Preface

book, concern problems of this type for functions of one or several variables. In case of several variables we will have, in general, to require that besides the initial estimates some boundary estimates be given in advance.

Differential inequalities treated in this book are the so-called non-stationary inequalities.

 Chapters I-VIII of the book deal with the theory of ordinary differential inequalities and with its applications to ordinary differential equations and to first order and second order partial differential equations of parabolic and hyperbolic type. The theory of ordinary differential inequalities was originated by Chaplygin [6] and by Kanko [13] and then developed by Wazewski [60]. The main applications of the theory concern questions such as: estimates of solutions of differential equations, estimates of the existence domain of solutions, estimates of the difference between two solutions, criteria of the uniqueness of the solution, estimates of the error for an approximate solution, stability and Chaplygin's method.

 Chapters IX-X concern partial differential inequalities of first and second order. First order partial differential inequalities were first treated by Harn [11] and by Nagumo [34]. Partial differential inequalities of second order, dealt with in this book, are of parabolic and hyperbolic type. First results on second order partial differential inequalities of parabolic type were obtained by Nagumo [35] and by Western [56].

 Chapter XI deals with differential inequalities in linear spaces. This chapter as well as §§ 31, 32 in Chapter V and §§ 66, 67 in Chapter X are written by Wlodzimierz Mak.

 We close these introductory remarks by the following one. From theorems that will be proved here on ordinary and partial differential inequalities, criteria of continuous dependence on initial values for solutions of corresponding equations can be derived. Now, since solutions of elliptic equations do not depend continuously on initial data, it is clear that theorems of the type described above cannot be expected to apply to partial differential equations or inequalities of elliptic type, i.e. to stationary equations or inequalities.

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 Jacek Szarski

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Chapter I

Monotone Functions

§ 1. Zygmund's lemma. We adopt the following terminology. A real function \(\varphi(t)\) defined in an interval \(\Delta\) is called increasing if for any two points \(t_1, t_2\) from \(\Delta\) such that

\[
t_1 < t_2
\]

we have

\[
\varphi(t_1) \leq \varphi(t_2).
\]

If for any two points of \(\Delta\) inequality (1.1) implies

\[
\varphi(t_1) < \varphi(t_2),
\]

then \(\varphi(t)\) is called strictly increasing. In a similar way we define a decreasing and a strictly decreasing function.

For a function \(\varphi(t)\), defined in some neighborhood of the point \(t_0\), we denote by \(D^+\varphi(t_0), D_+\varphi(t_0), D^-\varphi(t_0), D_-\varphi(t_0)\), respectively, its right-hand upper, right-hand lower, left-hand upper and left-hand lower Dini's derivatives at the point \(t_0\), i.e.

\[
D^+\varphi(t_0) = \limsup_{h \to 0^+} \frac{\varphi(t_0 + h) - \varphi(t_0)}{h},
\]

\[
D_-\varphi(t_0) = \liminf_{h \to 0^-} \frac{\varphi(t_0 + h) - \varphi(t_0)}{h},
\]

\[
D_+\varphi(t_0) = \limsup_{h \to 0^+} \frac{\varphi(t_0 + h) - \varphi(t_0)}{h},
\]

\[
D_-\varphi(t_0) = \liminf_{h \to 0^-} \frac{\varphi(t_0 + h) - \varphi(t_0)}{h},
\]

(the values \(+\infty\) and \(-\infty\) being not excluded). Symbols \(\varphi_+(t_0)\) and \(\varphi_-(t_0)\) will stand for the right-hand and left-hand derivative respectively.

The inequality \(a > 0\) will mean that either \(a\) is finite and positive or \(a = +\infty\). The meaning of the inequalities \(a > 0, a < 0, a \leq 0\) is defined in a similar way.
To begin with we will prove the following lemma.

**Zygmund’s Lemma.** Let \( \phi(t) \) be continuous in an interval \( \Delta \) and write

\[
Z_+ = \{ t \in \Delta : D_+ \phi(t) < 0 \}.
\]

Suppose that the set \( \phi(\Delta - Z_+)(\ast) \) does not contain any interval.

Under these assumptions \( \phi(t) \) is decreasing on \( \Delta \).

Proof. Suppose the contrary; then there would exist two points \( t_1, t_2 \in \Delta \) satisfying (1.1) and such that \( \phi(t_1) < \phi(t_2) \). Since, by our assumption, the set \( \phi(\Delta - Z_+) \) does not contain the interval \( [\phi(t_1), \phi(t_2)] \), there is a point \( y_0 \in [\phi(t_1), \phi(t_2)] \) such that

\[
(1.2) \quad y_0 \notin \phi(\Delta - Z_+).
\]

By Darboux’s property, the set

\[
E = \{ t \in (t_1, t_2) : \phi(t) = y_0 \}
\]

is not empty. Let us denote by \( t_0 \) its least upper bound. Then we have

\[
\phi(t_0) = y_0,
\]

and, by continuity,

\[
(1.3) \quad \phi(t) \geq y_0 \quad \text{for} \quad t_0 < t < t_2.
\]

Relations (1.2) and (1.3) imply that \( t_2 \in Z_+ \), and hence, by the definition of \( Z_+ \),

\[
(1.4) \quad D_+ \phi(t_2) > 0.
\]

On the other hand, by (1.3) and (1.4), it follows that

\[
D_+ \phi(t_2) \geq 0,
\]

which is a contradiction with (1.5). This completes the proof.

**Remark 1.** Since (1.3) and (1.4) imply \( D^+ \phi(t_2) \geq 0 \), it is obvious that the set \( Z_+ \) in Zygmund’s lemma can be replaced by the set

\[
Z^+ = \{ t \in \Delta : D^+ \phi(t) < 0 \}.
\]

**Remark 2.** The set \( Z_+ \) can be replaced by the set

\[
Z_+ = \{ t \in \Delta : D_- \phi(t) < 0 \}
\]

or by the corresponding set \( Z^- \). To prove Zygmund’s lemma with \( Z_+ \) replaced by \( Z_- \) or \( Z^- \), we have only to change the above argument by taking for \( t_4 \) the greatest lower bound of \( E \).

**Remark 3.** A similar lemma holds true for increasing functions.

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\( \ast \) \( \Delta \) being a subset of \( \Delta \), \( \phi(\Delta) \) denotes the image of \( \Delta \) by means of the mapping \( y = \phi(t) \).
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D.\varphi(t) \leq 0 \text{ for every } t \in A \text{ or } D.-\varphi(t) \leq 0 \text{ for every } t \in A, \text{ then } \varphi(t) \text{ is decreasing in } A. \text{ Now, if we assume that for every } t \in A \text{ we have either } D.\varphi(t) \leq 0 \text{ or } D.-\varphi(t) \leq 0, \text{ then } \varphi(t) \text{ is not necessarily decreasing. Indeed, for Weierstrass's functions } \varphi(t) \text{ (a continuous function without finite derivative at any point) we have for every } t \text{ either } D.\varphi(t) = -\infty \text{ or } D.-\varphi(t) = -\infty, \text{ and the function is not monotone.}

Similar results for increasing functions follow from those concerning decreasing functions by considering \(-\varphi(t)\) instead of \(\varphi(t)\).

We close this paragraph by an important theorem due to Dini.

Theorem 2.2. For \(\varphi(t)\) continuous in an interval \(A\) the following two propositions are true:

1° If any of its Dini's derivatives is \(\leq a\) \((< a)\) for \(t \in Z \cap A\), where \(A - Z\) is at most countable, then for any two different points \(t, s\) from \(A\) we have

\[
\frac{\varphi(t) - \varphi(s)}{t - s} \leq a \quad (< a).
\]

2° If any of its Dini's derivatives is \(\geq \beta\) \((> \beta)\) for \(t \in Z \cap A\), where \(A - Z\) is at most countable, then for any two different points \(t, s\) of \(A\) we have

\[
\frac{\varphi(t) - \varphi(s)}{t - s} \geq \beta \quad (> \beta).
\]

Proof. Since 2° follows from 1° by taking \(-\varphi(t)\) in place of \(\varphi(t)\), we prove proposition 1°. Suppose then, for instance, that

\[
D.\varphi(t) \leq a \quad (< a) \quad \text{in } Z \cap A.
\]

Fix \(s\) in \(A\) and put

\[
\psi(t) = \varphi(t) - \varphi(s) - at \quad \text{for} \quad t \in A.
\]

\(\psi(t)\) is then continuous in \(A\) and, by (2.2),

\[
D.\psi(t) = D.\varphi(t) - a \leq 0 \quad (< 0) \quad \text{in } Z.
\]

Since \(A - Z\) is at most countable, it follows, by Theorem 2.1 (Corollary 2.1), that \(\psi(t)\) is decreasing (strictly decreasing) in \(A\) and consequently

\[
\psi(t) \leq \psi(s) \quad \text{(}\varphi(t) < \varphi(s)\text{)} \quad \text{for} \quad t > s.
\]

Hence we get (2.1) for \(t > s\). Since \(s, t, \text{ and } t > s\) were arbitrary points in the interval \(A\), we conclude that (2.1) holds true for any two different points \(s, t\) of \(A\).

Next theorem is an immediate consequence of the preceding one.

Theorem 2.3. Let \(\varphi(t)\) be continuous in an open interval \(A\). Assume that one of its Dini's derivatives is finite and continuous at \(t_0 \in A\). Then \(\varphi'(t_0)\) exists.

§ 2. Condition for a continuous function to be monotone

Proof. Suppose, for instance, that \(D.\varphi(t)\) is finite and continuous at \(t_0\). Put \(D.\varphi(t_0) = \lambda\) and take an arbitrary \(\varepsilon > 0\). Then there is a \(\delta > 0\) so that

\[
1 - \varepsilon < D.\varphi(t) < 1 + \varepsilon \quad \text{for} \quad t \in \left(t_0 - \delta, t_0 + \delta\right).
\]

Hence, by Theorem 2.2, we get

\[
1 - \varepsilon < \frac{\varphi(t) - \varphi(t_0)}{t - t_0} < 1 + \varepsilon \quad \text{for} \quad t \in \left(t_0 - \delta, t_0 + \delta\right), \quad t \neq t_0.
\]

\(\varepsilon > 0\) being arbitrary, inequality (2.3) implies the conclusion of our theorem.

Corollary 2.2. For \(\varphi(t)\) continuous in an open interval \(A\) assume that one of its Dini's derivatives is finite and continuous on \(A\). Then \(\varphi'(t)\) exists and is continuous on \(A\).

§ 3. A sufficient condition for a function to be monotone.

As a further consequence of Zygmund's lemma we prove the following theorem.

Theorem 3.1. Let \(\varphi(t)\) be absolutely continuous in an interval \(A\) and assume that

\[
\varphi'(t) \leq 0 \quad \text{for almost every } t \in A.
\]

Then \(\varphi(t)\) is decreasing in \(A\).

Proof. Let \(\varepsilon > 0\) be arbitrary and put

\[
\psi(t) = \varphi(t) - \varepsilon t.
\]

\(\psi(t)\) is absolutely continuous in \(A\) and

\[
\psi'(t) = \varphi'(t) - \varepsilon \quad \text{for almost every } t \in A.
\]

Therefore, by (3.1), we have \(\psi'(t) < 0\) for almost every \(t \in A\) and hence the set \(A - Z_\varepsilon\), where

\[
Z_\varepsilon = \{t \in A : D.\varphi(t) < 0\},
\]

is of measure 0. \(\varphi(t)\) being absolutely continuous the set \(\psi(A - Z_\varepsilon)\) is of measure 0 too, and consequently does not contain any interval. Hence, by Zygmund's lemma, \(\varphi(t)\) is decreasing in \(A\) and \(\varepsilon > 0\) being arbitrary the same holds true for \(\varphi(t)\).

Remark 3.1. A similar theorem is true for increasing functions.

Remark 3.2. By an argument similar to that used in the proof of Theorem 3.1 we show the following result: If \(\varphi(t)\) is a generalized absolutely continuous function (see [45]) in an interval \(A\) and if its approximative derivatives (see [45]) is non-positive almost everywhere in \(A\), then \(\varphi(t)\) is decreasing in \(A\).