Z), \quad D^+(X \to Z), \quad D^+(Y \to Z), \quad D(Y \to Z),
\]
respectively, and
\[
\kappa_{AB} = \kappa_A + \kappa_B.
\]

**Proof.** First we prove the theorem for operators with a finite \( d \)-characteristic. Let \( Z_A = E_B \cap Z_A \), and let \( \dim Z_A = n_1 \). Evidently, the space \( E_A \) can be written as a direct sum
\[
Z_A = Z_A \oplus Z_A,
\]
where \( \dim Z_A = a_A - n_1 \), and the space \( Y \) as a direct sum
\[
Y = E_B \oplus Z_A \oplus Z_A,
\]
where \( \dim Z_A = n_2 \).

Since the space \( Z_A \oplus Z_A \) is isomorphic with the quotient space \( Y/E_B \), we have \( \dim(Z_A \oplus Z_A) = \beta_B \). Hence \( a_A - n_1 + n_2 = \beta_B \) and
\[
\alpha_A - \beta_B = n_1 - n_2.
\]

From formula (2.4) follows
\[
E_A = E_{AB} \oplus A Z_A.
\]

But \( \dim E_A = \dim Z_A = n_1 \); finally,
\[
\kappa_{AB} = \beta_B - a_B = \beta_A + n_2 - (a_A + n_1)
\]
\[
= \beta_B - \alpha_A + \beta_A - a_B = \kappa_A + \kappa_B.
\]

Now let us suppose \( a_A, a_B < +\infty \). Then \( n_1 < +\infty \) and consequently \( \kappa_{AB} = a_A + n_1 < +\infty \).

Supposing \( \beta_A, \beta_B < +\infty \), we have \( n_2 < \beta_B < +\infty \). Hence \( \kappa_{AB} = \beta_B - n_1 < +\infty \).

The following obvious theorem may be considered as converse to Theorem 2.1:

**Theorem 2.2.** Let \( A \in L(X \to Z), B \in L(X \to Y) \). By \( AB \) we understand the superposition of operators \( A \) and \( B \), whenever this superposition is defined. If \( AB \in D^+(X \to Z) \), then \( B \in D^+(X \to Y) \). If \( AB \in D^-(X \to Z) \), then \( A \in D^-(Y \to Z) \).

This immediately implies

**Corollary 2.3.** If \( A \in L(X \to Y), B \in L(Y \to X) \), and if
\[
AB \in D(Y), \quad BA \in D(X),
\]
then
\[
A \in D_L(X \to Y), \quad B \in D_L(Y \to X).
\]
Corollary 2.4. If \( T \in L_d(X) \) and if there exists a positive integer \( n \) such that \( I - T^n \in D_d(X) \), then \( I - T \in D_d(X) \).

Indeed, it is sufficient to take in Corollary 2.3
\[
A = I - T, \quad B = I + T + \ldots + T^{n-1}.
\]

It follows from the definitions of a monomorphism and an epimorphism that an operator \( A \) is a monomorphism if and only if \( \alpha_0 = 0 \), \( A \) is an epimorphism if and only if \( \beta_0 = 0 \), and \( A \) is an isomorphism if and only if \( \alpha_0 = \beta_0 = 0 \).

We say that an operator \( A \in L_d(X \rightarrow Y) \) is right-invertible (left-invertible) if there exists an operator \( B \in L_d(Y \rightarrow X) \) such that
\[
AB = I_Y \quad (resp. \quad BA = I_X),
\]
where \( I_X \) and \( I_Y \) are identity operators in spaces \( X \) and \( Y \), respectively.

If there is no danger of a misunderstanding, we shall denote the operators \( I_X \) and \( I_Y \) by the same letter \( I \).

Theorem 2.5. An operator \( A \in L_d(X \rightarrow Y) \) is

right-invertible \quad if and only if \quad \beta_0 = 0,

left-invertible \quad if and only if \quad \alpha_0 = 0.

Proof. Sufficiency. Let us suppose \( \alpha_0 = 0 \). This means that \( Z_A = \{0\} \) and that the operator \( A \) maps the whole space \( X \) onto the set \( E_A \). Let us decompose \( Y \) into a direct sum \( Y = E_A \oplus C \). We define an operator \( B \in L_d(Y \rightarrow X) \) as follows:
\[
B_y = \begin{cases} 0 & \text{for } y \in C, \\ x & \text{for } y = Ax, \ x \in X. \end{cases}
\]

Evidently, \( B(Atlas) = x \) for \( x \in X \). Hence \( BA = I_X \), and \( B \) is the left inverse of \( A \).

Now let us suppose \( \beta_0 = 0 \). This means that the operator \( A \) maps the space \( X \) onto the whole space \( Y \). We define an operator \( B \in L_d(Y \rightarrow X) \) by the following equality:
\[
B_y = x \quad \text{for } \ y = Ax, \ x \in X.
\]

Then we get \( A(By) = Ax = y \). Consequently, \( AB = I_Y \), and the operator \( B \) is the right inverse of \( A \).

Necessity. Let \( A \in L_d(X \rightarrow Y) \), and let us suppose that there exists an operator \( B \in L_d(Y \rightarrow X) \) such that \( BA = I_X \). Moreover, let us suppose that there is an element \( x \in Z_A, \ x \neq 0 \). Then
\[
(BA)x = B(Ax) = B(0) = 0,
\]
but this is impossible. Thus \( Z_A = \{0\} \) and \( \alpha_0 = 0 \).

§ 2. Dimensional characteristic of linear operators

Now, if there exists an operator \( B \in L_d(Y \rightarrow X) \) such that \( AB = I_Y \), then
\[
\beta_0 \leq \beta_{AB} = \beta_B = 0.
\]

Consequently, \( \beta_0 = 0 \).

Example 2.1. The operator \( A \) defined in the space \( C[0,1] \) of all continuous functions in the interval \([0,1]\) by the formula
\[
Ax = \int_0^t x(s)ds
\]
is left-invertible, because

\[
\frac{d}{dt} \int_0^t x(s)ds = x(t), \quad \text{and so} \quad \frac{d}{dt} A = I,
\]

and \( \alpha_0 = 0 \). It follows also from equality (2.7) that the operator \( d/dt \) is right-invertible. Hence
\[
\beta_{\frac{d}{dt}} = 0.
\]

Example 2.2. The operator \( p \) of multiplication by a function \( p(t) \in C[0,1] \), \( p(t) \) different from zero in \([0,1]\), considered as an operator in the space \( C[0,1] \), is invertible. Hence
\[
\alpha_p = \beta_p = \alpha_p = \beta_p = 0.
\]

§ 3. Finite-dimensional operators. Let two linear spaces \( X \) and \( Y \) be given. An operator \( K \in L(X \rightarrow Y) \) such that \( \dim E_X = n < +\infty \) is called a finite-dimensional operator. The number \( n \) is called the dimension of the operator \( K \). Each \( n \)-dimensional operator is of the form
\[
Kx = \sum_{i=1}^n \varphi_i(x) y_i,
\]
where the elements \( y_i \in Y \) are linearly independent, and \( \varphi_i \) are linear functionals defined over the space \( X \). We denote by \( K(X \rightarrow X) \) the set of all finite-dimensional operators. The set \( K(X \rightarrow X) \) will be denoted briefly by \( K(X) \). Let us remark that a sum of two finite-dimensional operators, and superpositions \( AK \) and \( KA \) of an arbitrary operator \( A \) and a finite-dimensional operator \( K \) are finite-dimensional operators.

Let \( K \in K(X) \). We may consider the operator \( I + K \), where \( I \) is the identity operator. We show that \( I + K \) has a finite \( d \)-characteristic. Indeed, let us consider the equation
\[
(I + K)x = x_0, \quad \text{where} \quad Kx = \sum_{i=1}^n \varphi_i(x) y_i.
\]

Equations in linear spaces
The solution of this equation must be of the form

\[ x = x_0 - \sum_{i=1}^{n} C_i y_i, \quad \text{where} \quad C_i = \varphi_i(x). \]

We apply functionals \( \varphi_j (j = 1, 2, \ldots, n) \) to both sides of equation (3.2). We obtain the following system of algebraic equations:

\[ C_j = - \sum_{i=1}^{n} K_{ij} C_i + L_j \quad (j = 1, 2, \ldots, n), \]

where \( K_{ij} = \varphi_j(y_i) \) and \( L_j = \varphi_j(x_0) \).

If the determinant of system (3.3),

\[
\begin{vmatrix}
K_{11} & K_{12} & \cdots & K_{1n} \\
K_{21} & K_{22} & \cdots & K_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
K_{n1} & K_{n2} & \cdots & K_{nn}
\end{vmatrix},
\]

is different from zero, then to any system of numbers \( (L_1, L_2, \ldots, L_n) \) there exists a solution of the system of equations (3.3). Hence the operator \( I + K \) is invertible and the solution of equation (3.2) may easily be determined.

On the other hand, if the determinant (3.4) is different from zero and if \( x_0 = 0 \), then equations (3.3) have only a zero solution. Since equations (3.3) are linearly independent, equation (3.2) also has a zero solution only.

If the determinant (3.4) is equal to zero, then it is not for all systems \( (L_1, L_2, \ldots, L_n) \) that a solution of system (3.2) exists. A necessary and sufficient condition for the existence of a solution of (3.2) is that the vector \( (L_1, L_2, \ldots, L_n) \) belong to a certain \( k \)-dimensional space, where the number \( k \) is the so-called rank of the matrix \( (K_{ij}) \). On the other hand, in this case the homogeneous equation (3.2) has \( n-k \) linearly independent solutions \( y_i \).

\[ \alpha_{i+k} = n-k \quad \text{and} \quad \beta_{i+k} = n-k \]

and we get the following

**Theorem 3.1.** If \( K \in K(X) \), then \( I + K \in D_b(X) \) and

\[ \alpha_{i+k} = 0. \]

The following theorem is a consequence of Theorems 3.1 and 2.1:

**Theorem 3.2.** If \( A \in D(X \to Y) \), \( A \in D'(X \to Y) \), and \( K \in K(X \to Y) \), then \( A + K \in D'(X \to Y) \), \( D'(X \to Y) \), \( D'(X \to Y) \), and \( D'(X \to Y) \), and \( D'(X \to Y) \), respectively.

\[ \alpha_{i+k} = \beta_{i+k} = 0. \]

Proof. First, we suppose \( A \in D(X \to Y) \) and decompose the space \( \mathcal{Z} \) into a direct sum of the space \( \mathcal{Z}_A \) and a certain space \( \mathcal{S} : X = \mathcal{Z}_A \oplus \mathcal{S}. \)

Evidently, by Theorem 2.5, the operator \( A \) defined as the restriction of the operator \( A \) to the space \( \mathcal{S} \), is left-invertible. Let \( A_i \) be the restriction of the operator \( A \) to the space \( \mathcal{S} \). Then

\[ A_i \in \mathcal{Z}_A \quad \text{and} \quad A_i = (I + E_i A_i^{-1}) A_i. \]

\( E_i A_i^{-1} \) is a finite-dimensional operator defined in the space \( \mathcal{Y} \) with values in \( \mathcal{Y} \). By Theorems 2.1, 2.5 and 3.1,

\[ \alpha_{i+k} = \alpha_{i+k}^+ = \alpha_{i+k}^- = \beta_{i+k} = \beta_{i+k}^+ = \beta_{i+k}^- = 0. \]

On the other hand, the operator \( A + K \) is an extension of the operator \( A + K \). We shall prove \( \alpha_{i+k} = \alpha_{i+k}^+ - \alpha_{i+k}^- \).

We consider three cases:

(i) \( K \mathcal{Z}_A \in \mathcal{E} \mathcal{A}_+ \mathcal{K}_1 \). Then \( E_{A_i} A_i = E_{A_i} A_i + K \mathcal{Z}_A \), and so \( \beta_{i+k} = \beta_{i+k}^+ - \beta_{i+k}^- \).

(ii) We have \( K \mathcal{Z}_A \in \mathcal{E} \mathcal{A}_+ \mathcal{K}_1 \) for each \( x \in \mathcal{Z}_A \), \( x \neq 0 \). We write \( \mathcal{Y} = \mathcal{Y} \mathcal{Z}_A \).

Obviously, \( \beta_{i+k} = \beta_{i+k}^+ - \beta_{i+k}^- \). On the other hand, \( a_{i+k} = a_{i+k}^+ - a_{i+k}^- \), since the number \( a_{i+k} \) is equal to the nullity of the operator \( K \), restricted to the space \( \mathcal{Z}_A \). Thus, \( \alpha_{i+k} = \alpha_{i+k}^+ - a_{i+k}^- \), because \( (A + K) x = K x \) for \( x \in \mathcal{Z}_A \).

(iii) In the general case, we decompose the space \( \mathcal{Z}_A \) into a direct sum of two spaces \( \mathcal{G} \) and \( \mathcal{G}_0 \): \( \mathcal{Z}_A = \mathcal{G}_0 \oplus \mathcal{G}_1 \), where \( \mathcal{K} \mathcal{E} \mathcal{A}_+ \mathcal{K}_1 \), and \( \mathcal{K} \mathcal{E} \mathcal{A}_+ \mathcal{K}_1 \), for every \( x \in \mathcal{G} \), \( x \neq 0 \). First, we consider the restriction \( A_i \mathcal{G}_0 + \mathcal{K}_0 \mathcal{A}_1 \mathcal{K}_1 \) of the operator \( A + K \) to the space \( \mathcal{G} \). Applying part (ii) of the proof we obtain \( \mathcal{K} \mathcal{E} \mathcal{A}_+ \mathcal{K}_1 \) and \( \mathcal{G}_0 + \mathcal{G}_0 \mathcal{E} \mathcal{A}_+ \mathcal{K}_1 \). On the other hand, applying part (i) of the proof we get \( \mathcal{K} \mathcal{E} \mathcal{A}_+ \mathcal{K}_1 \) and \( \mathcal{G}_0 \mathcal{E} \mathcal{A}_+ \mathcal{K}_1 \). Hence \( \mathcal{K} \mathcal{E} \mathcal{A}_+ \mathcal{K}_1 \) and \( \mathcal{G}_0 + \mathcal{G}_0 \mathcal{E} \mathcal{A}_+ \mathcal{K}_1 \).

These results together with formula (3.5) yield the theorem for \( A \in D(X \to Y) \).

Now let \( a_{i+k} = a_{i+k}^+ = a_{i+k}^- = 0. \) Formula (3.5) gives

\[ \alpha_{i+k} = \beta_{i+k} = + \infty, \]

and since the dimension of the space \( \mathcal{Z}_A \) is finite, we get \( \alpha_{i+k} = + \infty, \)

Finally, let us suppose \( a_{i+k} = a_{i+k}^+ = a_{i+k}^- = 0. \) Formula (3.5) implies \( \alpha_{i+k} = \beta_{i+k} = + \infty, \)

but the dimension of the space \( \mathcal{Z}_A \) is infinite. Hence

\[ \alpha_{i+k} = \beta_{i+k} = + \infty. \]
Equations with finite-dimensional operators are well known in the theory of integral equations as equations with degenerated kernels. We now give examples of such equations and determine their solutions.

Example 3.1. We shall investigate for what values of the real parameters $\lambda$ and $\mu$ the equation

$$\varphi(t) + \lambda \int_0^1 t \varphi(s)ds = \mu t - 1$$

has solutions belonging to the space $C[0,1]$.

The operator $K\varphi = \lambda \int_0^1 t \varphi(s)ds$ is one-dimensional. Hence the solution must be of the form

$$\varphi(t) = \mu t - 1 - \lambda C t,$$

where $C = \int_0^1 s \varphi(s)ds$.

This gives the equality

$$\int_0^1 s ((\mu t - 1) - \lambda C s)ds = C .$$

Integration reduces this equality to the following one:

$$(\frac{1}{2} \lambda + 1) C = \frac{1}{2} \mu - \frac{1}{2} .$$

(i) If $\lambda \neq -3$, equation (3.8) has a unique solution:

$$C = \frac{\frac{1}{2} \mu - \frac{1}{2}}{\frac{1}{2} \lambda + 1} = \frac{2 \mu - 3}{2 \lambda + 3} .$$

Applying (3.7) we find that in case $\lambda \neq -3$ for every value of $\mu$ equation (3.6) has a unique solution given by the formula

$$\varphi(t) = \mu t - 1 - \frac{\lambda}{2} \left(2 \mu - 3 \right) \int_0^1 s \varphi(s)ds = \mu t - 1 - \frac{3 (\lambda + 2 \mu)}{2 (\lambda + 3)} .$$

(ii) If $\lambda = -3$, a solution of equation (3.8) exists if and only if

$$\frac{1}{2} \mu - \frac{1}{2} = \frac{1}{2} \mu - \frac{1}{2} = 0 ,$$

that is if $\mu = \frac{1}{2}$. Then the homogeneous equation, i.e. the equation

$$\varphi(t) - \frac{3}{2} \int_0^1 t \varphi(s)ds = 0 ,$$

has only one linearly independent solution: $\varphi(t) = t$, and equation (3.6) has a solution (if and only if $\mu = \frac{1}{2}$) given by the formula

$$\varphi(t) = \frac{3}{2} t - 1 + 3 C t = 3 (C + \frac{1}{2}) t - 1 ,$$

where $C$ is an arbitrary constant.

Example 3.2. We investigate for what values of the real parameter $\lambda$ the equation

$$\varphi(t) - \int_0^1 \sin(t + s) \varphi(s)ds = \sin 2t - \frac{3}{2}$$

has a continuous solution in the interval $[0,\pi/2]$.

The operator $K\varphi = -\int_0^1 \sin(t + s) \varphi(s)ds$ is two-dimensional, for $\sin(t + s) = \sin t \cos s + \cos t \sin s$. Hence the solution must be of the form:

$$\varphi(t) = \sin 2t - \frac{3}{2} + C_1 \sin t + C_2 \cos t ,$$

where

$$C_1 = \int_0^{\pi/2} \cos t \varphi(t)ds, \quad C_2 = \int_0^{\pi/2} \sin t \varphi(t)ds .$$

We obtain the following two equalities:

$$C_1 = \int_0^{\pi/2} \cos \varphi \sin 2t ds = \frac{\pi}{2} C_1, \quad C_2 = \int_0^{\pi/2} \sin \varphi \cos 2t ds = \frac{\pi}{2} C_2 .$$

Since

$$\int_0^{\pi/2} \cos(\sin 2t - \frac{1}{2})ds = \int_0^{\pi/2} \sin(\sin 2t - \frac{1}{2})ds = 0 ,$$

we obtain a system of equations

$$(1 - \frac{\pi}{2}) C_1 - \frac{\pi}{2} \lambda C_2 = 0 ,$$

$$(1 - 2\pi) C_1 - \frac{\pi}{2} \lambda C_2 = 0 .$$

The determinant of this system

$$D_1 = \frac{\pi}{2} (1 - \pi) \lambda - 1 = \frac{\pi}{2}[(1 + \pi) \lambda - 2][(1 - \pi) \lambda - 2]$$

is equal to zero for $\lambda = 2/(1 \pm \pi)$. 
Hence:

(i) If \( \lambda \neq 2(1 \pm \pi) \), system (3.11) has no non-zero solutions, and equation (3.9) has a unique solution \( x(t) = \sin 2t \cdot \frac{1}{2} \).

(ii) If \( \lambda = 2(1 \pm \pi) \), then \( D_1 = 0 \). But the rank of the matrix of the determinant \( D_1 \) is equal to 1, and we can solve e.g. the first of the equations (3.11). We obtain

\[
\frac{\pi}{1 \pm \pi} (C_1 - C_2) = 0,
\]

that is \( C_1 = \pm C_2 \). Then the homogeneous equation

\[
x(t) = \lambda \int_0^\pi \sin(t + s) x(s) \, ds = 0
\]

has two non-zero linearly independent solutions:

\[
x_1(t) = \sin t + \text{const}; \quad x_2(t) = \sin t - \text{const}.
\]

The general solution of equation (3.9) is given by the formula

\[
x(t) = \sin 2t \cdot \frac{1}{2} + \frac{2}{1 \pm \pi} C (\sin t + \text{const}) + \frac{2}{1 \pm \pi} D (\sin t - \text{const}),
\]

where \( C \) and \( D \) are arbitrary constants.

§ 4. Perturbations of operators. Let a class \( \mathcal{U} \) of operators be given. An operator \( T \) is called an \( \mathcal{U} \)-perturbation of an operator \( A \in \mathcal{U} \), if \( A + T \in \mathcal{U} \).

If an operator \( B \) is an \( \mathcal{U} \)-perturbation of all operators \( A \in \mathcal{U} \), we call \( B \) an \( \mathcal{U} \)-perturbation.

**Theorem 4.1.** The set of \( \mathcal{U} \)-perturbations is additive, i.e. if operators \( T_1, T_2 \) are \( \mathcal{U} \)-perturbations, then the operator \( T_1 + T_2 \) is also an \( \mathcal{U} \)-perturbation.

**Proof.** Let \( A \in \mathcal{U} \). Then \( A + T_1 \in \mathcal{U} \), for \( T_1 \) is an \( \mathcal{U} \)-perturbation. But \( T_1 \) is also an \( \mathcal{U} \)-perturbation. Hence \( (A + T_1) + T_2 \in \mathcal{U} \). Thus, the operator \( T_1 + T_2 \) is an \( \mathcal{U} \)-perturbation.

**Theorem 4.2.** If the set \( \mathcal{U} \) is homogeneous, i.e. if for any scalar \( a \) and any operator \( A \in \mathcal{U} \) we have \( aA \in \mathcal{U} \), then the set of \( \mathcal{U} \)-perturbations is linear.

**Proof.** Let \( A \in \mathcal{U} \). Then \( \frac{1}{a} A \in \mathcal{U} \) for \( a \neq 0 \). If an operator \( T \) is an \( \mathcal{U} \)-perturbation, then

\[
\frac{1}{a} A + T = \frac{1}{a} (A + aT) \in \mathcal{U}.
\]

Hence \( A + aT \in \mathcal{U} \), and the operator \( aT \) is an \( \mathcal{U} \)-perturbation. Theorem 4.1 implies that the set of \( \mathcal{U} \)-perturbations is a linear set. 

In the last section (Theorem 3.2) we proved that finite-dimensional operators are perturbations of the class of operators with finite \( d \)-characteristics. Now we shall show the converse theorem to be also true.

**Theorem 4.3.** If the class \( D_0(X \to Y) \) of all operators with a finite \( d \)-characteristic is non-void, and if an operator \( K \in L(X \to Y) \) is a perturbation of the class \( D_0(X \to Y) \), then the operator \( K \) is finite-dimensional.

**Proof.** Let us suppose that the operator \( K \) is not finite-dimensional. There exists a sequence \( (y_n) \) of linearly independent elements belonging to the image \( EX \) of the space \( X \) by means of the operator \( K \). Let \( x_n \) be elements satisfying the condition \( K x_n = y_n \). Let \( X_n = X_n \oplus \mathbb{C} \). The space \( X_n \) has an infinite defect. Let \( A \) be an arbitrary operator with a finite \( d \)-characteristic which maps the space \( X \) into the space \( Y \). The set \( \mathcal{A} = \mathcal{A} \) also has an infinite defect, and only a finite number of elements \( y_n \) belongs to this set. Indeed, let us suppose that there exists a subsequence \( (y_m) \) such that \( y_m \notin \mathcal{A} \), i.e. there exists a sequence of elements \( (x_m) \) such that \( A^m x_m = y_m \). Then \( A^m x_m = 0 \) for \( k = m = m \). But elements \( x_m \) are linearly independent, and so \( a_m = + \infty \), which contradicts the assumption that the operator \( A \) has a finite \( d \)-characteristic.

We define a linear operator \( B \in L_0(X \to Y) \) in such a manner that

\[
Bx = \begin{cases} A x & \text{if } x \in \mathbb{C}, \\ y_n & \text{if } x = x_n \text{ and } y \in \mathbb{C}, \\ 0 & \text{if } x = x_n \text{ and } y \notin \mathbb{C}. \end{cases}
\]

It follows from the previous considerations that \( a_m < + \infty \). On the other hand, \( E_0 = E_0 \oplus E_1 = E_0 \). Hence \( \beta_1 = \beta_2 < + \infty \). However, the operator \( B - K \) does not have a finite \( d \)-characteristic, because \( (B - K)x_n = 0 \) for infinitely many elements of the sequence \( (x_n) \), and consequently \( a_m = + \infty \). Thus, the operator \( K \) is not a perturbation of the operator \( B \in D_0(X \to Y) \). This contradicts the assumption of the theorem. 

**Remark.** If the bases of spaces \( X \) and \( Y \) are of different powers, there exist no operators with finite \( d \)-characteristics which map the space \( X \) into the space \( Y \).

Let us remark that the inequality \( \beta_2 < + \infty \) was applied in the proof of Theorem 4.3 in order to show that \( \beta_2 < + \infty \). Hence the following theorem holds:

**Theorem 4.4.** If the class \( D_0^1(X \to Y) \) of all operators of finite nullity is non-void, and if an operator \( K \in L(X \to Y) \) is a perturbation of the class \( D_0(X \to Y) \), then the operator \( K \) is finite-dimensional.
§ 5. Algebras of operators and regularization to an ideal. Let $\mathcal{X}(X)$
$\subset L_0(X)$ be an algebra of operators, i.e. suppose that a linear combination
and a superposition of two operators belonging to $\mathcal{X}(X)$ also belong to
$\mathcal{X}(X)$. Moreover, we always assume the identity $I \in \mathcal{X}(X)$.

An example of an algebra $\mathcal{X}(X)$ yields the whole algebra $L_0(X)$. It is
easily verified that all finite-dimensional operators belonging to $\mathcal{X}(X)$
form a two-sided ideal in the algebra $\mathcal{X}(X)$. We shall denote this ideal
by $\mathcal{E}_0(X)$.

**Theorem 5.1 (On the Representation of Algebras).** Every algebra $\mathcal{X}$
can be represented as an algebra of operators over a certain linear space $X$.

**Proof.** We define the linear space $X$ as equal to the algebra $\mathcal{X}$: $X = \mathcal{X}$.
We associate with every $x \in X$ an operator $A_x$ defined as follows

$$A_x y = x y, \quad y \in X.$$

Obviously, $A_x A_y = A_{x+y} \quad A_x A_y = A_{xy}, \quad A_x = I$
(where $e$ is the unity of the algebra $\mathcal{X}$). Hence it follows immediately that
the set $(A_x : x \in X)$ is an algebra of operators over the space $X$.

If for a given operator $A \in \mathcal{X}(X) \subset L_0(X)$ there exists an operator
$R_A \in \mathcal{X}(X)$ such that $R_A A = I + T$ (or $A R_A = I + T$), where $T \in J$, and $J$
is a two-sided ideal contained in the algebra $\mathcal{X}(X)$, then this operator $R_A$
is called a left regularizer (or right regularizer) of the operator $A$ to the ideal $J$.

If a regularizer is both left and right, it is called a simple regularizer.

A left (right) regularizer of an operator $A$ to a zero ideal is usually
called a left inverse (right inverse) of the operator $A$. If there exists a

regularizer to a proper ideal $J$, it does not belong to $J$. Indeed, let us suppose
$R_A \in J$. Since $R_A A = I + T$, where $T \in J$, we have $I = - T + R_A A \cdot T$.
This would imply that the identity operator $I \in J$, which contradicts the assumption
that the ideal $J$ is proper.

If an operator $A$ has a left regularizer (right regularizer) to an ideal $J$,
then the coset $[R_A]$ is a left inverse (right inverse) of the coset $[A]$ in the
quotient algebra $\mathcal{X}(X)/J$. Hence, if the operator $A$ has a simple regularizer
to the ideal $J$, then

$$[R_A][A] = [A][R_A] = [I].$$

Thus $[A]^{-1} = [R_A]$. Let us remark that the last equality does not imply
that the operator $A$ is invertible.

This implies the following properties of regularizers:

**Property 5.1.** If an operator $A \in \mathcal{X}(X) \subset L_0(X)$ has a left regularizer $R_A$
and a right regularizer $R_A$, to an ideal $J \subset \mathcal{X}(X)$, then each of those regularizers
is simple, and $R_A = R_A \cdot J$.

**Property 5.2.** A simple regularizer to an ideal $J$ is unique in the
sense that two simple regularizers differ by a term belonging to the ideal $J$.

**Property 5.3.** If an operator $A \in \mathcal{X}(X)$ is of the form $A = B + T$,
where the operator $B$ has a left inverse (right inverse) $B_A \in \mathcal{X}(X)$ and $T \in J$
in $\mathcal{X}(X)$, then $A$ possesses a left regularizer (right regularizer) $R_A = B_A$ to the
ideal $J$. Conversely, if a left regularizer (right regularizer) of an operator $A$
has a left inverse (right inverse) $B \in \mathcal{X}(X)$, then $A = B + T$, where $T \in J$.

**Property 5.4.** If an operator $A \in \mathcal{X}(X)$ has a left regularizer (right
regularizer, simple regularizer) $R_A$ to a two-sided ideal $J \subset \mathcal{X}(X)$, then
ever to $T \in J$ there exists a left regularizer (right regularizer, simple regularizer)
$R_A T \subset$ of the operator $A + T$, and

$$R_A T = R_A.$$

Let us note that the following equality holds for a superposition of two operators:

$$R_A R_B = R_B R_A$$

(if the regularizers $R_A$ and $R_B$ exist). Indeed, we have e.g. for left regularizers

$$R_A R_B A B = R_B (I + T_A) B = R_B B + R_B T_A B = I + T_B + R_B T_A B,$$

where $T_A, T_B \in J$. Hence also $T_B + R_B T_A B \in J$. Thus, $R_B R_A$ is a regularizer
of the operator $A B$.

Evidently, if $A, B \in \mathcal{I}(X)$, then

$$(5.2) \quad Z_{AB} \supset Z_A \quad \text{and} \quad Z_{AB} \supset Z_B.$$

Let us suppose that the operator $A$ has a left regularizer (right
regularizer) $R_A$ to a certain two-sided ideal $J$. Then

$$R_A A = I + T_1 \quad (A R_A = I + T_2), \quad \text{where} \quad T_1, T_2 \in J.$$

Formulae (5.2) show that in order to investigate the kernel of an
operator $A$ it is sufficient to investigate the operator $A$ restricted to the space
$Z_{T_1 T_2} = Z_{A R_A} \supset Z_A$.

Similarly, to investigate the cokernel of an operator $A$ it is sufficient
to consider the operator $\hat{A}$ induced by the operator $A$ in the quotient
space $X/\{T_1, T_2\}$ for $X_{T_1 T_2} \subset Z_{A R_A}$ and $X_{T_1 T_2}$. In the case where $R_A T \subset (T_1, T_2) = \infty$,
this is an essential simplification, since it reduces the
investigation of infinite-dimensional spaces to the investigation of finite-
dimensional spaces. Therefore, in the sequel we shall investigate those
ideals of operators for which the operator $I + T$ has a finite nullity (deficiency)
for all $T \in J$.

Moreover, e.g. if an operator $A$ has a left regularizer $R_A$ which is
left-invertible, then the first of the formulae (5.2) implies $Z_A = Z_{R_A A}$.
Hence $a_A = a_{R_A A}$. Indeed, if $R_A$ is the left inverse of the operator $R_A$, then

$$Z_A = Z_{B_R R A} \supset Z_{R_A A} \supset Z_A.$$  

Analogously, if an operator $A$ has a right regularizer $R_A$ which is right-invertible, then $R_A = E_{R A A}$. Hence $\beta_A = \beta_{R A A}$. 

§ 6. Quasi-Fredholm ideals. A two-sided ideal $J$ of operators is called a quasi-Fredholm ideal if the operator $I + T$ has a finite $d$-characteristic for every $T \in J$. If, moreover, $\mu_{R A T} = 0$ for every $T \in J$, then $J$ is called a Fredholm ideal.

Let $X(\lambda)$ be an algebra of operators. The set $E(\lambda)$ of all finite-dimensional operators belonging to the algebra $X(\lambda)$ is a two-sided Fredholm ideal (Theorem 3.1).

It is possible to show that there exist quasi-Fredholm ideals which are not Fredholm ideals. This is proved by the following example given to the authors by G. Neubauer:

Example 6.1. Let $X$ be the space $\pi^s$ of all sequences. Let the operator $R$ transform the space $X$ into itself in the following manner:

$$Rz = y, \quad z = (\xi_1, \xi_2, \ldots), \quad y = (0, \xi_1, \xi_2, \ldots).$$

We write

$$B = R - I.$$  

Let $X(\lambda)$ be the algebra of all polynomials of the operator $B$ with complex coefficients. Each operator belonging to the algebra $X(\lambda)$ and different from zero has a finite $d$-characteristic. Indeed, let $q(B) \in X(\lambda)$.

If $q(B) = I$, this fact is obvious. If $q(B) \neq I$, this polynomial can be written in the form

$$q(B) = a_0 \sum_{i=1}^n (B - a_i I) = a_0 \sum_{i=1}^n (R - b_i I), \quad \text{where} \quad b_i = a_i + 1.$$  

Each of the operators $(R - b_i I)$ has a finite $d$-characteristic. Hence, by Theorem 3.1, the operator $q(B)$ has a finite $d$-characteristic. Thus every ideal in the algebra $X(\lambda)$ is quasi-Fredholm. Now we consider the ideal of all operators of the form $p = B \cdot q(B)$. This ideal is not a Fredholm ideal, for $q(B) = I$ implies $A = I + B = E$, whence $\beta_{R A} = \beta_1 = 1, \alpha_{R A} = 0$, and so $\mu_{R A} = 1 \neq 0$.

The following theorem is an immediate consequence of Theorem 2.2 and of Corollary 2.3:

Theorem 6.1. Let $X(\lambda)$ be an algebra of operators. If an operator $A \in X(\lambda)$ has a simple regularizer $R_A$ to a quasi-Fredholm ideal $J \in X(\lambda)$, then $A$ has a finite $d$-characteristic. If, moreover, $J$ is a Fredholm ideal, then $\kappa_A = -\kappa_{R A A}$. 

§ 6. Quasi-Fredholm ideals. An algebra of operators $X(\lambda) \subset L(\lambda) \subset X(\lambda)$ is called regularizable to a two-sided ideal $J \subset X(\lambda)$ if every operator with a finite $d$-characteristic and belonging to the algebra has a simple regularizer to the ideal $J$. An algebra regularizable to the ideal $R(\lambda)$ of all finite-dimensional operators belonging to the algebra $X(\lambda)$ is called briefly regularizable.

Theorem 6.2. If an algebra of operators $X(\lambda)$ is regularizable to a quasi-Fredholm ideal $J \subset X(\lambda)$, then every operator $T \in J$ is a perturbation of the class $D(\lambda) \subset X(\lambda)$. If, moreover, $J$ is a Fredholm ideal, then the perturbations $T + J$ do not change the index, i.e. we have

$$\kappa_{T + J} = \kappa_T$$

for every operator $T \in D(\lambda) \subset X(\lambda)$ and for every operator $T \in J$.

Proof. Let $A \in D(\lambda) \subset X(\lambda)$. By the assumption, there exists a simple regularizer $R_A$ of the operator $A$ to the ideal $J$. By Property 5.4, the regularizer $R_A$ is also a simple regularizer of the operator $A + T$ for all operators $T \in J$. By Theorem 6.1, it follows that the operator $A + T$ has a finite $d$-characteristic. Since $A$ is an arbitrary operator from the set $D(\lambda) \subset X(\lambda)$, the operators $T \in J$ are perturbations of the class $D(\lambda) \subset X(\lambda)$.

If, moreover, $J$ is a Fredholm ideal, then

$$\kappa_{T + J} = \kappa_T = -\kappa_{R A A} = -\kappa_{R_A A}. \quad \Box$$

Theorem 6.3. If $J$ is a quasi-Fredholm ideal in an algebra of operators $X(\lambda)$ regularizable to a quasi-Fredholm ideal $J_1 \subset X(\lambda)$, then every operator $T \in J$ is a perturbation of the class $D(\lambda) \subset X(\lambda)$. If, moreover, $J$ and $J_1$ are Fredholm ideals, then this perturbation does not change the index, i.e.

$$\kappa_{T + J} = \kappa_T = \kappa_{J_1}.$$  

Proof. Let $J_1 = J + J_1$. Evidently, $J_1$ is a linear set. We shall prove that $J_1$ is a two-sided ideal. Indeed, if $T \in J, T_1 \in J_1, A \in X(\lambda)$, then

$$A(T + T_1) = AT + AT_1 \in J + J_1 = J_1,$$

$$(T + T_1)A = TA + T_1 A \in J_1.$$  

The ideal $J_1$ is quasi-Fredholm, since Theorem 6.2 implies that if the operator $I + T$ has a finite $d$-characteristic then the operator $I + T + T_1$ also has a finite $d$-characteristic.

If $J$ and $J_1$ are Fredholm ideals, there $J_1$ is also a Fredholm ideal, because Theorem 6.2 implies $\kappa_{T + J_1} = \kappa_{T + J}$. Since the algebra $X(\lambda)$ is regularizable to the ideal $J_1$, there exists a simple regularizer $R_A$ to the ideal $J_1$, whence also to the ideal $J$. Property 5.4 implies that for every
operator $T \epsilon J$; in particular for $T \epsilon J$, the operator $R_A$ is a simple regularizer of the operator $A + T$ to the ideal $J$, i.e. $R_{A + T} = R_A$. Hence, by Theorem 6.4, the operator $A + T$ has a finite d-characteristic. If, moreover, $J$ is a Fredholm ideal, then $J$ is also a Fredholm ideal and Theorem 6.2 implies $\mu_{A + T} = \lambda_A$.

**Corollary 6.4.** If $J$ is a quasi-Fredholm ideal in a regularizable algebra $X(X)$, then every operator $T \epsilon J$ is a perturbation of the class $D(X) \cap X(X)$. If $J$ is a Fredholm ideal, then this perturbation does not change the index, i.e. $\mu_{A + T} = \lambda_A$ for $A \epsilon D(X) \cap X(X)$ and $T \epsilon J$.

An analogous theorem holds for operators with semifinite characteristics, namely:

**Theorem 6.5.** Let $x \epsilon X(X)$ be an algebra such that for every $A \epsilon D(X) \cap X(X)$ there exists a left regularizer (right regularizer) in the ideal $E_{A}(X)$. If $I + T \epsilon D'_X(X)$, $I + T \epsilon D'_X(X)$ for every $I \epsilon J$, the operators $T + I$ are perturbations of the class $D'_X(X) \cap X(X)$, $D'_X(X) \cap X(X)$, respectively.

Let $x \epsilon X(X) \subset L(X)$ be an algebra of operators, and let $J$ be a quasi-Fredholm operator in this algebra. In the definition of a quasi-Fredholm operator the ideal $J$ is supposed to be two-sided. Hence one can consider the quotient algebra $X_0 = X(X)/J$. The coset induced in the algebra $X_0$ by the operator $A \epsilon X(X)$ will be denoted by $[A]$. The coset $[I]$ is the unity of the algebra $X_0$. The radical $R(X_0)$ of the algebra $X_0$ is the set of elements $x \epsilon X_0$ such that the element $[I] + axb$ is invertible for arbitrary elements $a, b \epsilon X_0$. We write

$$J_0 = \{ U \epsilon X(X) : [U] \epsilon R(X_0) \}.$$

**Theorem 6.6.** The set $J_0$ is a quasi-Fredholm ideal.

**Proof.** It is well known (§ 0) that a radical is a two-sided ideal. Hence the set $J_0$ is also a two-sided ideal. If $U \epsilon J_0$, the definition of the radical implies that the element $[I] + axb$ is invertible in the algebra $X_0$. Hence there exists a coset $[V]$ such that $[V][I] + [U] = [V]$, this means that for every $V \epsilon [V]$, $I + [V] = I + [T]$, where $T_1, T_2 \epsilon J$.

Thus, the operator $I + U$ has a simple regularizer to the quasi-Fredholm ideal $J$. By Theorem 6.1, the operator $I + U$ has a finite d-characteristic.

**Corollary 6.7.** If the algebra $X(X)$ is regularizable to a quasi-Fredholm ideal $J$, then operators belonging to the ideal $J_0$ are perturbations of the class $D(X) \cap X(X)$.

§ 6. Quasi-Fredholm ideals

The proof is an immediate consequence of Theorems 6.3 and 6.6.

**Corollary 6.8.** If an algebra $X(X)$ is regularizable, and if

$$K_0 = \{ U \epsilon X(X) : [U] \epsilon R(X_0) \},$$

then operators belonging to the ideal $K_0$ are perturbations of the class $D(X) \cap X(X)$.

**Theorem 6.9.** If an algebra $X(X)$ is regularizable to a quasi-Fredholm ideal $J \subset X(X)$, then every quasi-Fredholm ideal $J \subset X(X)$ is contained in the ideal $J_0 = J \subset J_0$.

**Proof.** Let $U \epsilon J$. Given arbitrary operators $A_1, B \epsilon X(X)$ we have $A_0 = [A]$, $B_0 = [B]$. Since the algebra $X(X)$ is regularizable to the ideal $J$, the operator $I + A_0 U - B_0$ has a simple regularizer to the ideal $J$. Hence the element $[I] + [A] [U] [B]$ is invertible in the algebra $X_0 = X(X)/J$. But the operators $A$ and $B$ are arbitrary; this implies that the element $[U]$ belongs to the radical $R(X_0)$. Hence $U \epsilon J_0$.

**Corollary 6.10.** If an algebra $X(X)$ is regularizable to a quasi-Fredholm ideal $J$, and if, moreover, this algebra is regularizable, then $J_0 = K_0$.

Let an algebra $X(X)$ regularizable to a Fredholm ideal $J$ be given. Let us remark that the index $\lambda_A$ satisfies the following conditions:

(i) $\lambda_A$ is an integer-valued function defined in the set $D(X) \cap X(X)$;

(ii) $\lambda_{A + T} = \lambda_A$ for $T \epsilon J$ (Theorem 6.2);

(iii) $\lambda_{AB} = \lambda_A + \lambda_B$ (Theorem 2.1).

Properties (i), (ii), (iii) characterize in a certain sense the index of an operator. Namely, the following theorem holds:

**Theorem 6.11.** If $X(X)$ is an algebra regularizable to a Fredholm ideal $J$, and if a function $x(A)$ satisfies conditions (i), (ii), (iii) and $\lambda_A = 0$, then there exists an integer $p$ such that $x(A) = px_A$.

**Proof.** Let $C$ be an arbitrary operator with index $1$, belonging to the set $D(X) \cap X(X)$. By the assumption, there exists a simple regularizer $E_0$ of the operator $C$ to the ideal $J$. Hence $\mu_{E_0} = -\mu_0 = -1$.

Let $A \epsilon D(X) \cap X(X)$ and let $A$ have a positive index $\lambda_0$. By Theorem 6.1, we have $[R_0]^n A \epsilon D(X) \cap X(X)$ and the operator $[R_0]^n A$ has index zero. Hence, by the assumption, $x([R_0]^n A) = 0$. According to condition (iii)

$$x(A) = x([R_0]) = -x(R_0).$$
A. I. Operators with finite and semifinite dimensional characteristic

If \( n_d = n \leq 0 \), then we may prove \( \nu(A) = \nu(C) \) analogously. But

\[
\nu(B \cap C) = \nu(I + T) = \nu(I) = 0,
\]

where \( T \in J \).

Hence \( \nu(R_0) = \nu(C) \). Thus,

\[
\nu(A) = p\nu_d
\]

where \( p = \nu(C) \) for every \( A \in D(\mathcal{X}) \cap \mathcal{X}(\mathcal{X}) \).

Remark 6.12. The assumption that the function \( \nu(A) \) is defined in the whole set \( \mathcal{D}(\mathcal{X}) \cap \mathcal{X}(\mathcal{X}) \) in a non-essential way in Theorem 6.11. It is sufficient that this function is defined in a set \( W \) satisfying the following conditions:

(i) if \( A, B \in W \), then \( AB \in W \),

(ii) if \( A \in W \), then \( A + T \in W \) for every \( T \in J \),

(iii) if \( A \in W \), then there exists a simple regularizer \( R_A \in W \) of the operator \( A \) to the ideal \( J \).

In this case one cannot require the number \( p \) in Theorem 6.11 to be an integer. Indeed, if there exists an operator \( C \in W \) with index equal to 1, the proof of Theorem 6.11 can be performed without any changes. Let us suppose that there exists no operator \( C \in W \) with index 1. Let \( q \) be the least positive index of the operators belonging to the set \( W \). Then the index of every operator belonging to the set \( W \) is divisible by the number \( q \). Indeed, let us suppose that operators \( A, B \in W \) have indices \( q \) and \( q_1 \), respectively, and let \( s = sq_1 + r \), where \( 0 < r < q \). Then

\[
\nu(B(R_A)^s) \in W \quad \text{and} \quad \nu(B(R_A)^s) = \nu_B - \nu_{R_A} = s - \nu_q = r < q,
\]

which contradicts the definition of the number \( q \). Further arguments are analogous to those in the proof of Theorem 6.11, i.e., we consider an arbitrary operator \( A \in W \) with index \( q \) and an operator \( C \) with index \( q \).

The index of the operator \( A(R_C)^s \) is equal to zero, and

\[
\nu(A) = \nu(C) = \frac{p}{q} \cdot \nu_0 = \frac{p}{q} \cdot \nu_d,
\]

where \( p = \nu(C) \) is an integer.

§ 7. Decomposition of operators. There exists a connection between numbers \( a_\beta, \beta \) and the form of the operator. Namely, the following theorem holds:

**Theorem 7.1.** Let us suppose

\[
A \in \begin{cases} D^d(\mathcal{X} \to \mathcal{Y}) \\
D^d(\mathcal{X} \to \mathcal{Y}) \end{cases}
\]

then

\[
a_\beta \leq \beta_d
\]

if and only if the operator \( A \) can be written in the form

\[
A = S + K,
\]

where the operator \( K \) is finite-dimensional, and the operator \( S \in \mathcal{L}(\mathcal{X} \to \mathcal{Y}) \) is left-invertible (right-invertible).

Proof. We first prove the necessity of the condition. Let us decompose the space \( \mathcal{X} \) into a direct sum \( \mathcal{X} = Z_A \oplus \mathcal{C}_A \), and the space \( \mathcal{Y} \) into a direct sum \( \mathcal{Y} = E_\beta \oplus \mathcal{C}_B \). Let us suppose \( A \in D^d(\mathcal{X} \to \mathcal{Y}) \) and \( a_\beta \leq \beta_d \).

This means that \( \dim Z_A \leq \dim \mathcal{C}_A \). Hence there exists a finite-dimensional operator \( K \) which is a one-to-one map of the subspace \( Z_A \) in the subspace \( \mathcal{X} \). Thus, the operator \( S = A - K \) is defined in the space \( \mathcal{X} \) and maps \( \mathcal{X} \) to \( \mathcal{Y} \) one-to-one. We denote by \( E_\beta \) the image of the space \( \mathcal{X} \) by means of the operator \( S \). Obviously, the operator \( S^{-1} \) exists on the set \( E_\beta \). Let us denote by \( \tilde{S}^{-1} \) an arbitrary extension of the operator \( S^{-1} \) to the whole space \( \mathcal{X} \). Evidently, \( \tilde{S}^{-1}Sx = x \) for all \( x \in \mathcal{X} \). Hence the operator \( S \) is left-invertible, and \( A = S + K \).

Now let us suppose \( \beta \leq a_\beta \). Then \( \dim Z_A \geq \dim \mathcal{C}_A \). The operator \( A \) is a one-to-one map of the subspace \( \mathcal{C}_A \) into the subspace \( E_\beta \). Hence there exists a finite-dimensional operator \( K \) which maps part of the set \( Z_A \) into the whole space \( \mathcal{C}_A \) one-to-one. Thus the operator \( S = A - K \) maps part of the space \( \mathcal{X} \) onto the whole space \( \mathcal{Y} \). Let us consider the operator \( S^{-1} \). It is defined on the whole space \( \mathcal{X} \) and maps the space \( \mathcal{Y} \) into the space \( \mathcal{X} \). Hence \( \tilde{S}^{-1}y = y \) for \( y \in \mathcal{X} \). Thus the operator \( S \) is right-invertible, and \( A = S + K \).

Now we prove the sufficiency of the condition. If the operator \( S \) is left-invertible, Theorem 2.5 implies \( a_\beta = 0 \). If \( S \) is right-invertible, we obtain \( \beta \leq 0 \). Hence, by Theorems 2.2 and 3.2, \( a_\beta \leq \beta_d \) in the first case and \( \beta \leq a_\beta \) in the second case.

**Corollary 7.2.** If \( A \in D^d(\mathcal{X} \to \mathcal{Y}) \), then \( a_\beta = 0 \) if and only if

\[
A = S + K,
\]

where \( K \) is a finite-dimensional operator and \( S \) is an invertible operator.

**Remark 7.1.** It follows from the construction of the operator \( \tilde{S}^{-1} \) that if \( S \) is a right-invertible operator, then

\[
\tilde{S}^{-1}S = I, \quad \tilde{S}^{-1}S = I + K_a,
\]

where \( K_a \) is a finite-dimensional operator and \( \dim K_a = \dim E_{K_a} = -a_\beta \).

If \( S \) is left-invertible, then

\[
\tilde{S}^{-1}S = I, \quad \tilde{S}^{-1}S = I + K_1,
\]

where \( \dim K_1 = \dim E_{K_1} = +a_\beta \).

The following theorem is a consequence of Theorem 7.1:

**Theorem 7.3.** If \( A \in D^d(\mathcal{X} \to \mathcal{Y}) \) (resp. \( A \in D^d(\mathcal{X} \to \mathcal{Y}) \)), then there exists an element \( R_A \) such that the operator \( AR_A - I \) (resp. \( RA - I \)) is finite-dimensional.
§ 8. Eigenvalues, regular values, and the spectrum of an operator.

Let us consider the operator \( A = T_m M \), where \( T \in \mathcal{L}(X) \) and \( \lambda \) is a number. If
\[
A = \lambda I \quad \text{(res. } \lambda I = \lambda_0 I \text{)},
\]
the number \( \lambda \) is called a **regular value** of the operator \( T \). The set of all numbers \( \lambda \) which are not regular values is called the **spectrum** of the operator \( T \). If \( \sigma_{B \neq M} > 0 \), then such an element \( \lambda \) of the spectrum is called an **eigenvalue** of the operator \( T \). If \( \lambda_0 \) is an eigenvalue of the operator \( T \), then there exists an \( x \neq 0 \) such that \( Tx = \lambda_0 x \). All elements \( x \neq 0 \) possessing this property are called **eigenvectors** of the operator \( T \) corresponding to the eigenvalue \( \lambda_0 \). The space spanned by these vectors is called the **eigenspace**. A **principal vector** corresponding to the value \( \lambda_0 \) is an element \( x \) such that \( (T - \lambda_0 I)^n x = 0 \) for a positive integer \( n \). The space spanned by principal vectors is called the **principal space**.

The dimension of the principal space is called the multiplicity of the eigenvalue \( \lambda_0 \). Evidently, if there exist principal vectors, then there exist also eigenvectors. Indeed, if \( n \) is the least positive integer such that \( (T - \lambda_0 I)^n x = 0 \), then \( x = (T - \lambda_0 I)^{n-1} x \) is an eigenvector.

Every eigenvector is a principal vector. Therefore the dimension of the principal space is not less than the dimension of the eigenspace.

The principal space \( G_{\lambda_0} \) corresponding to a value \( \lambda_0 \) is called **splitting** if the space \( X \) can be written as a direct sum of subspaces
\[
(8.1) \quad X = G_{\lambda_0} \oplus N_{\lambda_0},
\]
where the subspace \( N_{\lambda_0} \) is invariant with respect to the operator \( T \), i.e., \( TN_{\lambda_0} \subset N_{\lambda_0} \), and \( N_{\lambda_0} = (T - \lambda_0 I) N_{\lambda_0} \).

If the principal space is finite-dimensional, then the decomposition (7.1) is unique. Indeed, if \( n \) is the multiplicity of the value \( \lambda_0 \), then
\[
(T - \lambda_0 I)^n G_{\lambda_0} = 0.
\]

Hence
\[
N_{\lambda_0} = (T - \lambda_0 I)^n G_{\lambda_0} \oplus (T - \lambda_0 I)^n N_{\lambda_0} = (T - \lambda_0 I)^n X = G_{\lambda_0} \oplus N_{\lambda_0}.
\]

Thus, if the principal space corresponding to the value \( \lambda_0 \) is finite-dimensional and splitting, then the operator \( T - \lambda_0 I \) has a finite d-characteristic.

The following theorem holds for regular values of powers of an operator:

**Theorem 8.1.** \( \lambda \) is a regular value of the operator \( T^n \) if and only if the \( n \)-th roots \( \lambda_1, \lambda_2, ..., \lambda_n \) of the number \( \lambda \) are regular values of the operator \( T \).

The proof is based on the following lemma:

**Equations in linear spaces**
Lemma. If commuting operators \( A_1, A_2, \ldots, A_n \) map the space \( X \) into itself, and if \( a_k = \beta_k = 0 \), where \( A = A_1 A_2 \ldots A_n \), then

\[
A_n = \beta_n = 0 \quad (i = 1, 2, \ldots, n).
\]

Proof. Let us suppose that \( a_k > 0 \) for a certain \( k \). But the operators \( A_k \) are commuting. Hence \( A = A_1 A_2 \ldots A_{k-1} A_{k+1} \ldots A_n A_k \). Consequently, \( a_k \geq \beta_k > 0 \), which is a contradiction.

On the other hand, let us suppose that \( \beta_k > 0 \) for a certain \( k \). Then \( A = A_k A_{k+1} \ldots A_n \). Hence \( \beta_k \geq \beta_k > 0 \), which is a contradiction.

Proof of Theorem 8.1. We can write

\[
T^\lambda = (T - \lambda_1 I)(T - \lambda_2 I) \ldots (T - \lambda_n I),
\]

and the sufficiency of the condition follows immediately. The necessity of the condition follows from the lemma.

The spectrum of an operator is called discrete if it is either finite or a countable sequence \((\lambda_k)\) convergent to zero.

From Theorem 8.1 follows immediately

**Theorem 8.2.** The following three conditions are equivalent:

1. Operator \( T^\lambda \) has a discrete spectrum.
2. For every \( n \) the operator \( T^\lambda \) has a discrete spectrum.
3. There exists a number \( n \) such that the operator \( T^\lambda \) has a discrete spectrum.

**Theorem 8.3.** If there exists a positive integer \( N \) such that for all \( n > N \) the operator \( T^\lambda \) has a discrete spectrum and the operator \( I - T^\lambda \) has a finite \( d \)-characteristic and index equal to zero, then the operator \( I - T^\lambda \) has a finite \( d \)-characteristic and index equal to zero.

Proof. By Theorem 8.2, the operator \( I - T^\lambda \) has a discrete spectrum.

By Corollary 8.4, the \( d \)-characteristic of the operator \( I - T^\lambda \) is finite. We write

\[
T^\lambda = (T - \xi_1 \phi_1 I)(T - \xi_2 \phi_2 I) \ldots (T - \xi_n \phi_n I),
\]

where \( \phi_1, \phi_2, \ldots, \phi_n \) are the \( n \)th roots of unity (\( \phi_n = 1 \)). Since the spectrum of the operator \( T \) is discrete, there exists a natural number \( p \) such that the operator \( T^\lambda \) has index zero, and numbers \( \phi_1, \phi_2, \ldots, \phi_p \), are not elements of the spectrum of the operator \( T \). Then the operator \( A_p = (T - \xi_1 \phi_1 I)(T - \xi_2 \phi_2 I) \ldots (T - \xi_p \phi_p I) \) is invertible and \( \eta_{dp} = 0 \). But \( I - T^\lambda = (I - T) A_p \). Hence, by Theorem 2.1,

\[
0 = \eta_{d-\varphi} = \eta_{d} + \eta_{dp} = \eta_{d-}\varphi.
\]

Remark. The assumption that the operator \( I - T^\lambda \) has index zero for sufficiently large \( n \) can be replaced in Theorem 8.3 by the assumption that this condition holds for an infinite sequence \((n_k)\) of relatively prime numbers. However, the last assumption cannot be weakened any more. Namely, one can show that to every sequence of the form \((n_k)\) (\( p \) being a fixed number) there exist a space \( X \) and an operator \( T \) with a discrete spectrum such that \( \eta_{d-\varphi} = 0 \) but \( \eta_{d-\varphi} \neq 0 \).

**Example 8.1.** Every real number \( \lambda \) is an eigenvalue of the operator \( \frac{d}{dt} \in L(\mathcal{C}[0, 1]) \). Indeed, the equation

\[
\frac{d}{dt}e^\lambda = \lambda e^\lambda
\]

has a solution for every value \( \lambda \). To the value \( \lambda \) there corresponds an eigenvector of the form

\[
x(t) = e^{\lambda t}.
\]

Hence the eigenspace is one-dimensional.

**Example 8.2.** Every complex number \( \lambda \) is an eigenvalue of the operator \( \frac{d^2}{dt^2} \in L(\mathcal{C}[0, 1]) \). As is well known, the following (linearly independent) eigenvectors correspond to the eigenvalue \( \lambda \):

\[
e^\lambda_1, e^\lambda_2 \]

where \( \lambda_1, \lambda_2 \) are square roots of the number \( \lambda \) if \( \lambda \neq 0 \),

\[
e^\lambda, 1 \]

if \( \lambda = 0 \).

Hence the eigenspace is two-dimensional.

**Example 8.3.** The integral operator defined in Example 3.1

\[
A x = \int_0^t t x(s) \, ds
\]

has only one eigenvalue \( \lambda = \frac{1}{2} \); all values \( \lambda \neq \frac{1}{2} \) are regular (compare the calculations in Example 3.1).
(2) If the multiplication is performable, then it is associative and distributive with respect to addition.

Let us remark that groups $P_2$ and $P_3$ with the operations defined above are rings (1).

In order to illustrate multiplication in the system $P$ we give the following diagram

Fig. 2. Pararing. Arrow 1 shows the order of elements of the product. Arrow 2 shows the group to which the product belongs

A system $P$ of four Abelian groups $(P_1, P_2, G_1, G_2)$ satisfying conditions (1) and (2) will be called a pararing and will be denoted by

$$P = \left( \begin{array}{c} P_1 \\ G_1 \\ P_2 \\ G_2 \end{array} \right).$$

If operations $xy$ and $yx$ are both performable, then either the products $xy$ and $yx$ both belong to one of the rings $P_i$, $i = 1, 2$, or one of the products belongs to the ring $P_1$ and other, to the ring $P_2$.

If two pararings are given:

$$P = \left( \begin{array}{c} P_1 \\ G_1 \\ P_2 \\ G_2 \end{array} \right) \quad \text{and} \quad P' = \left( \begin{array}{c} P'_1 \\ G'_1 \\ P'_2 \\ G'_2 \end{array} \right),$$

and if $P'_1 \subseteq P_1$, $G'_1 \subseteq G_1$ ($i = 1, 2$) and the operations in $P$ and $P'$ are consistent, then the pararing $P'$ is called a subpararing of the pararing $P$.

We shall write: $P' \subseteq P$.

A left ideal (right ideal) in a pararing $P$ is a pararing $J \subseteq P$ such that for any two elements $x \in J$, $y \in P$ for which the operation $xy$ (resp. $yx$) is performable the product $xy$ is in $J$ (resp. $yx$ is in $J$).

It follows from this definition that if a pararing $J$ is an ideal (left or right) in a pararing $P$, then $J$ is of the form:

$$J = \left( \begin{array}{c} P_1 \cap J \\ G_1 \cap J \\ P_2 \cap J \end{array} \right).$$

Evidently, the set $P_i \cap J$ is an ideal in the ring $P_i$ ($i = 1, 2$), and sets $G_i \cap J$ are bimodules over these rings.

In further considerations we shall not distinguish between a pararing $P$ and its set $\{P\}$ whenever no misunderstanding can arise. For instance, if $J$ is an ideal in a pararing $P$, we shall write simply:

$$J = \left( \begin{array}{c} P_1 \cap J \\ G_1 \cap J \\ P_2 \cap J \end{array} \right).$$

An ideal which is left and right simultaneously is called a two-sided ideal.

Evidently, the pararing $P$ itself and the system composed of neutral elements of all groups constituting $P$ are ideals. These ideals will be called trivial ideals. All other ideals will be called non-trivial or proper.

A pararing $P = \left( \begin{array}{c} P_1 \\ G_1 \\ P_2 \\ G_2 \end{array} \right)$ is called a paralgebra if the groups $P_1, P_2, G_1, G_2$ are linear spaces, and if for every two elements $x, y \in P$ for which the operation $xy$ is performable and for an arbitrary scalar $t$ we have

$$t(xy) = (tx)y = x(ty).$$

The following system gives an example of a paralgebra: Let two linear spaces $X$ and $Y$ be given. We take

$$P_1 = I_d(X), \quad P_2 = I_d(Y), \quad G_1 = I_d(X \to Y), \quad G_2 = I_d(Y \to X).$$
We write
\[ L_q(X \rightarrow Y) = \left( L_q(X) \otimes_{L_q(X \rightarrow Y)} L_q(Y) \right). \]

If the spaces \( X \) and \( Y \) are infinite-dimensional, then the set \( K_0(X \rightarrow Y) \) of all finite-dimensional operators belonging to \( L_q(X \rightarrow Y) \) is a non-trivial ideal.

If the rings \( P_t \) constituting a pararing \( P \) have unities \( e_t (t = 1, 2) \), then \( P \) is called a pararing with unities.

**Theorem 9.1.** Every paraalgebra \( P = \left( A_t, S_t, A_t \right) \) can be extended to a paraalgebra with unities.

**Proof.** Algebras \( A_t (i = 1, 2) \) can be extended to algebras \( \tilde{A}_t \) with unities by taking
\[
\tilde{A}_t = \left( (x, a); x \in A_t, a \text{ is a scalar} \right) \quad (i = 1, 2),
\]
and defining operations in sets \( \tilde{A}_t \) as follows
\[
\begin{align*}
(x, a) + (y, b) &= (x + y, a + b) \\
(\alpha(x, a)) &= (\alpha x, \alpha a) \\
(x, a)(y, b) &= (xy + xa + yb, ab) \\
\text{ (for } x, y, a, b \text{ scalars),}
\end{align*}
\]
(see Theorem 9.1).

We define multiplication between elements of spaces \( S_t \) and sets \( \tilde{A}_t \):
\[
\begin{align*}
(x, a) &\cdot (y, a) = (x + y, a + a) \\
&\text{ (for } x, y, a \text{ are elements of } A_t, \text{ and if the product xy (resp. } a \text{) is defined), then}
\end{align*}
\]
(9.1)

Let us remark that if the product \( xy \) is defined, the sums on the right-hand side of formulæ (9.2) are also defined.

It is easy to verify that the system \( \tilde{P} = \left( \tilde{A}_t, S_t, \tilde{A}_t \right) \) with operations defined by formulæ (9.1) and (9.2) is a paraalgebra and that \((0_t, 1)\) are unities of this paraalgebra, where \(0_t\) is the zero element of the algebra \( A_t\). Elements of algebras \( A_t \) can be identified with pairs of the form \((x, 0)\), where \(x \in A_t\).

Taking into account Theorem 9.1, in the sequel we shall consider only paraalgebras with unities, and we shall call them briefly paraalgebras.

A radical of a pararing \( P \) with unities is the set
\[
R(P) = \{ e \in P : \text{if } a, b \in P \text{ and } a \cdot e \cdot b \in P_t \ (t = 1 \text{ or } 2), \text{ then there exists an element } (e_t + a \cdot e \cdot b)^{-1} \}.
\]

**Theorem 9.2.** The radical \( R(P) \) of a pararing \( P \) with unities is a two-sided ideal.

**Proof.** Let \( x, y \in R(P) \), and let us suppose that the sum \( x + y \) exists, i.e. that elements \( x \) and \( y \) belong to the same group \( G_t \) or ring \( F_t \) simultaneously. Let \( a \) and \( b \) be two elements of the pararing \( P \) such that \( a(x + y)b \in P_t \ (t = 1 \text{ or } 2) \). It follows immediately from the definition of a pararing that the elements \( axb \) and \( a\cdot y \cdot b \) are in \( P(P) \), whence the element
\[
e_t + a(x + y)b = (a_t + axb)(e_t + a\cdot y \cdot b)^{-1} \cdot ab
\]
is invertible. Thus \( x + y \in R(P) \). Hence we have shown that the set \( R(P) \) is linear.

Let us suppose that \( x \in R(P) \). Let \( a \) and \( b \) be two elements such that the element \( ab \) is well defined, and let \( c \) and \( d \) denote two elements of the pararing \( P \) satisfying the condition \( e_t + c \cdot P_t \ (t = 1 \text{ or } 2) \). Then the element
\[
e_t + c \cdot d = e_t + (oa) \cdot d
\]
is invertible, since \( x \in R(P) \). Thus, \( x \cdot c \in R(P) \), since elements \( c \) and \( d \) are arbitrary. But elements \( e \) and \( d \) are also arbitrary; hence the set \( R(P) \) is an ideal. ■

Let a pararing \( P \) and a two-sided ideal \( J \) in \( P \) be given. Let us consider the following quotient groups:
\[
[\{P_t \} = \{P_t \} / P_t \cap J, \quad [G_t] = G_t / G_t \cap J \quad (t = 1, 2).
\]
The system \( \{[\{P_t \}] \} = \{[P_1] \} \cdot \{[G_1] \} \) is also a pararing. Indeed, if we write conventionally
\[
x + J = \begin{cases} x + G_t \cap J, \quad \text{where } x \in G_t \cap J \ (t = 1, 2), \\
x + P_t \cap J, \quad \text{where } x \in P_t \end{cases}
\]
the cosets induced by elements \( x, y \) are of the form \( x + J, y + J \). Consequently,
\[
[x, y] = (x + J)(y + J) = x + axJ + y + yJ = x + y + J = [xy].
\]
The pararing \( [P] \) is called the quotient pararing and is denoted by \( [P] = P/J \).

Let a pararing \( P \) and a two-sided ideal \( J \subseteq P \) be given. We say that an element \( a \in P \) has a left regularizer (right regularizer) to the left ideal (right ideal) \( J \subseteq P \) if there exists an element \( e_t \in P_t \) such that \( e_t a = a + J \) (resp. \( a e_t = a + J \), where \( T \subseteq J \). If \( e_t a = e_t + J \) (resp. \( a e_t = e_t + J \)), then \( e_t a \) is called a simple regularizer to the two-sided ideal \( J \) simultaneously, then \( e_t a \) is called a simple regularizer.

§ 9. Pararings and paraalgebras
§ 10. Paraalgebras of operators. Let two linear spaces \( X \) and \( Y \) be given, and let \( A(X) \) and \( A(Y) \) be algebras of operators which map the spaces \( X \) and \( Y \) into themselves, respectively. Let
\[
S_{l}(X \rightarrow Y) \subset L_{d}(X \rightarrow Y) \quad \text{and} \quad S_{d}(Y \rightarrow X) \subset L_{d}(Y \rightarrow X)
\]
be linear spaces of linear operators. If for any operators
\[
A \in S_{l}(X \rightarrow Y), \quad B \in S_{d}(Y \rightarrow X), \quad C \in A_{n}(X), \quad D \in A_{d}(Y)
\]
we have
\[
AB \in A_{d}(Y), \quad BA \in A_{d}(X),
\]
\[
DA, AC \in S_{l}(X \rightarrow Y), \quad BD, CB \in S_{d}(Y \rightarrow X),
\]
then the system
\[
P(X \rightarrow Y) = \begin{pmatrix}
A(X) & S_{l}(X \rightarrow Y) & A_{n}(X) \\
S_{d}(Y \rightarrow X) & D
\end{pmatrix}
\]
is a paraalgebra. Such a paraalgebra is called a paraalgebra of operators.

**Theorem 10.1.** Every paraalgebra \( P = \begin{pmatrix} P_{1} & G_{1} & F_{1} \\ G_{2} & F_{2} \end{pmatrix} \) is isomorphic with a certain paraalgebra of operators.

**Proof.** Let \( X = P_{1} \times G_{1}, \ Y = P_{2} \times G_{2}. \) We associate the operator
\[
p(x) = (p_{1}, p_{2}, g), \quad \text{where} \quad x = (p, g) \in X = P_{1} \times G_{1},
\]
with the element \( p_{1} \in P_{1}, \)
\[
g(y) = (p_{2}, p_{2}, g), \quad \text{where} \quad y = (p, g) \in Y = P_{2} \times G_{2},
\]
with the element \( p_{2} \in P_{2}, \)
\[
g_{2}(z) = (g_{2}, g_{2}), \quad \text{where} \quad z = (p, g) \in X, \quad y = (g_{2}, g_{2}) \in Y,
\]
with the element \( g_{2} \in G_{1}, \)
\[
g_{2}(y) = (g_{2}, g_{2}), \quad \text{where} \quad y = (p, g) \in X, \quad x = (g_{2}, g_{2}) \in X,
\]
with the element \( g_{2} \in G_{2}. \)

It is easily verified that this correspondence is an isomorphism. ■

In paraalgebras of operators the role of unity is played by the identity operators \( I_{X} \) and \( I_{Y} \) in space \( X \) and \( Y \), respectively. In the sequel we shall suppose that every paraalgebra of operators under consideration possesses unity.

We denote by \( K_{d}(X \rightarrow Y) \) the set of all finite-dimensional operators belonging to the paraalgebra \( P(X \rightarrow Y) \). We denote the ideal \( K_{d}(X \rightarrow Y) \) briefly by \( K_{d}(X \rightarrow Y) \). If every operator with a finite \( d \)-characteristic belonging to the paraalgebra \( P(X \rightarrow Y) \) has a simple regularizer to a two-sided ideal \( J \), then the paraalgebra \( P(X \rightarrow Y) \) is called regularizable to the ideal \( J \). If a paraalgebra \( P(X \rightarrow Y) \) is regularizable to the ideal \( K_{d}(X \rightarrow Y) \), we say briefly that \( P(X \rightarrow Y) \) is regularizable. It follows from Theorem 7.3 that the paraalgebra \( L_{d}(X \rightarrow Y) \) is regularizable. We write:

\[
D_{p}(X \rightarrow Y) \quad \text{the set of all operators from} \quad \text{nullity,}
\]
\[
D_{q}(X \rightarrow Y) \quad \text{the paraalgebra} \quad P(X \rightarrow Y) \quad \text{which} \quad \text{deficiency,}
\]
\[
D_{d}(X \rightarrow Y) \quad \text{have a finite} \quad d \quad \text{-characteristic.}
\]

We denote the respective sets in the paraalgebra \( L_{d}(X \rightarrow Y) \) by
\[
D_{p}(X \rightarrow Y), \quad D_{q}(X \rightarrow Y), \quad D_{d}(X \rightarrow Y).
\]

A proper ideal \( J \) in a paraalgebra \( P(X \rightarrow Y) \) is called a quasi-Fredholm ideal if for every \( T \in A_{n} \cap J \) \((i = 1, 2)\) the operator \( e_{i} + T \) has a finite \( d \)-characteristic. If, moreover, the indices are equal to zero:

\[
k_{i} = 0 \quad \text{for} \quad T \in A_{i} \cap J \quad (i = 1, 2),
\]

then \( J \) is called a Fredholm ideal.

**Theorem 10.2.** If an operator \( A \in P(X \rightarrow Y) \) has a simple regularizer \( K_{d} \) to a quasi-Fredholm ideal \( J \subset P(X \rightarrow Y) \), then \( A \) has a finite \( d \)-characteristic.

**Proof.** By the assumption, \( A \in K_{d} \) and \( A_{d} \) have the form \( e_{i} + T_{i} \) \((i = 1, 2)\), where \( T_{1}, T_{2} \in J \). Hence Corollary 2.3 and the definition of a quasi-Fredholm ideal complete the proof.

Now we repeat the construction of the maximal quasi-Fredholm ideal. Let \( P(X \rightarrow Y) \) be a paraalgebra of linear operators, regularizable to a quasi-Fredholm ideal \( J \subset P(X \rightarrow Y) \).

Let \( R(P_{d}) \) be the radical of the quotient paraalgebra \( P_{d} = P(X \rightarrow Y)/J \), and let
\[
J_{d} = \{ U \in P(X \rightarrow Y) : [U] \in R(P_{d}) \},
\]
where \([U]\) is the coset defined by the operator \( U \).

**Theorem 10.3.** \( J_{d} \) is the maximal quasi-Fredholm ideal in the paraalgebra \( P(X \rightarrow Y). \)

**Proof.** By Theorem 6.7, ideals \( J_{d} \cap A_{i} \) \((i = 1, 2)\) are quasi-Fredholm ideals in algebras \( A_{i} \), respectively. Hence, according to the definition of a quasi-Fredholm ideal, \( J_{d} \) is a quasi-Fredholm ideal in the paraalgebra \( P(X \rightarrow Y). \)

On the other hand, let \( J \) be a quasi-Fredholm ideal in the paraalgebra \( P(X \rightarrow Y) \) and let \( U \in J \). By the definition of a quasi-Fredholm ideal, the operator \( e_{i} + A UB \) has a finite \( d \)-characteristic for all \( A, B \in P(X \rightarrow Y) \) such that \( A UB \in A_{i}. \) Since the paraalgebra \( P(X \rightarrow Y) \) is regularizable to the ideal \( J \), the coset \([e_{i} + A UB]\) is invertible in the paraalgebra \( P_{d}. \) Since
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\( A, B \) are arbitrary, the coset \([U]\) generated by the operator \( U \) belongs to the radical \( \mathbf{R}(\mathbf{E}_0) \). Hence \( U \in \mathbf{J}_0 \).

**Example 10.1.** Let us consider the differential equation

\[ \frac{d}{dt}x(t) + p(t)x(t) = q(t). \]

A solution of this equation can be obtained by the so-called method of variation of a constant. This is just the method of construction of a simple regularizer of the operator \( A = \frac{d}{dt} + p \) (where \( p \) is the operator of multiplication by the function \( p(t) \)) to the ideal of finite-dimensional operators in the paraalgebra \( \mathcal{L}(C_0(0,1), C[0,1]) \). Indeed, the solution of this equation obtained by the method of variation of a constant can be written in the form:

(i)

\[ x = R_A q + s_0, \]

where \( s_0 \) is an arbitrary solution of the homogeneous equation:

\[ s_0 = c \exp \left\{ - \int_0^t p(s) ds \right\}. \]

\( c \) is an arbitrary constant, and the operator \( R_A \) is defined in the following manner:

\[ R_A y = \exp \left\{ - \int_0^t p(s) ds \right\} \int_0^t \exp \left\{ \int_s^t p(\tau) d\tau \right\} y(s) ds. \]

Since the constant \( c \) may be treated as a functional, one can write:

\[ s_0 = K_0 x, \quad \text{where} \quad s_0 = \exp \left\{ - \int_0^t p(s) ds \right\}, \]

and \( K_0 \) is a one-dimensional operator. Thus equality (i) gives

\[ x = R_A q + K_0 x = R_A x + K_0 x, \]

and so \( R_A = I - K_0 \).

Hence \( R_A \) is a left regularizer of the operator \( A \). On the other hand,

\[ AR_A x = \left( \frac{d}{dt} + p(t) \right) \exp \left\{ - \int_0^t p(s) ds \right\} \int_0^t \exp \left\{ \int_s^t p(\tau) d\tau \right\} x(s) ds \]

\[ = -p(t) \exp \left\{ - \int_0^t p(s) ds \right\} \int_0^t \exp \left\{ \int_s^t p(\tau) d\tau \right\} x(s) ds + \]

\[ + \exp \left\{ - \int_0^t p(s) ds \right\} \int_0^t \exp \left\{ \int_s^t p(\tau) d\tau \right\} x(t) + \]

\[ + p(t) \exp \left\{ - \int_0^t p(s) ds \right\} \int_0^t \exp \left\{ \int_s^t p(\tau) d\tau \right\} x(s) ds \]

\[ = x(t). \]

Thus \( AR_A = I \). Consequently \( R_A \) is also a right regularizer. Hence it is a simple regularizer to the ideal of finite-dimensional operators.

**§ 11. Semi-Fredholm ideals. Perturbations by means of operators belonging to some ideals.** In § 3 we have shown that if \( \mathcal{K} \) is a finite-dimensional operator, then it is a perturbation of all operators with finite \( \mathcal{K} \)-characteristics, and also of classes \( D^-(X \mapsto Y) \) and \( D^+(X \mapsto Y) \). If an operator \( \mathcal{K} \) is a \( D(X \mapsto Y) \)-perturbation, then it must be finite-dimensional (Theorem 4.3).

![Fig 3](image)

If we do not consider the set of all linear operators but only some subsets of this set, the situation may be different. Let a paraalgebra

\[ P(X \mapsto Y) = \left( \begin{array}{cc} A_1(X) & S_1(X \mapsto Y) \\ S_2(X \mapsto Y) & A_2(Y) \end{array} \right) \]

of operators over linear spaces \( X \) and \( Y \) be given. We say that an ideal \( J \subset P(X \mapsto Y) \) is a positive semi-Fredholm ideal if for every \( \mathcal{T} \in A_i \cap J \) (\( i = 1, 2 \)) the operator \( \mathcal{T} + \mathcal{I} \) is of finite nullity, i.e., if \( a_{\mathcal{T} + \mathcal{I}} < +\infty \).

Analogously, we say that an ideal \( J \) is a negative semi-Fredholm ideal if for every \( \mathcal{T} \in A_i \cap J \) the operator \( \mathcal{T} + \mathcal{I} \) is of finite deficiency, i.e., \( b_{\mathcal{T} + \mathcal{I}} < +\infty \).

We say that a paraalgebra \( P(X \mapsto Y) \) is left-regularizable (right-regularizable) to a left ideal (right ideal) \( J \subset P(X \mapsto Y) \) if every operator with finite nullity (deficiency) belonging to the paraalgebra \( P(X \mapsto Y) \) has a left regularizer (right regularizer) to the ideal \( J \).

**Theorem 11.1.** If a paraalgebra \( P(X \mapsto Y) \) is left-regularizable (right-regularizable) to a left (right) positive (negative) semi-Fredholm ideal \( J \subset P(X \mapsto Y) \), then for every operator \( \mathcal{A} \in P(X \mapsto Y) \) of finite nullity (deficiency) and for every operator \( \mathcal{T} \in J \) for which the sum \( \mathcal{A} + \mathcal{T} \) is defined, the operator \( \mathcal{A} + \mathcal{T} \) is of finite nullity (deficiency).

**Proof.** By hypothesis there exists an operator \( R_A \in P(X \mapsto Y) \) such
that $R_A = e_i + T$ (resp. $A R_A = e_i + T_i$), where $T + T_i$, $i = 1$ or 2. Hence for each operator $T_s \in J$ for which the sum $A + T$ is defined $R_A (A + T_s) = e_i + T + R_A T_s$ (resp. $(A + T) R_A = e_i + T + T_i R_A$).

Let us remark that operators on the right-hand side of the last equality are of finite nullity (deficiency), by the assumption regarding the ideal $J$. Hence, by Theorem 2.2, the operator $A + T_s$ is of finite nullity (deficiency).

**Theorem 11.2.** If a paralgebra $P(X \oplus Y)$ satisfies the assumptions of Theorem 11.1, then all operators belonging to a certain left (right) positive (negative) semi-Fredholm ideal $J_s \subseteq P(X \oplus Y)$ are perturbations of the class of operators of finite nullity (deficiency) belonging to the paralgebra $P(X \oplus Y)$.

**Proof.** Let $J = J_1 \oplus J_2$. In the same way as in Theorem 6.3 we show that the set $J$ is a left (right) positive (negative) semi-Fredholm ideal. Evidently, the paralgebra $P(X \oplus Y)$ is left-regularizable (right-regularizable) to the ideal $J_s$. Hence, by Theorem 11.1, operators belonging to the ideal $J_s$, and in particular operators belonging to the ideal $J_s$, are perturbations of the class of operators of finite nullity (deficiency) belonging to the paralgebra $P(X \oplus Y)$.

**Corollary 11.3.** If a paralgebra $P(X \oplus Y)$ is regularizable to a quasi-Fredholm ideal $J$, and if an operator $A \in P(X \oplus Y)$ has a finite $\delta$-characteristic, then, for every operator $T \in J$ for which the sum $A + T$ is defined, the operator $A + T$ has a finite $\delta$-characteristic. If $J$ is a Fredholm ideal, then, moreover,

$$x_{A + T} = x_A.$$

Indeed, if $J$ is a Fredholm ideal, then $x_{A + T} = -x_{R_A T} = -x_R = x_A$.

**Corollary 11.4.** Each quasi-Fredholm ideal $J$ in a paralgebra $L_2(X \oplus Y)$ is contained in the ideal $E_2(X \oplus Y)$ of all finite-dimensional operators.

**Proof.** The paralgebra $L_2(X \oplus Y)$ is regularizable. Hence, by Corollary 11.3, each operator $T \in J$ is a perturbation of the class $D_2(X \oplus Y)$ of operators with finite $\delta$-characteristics, belonging to the paralgebra $L_2(X \oplus Y)$. Consequently, by Theorem 4.3, the operator $T$ is finite-dimensional. Thus $J \subseteq E_2(X \oplus Y)$.

**Theorem 11.5.** If a paralgebra $P(X \oplus Y)$ of operators is regularizable, and if every operator $A \in P(X \oplus Y)$ is the sum of two operators: $A = A_1 + A_2$ of finite nullity (deficiency), then the set of perturbations of the class of operators of finite nullity (deficiency) is a left ideal (right ideal) in the paralgebra $P(X \oplus Y)$.

**Proof.** The linearity of the set of perturbations follows from Theorem 4.2. Let $V$ be a perturbation of the class of operators with finite $\delta$-characteristics, and let us suppose that $A + P(X \oplus Y)$ is an operator of finite nullity (deficiency) such that the sum $A + V$ exists. Let $B \in P(X \oplus Y)$ be an operator with a finite $\delta$-characteristic such that the superposition $B V$ (resp. $V B$) is meaningful.

The paralgebra $P(X \oplus Y)$ is regularizable. Hence there exists an operator $R_A$ such that $B R_A = e_1 + K$ (resp. $R_A B = e_1 + K$) $(i = 1$ or 2), where $K$ is a finite-dimensional operator. By Theorems 2.1 and 3.2, the operator

$$A + B V = (B R_A - K) A + B V = B (R_A A + V) - K A$$

(resp. $A + B V = A (R_A B - K) + V B = (A R_B + V) B - A K$)

is of finite nullity (deficiency). Hence the operator $B V$ (resp. $V B$) is a perturbation of the class of operators of finite nullity (deficiency). Since we supposed every operator from the paralgebra to be the sum of two operators of finite nullity (deficiency), by applying the linearity of the set of perturbations we obtain the theorem.

**Corollary 11.6.** If every operator belonging to a regularizable paralgebra $P(X \oplus Y)$ can be written as the sum of two operators with finite $\delta$-characteristics, then the perturbations of operators with finite $\delta$-characteristics belonging to this paralgebra form a two-sided ideal.

Evidently, the ideal obtained in this manner is a quasi-Fredholm ideal and, by Theorem 10.2, it is the maximal quasi-Fredholm ideal.

**Remark 11.1.** By Theorem 11.1, the assumption of regularizability of the paralgebra $P(X \oplus Y)$ to the ideal $E_2(X \oplus Y)$ of finite-dimensional operators belonging to the paralgebra $P(X \oplus Y)$ can be replaced in Theorem 11.5 and in Corollary 11.6 by the assumption of left regularizability (right regularizability) to a left (right) positive (negative) semi-Fredholm ideal contained in this paralgebra.

The following question arises: is it possible in every algebra to write an arbitrary operator as the sum of two operators with finite $\delta$-characteristics? In case of algebra $L_2(X)$ the following theorem holds.

**Theorem 11.8.** Every operator $A \in L_2(X)$ is the sum of two isomorphisms.

**Proof.** (1) If the space $X$ is finite-dimensional, the theorem is trivial. Let us suppose the space $X$ to be infinite-dimensional. The proof contains a few steps.

(1) The authors are indebted to G. Neubauer, who communicated this proof to them.
(a) Let us suppose \( \beta_d = 0 \), i.e. \( E_d = X \). Let us decompose the space \( X \) into a direct sum \( X = Z_d \oplus \mathbb{C} \). Obviously, the power of the basis of the space \( \mathbb{C} \) is the same as the power of the basis of the whole space \( X \). Hence the power of the basis of the subspace \( Z_d \) is not greater than the power of the basis of \( X \). Thus both bases can be numbered in the following manner:

- Basis of \( \mathbb{C} \): \( a_i \), \( i \in \mathbb{N} \), \( n = 1, 2, \ldots \)
- Basis of \( Z_d \): \( a_i \), \( i \in \mathbb{I} \cap I \)

Let us write

\[
y_{t_k} = \frac{1}{2} A x_{t_k}, \quad i \in \mathbb{I} \cap I, \quad n = 1, 2, \ldots
\]

and let us define operators \( B \) and \( C \) as follows:

\[
B x_k = - C x_k = y_{t_k} \quad \text{for} \quad i \in \mathbb{I} \cap I,
\]

\[
B x_k = y_{t_k} + y_{t_{k+1}}, \quad C x_k = y_{t_k} + y_{t_{k+1}} \quad \text{for} \quad i \in \mathbb{I} \cap I,
\]

\[
B x_k = C x_k = y_{t_k} \quad \text{for} \quad i \notin \mathbb{I} \cap I.
\]

It is easily verified that the subspaces \( E_B \) and \( E_C \) both contain all elements \( y_{t_k} \). Hence \( \beta_B = \beta_C = 0 \). Moreover, all elements \( B x_k \), \( B x_k \), \( C x_k \), \( C x_k \) are linearly independent, and the same holds for the elements \( C x_k \), \( C x_k \). Hence \( \alpha_B = \alpha_C = 0 \). Consequently, operators \( B \) and \( C \) are both isomorphisms. The sum \( (B + C)x_k = 2y_{t_k} = Ax_k \), and \( (B + C)x_k = 0 = Ax_k \). Thus \( B + C = A \).

(b) Let us suppose \( \alpha_d = 0 \), i.e. \( Z_d = \{0\} \). Let us decompose the space \( X \) into a direct sum \( X = E_d \oplus \mathbb{C} \). Obviously, the power of the basis of the subspace \( \mathbb{C} \) is not greater than the power of the basis of the space \( X \). Hence both bases can be defined as follows:

- Basis of \( X \): \( a_i \), \( i \in \mathbb{I} \cap I \), \( n = 1, 2, \ldots \)
- Basis of \( \mathbb{C} \): \( a_i \), \( i \notin \mathbb{I} \cap I \)

Let us write

\[
y_{t_k} = \frac{1}{2} A x_{t_k}, \quad n = 1, 2, \ldots
\]

Evidently, all elements \( y_{t_k} \), where \( i \in \mathbb{I} \), \( n = 0, 1, 2, \ldots \), are linearly independent. We define operators \( B \) and \( C \) by the following equalities:

\[
B x_{t_{k+1}} = y_{t_k} + y_{t_{k+1}} + y_{t_{k+2}}
\]

\[
B x_{t_{k+2}} = y_{t_k} + y_{t_{k+1}} + y_{t_{k+2}}
\]

\[
C x_{t_{k+1}} = -y_{t_k} + y_{t_{k+1}} - y_{t_{k+2}}
\]

\[
C x_{t_{k+2}} = -y_{t_k} + y_{t_{k+1}} - y_{t_{k+2}}
\]

\[
B x_k = C x_k = y_{t_k} \quad \text{for} \quad i \notin \mathbb{I} \cap I.
\]

If \( i \in \mathbb{I} \), then the subspace generated by the elements \( B x_k \) \( (C x_k \) respectively, \( n = 1, 2, \ldots, 2k+2 \), contains vectors \( y_{t_k}, \ldots, y_{t_{k+1}}, y_{t_{k+2}} \), \( y_{t_{k+3}}, \ldots, y_{t_{k+4}}, \ldots \) which are linearly independent. Hence the dimension of this subspace is equal to \( 2k+2 \). This implies the linear independence of elements \( B x_k \) \( (C x_k \) respectively, \( a_B = a_C = 0 \). On the other hand, we have \( y_{t_k} e B \) and \( y_{t_k} e B \). Hence \( \beta_B = \beta_C = 0 \), and operators \( B \) and \( C \) are both isomorphisms. Moreover, it is easily verified that

\[
(B + C)x_k = 2y_{t_k} = Ax_k.
\]

Thus \( B + C = A \).

In the general case it is always possible to decompose the space \( X \) into a direct sum in two ways:

\[
X = Z_d \oplus \mathbb{C}, \quad X = E_d \oplus \mathbb{C}.
\]

(c) If the powers of the bases of subspaces \( Z_d \) and \( \mathbb{C} \) are equal, there exists an isomorphism \( T \) of the subspace \( Z_d \) onto the subspace \( \mathbb{C} \). Let

\[
[Bx] = [C_0] = \frac{1}{2} A x \quad \text{for} \quad x \in \mathbb{C},
\]

\[
[C_0] = Ts = T \quad \text{for} \quad x \in Z_d.
\]

It is easily verified that operators \( B \) and \( C \) are both isomorphisms, and \( B + C = A \).

If the powers of the bases of subspaces \( Z_d \) and \( \mathbb{C} \) are different, then either the power of the basis of \( Z_d \) is greater than the power of the basis of \( \mathbb{C} \), or conversely. In the first case we write \( Z_d \) as a direct sum \( Z_d = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \), where the power of the basis of the subspace \( \mathbb{C} \) is equal to the power of the basis of \( \mathbb{C} \).

Let \( X = \mathbb{C} \oplus \mathbb{C} \) and let \( A \) be the restriction of the operator \( A \) to the space \( X \). Arguments analogous to those applied in the case (a) show that the operator \( A \) can be written as the sum of two isomorphism \( B \) and \( C \) of the space \( X \) onto the space \( E_d \). Next, by applying the arguments of (c) we can extend these operators to isomorphisms of the space \( X \) onto itself.

The second case can be reduced to (b) and (c) in a similar manner.

Remark 11.9. The assumption that the operator \( A \) maps the space \( X \) onto itself was not essential in the above proof. Essential was the role played by the powers of the bases. Hence we may say that if the powers of the bases of the space \( X \) and \( Y \) are equal, then every operator \( A \in L(X,Y) \) is the sum of two isomorphisms of the space \( X \) onto the space \( Y \).
If the powers of the bases of spaces $X$ and $Y$ are not equal, then such a representation of $A$ cannot exist, for there exist no isomorphisms of the space $X$ onto the space $Y$.

In the case of an arbitrary algebra of operators $X(X)$ the theorem on the representation of an operator as the sum of two operators with finite $d$-characteristics does not hold. This can be verified by the following example:

**Example 11.10.** Let $X$ be the space of all continuous functions $x(t)$ on a real variable defined in the whole complex plane. Let $X(X)$ be the algebra of operators $P$ of multiplication by a continuous polynomial $p(t)$. If the polynomial $p(t)$ is not a constant, then the corresponding operator $P$ is of infinite deficiency $\beta_P$. Indeed, by the fundamental theorem of algebra, there exists a number $a_0$ such that $p(a_0) = 0$. Let us remark that if $0 < \beta < 1$, then there exists a constant $c > 0$ such that

$$|x(a) - x(a_0)| \leq c|a - a_0|$$

for $|a - a_0| < 1$.

Consequently, writing

$$x(t) = a - a_0^t, \quad 0 < a < 1,$$

we obtain

$$\alpha = \lim \{E_n, \alpha_n, \beta > a\}.$$

Hence $\beta_P = +\infty$.

Thus

$$D_2(X) = \{\alpha; \ 0 = \alpha \text{ is a complex number}\}.$$

Hence, if the operator $P \in X(X)$ is not of the form $aI$, it cannot be written as a sum of two operators with finite $d$-characteristics belonging to this algebra.

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**CHAPTER II**

**ALGEBRAIC AND ALMOST ALGEBRAIC OPERATORS**

In this chapter we shall investigate algebraic and almost algebraic operators. We shall show later (Part D) that this class has some very important applications. In case of polynomials of algebraic and almost algebraic operators it is easy to determine effectively the simple regularizer and, consequently, the solutions of the respective equations. In this chapter Sections 1-6 are of auxiliary character. Fundamental theorems are given in § 5 and 6. Section 7 has a special character: it contains more general theorems for equations in algebraic operators with constant coefficients.

§ 1. Hermite's interpolation formula. Partition of unity. We now give the following lemma by Hermite on interpolation with multiple knots.

**Lemma 1.1.** (Hermite [1]) There exists exactly one polynomial $W(t)$ of degree $N - 1$ assuming (together with its derivatives of order $k$) given values $y_{n_k}$ at $n$ different points $t_i$ ($i = 1, 2, \ldots, n$; $k = 0, 1, \ldots, r_1 - 1$; $r_1 + r_2 + \ldots + r_n = N$):

$$W^{(k)}(t_i) = y_{n_k}.$$

The polynomial $W(t)$ is given by the following formula:

$$W(t) = \sum_{m=0}^{N} \frac{P(t)}{(t - t_m)^m} \sum_{k=0}^{r_m - 1} \frac{[(t - t_m)^n]}{m!} \frac{(t - t_m)^k}{k!}.$$

Here

$$P(t) = \sum_{m=0}^{N} (t - t_m)^m,$$

$$\left(f(t)\right)^{(k)}_{|x=0} = \sum_{m=0}^{N} \frac{d^m f(t)}{m!} \left|_{t=x}^{t=0}.$$

**Proof.** The polynomial under consideration is of the form

$$W(t) = \sum_{m=0}^{N} \sum_{k=0}^{r_m - 1} y_{n_k} L_{mk}(t)$$

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if the polynomials $L_{ia}(t)$ of degree $N-1$ \((i = 1, 2, \ldots, n; \ m = 0, 1, \ldots, r_i-1)\) satisfy the following conditions:

1. \(L_{ia}(t_i) = 0 \quad (m \neq i, \ j = 0, 1, \ldots, r_i-1, \ m = 0, 1, \ldots, r_i-1)\),

2. \(L_{ia}(t) = \delta_{ik} \quad (j = 0, 1, \ldots, r_i-1, \ m = 0, 1, \ldots, r_i-1)\),

where \(\delta_{ik}\) is the Kronecker symbol, i.e.,

\[
\delta_{ik} = \begin{cases} 
0 & \text{if } i \neq k, \\
1 & \text{if } i = k.
\end{cases}
\]

Now, we determine such polynomials $L_{ia}$. If we take into account property (1), the polynomial $L_{ia}$ should have the root $t_i$ of multiplicity $r_i$ for $m \neq i$, and the root $t_i$ of multiplicity 1. Hence $L_{ia}(t)$ must be of the form

\[
L_{ia}(t) = (t-t_i)^{r_i} \cdots (t-t_{i+1})^m (t-t_{i+2})^m \cdots (t-t_n)^m L_{ia}(t),
\]

where $L_{ia}(t)$ is a polynomial of degree

\[
N-1 - (r_1 + \cdots + r_{i-1} + k + r_{i+1} + \cdots + r_n) = r_i - k - 1.
\]

Applying the notation adopted above we may write the polynomial $L_{ia}(t)$ in the form

\[
L_{ia}(t) = \frac{P(t)}{(t-t_i)^{r_i}} L_{ia}(t).
\]

In order to determine the polynomials $L_{ia}(t)$ we apply property (2), which implies that the expansion of the polynomial $L_{ia}(t)$ in a Taylor series in a neighbourhood of the point $t_i$ is of the form

\[
L_{ia}(t) = \frac{1}{k!} (t-t_i)^k [1 + \sigma (t-t_i)^{k+1} + \cdots].
\]

The rational function \(\frac{1}{k!} \frac{P(t)}{(t-t_i)^m} \) is regular in a neighbourhood of the point $t_i$. Hence it can be expanded in a Taylor series with respect to powers of $t-t_i$. On the other hand, $L_{ia}(t)$ must be a polynomial of degree $r_i - k - 1$. Thus, it must be the sum of those terms of this expansion which are of degree not greater than $r_i - k - 1$, i.e.,

\[
L_{ia}(t) = \frac{1}{k!} \left[ \frac{P(t)}{(t-t_i)^m} \right]_{t=t_i-1+\lambda}.
\]

Conversely, if $L_{ia}$ satisfies this equality, conditions (1) and (2) hold. Consequently,

\[
L_{ia}(t) = \frac{P(t)}{(t-t_i)^{r_i}} L_{ia}(t) = \frac{P(t)}{(t-t_i)^m} \frac{(t-t_i)^k}{k!} \left[ \frac{P(t)}{(t-t_i)^m} \right]_{t=t_i-1+\lambda}.
\]

Hence we obtain the required form of the polynomial $W(t)$. \(\square\)

\section{Hermite's interpolation formula}

This is the Hermite interpolation formula. Let us remark that

\[
W(t) = \prod_{m=1}^{n} \frac{(t-t_m)^n}{(t-t_m)^n}.
\]

By the assumptions of the preceding lemma the following lemma also holds:

\begin{lemma}
(On the partition of unity). If we write

\[
p_i(t) = \frac{(t-t_i)^n}{P(t)_{|t=t_i}}, \quad (i = 1, 2, \ldots, n),
\]

then

\[
1 = \sum_{i=1}^{n} p_i(t),
\]

and this representation is unique (if $t_i$ and $r_i$ are fixed).
\end{lemma}

\begin{proof}
We take $W(t) = 1$; hence $W(t_i) = 1$, $W(t_i) = 0$ for $k \geq 1$ and $i = 1, 2, \ldots, n$. Thus, by the Hermite interpolation formula, we have

\[
1 = \sum_{i=1}^{n} \prod_{m=1}^{n} \frac{(t-t_m)^n}{P(t)_{|t=t_i}},
\]

If $t_i$ are single knots, Hermite's lemma yields the Lagrange interpolation formula. Indeed, in this case

\[
\frac{(t-t_i)^n}{P(t)_{|t=t_i}} = \frac{P(t)_{|t=t_i}}{P(t)_{|t=t_i}} = \frac{(t-t_i)^n}{P(t)_{|t=t_i}},
\]

\[
= \prod_{m=1}^{n} \frac{(t-t_m)^n}{P(t)_{|t=t_i}},
\]

Hence we have the following formula for the polynomial $W(t)$:

\[
W(t) = \sum_{i=1}^{n} y_i \prod_{m=1}^{n} \frac{(t-t_m)^n}{(t-t_m)^n}.
\]

This is the Lagrange interpolation formula. Let us also remark that the last formula gives the following partition of unity in case of single knots:

\[
1 = \sum_{i=1}^{n} \prod_{m=1}^{n} \frac{(t-t_m)^n}{(t-t_m)^n}.
\]
§ 2. Algebraic and almost algebraic elements in a linear ring. Let $X$ be an algebra with unity $I$ (over the field of complex numbers). If there exists a polynomial
\[ P(t) = p_0 + p_1 t + \ldots + p_n t^n \]
in variable $t$ with complex coefficients, satisfying the condition
\[ P(S) = T, \]
where $S \in X$ and $T$ is an element of a two-sided ideal $J \subset X$, then we say that the element $S$ is *almost algebraic* with respect to the ideal $J$. Whenever there is no danger of confusion we shall call $S$ briefly an *almost algebraic element*.

Without loss of generality we shall assume once for all that $p_n = 1$. If $S$ satisfies the polynomial identity $P(S) = T$ with a polynomial of degree $N$ and does not satisfy any identity of degree less than $N$, we say that the almost algebraic element $S$ is of *order* $N$. In this case we call $P(t)$ the *characteristic polynomial* of the element $S$, and the roots of this polynomial the *characteristic roots* of the element $S$. Since $p_n = 1$, we assume
\[ P(t) = \prod_{m=1}^{n} (t - t_m)^{m}, \]
where all complex numbers $t_m$ are different and $r_1 + r_2 + \ldots + r_n = N$. If $T = 0$ in identity (2.1), we call the element $S$ *algebraic*.

Evidently, if an element $S$ is almost algebraic, then the coset $[S]$ defined by this element in the quotient ring $[X] = X/ J$ is algebraic and has the same characteristic polynomial and the same characteristic roots as $S$. This follows from the fact that the sum and the product of cosets correspond to the sum and the product of elements generating those cosets.

In the sequel we shall see that the properties of the characteristic polynomial determine in a certain sense the properties of the element $S$. Our further considerations will be based on those properties.

The following two cases are of great importance in various applications: $P(t) = p - 1$ and $P(t) = t^n - 1$. In the first case we say that the element $S$ is in an *involution*, because it satisfies the equation $S^2 = I$. In the second one, we call $S$ an *involution of order* $N$, because it satisfies the equation $S^N = I$. In these cases the characteristic roots are simple, i.e. $n = N$ and $r_1 = \ldots = r_n = 1$, and they are the $N$th roots of unity, i.e.
\[ t_m = e^{2\pi i m/N} \quad (m = 0, 1, \ldots, N - 1). \]

Suppose we are given an algebraic element $S$ of order $N$ with the characteristic polynomial
\[ P(t) = \prod_{m=1}^{n} (t - t_m)^{m} \quad (r_1 + r_2 + \ldots + r_n = N). \]

Fig. 4. Classification of almost algebraic operators

We assume, of course, once for all that the numbers $t_i$ are different from each other. We write
\[ P_i = p_i(S) \quad (i = 1, 2, \ldots, n), \]
where $p_i(t)$ are polynomials defined by formulae (1.2). Elements $P_i$ have the following properties important in further considerations:

**Property 2.1.** The sum of elements $P_i$ is equal to the unity of the ring
\[ \sum_{i=1}^{n} P_i = I. \]

This property follows immediately from the definition of polynomials $p_i(t)$ and from Lemma 1.2 on the partition of unity.

**Property 2.2.** Elements $P_i$ are idempotent and disjoint, i.e.
\[ P_i P_j = \begin{cases} P_i & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases} \quad (i, j = 1, 2, \ldots, n). \]

Indeed, let $i \neq j$. Taking into account the fact that all polynomials of the element $S$ whose coefficients are numbers are commutative, we obtain
\[ P_i P_j = g(S)g(S) \prod_{m=1}^{n} (S - t_m I)^m \prod_{k=1}^{n} (S - t_k I)^k \]
\[ = g(S)g(S) \prod_{m=1}^{n} (S - t_m I)^m P(S) = 0. \]
Hence, by formula (2.4), we get

\[(2.6) \quad P_i = P_i \sum_{j=1}^{n} P_j = \sum_{j=1}^{n} P_i P_j = P_i \quad (i = 1, 2, \ldots, n) .\]

Let us remark that formula (2.6) implies that each element \(P_i\) is also algebraic, has the characteristic polynomial \(P_i - t_i\) and, consequently, is of order 2 and has characteristic roots 0, 1.

**Property 2.3.** The following equality holds for every fixed \(i\):

\[(2.7) \quad (S - t_i I)^n P_i = 0 .\]

Indeed,

\[(S - t_i I)^n P_i = g_i(S)(S - t_i I)^n \prod_{m=1}^{n} (S - t_m I)^n = g_i(S) P(S) = 0 .\]

Let us now suppose that the characteristic roots of the element \(S\) are single, i.e. \(t_1 = \ldots = t_n = 1\) and \(N = n\). By formula (2.7), we have \((S - t_i I) P_i = 0\), that is

\[(2.8) \quad S P_i = t_i P_i .\]

Hence

\[(2.9) \quad S = \sum_{i=1}^{n} t_i P_i .\]

**Remark 2.1.** If \(S^N = I\), then the elements \(P_i\) are of the form

\[P_i = \frac{1}{N} (I + S) , \quad P_n = \frac{1}{N} (I - S) .\]

**Remark 2.2.** If \(S^N = I (N \geq 2)\), then the elements \(P_i\) can be written in the following form:

\[P_n = \frac{1}{N} \sum_{k=1}^{N} e^{-\frac{2\pi i k}{N}} = \sum_{k=1}^{N} e^{-\frac{2\pi i k}{N}} = 0 .\]

\[Q(S) = \sum_{i=1}^{n} Q(t_i) P_i .\]

for an arbitrary polynomial \(Q(t)\) with complex coefficients. Hence the element \(Q(S)\) is also algebraic and has characteristic roots \(Q(t_i)\).

**Remark 2.3.** If the algebraic elements \(S, S_1, S_2\) commute, then their sum and their superposition are also algebraic. Moreover, if the characteristic roots of these elements are

\[t_1, \ldots, t_N \quad \text{and} \quad t_1', \ldots, t_N' ,\]

then their sum and their superposition have characteristic roots

\[t_1 + t_1' \quad \text{and} \quad t_k + t_k' \quad (k = 1, 2, \ldots, N, m = 1, 2, \ldots, N) ,\]

respectively (in this case each root is taken the number of times equal to its multiplicity).

**Theorem 2.1.** If \(S\) is an algebraic element in an algebra \(A\), then \(AS - SA \neq aI\) for each scalar \(a \neq 0\) and for any \(A \in \mathcal{X}\).

**Proof.** Let \(P(t)\) be the characteristic polynomial of the element \(S\), and let us suppose that \(AS - SA = aI (a \neq 0)\). We shall show by induction that

\[AS^{n+1} - S^{n+1}A = anS^n \quad \text{for} \quad n = 1, 2, \ldots .\]

Indeed,

\[AS^{n+1} - S^{n+1}A = AS^n - S^nA = aS^n + S(AS^n - S^nA) = aS^n + S(a(n-1)S^{n-1}) = a[S^n + (n-1)S^{n-1}] = anS^n .\]

Hence, given any polynomial \(Q(t)\), we have

\[A Q(S) - Q(S)A = aQ(S) ,\]

where \(Q'\) denotes the derivative of the polynomial \(Q\). In particular,

\[P(S) = \frac{1}{N} = 0 \quad \text{implies} \quad 0 = AP(S) - P(S)A = aP(S) .\]

Hence \(P(S) = 0\), which contradicts the assumption that \(P(t)\) is the characteristic polynomial of the element \(S\). Thus, \(AS - SA \neq aI\).

**Remark 2.5.** Let \(S\) be an almost algebraic element, and let \(P(S) = T\). If there exists a polynomial \(Q(t)\) such that \(Q(S)T = 0\) (or \(TQ(S) = 0\)), then \(S\) is an algebraic element. Indeed, we have

\[Q(S)P(S) = Q(S)T = 0 \quad \text{(resp.} \quad P(S)Q(S) = TQ(S) = 0) .\]

**Remark 2.6.** If \(S\) is an almost algebraic element: \(P(S) = T\) and there exists a polynomial \(Q(t)\) such that \(Q(T) = 0\), then \(S\) is an algebraic element, since \(QP(S) = Q(T) = 0\).

**Remark 2.7.** If \(S\) is an almost algebraic element: \(P(S) = T\), then \(ST = TS\). Indeed, \(ST = SP(S) = P(S)S = TS\).

**Corollary 2.2.** If \(S\) is an almost algebraic element, \(P(S) = T\), then \(AS - SA \neq aI + Ta\) for each scalar \(a \neq 0\), any \(A \in \mathcal{X}\) and any \(T \in \mathcal{J}\).

This follows immediately from Theorem 2.1 if we consider the quotient ring \(XJ\).

\[\S 3. \text{Properties of polynomials in algebraic and almost algebraic elements.}\]

We now give some properties of polynomials in algebraic operators
necessary in further considerations. As before, let $S$ be an algebraic element of order $N$ with the characteristic polynomial $P(t)$. Let us consider an element of the form

$$A(S) = \sum_{n=0}^{M} A_m S^n, \quad \text{where} \quad A_m \in \mathbb{K} \quad (m = 0, 1, \ldots, N).$$

We write

$$A(t) = \sum_{m=0}^{M} A_m t^m \quad (t \text{ is a scalar}),$$

$$A^{(m)}(t) = A(t), \quad m \in \mathbb{N}$$

$$A^{(m)}(t) = m! \sum_{k=0}^{m} \binom{m}{k} A_k t^{m-k} = \frac{d^m A(t)}{dt^m} \bigg|_{t=t_i}$$

$$\quad (m = 1, 2, \ldots, M; \quad i = 1, 2, \ldots, n),$$

where $\frac{d}{dt} R(t)$ is the derivative of the polynomial $R(t)$.

We prove the following formulæ:

(3.3) $$A(S) = \sum_{m=0}^{M} \frac{1}{m!} A^{(m)}(t_i) (S - t_i I)^m,$$

(3.4) $$A(S) P_i = \sum_{m=0}^{M} \frac{1}{m!} A^{(m)}(t_i) (S - t_i I)^m P_i, \quad (i = 1, 2, \ldots, n)$$

(3.5) $$A(S) = \sum_{i=1}^{n} \sum_{m=0}^{M} \frac{1}{m!} A^{(m)}(t_i) (S - t_i I)^m P_i.$$

Formula (3.3) may be derived in the following manner:

$$A(S) = \sum_{n=0}^{M} A_m S^n = \sum_{n=0}^{M} A_m [(S - t_i I) + t_i I]^n$$

$$= \sum_{n=0}^{M} A_m \sum_{k=0}^{n} \binom{n}{k} t_i^{n-k} (S - t_i I)^k$$

$$= \sum_{k=0}^{M} \left( \sum_{n=k}^{M} A_n \binom{n}{k} t_i^{n-k} \right) (S - t_i I)^k$$

$$= \sum_{n=0}^{M} \frac{1}{m!} A^{(m)}(t_i) (S - t_i I)^m.$$

To prove formula (3.4) let us remark that, by formula (2.7),

$$A(S) P_i = \sum_{m=0}^{M} \frac{1}{m!} A^{(m)}(t_i) (S - t_i I)^m P_i = \sum_{m=0}^{M} \frac{1}{m!} A^{(m)}(t_i) (S - t_i I)^m P_i.$$

Formula (3.5) follows from the preceding ones and from formula (2.4):

$$A(S) = A(S) \sum_{i=1}^{n} P_i = \sum_{i=1}^{n} A(S) P_i = \sum_{i=1}^{n} \sum_{m=0}^{M} \frac{1}{m!} A^{(m)}(t_i) (S - t_i I)^m P_i.$$

Let us remark that, by formulæ (3.4) and (3.5), it is sufficient to consider polynomials $A(S)$ of degrees not greater than $N - 1$. If we consider $A(S) P_i$, it is even sufficient to limit oneself to polynomials of degrees not greater than $r_i - 1$. All terms of higher degrees can be reduced by means of the identity $P(S) = 0$. In the sequel we shall proceed in this manner.

If one of the characteristic roots is equal to zero, e.g. $t_i = 0$, we have $A^{(m)}(0) = m! A_m$ and

(3.6) $$A(S) P_i = \sum_{m=0}^{N-1} A_m S^m P_i.$$

Hence in this case decompositions (3.3) and (3.5) are not essential.

If the characteristic roots of an element $S$ are single, formula (3.5) assumes a very simple form (compare Example 2.3):

(3.7) $$A(S) = \sum_{i=1}^{n} A(t_i) P_i.$$

Now we consider the polynomial

(3.8) $$A(S) = \sum_{m=0}^{N-1} S^m A_m$$

under the preceding assumptions.

If the coefficients $A_m \in \mathbb{K}$ are commutative with the element $S$ (e.g. if the coefficients of the polynomial $A(S)$ are numbers), polynomials $A(S)$ and $A(A(S))$ are identical. In other cases we have the following equality:

(3.9) $$A(A(S)) = A(S) + \sum_{m=0}^{N-1} (S^m A_m - A_m S^m).$$

Of course, formulæ analogous to those deduced above are valid also for polynomials $A(A(S))$, only the order arrangement of factors containing the element $S$ and not containing $S$ must be changed into the inverse one.
The bilinear form
\[ [A, B] = AB - BA \]
is called a commutator of elements \( A \) and \( B \). The following formulae are easily verified:
\[
[B, A] = -[A, B], \quad [A + B, C] = [A, C] + [B, C],
\]
\[
\]

**Theorem 3.1.** If \( S \) is an algebraic element of an algebra \( X \) and if
\[
b_m = \frac{b(S)}{S^m}, \quad A_m = \frac{A(S)}{S^m},
\]
where the coefficients \( b_m \) are complex numbers, \( A_m \in X \) and \([A_m, S] \in J\), \( J \) being a two-sided ideal in the algebra \( X \), then
\[
[b_m(S), A_m(S)] = J.
\]

**Proof.** Let \( 0 \leq m \leq N-1 \). The properties of commutators imply
\[
[S, A_m S] = -[A_m S, S] = -A_m [S^m, S], J
\]
\[
= -[A_m, S] S^m \in J.
\]
Hence
\[
[S, A(S)] = \sum_{m=0}^{N-1} [S, A_m S] e J.
\]
Let us suppose that \([S^m, A(S)] \in J\). Then
\[
[S^{m+1}, A(S)] = [S^m, A(S)] + [S, A(S)] S^m \in J.
\]
Thus, \([S^m, A(S)] \in J\) for \( n = 1, 2, \ldots \). Consequently,
\[
[b_m(S), A_m(S)] = \sum_{m=0}^{N-1} b_m S^m, A(S) = \sum_{m=0}^{N-1} b_m [S^m, A(S)] \in J.
\]

**Remark.** In general, this theorem is not true if the coefficients \( b_m \) are not numbers. In that case it would be necessary to suppose additionally that \( [b_m, A_1] = 0 \) for all indices \( m \) and \( k \).

**Theorem 3.2.** If \( S \) is an algebraic element in an algebra \( X \) with the characteristic polynomial \( P(t) = \prod_{m=1}^{N} (t - \lambda_m)^{r_m} \) \((r_1 + \cdots + r_N = N)\), and if
\[
A(S) = \sum_{m=0}^{N-1} A_m S^m, \quad B(S) = \sum_{m=0}^{N-1} B_m S^m,
\]
where \( A_m, B_m \in X \) \((m = 0, 1, \ldots, N-1)\) and \([A_m, S] \in J \subseteq X\), \( J \) being a two-sided ideal, then
\[
B(S) A(S) = C(S) + T_{BA},
\]
where
\[
C(S) = \sum_{m=0}^{N-1} C_m S^m = \sum_{m=0}^{N-1} \sum_{i=0}^{m-1} \frac{1}{i!} \sum_{j=0}^{i} \frac{1}{j!} A^{(j)}(t) (S - t_i I)^j P_i,
\]
\[
C^{(j)}(t_i) = \sum_{i=0}^{j} \left( \prod_{k=1}^{i} \frac{1}{k!} \right) B^{(k)}(t_i) A^{(j-k)}(t_i), \quad (i = 0, 1, \ldots, n; j = 0, 1, \ldots, m-1),
\]
\[
T_{BA} = \sum_{m=0}^{N-1} \sum_{i=0}^{m-1} \frac{1}{i!} \left( \prod_{k=1}^{i} \frac{1}{k!} \right) A^{(m)}(t_i) (S - t_i I)^i P_i A^{(m)}(t_i) (S - t_i I)^m P_i + \sum_{i=0}^{m} B(S) [A(S), P] P_i e J.
\]

**Proof.** Since \( P_i = P_i \) \((i = 1, 2, \ldots, n)\), we have
\[
B(S) A(S) = \sum_{i=0}^{n-1} B(S) A(S) P_i\]
\[
= \sum_{i=0}^{n-1} [B(S) A(S) P_i - B(S) A(S) P_i + B(S) A(S) P_i]
\]
\[
= \sum_{i=0}^{n-1} B(S) [A(S), P] P_i + \sum_{i=0}^{n-1} B(S) P_i A(S) P_i.
\]
However, polynomials \( P_i \) and \((S - t_i I)^k\) are commutative. Hence, by formula (2.4),
\[
B(S) P_i A(S) P_i
\]
\[
= \left( \prod_{k=0}^{m-1} \frac{1}{k!} A^{(m)}(t_i) (S - t_i I)^i P_i \left( \sum_{m=0}^{N-1} \frac{1}{i!} A^{(m)}(t_i) (S - t_i I)^i P_i \right) \right) P_i
\]
\[
= \left( \prod_{k=0}^{m-1} \frac{1}{k!} B^{(k)}(t_i) A^{(m)}(t_i) (S - t_i I)^i P_i + \sum_{i=0}^{m} \left( \prod_{k=0}^{m} \frac{1}{k!} A^{(m)}(t_i) (S - t_i I)^i P_i \right) \right) P_i.
\]
4. Regularisation of polynomials in algebraic and almost algebraic elements.

We show that, under some assumptions, regularizers of polynomials in algebraic and almost algebraic elements can be obtained effectively, and that they are also polynomials of the same type with coefficients determined by coefficients of the given polynomials.

4.1. Let $S$ be an algebraic element in an algebra $\mathcal{X}$ with the characteristic polynomial $P(t) = \prod_{i=0}^{n} (t - r_i)^n$, $r_1, \ldots, r_n \in \mathcal{X}$, and let $J$ be a two-sided ideal in the algebra $\mathcal{X}$. If a polynomial $A(S) = \sum_{m=0}^{N} A_m S^m$ with coefficients $A_m \in \mathcal{X}$ satisfies the following conditions:

1. $A_m S - S A_m \in J$ for $m = 0, 1, \ldots, N - 1$,

2. there exist elements $[A(t_i)]^{-1}$ for $i = 1, 2, \ldots, n$,

then there exists a simple regularizer $R_{\text{reg}}$ of the element $A(S)$ to the ideal $J$, given by the formula

$$R_{\text{reg}} = \sum_{m=0}^{N} B_m S^m,$$

where

$$B_m = \sum_{i=1}^{n} \sum_{j=1}^{N} \frac{(-1)^{m-j}}{j! (m-j)!} \Gamma^m_j P_t = \sum_{i=1}^{n} \sum_{j=1}^{N} \frac{(-1)^{m-j}}{j! (m-j)!} \Gamma^m_j P_t,$$

and $B_m(S) = [A(t_i)]^{-1}$ for $i = 1, 2, \ldots, n$.

Proof. First we determine the left regularizer $R_S$ of the element $A(S)$. By Theorem 3.2, the element $R_S(S)$ is a left regularizer of the element $A(S)$ if $R_m = I$, $R_m = 0$, else $R_m = 0$, i.e., if $R_m(t_i) = \delta_m I$ for $m = 0, 1, \ldots, N - 1$, where $\delta_m$ is the Kronecker symbol. Hence we obtain the following system of equations:

$$\sum_{i=1}^{n} \sum_{j=1}^{N} \frac{(-1)^{m-j}}{j! (m-j)!} \Gamma^m_j P_t = \sum_{i=1}^{n} \sum_{j=1}^{N} \frac{(-1)^{m-j}}{j! (m-j)!} \Gamma^m_j P_t,$$

and $[A(S), P_t] = [[S - t_i] P_t, A^{(m)}(t_i)] \in J$.

Thus all terms of the element $T_{\text{reg}}$ belong to the ideal $J$. Consequently, $T_{\text{reg}} \in J$. 

4.2. Regularisation of polynomials in algebraic and almost algebraic elements.

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Hence

$$B(S) = \sum_{i=1}^{n} \left[ \sum_{m=0}^{n-1} \frac{1}{m!} B^{\alpha m}(t_i)(S-t_i)^m \right] P_i$$

$$= \sum_{i=1}^{n} \left[ \sum_{m=0}^{n-1} \frac{1}{m!} B^{\alpha m}(t_i) \left( \sum_{j=0}^{m} \frac{(-1)^m}{j!(m-j)!} t_i^{-j} S^j \right) P_i \right]$$

$$= \sum_{i=1}^{n} \left[ \sum_{m=0}^{n-1} \frac{(-1)^m}{j!(m-j)!} t_i^{-j} B^{\alpha m}(t_i) \right] S^j P_i.$$  

Thus

$$B_j P_i = \sum_{m=0}^{n-1} \frac{(-1)^m}{j!(m-j)!} t_i^{-j} B^{\alpha m}(t_i) P_i \quad (j = 0, 1, \ldots, r_i-1),$$

for every fixed $i$. Evidently, this implies

$$B_{j} = \sum_{m=0}^{n-1} \frac{(-1)^m}{j!(m-j)!} t_i^{-j} B^{\alpha m}(t_i) P_i \quad (j = 0, 1, \ldots, N-1).$$

The right regularizer $B'(S)$ is determined in an analogous manner by changing the order of elements $A(S)$ and $B(S)$ in formulae (4.2).

By Property 5.1, 1 (c),

$$T_R = B'(S) - B(S) \in \mathcal{J},$$

and each of these regularizers is simple. Thus, we take $R_{A(0)} = B(S)$.

Remark. If the characteristic roots of the elements $A(S)$ are single, then the existence of left inverses (right inverses) of elements $A(t_i)$ is a sufficient condition for the existence of a left regularizer (right regularizer). Indeed, in this case system (4.2) is reduced to the first equation, and our assumption is sufficient to solve this equation.

**Corollary 4.2.** If the assumptions of Theorem 4.1 are satisfied, and if

$$[A_m, S] = 0 \quad \text{for} \quad m = 0, 1, \ldots, N-1,$$

then the element $[A(S)]^{-1}$ exists and

$$[A(S)]^{-1} = R_{A(0)}.$$

Proof. Indeed, in this case we can assume $J = (0)$. Then $B'(S) - R_{A(0)} = 0$. Moreover,

$$R_{A(0)} A(S) = I \quad \text{and} \quad A(S) R_{A(0)} = I.$$  

§ 4. Regularization of polynomials

In some cases the regularizer $R_{A(0)}$ can be given in a simpler form, not by means of a recurrence formula.

**Theorem 4.3.** If the assumptions of Theorem 4.1 are satisfied, and if all elements $A_1, \ldots, A_{N-1}$ are commutative, then the simple regularizer of the element $A(S)$ to the ideal $J$ is of the form:

$$R_{A(0)} = \sum_{i=1}^{N-1} B_i S^i,$$

where

$$B_i = \sum_{m=0}^{N-1} \frac{1}{j!(m-j)!} \left( (-1)^m - (-1)^m t_i^{-j} \sum_{m=0}^{N-1} D_{n+m}(v) P_i \right),$$

and $D_{n+m}$ is a minor of the determinant

$$D(t_i) = \det_{0 \leq k, l < N-1} A_{n+m}(v),$$

where

$$A_{n+m}(v) = \begin{cases} m \choose k \end{cases} A^{m-k}(t_i) \quad \text{for} \quad m \geq k, \\ 0 \quad \text{for} \quad m < k,$$

obtained by cancelling the $(m+1)$st column and the $(k+1)$st row.

Proof. Let us consider the system of equations (4.2). Since the elements $A_1, \ldots, A_{N-1}$ are commutative, this system can be solved by means of determinants. The determinant $D(t_i)$ of this system has elements $A_{n+m}(t_i) = A(t_i)$ on the principal diagonal, and all the elements under this diagonal are equal to zero. Hence $D(t_i) = (A(t_i))^{N-1}$, and

$$B^{m}(t_i) = (A(t_i))^{N-1} \sum_{k=0}^{N-1} (-1)^{k+m} D_{n+m}(t_i) \delta_{mk}$$

$$= (-1)^m (A(t_i))^{N-1} D_{n+m}(t_i) \quad (m = 0, 1, \ldots, N-1).$$

Thus,

$$B_i = \sum_{m=0}^{N-1} \sum_{j=0}^{N-1} \frac{(-1)^m}{j!(m-j)!} (A(t_i))^{N-1} t_i^{-j} D_{n+m}(t_i) P_i$$

$$= \sum_{j=0}^{N-1} \sum_{m=0}^{N-1} \frac{1}{j!(m-j)!} t_i^{-j} D_{n+m}(t_i) P_i \quad (j = 0, 1, \ldots, N-1).$$

Remark. If the characteristic roots of an element $S$ are single, we need not assume the commutativity of elements $A_1, \ldots, A_{N-1}$ for in this case the system (4.2) is reduced to one equation $B^{m}(t_i) A^{m}(t_i) = I$.

Hence the following corollary holds:
Corollary 4.4. If the assumptions of Theorem 4.1 are satisfied and if all characteristic roots of the elements \( S \) are single, then the simple regularizer of the element \( A(S) \) is the form

\[
R_{A(S)} = \sum_{n=1}^{\infty} [A(t_n)]^{-1}P_n = \sum_{k=0}^{n-1} R_k S^k,
\]

where

\[
R_k = \frac{1}{\nu} \sum_{m=1}^{n} (-1)^{k+m+1} \det[A(t_m)]^{-1} \quad (k = 0, 1, \ldots, n-1),
\]

\( \nu \) is the Vandermonde determinant of numbers \( t_1, \ldots, t_n \), and \( d_{mk} \) is the minor of that determinant obtained by cancelling its \( m \)-th row and \( (k+1) \)-st column.

Proof. This follows directly from the proof of Theorem 4.1, if we take \( r_m = 1 \) \( (m = 1, \ldots, n) \) and apply the equality

\[
R(t_m) = [A(t_m)]^{-1} = \sum_{m=1}^{r_m} t_m R_k \quad (m = 1, \ldots, n).
\]

---

5. Algebraic and almost algebraic operators. An operator \( S \) is algebraic resp. almost algebraic if it is an algebraic resp. almost algebraic element of an algebra \( \mathbb{X}(X) \subset L_0(X) \). Obviously, all theorems proved previously also hold in the case of the algebra \( \mathbb{X}(X) \). Let us remark that in this case operators \( P_i \) defined by formulae (2.3) are disjoint projection operators by formulae (2.5).

Property 5.1. If an operator \( S \in \mathbb{X}(X) \) is algebraic, then the space \( X \) is a direct sum of projections:

\[
X = \bigoplus_{i=1}^{n} X_i, \quad X_i = P_i X = \{P_i x, \ x \in X\}.
\]

To prove this property let us remark that, by Property 2.1, each element \( x \in X \) can be written as a sum

\[
x = \sum_{i=1}^{n} x_i, \quad \text{where} \quad x_i = P_i x \in X_i \quad (i = 1, 2, \ldots, n).
\]

By Property 2.2, this representation is unique.

From Properties 2.3 and 5.1 follows

\[
(\mathbb{S}-t_i)^m x_i = 0
\]

for arbitrary \( x_i \in X_i \) \( (i = 1, 2, \ldots, n) \). The last formula shows that the operator \( S \) is an algebraic operator on each of the spaces \( X_i \), with only one root \( t_i \), and of order \( r_i \).

Theorem 5.1. If \( S \in \mathbb{X}(X) \), then the following conditions are equivalent:

(a) \( S \) is an algebraic operator with the characteristic polynomial

\[
P(t) = \prod_{i=1}^{n} (t-t_i)^{r_i},
\]

and of order \( N = t_1 + \ldots + t_n \).

(b) There exist \( n \) linear operators \( P_i \) such that

\[
P_i P_j = \delta_{ij} P_i; \quad \sum_{i=1}^{n} P_i = I \quad \text{and} \quad (\mathbb{S}-t_i)^m P_i = 0
\]

\( (i, j = 1, 2, \ldots, n) \)

(\( \delta_{ij} \) means the Kronecker symbol).

(c) The space \( X \) is the direct sum of \( n \) subspaces \( X_i \) such that

\[
(\mathbb{S}-t_i)^m x = 0 \quad \text{for} \quad x \in X_i.
\]
Proof. The implications (a) \(\Rightarrow\) (b) \(\Rightarrow\) (c) immediately follow from properties (2.1)-(2.3) and 5.1. Hence we prove only that condition (a) follows from condition (c). Indeed, let the space \(X_n\) be the direct sum of \(n\) spaces \(X_m\) such that \(P(S_m) = 0\) for \(x \in X_m\). We take

\[
P(S) = \sum_{m=1}^{N} (S_m - t_m I)^{m}.
\]

By hypothesis, \(x = \sum_{i=1}^{n} x_i\), where \(x_i \in X_i\). Hence

\[
P(S)x = \sum_{i=1}^{n} P(S)x_i = \sum_{i=1}^{n} \left| \prod_{m=1}^{n} (S_m - t_m I)^{m} x_i \right| = 0
\]

for every \(x \in X\). Consequently, the operator \(S\) is algebraic of order \(N = t_1 + \ldots + t_n\) with the characteristic polynomial \(P(t)\). Thus, conditions (a), (b), (c) are equivalent.

If the characteristic roots of an algebraic operator \(S\) are single, then, formula 5.2,

\[
Sx_i = \lambda_i x_i
\]

for arbitrary \(x_i \in X_i\) \((i = 1, 2, \ldots, n)\). Hence any of the spaces \(X_i\) is an eigenspace for the operator \(S\).

One can give another characterization of algebraic operators. Let \(x \in X\) and let us write:

\[
X_0(x) = \operatorname{lin}(x, Sx, \ldots, S^N x) \quad (n = 0, 1, \ldots), \quad X_n(x) = \operatorname{lin}(x, Sx, S^2 x, \ldots, S^n x) .
\]

Evidently, \(\dim X_n(x) \leq n+1\) for an arbitrary \(x \in X\).

**Theorem 5.2.** (Kaplansky [2]) An operator \(S : X \rightarrow X\) is algebraic of order \(N\) if and only if

\[
b_0 = \sup_{x \in X} \dim X_0(x) = N .
\]

**Proof. necessity.** Let us suppose that the operator \(S\) is algebraic of order \(N\). Let its characteristic polynomial be

\[
P(S) = P_1 I + P_2 S + \ldots + P_N S^N = 0 \quad (P_N = 1) .
\]

If follows from the definition of an algebraic operator that

\[
S^n x = \operatorname{lin}(x, Sx, \ldots, S^{n-1} x) = X_{n-1}(x) \quad (m \geq N)
\]

for an arbitrary \(x \in X\). This immediately implies \(b_0 \leq N\). The fact that \(P(t)\) is a characteristic polynomial implies \(N \geq b_0\). Hence \(N = b_0\).

**Sufficiency.** We write

\[
X_n = \{x \in X : \text{there exists a natural number } n_s \text{ such that} \ (S - aI)^{m} x = 0 \}
\]

\[
(S - aI)^{m} x = 0 .
\]

First, we prove that \(X_0 \cap X_0 = \{0\}\) for \(a \neq b\). Let us remark that operators \((S - aI)\) and \((S - bI)\) are commutative. Hence

\[
(S - aI)X_0 \subset X_0 \quad \text{for} \quad a \neq b .
\]

Let us suppose that there exists an element \(x \neq 0, x \in X_0 \cap X_0\). Hence there is a number \(n_s\) such that \((S - aI)^{n_s} x = 0\), but \(z = (S - aI)^{n_s-1} x \neq 0\). From condition (5.4) follows \(z \in X_0\).

On the other hand, \(S^{n_s} = a^{n_s} z_s\); hence

\[
(S - bI)^{m} z_s = (a - b)^{m} z_s \neq 0 \quad \text{for} \quad m = 0, 1, \ldots
\]

Thus, \(z_s \in X_0\), which contradicts \(z_s \in X_0\). Hence \(x = 0\).

Since \(b_0 = N\), the elements \(x, Sx, \ldots, S^N x\) are linearly dependent for an arbitrary \(x \in X\), \(x \neq 0\). Hence there exists a polynomial \(P(t)\) of degree \(N\) such that \(P(S)x = 0\). Let us factorize the polynomial \(P(S)\):

\[
P(S) = \prod_{i=1}^{n} (t - a_i) \quad (r_1 + \ldots + r_n = N) .
\]

Hence, by Theorem 5.1, \(x = \sum_{i=1}^{n} x_i\), where \((S - a_i I)^{r_i} x_i = 0\). Thus, \(x = X_{a_i}\).

We now show that the condition \(X_{a_i} \neq 0\) may hold only for a finite number of \(a_i\). Indeed, let us suppose that there are more than \(N\) numbers \(a_i\) satisfying the condition \(X_{a_i} \neq 0\). Let \(a_i \in X_{a_i}\) be elements different from zero such that \(S a_i = a_i x_i\), and let \(x = \sum_{i=1}^{N} x_i\). By hypothesis, \(b_0 = N\). Hence there exists a non-trivial polynomial \(W(t)\) of degree not greater than \(N\) for which \(W(S)x = 0\). But

\[
W(S) = \sum_{i=1}^{N} W(S) x_i = \sum_{i=1}^{N} W(a_i) x_i .
\]

Thus, \(W(a_i) = 0\) for \(i = 1, 2, \ldots, N + 1\). This proves the polynomial \(W(t)\) to be identically equal to zero, which is a contradiction. Hence there exist at most \(N\) spaces \(X_{a_i} \neq 0\).

It follows that every element of the space \(X\) can be written in the form

\[
x = \sum_{i=1}^{n} x_i, \quad \text{where} \quad x_i \in X_{a_i} \quad (i = 1, 2, \ldots, n \leq N) .
\]
2. There exist simple regularizers $E(t_i)$ of operators $A(t_i)$ to the ideal $J$, then the operator $A(S) + T$ has a finite $d$-characteristic for every $T \in J$. If, moreover, $J$ is a Fredholm ideal and the algebra $X(X)$ is regularizable to a certain Fredholm ideal $I \subseteq X(X)$, then

$$x_{A(S) + T} = x_{A(S)}$$

for every $T \in J$.

**Proof.** By Corollary 4.5 and by the assumptions, the operator $A(S)$ has a simple regularizer $R_{A(S)}$ to the ideal $J$. By Property 5.4, the operator $A(S) + T$ also has a simple regularizer to the ideal $J$, and $R_{A(S) + T} = R_{A(S)}$. Hence, by Theorem 6.1, I, the operator $A(S) + T$ has a finite $d$-characteristic for every operator $T \in J$.

If, moreover, $J$ and $I$ are both Fredholm ideals, and of the algebra $X(X)$ is regularizable to the ideal $J$, it follows from Theorem 6.3, I, that every operator $T \in J$ is a perturbation of the class $D_{A(S)} \cap X(X)$ of all operators from the algebra $X(X)$ with a finite $d$-characteristic, which does not change the index. Hence $x_{A(S) + T} = x_{A(S)}$ for all $T \in J$. ■

**Theorem 6.2.** Let $S$ be an algebraic operator in the algebra $X(X)$, with the characteristic polynomial $P(t) = \prod_{m=1}^{n}(t-e_{m})^{e_{m}}$, and let a polynomial $A(S)$ satisfy the following conditions:

1. $A_{m} \in X(X)$, $[A_{m}, S] \in J \subset X(X)$ ($m = 0, 1, \ldots, N-1$), where $J$ is a quasi-Fredholm ideal.

2. There exist operators $\tilde{A}(t_{i})^{(-1)}$ ($i = 1, 2, \ldots, n$).

Then the operator $A(S) + T$ has a finite $d$-characteristic for every $T \in J$. If, moreover, $J$ is a Fredholm ideal, and if the algebra $X(X)$ is regularizable to a certain Fredholm ideal $I$, then

$$x_{A(S) + T} = x_{A(S)}$$

for every $T \in J$.

**Proof.** It follows from Theorem 4.1 that the operator $A(S)$ has a simple regularizer $R_{A(S)}$, and the proof follows the same lines as that of Theorem 6.1. ■

**§ 7. Polynomials in algebraic operators with constant coefficients.** If an operator $S \in A_{q}(X)$ is algebraic, and if

$$a(S) = a_{0}S + \cdots + a_{N-1}S^{N-1},$$

where $a_{0}, \ldots, a_{N-1}$ are complex numbers, then of course $a_{0}S - S = 0$ for $i = 0, 1, \ldots, N-1$. Hence, if $a_{0}S \neq 0$ ($i = 1, 2, \ldots, n$), then, by Corollary 4.2, the operator $[a(S)]^{-1}$ exists and

$$[a(S)]^{-1} = R_{A(S)}.$$

However, in case of constant coefficients one can apply Theorem 5.1 and obtain results also if some $a(t_{i})$ are equal to zero.
Let us suppose that \( S \in L_d(X) \) is an algebraic operator with the characteristic polynomial \( P(t) = \prod_{i=1}^{n} (t - a_i)^{m_i} \) with \( a_i \in \mathbb{R} \).

We write
\[
E_{m}(t_i) = Z_{(S-t_i)\mathbb{N}} = \{ x \in X : (S-t_i)^m x_i = 0 \}
\]
for \( m = 0, 1, 2, \ldots \). It is easily seen that \( E_{0}(t_i) = \{ 0 \} \) and \( E_{m}(t_i) \cap E_{m+1}(t_i) = \{ 0 \} \).

Lemma 7.1. \( E_{m}(t_i) \neq E_{m+1}(t_i) \) for \( m = 0, 1, 2, \ldots \).

Proof. Let us suppose that there exists an index \( m_k \) such that
\[
E_{m_k}(t_i) = E_{m_k+1}(t_i) = \{ 0 \}.
\]
By Theorem 1.3, we obtain
\[
E_{m}(t_i) = E_{m+1}(t_i) \text{ for all } m_k \leq m \leq r_i.
\]
In particular,
\[
E_{m_k}(t_i) = E_{m_k+1}(t_i) = X_{i}.
\]

Hence \( m_k = r_i \), which contradicts the assumption. Thus we have
\[
E_{m_k}(t_i) \neq E_{m_k+1}(t_i) \text{ for } m = 0, 1, 2, \ldots , r_i - 1.
\]

Theorem 7.1. Let \( P_y \in E_0(t_i) \), where \( r_i \) is an arbitrarily fixed integer in the interval \( 0 \leq r_i \leq r_i - 1 \).

\[
y = \sum_{i=1}^{r_i} P_y, \text{ if and only if }\]
\[a^{(0)}(t_i) = 0 \]
for \( k = 0, 1, 2, \ldots , r_i - 1 \). This is equivalent to the solution \( a(S)y = 0 \) of the system of independent equations:
\[
a(S)y = 0 \quad (i = 1, 2, \ldots , n).
\]

Let us write \( y_i = P_y \), and let us suppose that conditions (7.1) are satisfied. If \( y \in E_0(t_i) \), then
\[
a(S)y = a(S)y = \sum_{m=0}^{r_i-1} \frac{1}{m!} a^{(m)}(t_i)(S-t_i)^m y_i
\]

\[
= \sum_{m=0}^{r_i-1} \frac{1}{m!} a^{(m)}(t_i)(S-t_i)^m y_i
\]

\[
= \sum_{m=0}^{r_i-1} \frac{1}{m!} a^{(m)}(t_i)(S-t_i)^m y_i = 0.
\]

Hence condition (7.1) is necessary.

Theorem 7.2. If an operator \( a(S) \) satisfies condition (7.1), then a necessary condition for the equation \( a(S)x = x \) to have a solution is
\[
(S-t_i)^{k-n} P_{x_i} x = 0 \quad (i = 1, 2, \ldots , n) \quad (x \in X).
\]

Proof. By Theorem 5.1, the equation \( a(S)x = x \) is equivalent to the system of independent equations:
\[
a(S)P_{x} = P_{x} \quad (i = 1, 2, \ldots , n).
\]

If \( x = P_{x} \) is a solution of the equation \( a(S)x = P_{x} \), then condition (6.1) implies
\[
(S-t_i)^{k-n} P_{x} x = (S-t_i)^{k-n} a(S)x
\]

\[
= (S-t_i)^{k-n} \sum_{k=0}^{r_i} \frac{1}{k!} a^{(k)}(t_i)(S-t_i)^k x_i
\]

\[
= \left[ \sum_{k=0}^{r_i} \frac{1}{k!} a^{(k)}(t_i)(S-t_i)^{k-n}(S-t_i)^k x_i \right] (S-t_i)^{k-n} x_i = 0.
\]
Hence condition (7.2) is necessary.

**Theorem 7.3.** If conditions (7.1) and (7.2) are satisfied, then the equation \( a(S)x = x_0 \) has a solution \( x \) if and only if

\[
(7.3) \quad (S - t_i I)^{n-m} P_{x_0} = \left[ \frac{x_i^{1}}{a^{(0)}(t_i)} \right]^{n-m} \sum_{k=0}^{m-1} (-1)^{k-m} \frac{a^{(k)}(t_i)}{a^{(0)}(t_i)} (S - t_i I)^k P_{x_0}
\]

\[
(m = 0, 1, ..., r_t - r_i - 1; \ i = 1, 2, ..., n),
\]

where \( a^{(k)}(t_i) \) is the determinant obtained by cancelling the \( (m+1) \)-st column and the \( (k+1) \)-st row in the determinant

\[
d_{m}(t_i) = \det_{\delta \in \mathbb{C}, 0 < \delta < \infty} a_{m}(t_i),
\]

\[
a_{m}(t_i) = \begin{cases} 0 & \text{for } k > m, \\ \frac{1}{m!} a^{(m)}(t_i) & \text{for } k \leq m. \end{cases}
\]

**Proof.** By Theorem 5.1, the equation \( a(S)x = x_0 \) is equivalent to the system of \( n \) independent equations

\[
a(S) P_{x} = P_{x_0} \quad (i = 1, 2, ..., n).
\]

Applying the operators \( (S - t_i I)^m \) \( (m = 0, 1, ..., r_t - r_i - 1) \) to both sides of the equation \( a(S) P_{x} = P_{x_0} \), we obtain the following system of equations:

\[
(7.4) \quad \sum_{k=0}^{m-1} \frac{1}{k!} \frac{a^{(k)}(t_i)}{a^{(0)}(t_i)} (S - t_i I)^k P_{x} = (S - t_i I)^m P_{x_0}
\]

\[
(m = 0, 1, ..., r_t - r_i - 1).
\]

Now, let us write

\[
u_j = (S - t_i I)^{m-j} P_{x}, \quad v_j = (S - t_i I)^{j} P_{x_0}
\]

\[(j = 0, 1, ..., r_t - r_i - 1).
\]

System (7.4) can be written in the following form:

\[
\sum_{k=0}^{m-1} \frac{1}{k!} \frac{a^{(k)}(t_i)}{a^{(0)}(t_i)} u_{k+m-n} = v_n \quad (m = 0, 1, ..., r_t - r_i - 1)
\]

or in the more convenient form:

\[
\sum_{k=0}^{m-1} \frac{1}{(m-j)!} \frac{a^{(m-j)}(t_i)}{a^{(0)}(t_i)} u_{m-j} = v_m \quad (m = 0, 1, ..., r_t - r_i - 1).
\]

All the elements of the principal diagonal of the determinant \( d_{m}(t_i) \) are equal to \( \frac{1}{m!} a^{(m)}(t_i) \neq 0 \), but the elements below the principal diagonal are equal to zero. Hence

\[
d_{m}(t_i) = \left[ \frac{x_i^{1}}{a^{(0)}(t_i)} \right]^{m-n} \neq 0.
\]

The solution of system (7.5) assumes the form

\[
u_{m} = \frac{1}{d_{m}(t_i)} \sum_{k=0}^{m-1} (-1)^{k-m} a_{m-k}(t_i) v_{k}
\]

\[(m = 0, 1, ..., r_t - 1).
\]

Hence conditions (7.3) follow immediately. Conversely, if there exists an element \( x \) satisfying conditions (7.3), one can make the same substitution and solve system (7.6) (with respect to elements \( v_{x} \)). The determinant of system (3.2) is different from zero, since it is the determinant of the linear transformation inverse to the linear transformation defined by formula (7.5). Hence system (7.6) has a solution \( (v_{1}, ..., v_{r_t - 1}) \) defined by formula (7.5). It follows that for \( m = 0 \) the equation

\[
\sum_{k=0}^{m-1} \frac{1}{k!} \frac{a^{(k)}(t_i)}{a^{(0)}(t_i)} (S - t_i I)^k P_{x} = P_{x_0} \quad (i = 1, 2, ..., n)
\]

is satisfied. But we supposed condition (7.1) to be satisfied. Hence \( x \) is a solution of the equation \( a(S)x = x_0 \).
§ 1. Conjugate spaces and conjugate operators. We denote by \( X' \) the space of all linear functionals defined on the space \( X \). A space \( \mathcal{S} \subseteq X' \) is total if the condition \( \xi(x) = 0 \) for all \( \xi \in \mathcal{S} \) implies \( x = 0 \). Evidently, the space \( X' \) is total. Indeed, let \( x \) be an arbitrary element of the space \( X \).\( x \neq 0 \). Let us denote by \( X_0 \) the one-dimensional space spanned by the element \( x \), i.e., \( X_0 = \{ y : y = ax, a \text{ being a scalar} \} \). We consider a functional \( f \) defined on the space \( X_0 \) by the formula \( f(ax) = a \). Evidently, it is a functional. By Theorem 1.1, I, this functional can be extended to a linear functional \( f \) defined on the whole space \( X \). Hence \( f \in X' \) and \( f(x) \neq 0 \). Since the element \( x \in X \) is arbitrary, the space \( X' \) is total.

Let us remark that elements \( x \in X \) can be treated as functionals defined on a total space \( \mathcal{S} \) by the formula

\[
F(x) = \xi(x).
\]

Thus if we denote by \( \mathcal{S} \) the space of all linear functionals defined on the space \( X \), the space \( X' \) can be mapped into the space \( \mathcal{S} \) monomorphically. This monomorphism will be called the natural embedding and will be denoted by \( i \). The image \( i(X) \) of the space \( X \) by means of this monomorphism is a total space of functionals on the space \( \mathcal{S} \), since \( i(x) = 0 \) for all \( x \neq 0 \).

Each total subspace of the space \( X' \) is called a conjugate space to \( X \) (adjoint space, dual space).

Let \( H \subseteq X' \) be a conjugate space. To any operator \( A \in L(X \rightarrow Y) \) corresponds an operator \( A^* \) defined on the space \( H \) with values in the space \( X' \):

\[
(\eta A)x = \eta(Ax) \quad (\text{for all } x \in H \text{ and for all } \eta \in H).
\]

The operator \( \eta A \) is called the \textit{conjugate operator} (adjoint operator) to the operator \( A \). We shall denote it by \( A^* \), i.e., \( A^* \eta = \eta A \). Let us remark that \( (A + B)^* = A^* + B^* \) and \( I^* = I \).

Let \( \mathcal{S} \subseteq X' \) be an arbitrary conjugate space. We consider the operator \( A^* \) as defined for those functionals \( \eta \in H \) for which \( A^* \eta = \eta A \). In this manner to every operator \( A \in L(X \rightarrow Y) \) there corresponds an operator \( A^* \in L(H \rightarrow \mathcal{S}) \). With this general formulation the operator \( A^* \) may happen to be defined only at the element \( 0 \).

In the sequel we shall consider only operators \( A \in L(X \rightarrow Y) \) for which \( A \mapsto L(H \rightarrow \mathcal{S}) \), i.e., operators \( A \in L(X \rightarrow Y) \) such that \( A^* \eta \in \mathcal{S} \) for every \( \eta \in H \). We denote the set of such operators by \( L(X \rightarrow Y, H \rightarrow \mathcal{S}) \).

Obviously, this is a linear space. Writing \( A \in L(X \rightarrow Y, H \rightarrow \mathcal{S}) \) we shall suppose automatically that we consider the conjugate operator \( A^* \) as an operator mapping the space \( H \) into the space \( \mathcal{S} \). We denote the space \( L(X \rightarrow Y, H \rightarrow \mathcal{S}) \) by \( L(X \rightarrow Y, H \rightarrow \mathcal{S}) \). Evidently, the space \( L(X \rightarrow Y, H \rightarrow \mathcal{S}) \) is an algebra, since \((AB)^* = B^* A^* \).

**Theorem 1.1.** A finite-dimensional operator \( E \subseteq \mathcal{S} \), where functionals \( f_i \in X' \) and elements \( x_i \in X \) are linearly independent, belongs to the set \( L(X \rightarrow Y, H \rightarrow \mathcal{S}) \) if and only if \( f_i(x) = 0 \) for all \( i \in \{1, 2, \ldots, n\} \).

**Proof.** Let \( f \in \mathcal{S} \). If \( f_i(x) = 0 \) for all \( i \in \{1, 2, \ldots, n\} \). On the other hand, if \( A^* f = \sum f_i(x_i) f_i \) for all \( f \in \mathcal{S} \), then one can find functionals \( \xi_i \in H \) such that \( \xi_i(x_i) = b_{ij} \), for the elements \( x_i \) are linearly independent and the space \( H \) is total. Hence \( A^* f = f \) belongs to \( \mathcal{S} \) by hypothesis.

Evidently, if \( A \in L(X \rightarrow Y, H \rightarrow \mathcal{S}) \), then \( A^* \in L(H \rightarrow \mathcal{S}, X \rightarrow Y) \). If a set \( E \) is a subset of the space \( Y \), we write

\[
E^\perp = \{ \eta \in H : \eta(y) = 0 \text{ for all } y \in E \}.
\]

The set \( E^\perp \) is called the \textit{H-orthogonal complement} of the set \( E \).

**Theorem 1.2.** If \( A \in L(X \rightarrow Y, H \rightarrow \mathcal{S}) \), then \( a_{H \perp} \leq \beta_A \).

**Proof.** Evidently,

\[
a_{H \perp} = \dim E^\perp \leq \beta_A.
\]

On the other hand, every functional \( \eta \in E^\perp \) induces a functional in the quotient space \( Y/E_A \). If \( \beta_A = \dim Y/E_A < +\infty \), then the dimension of the space of functionals \( (Y/E_A)^* \) is equal to the number \( \beta_A \). Hence \( a_{H \perp} \leq \beta_A \). If \( \beta_A = +\infty \), then the inequality in the theorem holds automatically.

We say that a subspace \( B \subseteq Y \) is \textit{H-describable} if \((B^\perp)^\perp = B \), where \((B^\perp)^\perp = \{ y \in Y : \eta(y) = 0 \text{ for all } \eta \in B^\perp \} \).

An operator \( A \in L(X \rightarrow Y, H \rightarrow \mathcal{S}) \) is said to be \textit{H-describable} if the set \( E_A = H \)-describable.

If an operator \( A \) is \( H \)-describable then, of course, \( \dim E^\perp_A = \beta_A \). Hence we have
A. III. $\Phi_X$-operators

**Theorem 1.3.** If an operator $A \in L_0(X \to Y, H \to \mathbb{F})$ is $H$-resolvable, then $a_A = \beta_A$.

**Corollary 1.4.** If an operator $A \in L_0(X \to Y, H \to \mathbb{F})$ is an epimorphism then the conjugate operator $A'$ is a monomorphism.

Let $E$ be a subspace of a linear space $X$. Let $\Phi_E$ be a map of the space $X$ into the quotient space $X/E$ such that to every element $x \in X$ there corresponds the coset $[x] = x + E$ containing $x$, i.e.

$$\Phi_E x = x + E \quad (x \in X).$$

Evidently, the operator $\Phi_E$ is linear.

**Corollary 1.5.** Let $E$ be a subspace of a linear space $X$, and let $\mathbb{F}$ be a conjugate space to $X$. If $H \subset (X/E)'$ is a subspace satisfying the condition $\mathcal{F} H \subset \mathbb{F}$, then the operator $\Phi_E$ is a monomorphism of the space $H$ into the space $E'$. \[\]

**Proof.** Evidently, the map $\Phi_E$ is an epimorphism. Hence the operator $\Phi_E^*$ is a monomorphism. Let $\xi \in \Phi_E^* H$, $\xi \in E'$. There exists an element $x \in E$ such that $\xi(x) \neq 0$, but $\xi = \Phi_E y$. Thus,

$$\xi(x) = \eta(\Phi_E x) = \eta(0) = 0,$$

which contradicts the condition $\xi(x) \neq 0$. \[\]

Each operator $A \in L_0(X \to Y)$ is $Y$-resolvable. Indeed, let $y_A$ be an arbitrary element of the space $Y$ not belonging to the set $E_A$. Let $x_A = \text{lin} (y_A + E_A)$, i.e. $x_A = (y_A + x : x \in E_A)$, $x$ being a scalar. We define the following functional: $\eta_A(y) = x$ for all $y \in X$. By Theorem 1.1, 1, the functional $\eta_A$ can be extended to the whole space $Y$. Let us remark that this extension is a functional equal to zero on the set $E_A$, just as the functional $\eta_A$. Hence this extension belongs to the set $E_A$. Thus to every $y \in E_A$ there exists a functional $\eta_A(y) \in E_A$ such that $\eta_A(y_A) = 1$. Hence $E_A = \{y \in Y : \eta(y) = 0\}$ for all functionals $\eta \in E_A$. \[\]

Hence we obtain the following

**Corollary 1.6.** If $A \in L_0(X \to Y, Y' \to X')$, then $a_A = \beta_A$.

Theorem 1.3 does not hold without the assumption of $H$-resolvability even if $X = Y$ and $\mathbb{F} = H$. This is shown by the following example:

**Example 1.1.** Let $X = Y = C^0[0, 1]$ be the space of functions $x(t)$ infinitely many times differentiable in the interval $[0, 1]$. Let $E = H$ be the space of functionals $\xi$ of the form $\xi(x) = \int_0^1 x(t, \xi(t)) dt$, where $\xi(t) \in C^0[0, 1]$ and $\xi^{(n)}(0) = 0$ ($n = 0, 1, 2, \ldots$). Here $\xi^{(n)}(t)$ denotes the $n$th derivative of the function $\xi(t)$. Finally, let $A(x(t)) = y(t) = \int_0^1 x(s) ds$.

§ 1. Conjugate spaces and conjugate operators

Evidently, $a_A = 0$ and $\beta_A = 1$. We calculate $\eta(Ax)$, where the functional $\eta$ is defined by means of function $\eta(t)$:

$$\eta(Ax) = \int_0^1 \eta(x(s) ds) = \int_0^1 \eta(y(t)) dt = \int_0^1 \eta(y(t)) dt$$

Hence the conjugate operator $A'$ maps the functional $\eta$ of the form

$$\eta(x) = \int_0^1 \eta(x(s) ds)$$

in the functional $\xi(t)$ of the form

$$\xi(t) = \int_0^1 \xi(t) y(t) dt,$$

where $\xi(t) = \int_0^1 \eta(s) ds$.

This operator is a one-to-one map of the space $H$ onto itself. Hence $a_d = -\beta_d = 0$ and $a_d = -\beta_d$.

Let $A \in L_0(X \to Y, H \to \mathbb{F})$. According to our convention the conjugate operator $A^*$ maps the space $H$ into the space $\mathbb{F}$. Spaces $X$ and $Y$ can be treated as spaces of functionals over spaces $H$ and $\mathbb{F}$ respectively. Then it immediately follows from the definition of the conjugate operator $A^*$ that the operator $A''$ conjugate to $A$ is the operator $A$ itself, and $A'' = L_0(H \to \mathbb{F}, X \to Y)$. Applying these arguments and changing the roles of operators $A$ and $A'$ in Theorem 1.3 we obtain

**Theorem 1.7.** If $A \in L_0(X \to Y, H \to \mathbb{F})$, then $a_A = \beta_A$.

In order to get a theorem dual to Theorem 1.3 we must explain what it means that the operator $A'$ is $X$-resolvable. We write

$$H = \{x \in X : \xi(x) = 0\}$$

Then

$$E_A = \{x \in X : \xi(x) = 0\}$$

But $\xi(e_A)$ implies $\xi = A^*$. Hence $\xi(x) = \eta(Ax)$ and since the space $H$ is total, we obtain $H = \{x \in X : Ax = 0\} = Z_A$.

Thus the following theorems hold:

**Theorem 1.8.** If $A \in L_0(X \to Y, H \to \mathbb{F})$ and $E_A = Z_A$, then $a_A = \beta_A$.

**Theorem 1.9.** If $A \in L_0(X \to Y, Y' \to X')$, then $a_A = -\beta_A$.

**Proof.** Let us decompose the spaces $X$ and $Y$ into direct sums:

$$A = Z_A \oplus G, \quad Y = E_A \oplus G.$$

The operator $A$ is a one-to-one map of the space $G$ onto the space $E_A$. Hence the operator $A'$ is a one-to-one map of the space of all functionals
§ 2. \( d_A \)-characteristic and \( \Phi_H \)-operators. It follows from the considerations of the previous section that we do not always have \( \kappa_A = -\kappa_A \).

Therefore, the \( H \)-index as the difference of numbers \( \beta_A = a_A \) and \( a_A \).

Hence, the \( H \)-index \( x_A \) of an operator \( A \in L(H) \) is defined by the equality

\[ x_A = \beta_A - a_A. \]

Since \( \beta_A \leq \beta_A \), we have \( x_A \leq \kappa_A \). The pair of numbers \((\beta_A, x_A)\) is called the \( \Phi_H \)-characteristic of the operator \( A \).

Evidently, the pair of numbers \((\beta_A, x_A)\) is the \( \Phi \)-characteristic of the operator \( A \) by definition.

Hence \( x_A = -x_A \).

![Diagram](image)

**Fig. 5**

As is shown by Example 1.1, the \( \Phi_A \)-characteristic is not always equal to the \( \Phi \)-characteristic, even if \( \Phi = \Phi_H \). By this example one can also show that a theorem on the index of a superposition analogous to Theorem 2.1, I, does not hold for the \( H \)-index. Indeed, if we take

\[ B = d/dt \text{ in Example 1.1}, \]

we get \( BA = -I \).

Hence \( x_B = x_B = 0 \).

On the other hand, \( x_B = x_B = 0 \).

Thus \( x_B = x_B = x_B + x_B \).

**Theorem 2.1.** If a finite-dimensional operator \( A \in L(H) \) is equal to its \( \Phi \)-characteristic, then the operator \( A \) is called a \( \Phi_H \)-operator.

**Theorem 2.2.** If a finite-dimensional operator \( K \in L(H) \) belongs to \( L(H, E) \), then the operator \( I + K \) is a \( \Phi_H \)-operator.

**Proof.** Let \( K = \sum_{i=1}^{n} \xi_i \xi_i(a_i) \). We have shown that (Theorem 3.1, I)

\[ a_{i+1} = \beta_{i+1} = n - k, \]

where \( k \) is the rank of the matrix \( \xi_i(a_i) \).

If we consider the conjugate operator \( I + K' \), where \( k' = \sum_{i=1}^{n} \xi_i(a_i) \xi_i(a_i) \), then of course \( a_{i+1} = \beta_{i+1} = n - k' \), where \( k' \) is the rank of the matrix \( \xi_i(a_i) \).

Hence \( \beta_{i+1} \) (equal to \( a_{i+1} \), by definition) is equal to \( a_{i+1} \).

We say that a subspace \( X_0 \) is described by a family \( \Sigma \) (not necessarily linear) of linear functionals defined on a space \( X \) if \( \xi \in X_0 \) holds if and only if \( \xi \in X_0 \) for all functionals \( \xi \in \Sigma \).

In other words, a subspace \( X_0 \) is described by a family \( \Sigma \) and only if \( \Sigma \) describes \( X_0 \).

An operator \( \alpha \in L(X, Y) \) is \( H \)-resolvable if and only if the set \( E_\Sigma \) of its values can be described by a family \( H \subset H \subset H \).

It is easily verified that an operator \( \alpha \in L(X, Y, Z) \) with a finite \( d \)-characteristic is a \( \Phi_H \)-operator if and only if the set \( E_\Sigma \) can be described by a finite system of functionals.

If a subspace \( X_0 \subset X \) can be described by a finite system of functionals \( \Sigma \subset \Sigma \), then of course every space \( X \) containing \( X_0 \subset X \subset X \subset X \) can also be described by a finite system of functionals \( \Sigma \subset \Sigma \subset \Sigma \).

**Theorem 2.3.** If \( A \in L(X, Y) \), then to every conjugate subspace \( E \) there exists a \( \Phi_H \)-operator \( B \in L(Y, X) \) such that the operators \( AB \) and \( BA \) are finite-dimensional in spaces \( Y \) and \( X \), respectively.

**Proof.** By hypothesis, the dimension of the space \( Z_0 \) is finite: \( \dim Z_0 = n \).

Let us consider a system of functionals \( \{f_1, ..., f_n\} \subset \Sigma \) to the space \( Z_0 \) linearly independent. Let \( C = \{x \in X: f_i(x) = 0, i = 1, 2, ..., n\} \).

We write the space \( X \) as a direct sum \( X = Z_0 \oplus C \).

The operator \( A \) restricted to the space \( C \) is invertible and maps \( C \) into \( C \).

Let \( A^{-1} \) be its inverse defined on the space \( E_2 \).

We write the space \( Y \) as a direct sum \( Y = E_1 \oplus E_2 \).

We define the operator \( B \) as follows:

\[ B = \begin{cases} \frac{A^{-1}}{y} & \text{for } y \in E_2, \\ 0 & \text{for } y \in E_1. \end{cases} \]

Since \( E_2 = C \) can be described by a finite system of functionals, \( B \) is a \( \Phi_H \)-operator.

On the other hand, \( AB \) is a projection \( P_1 \) on the subspace \( E_1 \), and \( BA \) is a projection \( P_2 \) on the subspace \( E_2 \).

Since \( A \) and \( B \) are finite, these operators differ from identity by a finite-dimensional operator only.

**Remark.** Since the space \( C \) can be described by a finite system of functionals belonging to the space \( C \), we have \( P_2 = I - K \), where
K ⊆ K(X, E). Similarly, if A is a \( \Phi_H \)-operator, then \( F_A = I - K \), where \( K = K(Y, H) \).

Let us remark that the operator B defined in Theorem 2.2 satisfies the following equalities:

\[
AB = A \quad \text{and} \quad BAB = B.
\]

Each operator B satisfying equalities (2.1) is called almost inverse to the operator A. It follows from the form of these equalities that if an operator B is almost inverse to an operator A, then the operator A is almost inverse to the operator B. Hence the following corollary can be formulated:

**Corollary 2.3.** If \( A \in D_0(X \rightarrow Y) \), then to every conjugate space \( E \) there exists a \( \Phi_X \)-operator \( B \in D_0(Y \rightarrow X) \) which is almost inverse to the operator A.

**Theorem 2.4.** If a \( \Phi_H \)-operator B belongs to \( L_0(X \rightarrow Y) \) and if a \( \Phi_X \)-operator \( A \) belongs to \( L_0(Y \rightarrow X) \), where \( H A \subset \Sigma \), then the superposition \( AB \) is also a \( \Phi_X \)-operator and

\[
\alpha_{AB} = \alpha_A + \alpha_B.
\]

**Proof.** Let us decompose the space Y into a direct sum of form (9.4), Chapter I, i.e., \( Y = E_0 \oplus E_1 \oplus \ldots \oplus E_m \). Since B is a \( \Phi_Y \)-operator, the subspace \( E_0 \) can be described by a finite system of functionals of dimension equal to the dimension of the space \( E_0 \oplus E_1 \oplus \ldots \oplus E_m \). Hence every space \( Y_0 \subset E_0 \) can be described by a finite system of functionals. In particular, the space \( E_0 \oplus E_2 \) can be described by a finite system of functionals \( \xi \) (i.e., i = 1, 2, ..., m). Since A is a \( \Phi_X \)-operator, the set \( E_2 \) can be described by a finite system of functionals \( \eta \). The inclusion \( H A \subset \Sigma \) and formula (2.9), Chapter I, imply

\[
E_A = E_0 \oplus E_2 \oplus \cdot \cdot \cdot \oplus E_m.
\]

Hence the set \( E_{AB} \) can be described by the system \( \xi_0, \ldots, \xi_m, \xi_1, \ldots, \xi_m \). Thus \( \alpha_{AB} = \alpha_A + \alpha_B \), and Theorem 2.1, I, gives (2.2). □

The following theorem is, in a sense, converse to Theorem 2.4:

**Theorem 2.5.** Let \( B \in L_0(X \rightarrow Y) \) and \( A \in L_0(Y \rightarrow Z) \). If \( \beta_{AB} < +\infty \) and \( \rho_{AB} = \rho_A^2 + \rho_B^2 \), then \( \beta_A < +\infty \) and \( \beta_B < +\infty \).

**Proof.** By Theorem 2.3, I, \( \beta_{AB} < +\infty \) implies \( \beta_A < +\infty \).

By hypothesis, the set \( E_A \) can be described by a finite system of functionals. Hence every space containing \( E_{AB} \), in particular \( E_A = E_0 \oplus \cdot \cdot \cdot \oplus E_m \), (see 2.5, Chapter I), can be described by a finite system of functionals. Thus \( \beta_A = \beta_B^2 \). □

From Theorem 2.5 and Corollary 2.3, I, follows at once

**Corollary 2.6.** If \( A \in L_0(X \rightarrow Y) \) and \( B \in L_0(Y \rightarrow X) \), and if \( AB \) is a \( \Phi_X \)-operator and \( BA \) is a \( \Phi_Y \)-operator, then \( A \) and \( B \) are a \( \Phi_X \)-operator and a \( \Phi_Y \)-operator, respectively.

**Corollary 2.7.** If \( T \in L_0(X) \) and if there exists a positive integer \( m \) such that the operator \( I - T^* \cdot \cdot \cdot + T^{m-1} \) is a \( \Phi_X \)-operator, then the operator \( I - T \) is also a \( \Phi_X \)-operator.

Indeed, it is sufficient to apply the previous corollary with \( A = I - T \), \( B = I + T + \cdot \cdot \cdot + T^{m-1} \). □

**§ 3. \( (E, H) \)-quasi-Fredholm ideals.** Let us write

\[
L_0(X \rightarrow Y, H \rightarrow E) = \left( L_0(X, E), L_0(Y \rightarrow X, H \rightarrow E), L_0(Y, H) \right).
\]

Evidently, the system \( L_0(X \rightarrow Y, H \rightarrow E) \) is a paraideal of operators. Suppose we are given two arbitrary algebras

\[
A_0(X, E) \subset L_0(X, E), \quad A_0(Y, H) \subset L_0(Y, H)
\]

and spaces of operators

\[
S_0(X \rightarrow Y, H \rightarrow E) \subset L_0(X \rightarrow Y, H \rightarrow E),
\]

\[
S_0(Y \rightarrow X, H \rightarrow E) \subset L_0(Y \rightarrow X, H \rightarrow E).
\]

Let us consider the paraideal

\[
P(X \rightarrow Y, H \rightarrow E) = \left( A_0(X, E), S_0(X \rightarrow Y, H \rightarrow E), A_0(Y, H) \right).
\]

We denote by \( E_0(X \rightarrow Y, H \rightarrow E) \) the set of all finite-dimensional operators belonging to the paraideal \( P(X \rightarrow Y, H \rightarrow E) \).

We say that an ideal \( I \subset P(X \rightarrow Y, H \rightarrow E) \) is a \( (E, H) \)-quasi-Fredholm ideal if

\[
\begin{cases}
I + T & \text{is a } \Phi_X \text{-operator for all } T \in J \cap A_0(X, E), \\
I + T & \text{is a } \Phi_H \text{-operator for all } T \in J \cap A_0(Y, H).
\end{cases}
\]

**Theorem 3.1.** If an operator \( A \in P(X \rightarrow Y, H \rightarrow E) \) has a simple regulariser \( R_A \) to a \( (E, H) \)-quasi-Fredholm ideal \( J \subset P(X \rightarrow Y, H \rightarrow E) \), then \( A \) is either a \( \Phi_X \)-operator or a \( \Phi_H \)-operator.

**Proof.** Indeed, under the assumptions of the theorem, operators \( A B_A \) and \( B_A A \) are \( \Phi_X \)-operators or \( \Phi_H \)-operators. By Corollary 2.5, both operator \( A \) and operator \( R_A \) are \( \Phi_X \)-operators or \( \Phi_H \)-operators, respectively. □
A. III. \(\Phi_2\)-operators

**Theorem 3.2.** Every quasi-Fredholm ideal contained in a regularizable paraalgebra \(P(X = Y, H = \mathbb{E}) \subseteq (\mathbb{E}, H)\)-quasi-Fredholm.

**Proof.** Let either \(T \in J \cap A(H, H)\) or \(T \in J \cap \mathcal{A}(H)\). Hence the operator \(a + T\) has a finite \(d\)-characteristic. If the assumption of regularizability of the paraalgebra \(P(X = Y, H = \mathbb{E})\) implies that the operator \(a + T\) has a simple regularizer to the ideal \(K_{a + T}(X = Y, H = \mathbb{E})\). But this ideal is \((\mathbb{E}, H)\)-quasi-Fredholm (Theorem 2.1). By Theorem 3.1, the operator \(a + T\) is a \(\Phi_2\)-operator (or a \(\Phi_\mathbb{E}\)-operator).

**Theorem 3.3.** If an algebra of operators \(\mathcal{K}(X)\) contains the ideal \(K(X, \mathbb{E})\), and if the operator \(I + K\) is a \(\Phi_2\)-operator for every finite-dimensional operator \(K \in \mathcal{K}(X)\), then \(\mathcal{K}(X) \subseteq L(X, \mathbb{E})\).

**Proof.** Let us suppose that the operator \(T \in \mathcal{K}(X)\) does not preserve the conjugate space \(\mathbb{E}\), i.e. there exists a functional \(\xi \in \mathbb{E}\) such that \(\eta = T\xi \notin \mathbb{E}\). Let the operator \(K \in \mathcal{K}(X, \mathbb{E})\) be of the form \(K \xi = \xi \eta \xi\).

Then

\[PT = \xi(T\xi)\eta = (\eta(\eta)\eta)\eta\eta = 0\]

We choose \(\eta \xi\) such in a manner that \(\eta(\eta)\eta \neq 0\). We take

\[K\xi = \eta(\eta)\eta\eta\eta\eta\eta\eta = 0\]

Then \(K\) is a finite-dimensional operator belonging to \(\mathcal{K}(X)\). But

\[\mathcal{K}(X) \subseteq L(X, \mathbb{E})\]

Indeed,

\[\eta(\eta - \xi)\xi = \eta(\eta)\eta\eta\eta\eta\eta - \eta(\eta)\eta\eta\eta\eta\eta = 0\]

However, \(\eta \notin \mathbb{E}\). Hence \(I + K\) is not a \(\Phi_2\)-operator, which contradicts the assumption. Thus, every operator \(T \in \mathcal{K}(X)\) preserves the conjugate space \(\mathbb{E}\).

Let a paraalgebra \(P(X = Y, H = \mathbb{E})\) be given. Denote by \(P'(H = \mathbb{E}, X = Y)\) the set of operators conjugate to operators belonging to the paraalgebra \(P(X = Y, H = \mathbb{E})\). It is easily verified that this set is a paraalgebra. If \(J\) is an ideal in the paraalgebra \(P(X = Y, H = \mathbb{E})\), then the set \(J'\) of operators conjugate to operators belonging to the ideal \(J\) is an ideal in the paraalgebra \(P'(H = \mathbb{E}, X = Y)\). Indeed, if \(T', T'' \in J'\), \(A \in P'(H = \mathbb{E}, X = Y)\), then

\[a_1 T' + a_2 T'' = (a_1 T_1 + a_2 T_2)\in J', \quad (a_1, a_2\text{ numbers}), \]

\[A T' = (A T_1)\in J', \quad T'' A = (T''_1 T''_2)\in J'.\]

Hence, if \(R\) is a left regularizer of the operator \(A\) to the ideal \(J\), by applying the equality

\[A R = (R A) = (I + T') = I + T', \quad \text{where} \quad T' \in J',\]

we find that \(R'\) is a right regularizer of the operator \(A'\) to the ideal \(J'\) (an analogous implication holds for a right regularizer). Taking into account the fact that two simple regularizers to an ideal differ by a term belonging to the ideal only (Property 5.2, I), we find that a simple regularizer of an operator \(A\) conjugate to \(A\) satisfies the equality

\[R (A) = (R A)\]

§ 4. Perturbations of \(\Phi_2\)-operators. The following theorem, analogous to Theorems 3.2, I, and 4.2, I, hold for perturbations of \(\Phi_2\)-operators:

**Theorem 4.1.** If \(A \in D(X = Y)\) is a \(\Phi_2\)-operator, then every operator \(K \in K(X = Y, H = \mathbb{E})\) is an \(\Phi_2\)-perturbation of the operator \(A\).

**Proof.** Since \(K \in K(X = Y, H = \mathbb{E})\), we have \(K \xi = \sum g_i(x)\xi_i\), where \(g_i \in \mathbb{E}\). Evidently, \(A + K \in D(X = Y)\) (Theorem 3.2, I). Let \(C = (x: g_i(\Delta x) = 0\text{ for } i = 1, 2, \ldots, n)\). If \(x \notin C\),

\[(A + K) = \Delta x\]

But

\[A C = \Delta x \quad (y: g_i(y) = 0\text{ for } i = 1, 2, \ldots, n)\]

On the other hand, \(A\) is a \(\Phi_2\)-operator, by hypothesis. Hence \(A C\) can be described by a finite system of functionals. Consequently, \(A C\) can also be described by a finite system of functionals. But \(E_{\Delta x} \subset A C\). Hence \(E_{\Delta x} \subset A C\). This proves \(A + K\) to be a \(\Phi_2\)-operator.

**Theorem 4.2.** If \(J\) is an \((\mathbb{E}, H)\)-quasi-Fredholm ideal in a paraalgebra \(P(X = Y, H = \mathbb{E})\) regularizable to this ideal, the operators belonging to \(J\) are perturbations of the class of all \(\Phi_2\)-operators and \(\Phi_\mathbb{E}\)-operators belonging to the paraalgebra \(P(X = Y, H = \mathbb{E})\).

**Proof.** Let \(A \in P(X = Y, H = \mathbb{E})\) be a \(\Phi_2\)-operator (or a \(\Phi_\mathbb{E}\)-operator). By hypothesis, there exists a simple regularizer \(A\) of the operator \(A\) to the ideal \(J\), i.e.

\[R = a_1 + T\]

\[A R = a_1 + T, \quad T \in J, \quad (i = 1 \text{ or } 2)\]

Let \(T \in J\). If the addition \(A + T\) is performable, then

\[R(A + T) = R(a_1 + T) + T\]

But the set \(J\) is an ideal. Hence \(R a_1 + T, T R a_1 + T\in J\). Since \(J\) is a \((\mathbb{E}, H)\)-quasi-Fredholm ideal, the operators \(R a_1 + T\) and \((A + T) R a_1\) are \(\Phi_2\)-operators or \(\Phi_\mathbb{E}\)-operators. By Corollary 2.5, the operator \(A + T\) is a \(\Phi_2\)-operator or a \(\Phi_\mathbb{E}\)-operator, respectively.

**Theorem 4.3.** Theorem 4.2 remains true if we replace the assumption of regularizability of the paraalgebra \(P(X = Y, H = \mathbb{E})\) to the ideal \(J\) by the
assumption of regularizability of this paralgebra to an arbitrary \((\mathcal{E}, H)\)-quasi-Fredholm ideal \(J_1\), contained in \(P(X = Y, H = H)\).

Proof. Let \(\tilde{J} = J + J_1\). As in the proof of Theorem 4.3, I, we show \(\tilde{J}\) to be a \((\mathcal{E}, H)\)-quasi-Fredholm ideal. By Theorem 4.2, operators belonging to the ideal \(\tilde{J}\), in particular operators belonging to the ideal \(J\), are perturbations of the class of all \(\Phi_\eta\) and \(\Phi_\mu\)-operators belonging to the paralgebra \(P(X = Y, H = H)\).

Remark. Since in case of \(\Phi_\eta\)-operators the \(E\)-index is equal to the index, by applying Theorem 4.2, I, one can prove that if \(J\) and \(J_1\) are \((\mathcal{E}, H)\)-Fredholm ideals, then perturbations by means of operators belonging to the ideal \(J\) do not change the \(E\)-index and the \(H\)-index.

§ 5. Theorems on reduction of the space of functionals. As is shown by Example 1.1, not all \(\eta\)-characteristics are equal if we vary the conjugate space \(\mathcal{E}\). The following theorem shows that some changes of spaces \(X\) and \(\mathcal{E}\) leave the \(\eta\)-characteristic unchanged.

\[
\mathcal{X}_A - \mathcal{X}_E = \mathcal{E}_A - \mathcal{E}_E.
\]

Fig. 6

**Theorem 5.1 (First Theorem on Reduction).** Let a linear space \(X\) and a conjugate space \(\mathcal{E}\) be given. Suppose that an operator \(T \in L(X)\) is such that the operator \(\Lambda = I + T\) has a finite \(\eta\)-characteristic. Let \(X_0\) be an arbitrary subspace of the space \(X\), containing \(TX\), and \(\mathcal{E}_0\) an arbitrary subspace of the space \(\mathcal{E}\), containing \(\mathcal{E}T\). Then the operator \(\Lambda\) restricted to the space \(X_0\) has a finite \(\eta\)-characteristic equal to the \(\eta\)-characteristic of the operator \(\Lambda\) on the whole space \(X\).

This follows immediately from the fact that all solutions of the equation \((I + T)x = 0\) in the space \(X\) belong to \(X_0\). Similarly, all solutions of the equation \(\mathcal{E}(I + T) = 0\) in the space \(\mathcal{E}\) belong to \(\mathcal{E}_0\).

Applying Theorem 5.1 one can prove the following

**Theorem 5.2 (Second Theorem on Reduction).** Let \(X_0\) be a subspace of a linear space \(X\), and let \(\mathcal{E}_0\) be a subspace of a space \(\mathcal{E}\) conjugate to \(X\). If an operator \(\Lambda \in L(X_0, \mathcal{E}_0)\) has a simple regularizer \(R_\Lambda\) such that

\[
AR_\Lambda = I + T, \quad R_\Lambda \Lambda = I + T_1,
\]

where the operators \(T, T_1\) can be extended to operators \(\tilde{T}, \tilde{T}_1 \in L(X, \mathcal{E})\), and the operators \(I + \tilde{T}, I + \tilde{T}_1\) are \(\Phi_\eta\)-operators, then the operator \(\Lambda\) is a \(\Phi_\eta\)-operator.

Proof. Operators \(I + \tilde{T}, I + \tilde{T}_1\) may be considered on the whole space \(X\). By hypothesis, \(I + \tilde{T}\) and \(I + \tilde{T}_1\) are \(\Phi_\eta\)-operators. By the First Theorem on Reduction (Theorem 5.1), the operators \(I + T\) and \(I + T_1\) are \(\Phi_\eta\)-operators. Hence the operators \(AR_\Lambda\) and \(R_\Lambda \Lambda\) are also \(\Phi_\eta\)-operators. By Corollary 2.5, the operator \(\Lambda\) is a \(\Phi_\eta\)-operator.
CHAPTER IV

DETERMINANT THEORY OF $\Phi_n$-OPERATORS

§ 1. Almost inverse operators. Let us remember that an operator $B \in I_0(X \rightarrow X)$ is called almost inverse to an operator $A \in I_0(X \rightarrow X)$ if

$$ABA = A, \quad BAB = B$$

(compare § 2, III).

Theorem 1.1. If an operator $B$ is almost inverse to an operator $A \in I_0(X \rightarrow Y)$, then the general form of the solution of the equation

$$Ax = x_0, \quad x_0 \in E_A,$$

is

$$x = x_0 + Bx_0, \quad x_0 \in Z_A.$$

Proof. Let $x \in X$ be a solution of equation (1.1). Then $Ax = ABx = ABx_0$. But the equality $Ax = ABx_0$ holds if and only if $x = x_0 + Bx_0$.

If $A$ is a $\Phi_n$-operator mapping the space $X$ onto itself, then Theorem 1.1 can be formulated in another way. Let $(z_1, \ldots, z_n)$ be a basis of the space $E_A$, and $(z_1, \ldots, z_m)$ a basis of the space $E_{\mu}$, where $E = A' \xi$ ($\xi \in \mathbb{S}$). Let $\eta_i$ ($i = 1, 2, \ldots, n$) be linearly independent functionals, and $y_i$ ($i = 1, 2, \ldots, m$) linearly independent elements such that

$$\eta_i(y_j) = \delta_{ij} \quad (i, j = 1, 2, \ldots, n) \quad \text{and} \quad \eta_i(y_j) = 0 \quad \text{if} \quad y_j \in E_A,$$

$$\xi_i(y_j) = \delta_{ij} \quad (i = 1, 2, \ldots, m) \quad \text{and} \quad \xi_i(y_j) = 0 \quad \text{if} \quad x_i \in E_{\mu}.$$  

Evidently, each element of the space $X$ can be written in the form

$$x = \sum_{i=1}^{n} a_i y_i + y_0, \quad y_0 \in E_A,$$

and each element of the space $\mathbb{S}$ can be written in the form

$$\xi = \sum_{i=1}^{m} a_i \eta_i + \eta_0, \quad \eta_0 \in E_A,$$

(coefficients $a_i$ being scalars).

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Theorem 1.2. If a $\Phi_n$-operator $A$ maps the space $X$ onto itself, and if $B$ is an operator almost inverse to $A$, then

(1) each solution of the equation

$$Ax = x_0, \quad \xi_i(x_0) = 0 \quad (i = 1, 2, \ldots, m)$$

is of the form

$$x = a_0 z_0 + \ldots + a_n z_0 + Bz_0;$$

the element $Bz_0$ is the only solution of the equation (1.5) satisfying the conditions

$$\eta_i(Bz_0) = 0 \quad (i = 1, 2, \ldots, m);$$

(2) each solution of the equation

$$\xi A = \xi_0, \quad \xi_i(a_0) = 0 \quad (i = 1, 2, \ldots, n),$$

is of the form

$$x = a_0 z_0 + \ldots + a_m z_m + \xi_0 B;$$

the element $\xi_0 B$ is the only solution of equation (1.6) satisfying the conditions

$$\xi_i(B) = 0 \quad (i = 1, 2, \ldots, m).$$

§ 2. Determinant system of a $\Phi_n$-operator. Let $X_1, \ldots, X_k$ be linear spaces. A $k$-linear functional is a map of the product $X_1 \times \ldots \times X_k$ into the set of scalars, $f(x_1, \ldots, x_k)$, such that for any fixed $k-1$ elements $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k$, the function $f(x_1, \ldots, x_{i-1}, x', x_{i+1}, \ldots, x_k)$ is a linear functional of the element $x_i$ ($i = 1, 2, \ldots, k$). We write

$$X^n = X \times X \times \ldots \times X.$$

Let $A \in I_0(X, \mathbb{S})$; then the expression $\xi(Ax)$ is a bilinear functional defined on the product $X \times \mathbb{S}$. We say that a bilinear functional $f$ belongs to the space $I_0(X, \mathbb{S})$ if there exists an operator $A \in I_0(X, \mathbb{S})$ for which

$$f(x, \xi) = \xi(Ax) \quad (x \in X, \xi \in \mathbb{S}).$$

The set of all $\Phi_n$-operators belonging to the ring $I_0(X, \mathbb{S})$ will be denoted by $\Phi_n(X, \mathbb{S})$.

The order of an operator $A \in \Phi_n(X, \mathbb{S})$ is the number $r_A = \min(a_0, a_{\mu})$.

We write

$$d_A = |d_A| = |a_{\mu} - a_0|.$$

If an operator $A \in \Phi_n(X, \mathbb{S})$ has a non-positive index, $a_{\mu} = a_0 \leq 0$, then every infinite sequence

$$(D_0, D_1, \ldots)$$

is a linearly independent sequence in $\mathbb{S}$.
satisfying the following five conditions is called a determinant system for the operator $A$:

$(1) \ D_k$ is a $(2n + d)$-linear functional defined on the product $S^{n+d} \times X^n$; we denote the value of this functional at the point $(\xi_1, \ldots, \xi_{n+d}, x_1, \ldots, x_n)$ by

$$
D_k(\xi_1, \ldots, \xi_{n+d}, x_1, \ldots, x_n)
$$

in particular, $D_k$ is a $d$-linear functional, and if $d = 0$, $D_k$ is a number.

$(2) \ D_k$ is skew-symmetric with respect to variables $\xi_i$ and $x_i$, i.e. for every permutation $p = (p_1, \ldots, p_{n+d})$ of numbers $1, \ldots, n + d$ and every permutation $q = (q_1, \ldots, q_n)$ of numbers $1, \ldots, n$, the following equalities hold:

$$
D_k(\xi_{p_1}, \ldots, \xi_{p_{n+d}}, x_1, \ldots, x_n) = \text{sgn} \, p \cdot D_k(\xi_{q_1}, \ldots, \xi_{q_n}, x_1, \ldots, x_n),
$$

$$
D_k(\xi_{q_1}, \ldots, \xi_{q_n}, x_{p_1}, \ldots, x_{p_{n+d}}) = \text{sgn} \, q \cdot D_k(\xi_{q_1}, \ldots, \xi_{q_n}, x_1, \ldots, x_n),
$$

where $\text{sgn} \, p = 1$ if the permutation $p$ is even, and $\text{sgn} \, p = -1$ if it is odd.

$(3) \ If \ n > 0$, then $D_k$ considered as a bilinear functional of arbitrarily chosen variables $x_i$ and $\xi_i$ belongs to the space $L^d(\Omega, X)$.

$(4) \ There \ exists \ a \ constant \ r \geq 0$ such that $D_k$ is not identically equal to zero.

$(5) \ The \ following \ equalities \ hold \ for \ n = 0, 1, \ldots$:

\[ D_{n+1} \left( \xi_1, A \xi_2, \xi_3, \ldots, \xi_{n+d} \right) \]

\[ = \sum_{\xi_0=0}^n (-1)^n \xi_0(\xi_0) \cdot D_k(\xi_1, \xi_2, \ldots, \xi_{n+d}), \]

\[ D_{n+1} \left( \xi_1, \xi_2, \xi_3, \ldots, \xi_{n+d} \right) \]

\[ = \sum_{\xi_0=0}^n (-1)^n \xi_0(\xi_0) \cdot D_k(\xi_1, \xi_2, \xi_3, \ldots, \xi_{n+d}). \]

If $a > 0$, the determinant system $(D_k)$ is defined analogously, but we change the roles of spaces $X$ and $\Omega$, i.e. we consider the product $S^X X^{n+d}$.

§ 2. Determinant system of a $\Phi_\alpha$-operator

The constant $r$ determined by condition (4) is called the order of the determinant system $r = r(D_k)$, and the number $d = d(D_k)$ is called the defect of the determinant system. $D_k$ is called the determinant of the operator $A$, and $D_k$ for $n = 1$ the minors of order $n$ of the operator $A$.

The following properties are easily verified:

**Property 2.1.** If $x_i = x_j$ or $\xi_i = \xi_j$ for $i \neq j$, then

$$
D_k(\xi_1, \ldots, \xi_{n+d}, x_1, \ldots, x_n) = 0.
$$

**Property 2.2.** If $(D_k)$ is a determinant system of an operator $A \in \Phi_\alpha(X, \Omega)$ and $\alpha \neq 0$, then $(cD_k)$ is also a determinant system for the operator $A$, and $\left\{ \frac{1}{\alpha} \right\}_\alpha$ is a determinant system for the operator $cA$.

**Property 2.3.** If $(D_k)$ is a determinant system for the operator $A \in \Phi_\alpha(X, \Omega)$ and if the operator $B \in L^d(\Omega, X)$ has an inverse $B^{-1} \in L^d(\Omega, X)$, then

$$
D_k(\xi_1, B^{-1} \xi_2, \ldots, \xi_{n+d})
$$

is a determinant system for the operator $AB$ and

$$
D_k(B^{-1} \xi_1, \ldots, B^{-1} \xi_{n+d})
$$

is a determinant system for the operator $BA$.

**Theorem 2.1.** If an operator $A \in \Phi_\alpha(X, \Omega)$ has an almost inverse operator $B \in L^d(\Omega, X)$ and if $r = 0$, then the system $(\theta_k)$ defined by the formula

$$
\theta_k(\xi_1, \ldots, \xi_{n+d}), (\xi_1, \ldots, \xi_{n+d})
$$

is a basis of the space $L^d(\Omega)$ (in this case $d = d(\theta_k)^* = -s \alpha$s). The system $(\theta_k)$ is a system of generalized inverses of $A$.

**Proof.** It is easily seen that the system $(\theta_k)$ satisfies conditions (1)-(4) defining the determinant system. We show that condition (5) is also satisfied. It follows from the contraction of the almost inverse operator $B$ (Corollary 2.3, III) that the following equalities hold:

$$
AB = I, \quad BA = I - K_a.
$$
where the $a_d$-dimensional operator $\mathcal{K}_d$ is a projection operator on the subspace $\mathcal{Z}_d$. Hence we obtain condition (2.1) expanding the determinant

$$
\left| \begin{array}{cc}
\xi \left( a_1 \right) & \xi \left( a_2 \right) \\
\xi \left( B_{a_1} \right) & \xi \left( B_{a_2} \right) \\
\xi \left( a_3 \right) & \xi \left( a_4 \right) \\
\xi \left( a_5 \right) & \xi \left( a_6 \right)
\end{array} \right| = \cdot \cdot \cdot 
$$

with respect to the first row, and condition (2.2) expanding the determinant

$$
\left| \begin{array}{cc}
\xi \left( a_1 \right) & \xi \left( B_{a_1} \right) \\
\xi \left( a_2 \right) & \xi \left( B_{a_2} \right) \\
\xi \left( a_3 \right) & \xi \left( a_4 \right) \\
\xi \left( a_5 \right) & \xi \left( a_6 \right)
\end{array} \right| = \cdot \cdot \cdot 
$$

with respect to the first column.

In an analogous manner we obtain the dual theorem:

**Theorem 2.2.** If an operator $A \in \mathcal{F}_d(X, E)$ has an almost inverse operator $B \in \mathcal{L}_d(X, E)$ and if $r_d = 0$, then the system $\left( \theta_n \right)$ defined by the formula

$$
\theta_n \left( \xi_1, ..., \xi_n \right) = \left| \begin{array}{cc}
\xi_n \left( B_{a_1} \right) & \xi_n \left( B_{a_2} \right) \\
\xi_n \left( a_3 \right) & \xi_n \left( a_4 \right) \\
\xi_n \left( a_5 \right) & \xi_n \left( a_6 \right)
\end{array} \right| 
$$

is a determinant system for the operator $A$ if $a_d = 0$. Here $(\xi_1, ..., \xi_n)$ is a basis of the space $\mathcal{Z}_d'$ (in this case $d = a_d = a_d'$).

**Corollary 2.3.** If $A \in \mathcal{F}_d(X, E)$ and if $r_d = 0$ and $r_d = 0$, then the operator $A^{-1} \in \mathcal{L}_d(X, E)$ exists, and the determinant system for the operator $A$ is of the form

$$
D_a = 1; \quad D_a \left( \xi_1, ..., \xi_n \right) = \det \xi \left( A^{-1}x_0 \right) \quad (a = 1, 2, ..., n).
$$

In particular, the determinant system for the identity operator is the following one:

$$
I_d = 1; \quad I_d \left( \xi_1, ..., \xi_n \right) = \det \xi \left( x_0 \right) : \quad (3.1)
$$

### § 3. Determinant System of a $\mathcal{F}_d$-Operator

**Corollary 2.4.** If $\mathcal{S} \in \mathcal{L}_d(X)$ is an algebraic operator with the characteristic polynomial $P(t) = \prod_{i=1}^n \left( t - \alpha_i \right)$, it is an operator

$$
\Delta \left( \mathcal{S} \right) = \sum_{n=0}^{N-1} A_n \mathcal{S}^n
$$

satisfies the following conditions:

(i) $A_n \mathcal{S} - \mathcal{S} A_n = 0$,

(ii) there exist operators $\left[ \Delta \left( \mathcal{S} \right) \right]^{-1} (i = 1, 2, ..., n)$,

then there exists a determinant system for the operator $\Delta \left( \mathcal{S} \right)$ and it is of the form

$$
D_a = 1; \quad D_a \left( \xi_1, ..., \xi_n \right) = \det \xi \left( \mathcal{R} \left( \mathcal{S} a_i \right) \right) \quad (n = 1, 2, ..., n),
$$

where $\mathcal{R} \left( \mathcal{S} \right)$ is a simple regularizer of the operator $\Delta \left( \mathcal{S} \right)$.

**Proof.** By Corollary 4.6, $\Pi$ implies the existence of the operator $\Delta \left( \mathcal{S} \right)^{-1} = \mathcal{R} \left( \mathcal{S} \right)$ (the operator $\mathcal{R} \left( \mathcal{S} \right)$ being determined effectively). By Theorem 3.1, $\Pi$, we have $\Delta \left( \mathcal{S} \right) \in \mathcal{F}_d(X, E)$ for an arbitrary family $\mathcal{E}$. By Corollary 2.3, the operator $\Delta \left( \mathcal{S} \right)$ has a determinant system ($D_a$) of the form described above.

**§ 3. Connection between the determinant system and the solutions of equations.** We shall investigate the question which operators possess a determinant system.

**Theorem 3.1.** If an operator $A \in \mathcal{L}_d(X, E)$ has a determinant system ($D_a$), then $A \in \mathcal{F}_d(X, E)$. Moreover,

$$
r_d = r \left( D_a \right) = r; \quad d = d \left( D_a \right) = d
$$

and

$$
\mathbf{a} = \mathbf{a} + \mathbf{d}, \quad \mathbf{a}_d = \mathbf{a}_d,
$$

and if $\eta_1, ..., \eta_r \in \mathcal{E}$, $y_1, ..., y_r \in X$ are elements satisfying the condition

$$
D_r \left( \eta_1, ..., \eta_r \right) \neq 0,
$$

then there exist elements $\xi_1, ..., \xi_n \in \mathcal{E}$ and $\xi_1, ..., \xi_n \in X$ such that

$$
(3.1) \quad \xi \left( \mathcal{S} \right) = \left[ \begin{array}{c}
\eta_1 \left( y_1 \right) \\
\eta_2 \left( y_2 \right) \\
\vdots \\
\eta_r \left( y_r \right)
\end{array} \right]
$$

for every $\mathbf{a} \in X$. 

**Proof.** By Corollary 2.3, $A \in \mathcal{F}_d(X, E)$ and $\left[ \Delta \left( \mathcal{S} \right) \right]^{-1} = \mathcal{R} \left( \mathcal{S} \right)$ (the operator $\mathcal{R} \left( \mathcal{S} \right)$ being determined effectively). By Theorem 3.1, we have $\Delta \left( \mathcal{S} \right) \in \mathcal{F}_d(X, E)$ for an arbitrary family $\mathcal{E}$. By Corollary 2.3, the operator $\Delta \left( \mathcal{S} \right)$ has a determinant system ($D_a$) of the form described above.
and

$$D_2 \begin{pmatrix} \eta_1, ..., \eta_{r+d} \\ \gamma_1, ..., \gamma_r \end{pmatrix} = \begin{pmatrix} \xi_{i+1}, ..., \xi_{i+r} \\ \xi_{i+1}, ..., \xi_{i+r} \end{pmatrix}$$

for every $\xi \in \mathcal{E}$.

Elements $\xi_i (i = 1, 2, ..., r)$ form a basis of the space $Z_n$ and elements $x_i (i = 1, 2, ..., r + d)$ a basis of the space $Z_d$. The bilinear functional defined by the formula

$$D_2 \begin{pmatrix} \eta_1, ..., \eta_{r+d} \\ \gamma_1, ..., \gamma_r \end{pmatrix} = \begin{pmatrix} \xi_{i+1}, ..., \xi_{i+r} \\ \xi_{i+1}, ..., \xi_{i+r} \end{pmatrix}$$

defines an operator $B \in L_2(X, \mathcal{E})$ almost inverse to $A$.

The equation $Ax = x_0$ has a solution if and only if $\xi(x_0) = 0$

$$(i = 1, 2, ..., r)$$

This solution is of the form

$$x = c_1 x_1 + c_2 x_2 + ... + c_r x_r + Bx_0,$$

where $\eta_i(x_0) = 0$ (i = 1, ..., r + d).

The equation $Ax = \xi$ has a solution if and only if $\xi(x_0) = 0$ (i = 1, 2, ..., r + d). This solution is of the form

$$\xi = c_1 \xi_1 + c_2 \xi_2 + ... + c_r \xi_r + B,$$

where $\xi_i(Bx_0) = 0$ (i = 1, 2, ..., r).

Proof. Condition (3) of the preceding section ensures the existence of elements $\xi_i$ and $x_i$. By (3.1) and by condition (2), $\xi_i(y_j) = \delta_{ij}$ for $i, j = 1, 2, ..., r$. Moreover, by (3.2) and by condition (2), $\eta_j(x_i) = \delta_{ij}$ for $i, j = 1, 2, ..., r + d$. Hence the elements $\xi_1, ..., \xi_r$ and $x_1, ..., x_r$ are linearly independent, and also the elements $\eta_1, ..., \eta_{r+d}$ and $\eta_{r+1}, ..., \eta_{r+d}$ are linearly independent. By (3.1), $\xi_i$ are solutions of the equation $Ax = 0,$ since conditions (2) and (2.2) for $n = r - 1$ imply $\xi(Az) = 0$ for every $z \in \mathcal{X}.$ Analogously, by conditions (3.2), (2) and (2.1) with $n = r - 1,$ it follows that $\xi(Ax_0 = 0$ for every $\xi \in \mathcal{E}$, since $\mathcal{E}$ is total, this implies $Ax_0 = 0.$

Replacing $\xi$ by $\xi_i$ in formula (3.3) and applying condition (2.1) for $n = r$ we obtain $AB = I - K_2$, where $K_2 x = \sum \xi_i x_i y_i$ is an $r$-dimensional operator. Analogously, replacing $x$ by $Ax$ and applying formula (2.2) for $n = r$ we obtain $BA = I - K_2$, where $K_2 x = \sum \eta_j(x) y_j$. Hence the operator $B$ is almost inverse to the operator $A$, by Corollary 1.1.

### 3. Connection between the determinant system and solutions of equations

Since the operator $\mathfrak{A}$ has a finite $d$-characteristic: $a_d = r + d$ and $a_{d-1} = r$, and since the set $E_d$ is described by the functionals $\xi_i (i = 1, ..., r)$ (§ 5, 1), $\mathfrak{A}$ is a $\Phi_d$-operator. By Theorem 1.3, we obtain the part of the theorem concerning the equation $Ax = x_0$. Taking into account the fact that $\mathfrak{A}$ is a $\Phi_d$-operator, we get the part of the theorem concerning the equation $Ax = x_0$.

Also the converse theorem holds:

**Theorem 3.2.** Every operator $\mathfrak{A} \in \Phi_d(X, \mathcal{E})$ has a unique determinant system $(D_n)$ if we do not take into account a constant factor different from zero. Moreover,

$$d = d(D_n)^d = |a_d|, \quad r = r(D_n)^r = r_d.$$

This system is defined in the following manner (for $a_d \neq 0$):

$$(3.4) \quad D_n = 0 \quad \text{for} \quad n = 0, 1, ..., r - 1,$$

$$D_n \begin{pmatrix} \xi_1, ..., \xi_{r+d} \\ \gamma_1, ..., \gamma_r \end{pmatrix} = \left[ \begin{array}{c|c} \begin{vmatrix} \xi_{r+1} & \cdots & \xi_{r+d} \\ \gamma_1 & \cdots & \gamma_r \end{array} \end{array} \right] \begin{pmatrix} \xi_1, ..., \xi_{r+d} \\ \gamma_1, ..., \gamma_r \end{pmatrix},$$

where the operator $B$ is almost inverse to the operator $A$, $\{x_1, ..., x_{r+d}\}$ is a basis of the space $Z_d$, $\{x_1, ..., x_r\}$ is a basis of the space $Z_d$, and functionals $\eta_1, ..., \eta_{r+d}$ and elements $\gamma_1, ..., \gamma_r$ satisfy the conditions:

$\eta_j(x_i) = \delta_{ij}$ for $i, j = 1, 2, ..., r + d$; $\xi_i(y_j) = \delta_{ij}$ for $i, j = 1, 2, ..., r$.

Summation is extended over all permutations $p = (p_1, ..., p_{r+d})$ and $q = (q_1, ..., q_r)$ of numbers $1, 2, ..., r, r + d + k$ and $1, 2, ..., r, r + k$, respectively, such that

$$p_1 < p_2 < \cdots < p_{r+d} < p_{r+d+1} < \cdots < p_{r+d+k}$$

and the same for the permutation $q$.

Proof. The system given above satisfies conditions (1)-(4) of a determinant system in an obvious manner, since $D_n \begin{pmatrix} \xi_1, ..., \xi_{r+d} \\ \gamma_1, ..., \gamma_r \end{pmatrix} = 1.$

It remains to prove condition (5). From the first of the conditions (3.4) it follows immediately that condition (5) holds for $n = 0, 1, ..., r - 2$. It is easily seen that this condition is satisfied also for $n = r - 1$. Since

$$(3.5) \quad D_n \begin{pmatrix} \xi_1, ..., \xi_{r+d} \\ \gamma_1, ..., \gamma_r \end{pmatrix} = 0$$

for $n = r - 1$, and $a_d = r + d$ and $a_{d-1} = r$, and since the set $E_d$ is described by the functionals $\xi_i (i = 1, ..., r)$ (§ 5, 1), $\mathfrak{A}$ is a $\Phi_d$-operator. By Theorem 1.3, we obtain the part of the theorem concerning the equation $Ax = x_0$. Taking into account the fact that $\mathfrak{A}$ is a $\Phi_d$-operator, we get the part of the theorem concerning the equation $Ax = x_0$.■
if at least one of the points \( \xi_1, \ldots, \xi_{n+1} \) belongs to the space \( E_{x+1} \), or if at least one of the points \( x_{1}, \ldots, x_{n} \) belongs to the space \( E_{x} \). The proof of condition (5) for \( n > r \) is based on the following formula:

\[
(3.6) \quad \sum_{p} \text{sgn } p \left( \frac{\det \xi(p_{n})}{\det \xi(p_{1})} \right) \cdot \sum_{q} \text{sgn } q \left( \frac{\det \xi(q_{n})}{\det \xi(q_{1})} \right) = \sum_{q} \text{sgn } q \left( \frac{\det \xi(q_{n})}{\det \xi(q_{1})} \right) = \sum_{q} \text{sgn } q \left( \frac{\det \xi(q_{1})}{\det \xi(q_{n})} \right),
\]

where the summation is extended over all permutations \( p = (p_{1}, \ldots, p_{n+1}) \) and \( q = (q_{1}, \ldots, q_{n}) \) of numbers \( 1, 2, \ldots, n+1 \) such that

\[
P_{1} < P_{2} < \ldots < P_{n}; \quad P_{n+1} < \ldots < P_{n+d};
\]

\[
q_{1} < q_{2} < \ldots < q_{n}; \quad q_{n+1} < \ldots < q_{n+d};
\]

respectively.

Hence we obtain the following equality:

\[
D_{n+1} \left( \xi_{1}, \xi_{2}, \ldots, \xi_{n+1} \right) = \sum_{p} \text{sgn } p \cdot \sum_{q} \text{sgn } q \left( \frac{\det \xi(p_{n})}{\det \xi(p_{1})} \right) \cdot D_{n} \left( \xi(p_{1}), \ldots, \xi(p_{n}) \right) \cdot D_{n} \left( \xi(q_{1}), \ldots, \xi(q_{n}) \right)
\]

where \( p', p'', q \) are arbitrary permutations of numbers \( 0, 1, \ldots, n+1-k \) of the following form:

\[
p' = (P_{1}, P_{2}, \ldots, P_{n}, P_{n+1}, \ldots, P_{n+d+1});
\]

\[
p'' = (P_{1}, P_{2}, \ldots, P_{n}, P_{n+1}, \ldots, P_{n+d+1});
\]

\[
q = (q_{1}, q_{2}, \ldots, q_{n}, q_{n+1}, \ldots, q_{n+d+1});
\]

As follows from Corollary 2.3, III, the operator \( B \) almost inverse to the operator \( A \) satisfies the equality \( AB = I - K_{1} \), where the operator \( K_{1} \) is defined by the formula:

\[
K_{1} \alpha = \sum_{i=1}^{n} \xi_{i}(\alpha) y_{i}, \quad \text{where} \quad \xi_{i}(\alpha) = \delta_{0}, \quad (i, j = 1, 2, \ldots, \tau)
\]

(since if \( \mu_{j} \leq 0, \tau = \mu_{0} = \beta_{k} \)), and elements \( y_{i} \) are defined by formula (1.4); moreover, \( \xi_{1}, \ldots, \xi_{\tau} \) denotes a basis of the space \( Z_{a} \). Hence, by formulæ

\[
(3.5) \text{ and } (3.6), \text{ we obtain }
\]

\[
D_{n+1} \left( \xi_{1}, \ldots, \xi_{n+1}, \xi_{n+2}, \ldots, \xi_{n+d+1} \right) \cdot \left( \begin{array}{c}
\xi_{1}(A) y_{1} \cdot \xi_{1}(A) y_{2} \cdot \ldots \cdot \xi_{1}(A) y_{n} \\
\xi_{2}(A) y_{1} \cdot \xi_{2}(A) y_{2} \cdot \ldots \cdot \xi_{2}(A) y_{n} \\
\vdots \\
\xi_{n+1}(A) y_{1} \cdot \xi_{n+1}(A) y_{2} \cdot \ldots \cdot \xi_{n+1}(A) y_{n} \\
\xi_{n+2}(A) y_{1} \cdot \xi_{n+2}(A) y_{2} \cdot \ldots \cdot \xi_{n+2}(A) y_{n} \\
\vdots \\
\xi_{n+d+1}(A) y_{1} \cdot \xi_{n+d+1}(A) y_{2} \cdot \ldots \cdot \xi_{n+d+1}(A) y_{n}
\end{array} \right)
\]

\[
= \sum_{p} \text{sgn } p' \cdot \sum_{q} \text{sgn } q \left( \frac{\det \xi(p_{n})}{\det \xi(p_{1})} \right) \cdot D_{n} \left( \xi(p_{1}), \ldots, \xi(p_{n}) \right) \cdot \left( \begin{array}{c}
\xi_{1}(A) y_{1} \cdot \xi_{1}(A) y_{2} \cdot \ldots \cdot \xi_{1}(A) y_{n} \\
\xi_{2}(A) y_{1} \cdot \xi_{2}(A) y_{2} \cdot \ldots \cdot \xi_{2}(A) y_{n} \\
\vdots \\
\xi_{n+1}(A) y_{1} \cdot \xi_{n+1}(A) y_{2} \cdot \ldots \cdot \xi_{n+1}(A) y_{n} \\
\xi_{n+2}(A) y_{1} \cdot \xi_{n+2}(A) y_{2} \cdot \ldots \cdot \xi_{n+2}(A) y_{n} \\
\vdots \\
\xi_{n+d+1}(A) y_{1} \cdot \xi_{n+d+1}(A) y_{2} \cdot \ldots \cdot \xi_{n+d+1}(A) y_{n}
\end{array} \right)
\]

\[
\left( \begin{array}{c}
\xi_{1}(A) y_{1} \cdot \xi_{1}(A) y_{2} \cdot \ldots \cdot \xi_{1}(A) y_{n} \\
\xi_{2}(A) y_{1} \cdot \xi_{2}(A) y_{2} \cdot \ldots \cdot \xi_{2}(A) y_{n} \\
\vdots \\
\xi_{n+1}(A) y_{1} \cdot \xi_{n+1}(A) y_{2} \cdot \ldots \cdot \xi_{n+1}(A) y_{n} \\
\xi_{n+2}(A) y_{1} \cdot \xi_{n+2}(A) y_{2} \cdot \ldots \cdot \xi_{n+2}(A) y_{n} \\
\vdots \\
\xi_{n+d+1}(A) y_{1} \cdot \xi_{n+d+1}(A) y_{2} \cdot \ldots \cdot \xi_{n+d+1}(A) y_{n}
\end{array} \right)
\]

\[
\left( \begin{array}{c}
\xi_{1}(A) y_{1} \cdot \xi_{1}(A) y_{2} \cdot \ldots \cdot \xi_{1}(A) y_{n} \\
\xi_{2}(A) y_{1} \cdot \xi_{2}(A) y_{2} \cdot \ldots \cdot \xi_{2}(A) y_{n} \\
\vdots \\
\xi_{n+1}(A) y_{1} \cdot \xi_{n+1}(A) y_{2} \cdot \ldots \cdot \xi_{n+1}(A) y_{n} \\
\xi_{n+2}(A) y_{1} \cdot \xi_{n+2}(A) y_{2} \cdot \ldots \cdot \xi_{n+2}(A) y_{n} \\
\vdots \\
\xi_{n+d+1}(A) y_{1} \cdot \xi_{n+d+1}(A) y_{2} \cdot \ldots \cdot \xi_{n+d+1}(A) y_{n}
\end{array} \right)
\]

where the summation is extended over all permutations \( p = (p_{1}, \ldots, p_{n+d+1}) \) of numbers \( 1, \ldots, n+1-k \) such that \( p_{1} < \ldots < p_{n}; \quad p_{n+1} < \ldots < p_{n+d+1}; \quad p_{n+d+1} < \ldots < p_{n+1} \), and all permutations \( q'' = (q''_{1}, \ldots, q''_{n+d+1}) \) of numbers \( 0, 1, \ldots, k+1, 1, \ldots, k+1 \) such that \( q''_{0} < \ldots < q''_{k}; \quad q''_{k+1} < \ldots < q''_{k+1} \). Thus we have com-
pleted the proof of equality (2.1) of condition (5). Equality (2.2) is proved in an analogous manner.

We have to prove the last part of the theorem, namely that the determinant system \( (D_n)^+ \) is unique if we do not take into account a constant factor different from zero. As before we limit ourselves to the case \( n < 0 \), i.e., to the consideration of systems \( (D_n)^+ \). Let \( (D_n)^+ \) be an arbitrary determinant system of an operator \( \Lambda \). We show that there exists a constant \( c \neq 0 \) such that

\[
D_n = cD_n' \quad \text{for} \quad n = 0, 1, \ldots ,
\]

where \( D_n \) are defined by formulae (3.4).

By Theorem 3.1,

\[
r(D_n)^+ = r(D_n)^+, \quad d(D_n)^+ = d(D_n)^+.
\]

Hence

\[
\tag{3.7}
D_n = 0 \quad \text{for} \quad n = 0, 1, \ldots , r-1 ,
\]

and \( D_r \neq 0 \). If one of the points \( x_1, \ldots , x_n \) belongs to the subspace \( E_d \), or if one of the points \( \xi_1, \ldots , \xi_r \) belongs to the subspace \( E_d' \), formula (3.7) and properties (2) and (5) for \( n = r-1 \) imply \( D_r = 0 \).

It is easily verified that \( X = E_k \oplus E_d, \ Y = E_k \oplus E_{d'}, \) where \( B \) is the almost inverse operator of \( \Lambda \). Hence we obtain the following unique representation:

\[
x_i = x_i' + x_{i'}', \quad \text{where} \quad x_i' \in E_d, \ x_{i'}' \in E_{d'}.
\]

\[
\xi_i = \xi_i' + \xi_{i'}', \quad \text{where} \quad \xi_i' \in E_d, \ \xi_{i'}' \in E_{d'}.
\]

This decomposition yields the following identity:

\[
D_r \left( x_1', \ldots , x_{r+d} \right) = B \left( \xi_1', \ldots , \xi_{r+d} \right) ,
\]

and an analogous identity for \( D_r' \). But \( D_r \) and \( D_r' \) are \((2r+d)\)-linear functionals defined in a \((2r+d)\)-dimensional space \( Z_d \times Z_{d'} \), and skew-symmetric with respect to variables \( \xi_i', \ldots , \xi_{r+d} \) and \( x_{i'}, \ldots , x_{r+d} \). Hence \( D_r \) and \( D_r' \) differ at most by a constant factor \( c \neq 0 \) (since \( D_r \neq 0 \neq D_r' \)).

If \( n > r \), the proof will be performed by induction. Let us suppose that \( D_n = cD_n \) for \( n > r \). We shall prove that

\[
D_{n+1} \left( x_1', \ldots , x_{n+d} \right) = cD_{n+1} \left( \xi_1', \ldots , \xi_{n+d+1} \right) .
\]

Since \( D_n \) and \( D_n' \) are linear with respect to each of the variables, it is sufficient to prove the last equality for the case where each of the points \( \xi_1', \ldots , \xi_{n+d} \) either belongs to the set \( E_d' \) or is equal to one of the points \( \eta_{n+1}, \ldots , \eta_{n+d} \) denoting a basis of the subspace \( E_{d'} \).

\section{Connection between the determinant system and solutions of equations}

In the first case property (3.9) follows from condition (2.1) for \( D_{n+1} \) and \( D_{n+1}' \). If the sequence \( \xi_1, \ldots , \xi_{n+d} \) contains only points \( \eta_{n+1}, \ldots , \eta_{n+d} \), then at least one of these points must appear twice, and since \( D_{n+1} \) and \( D_{n+1}' \) are skew-symmetric, both \( D_{n+1} \) and \( D_{n+1}' \) are equal to zero.

\textbf{Corollary 3.3.} If an operator \( \Lambda \in \Phi_k(X, Z) \) has index \( x_A \leq 0 \) and if \( r_A = 0 \), then the determinant system \( (D_n)^+ \) for the operator \( \Lambda \) is of the following form:

\[
D_r \left( x_1', \ldots , x_{r+d} \right) = \sum_{p} \text{sign} p \cdot D_r(\xi_{p+1}, \ldots , \xi_{p+d})(\text{det} \xi_{p+d}(E_{d'}))
\]

\[
(n = 0, 1, \ldots ,)
\]

where \( B \) is an almost inverse operator of the operator \( \Lambda \), and the summation is extended over all permutations \( p = (p_1, \ldots , p_{n+d}) \) of numbers \( 1, \ldots , n+d \) such that \( p_1 < \ldots < p_d, \ p_{d+1} < \ldots < p_{n+d} \).