CHAPTER VIII

ELLiptic Functions

§ 1. General remarks about periodic functions. In Chapter I, §§ 8, 9, we mentioned that the function \( \sin z \) has the period \( 2\pi \), and the function \( \varepsilon \) the period \( 2\pi i \). In general, by a period of a function \( F(z) \) meromorphic in any region \( G \) we shall mean a number \( \omega \) such that:

1° for every point \( z \), if one of the points \( z, z + \omega \) belongs to \( G \), then so does the other;

2° if \( z \in G \), then \( F(z + \omega) = F(z) \).

For a constant function every number is a period. A function meromorphic in a region \( G \) is said to be periodic if it has at least one period different from 0.

From the definition of a period it follows that if \( F(z) \) has a period \( \omega \), then it also has the period \( m\omega \), where \( m \) is an arbitrary integer. More generally, if \( \omega_1, \omega_2, \ldots, \omega_p \) are periods of the function \( F(z) \), then every number \( m_1\omega_1 + m_2\omega_2 + \ldots + m_p\omega_p \), where \( m_1, m_2, \ldots, m_p \) are arbitrary integers, is also a period.

Differentiating \( F(z + \omega) = F(z) \), we get \( F'(z + \omega) = F'(z) \). Consequently, the derivative of a periodic function \( F(z) \) is also a periodic function, and every period of the function \( F(z) \) is also a period of the function \( F'(z) \).

However, a primitive function of a periodic function need not be periodic. For example, the function \( F(z) = z \) is not periodic, in spite of the fact that its derivative \( F'(z) = 1 \), being a constant, is a periodic function.

Let us now consider an arbitrary periodic function \( F(z) \) and the set \( \Omega \) of all its periods. If \( F(z) \) is not a constant, then \( \Omega \) does not have finite points of accumulation. In fact, otherwise there would exist a sequence \( \{\omega_n\} \) of distinct periods, converging to a finite number \( a \), and hence there would exist periods \( \omega_n = \omega_n - a \neq 0 \) with arbitrarily small absolute values. Since \( F(z + \omega_n) = F(z) \), an arbitrary point \( z \), at which the function \( F \) is holomorphic, would be the limit of the sequence of the points \( z + \omega_n \), at which the function would assume the same value as at the point \( z \). Therefore \( F(z) \) would be a constant, contrary to the hypothesis.

As to the distribution of the points of the set \( \Omega \), we distinguish two cases:

a) The set \( \Omega \) lies on some straight line \( p \) (passing, obviously, through the point 0). Let \( \omega \neq 0 \), then, be a point of the set \( \Omega \) situated nearest the point 0. Since the set \( \Omega \) does not have any finite points of accumulation and is symmetric with respect to the point 0, there exist exactly two numbers having the given property and differing only in sign; we choose any one of them as \( \omega \). We easily ascertain that all the elements of the set \( \Omega \) are of the form \( n\omega \), where \( n = 0, \pm 1, \pm 2, \ldots \).

In fact, if a period \( \omega \) not of this form existed, then, since it would lie on the straight line \( p \), we should have \( \omega = (m + \theta)\omega \), where \( m \) is an integer, and \( 0 < \theta < 1 \). The number \( \theta \omega = \omega - m\omega \), being the difference of two periods, would be a period different from 0 and lying closer to the point 0 than \( \omega \), contrary to the assumption concerning \( \omega \).

The number \( \omega \) defined to within a sign, is called a primitive period of the function \( F(z) \). For example, by theorem 6.10, Chapter I, \( 2\pi i \) is a primitive period of the function \( \varepsilon \).

b) The set \( \Omega \) does not lie on one straight line. Let, as before, \( \omega \neq 0 \) be a point of the set \( \Omega \) nearest the point 0. In view of the symmetry of the set \( \Omega \) with respect to the point 0, there exist at least two points with this property. Let \( p \) denote the straight line \( \theta\omega \). The same reasoning as in case a) indicates that all the elements of the set \( \Omega \) lying on the straight line \( p \) are of the form \( m\omega \), where \( m = 0, \pm 1, \pm 2, \ldots \). Let us now consider those points of the set \( \Omega \) which do not lie on the straight line \( p \), and let us choose from them a point \( \omega' \) nearest the point 0. It is clear that \( |\omega'| \geq |\omega| \).

We shall prove that all the elements of the set \( \Omega \) are of the form \( m\omega + n\omega' \), where \( m \) and \( n \) are arbitrary integers.

We already know that numbers of this form are periods. All that remains to be shown is that there are no other periods.

The numbers \( m\omega + n\omega' \) are the vertices of a net of parallelograms covering the plane (see the Fig. 32, p. 315). If some point
w of the set \( \Omega \) were not of the form \( m\omega + n\omega' \), then it would lie inside or on the perimeter of one of these parallelograms, but not on a vertex. Hence, we should have

\[
w = (m+\theta)\omega + (n+\theta')\omega',
\]

where the numbers \( \theta \) and \( \theta' \), not both simultaneously zero, would satisfy the inequalities \( 0 \leq \theta < 1 \) and \( 0 < \theta' < 1 \). The number \( w' = w - (m\omega + n\omega') \) would be a period lying inside or on the perimeter of the parallelogram with vertices \( 0, \omega, \omega + \omega', \omega' \), but not on any one of these vertices. Now, it cannot lie either inside or on the perimeter of the triangle with vertices \( 0, \omega, \omega' \), since in view of the inequality \( |w| < |\omega'| \) it would belong to the open circle \( K(0; |\omega'|) \).

We should therefore have \( |w'| < |\omega'| \), which is contrary to the definition of the number \( \omega' \). And if \( w' \) lay in the triangle with vertices \( \omega, \omega' - w' \), then the number \( w = \omega + \omega' - w' \), which is a period, would lie in the triangle \( 0, \omega, \omega' \), and we should again come to a contradiction.

A pair of periods \( \omega, \omega' \), such that every period is of the form \( m\omega + n\omega' \), where \( m \) and \( n \) are integers, is called a pair of primitive periods. In contrast to the primitive periods in case a), there are infinitely many primitive pairs in case b). For example, if \( k \) is an integer, then, together with \( \omega, \omega' \), the pair \( \omega, k\omega + \omega' \) is also a pair of primitive periods.

Case b) can be characterized in the following manner: the function has two periods with a non-real quotient.

Summarizing, we can say:

(1.1) If a function \( F(z) \), meromorphic in a region \( G \) and different from a constant, is periodic, then one of the following two possibilities occurs:

a) there exists a period \( \omega \) (primitive period) such that every other period is an integral multiple of \( \omega \); this period is determined to within a sign;

b) there exists a pair of periods \( \omega, \omega' \) different from zero, with a non-real quotient (a pair of primitive periods), such that every period of the function \( F(z) \) is of the form \( m\omega + n\omega' \), where \( m \) and \( n \) are arbitrary integers; there are infinitely many pairs \( \omega, \omega' \) having this property.

In a case a) the function \( F(z) \) is called simply periodic, in a case b) doubly periodic. Functions reducing to a constant will also be called doubly periodic, and by a pair of primitive periods of such a function we shall mean an arbitrary pair of numbers \( \omega, \omega' \) different from zero with a non-real quotient.

Let us suppose that a function \( F(z) \), holomorphic in a region \( G \), has a period \( \omega \). Through the points \( n\omega \) (for \( n = 0, \pm 1, \pm 2, \ldots \)) let us draw a family of parallel straight lines \( g_n \), different from the straight line \( 0\omega \). In this way the entire plane will be divided into a series of parallel strips \( S_n \), contained between \( g_n \) and \( g_{n+1} \), respectively. If we agree e.g. to include \( g_n \) in \( S_n \), excluding \( g_{n+1} \), however, then every point of the open plane will belong to exactly one strip \( S_n \). The strips are called period-strips. If \( S \) is one of these, then we obviously have \( G \cdot S = \emptyset \) (see p. 356, condition 1 of the definition) and for the investigation of a function in the entire region \( G \) it is sufficient to limit ourselves to the set \( G \cdot S \). For periodic functions, the most frequently considered region \( G \) is a strip bounded by two parallel straight lines. Obviously, the lines bounding this strip must be parallel to the straight line \( 0\omega \). As extreme cases we obtain here as \( G \) the half-plane bounded by a straight line parallel to \( 0\omega \) or the entire open plane.

In the case of doubly periodic functions having a pair of primitive periods \( \omega, \omega' \), we consider the net of parallelograms covering the plane, with vertices at the points \( m\omega + n\omega' \), where \( m \) and \( n \) are integers. Let us consider one such parallelogram. All of its points are of the form \( (m+\theta)\omega + (n+\theta')\omega' \), where \( 0 < \theta < 1 \), \( 0 < \theta' < 1 \). Let us remove from this parallelogram the points corresponding to \( \theta = 1 \) or \( \theta' = 1 \), i.e. let us include in the parallelogram, besides the interior points; only the vertex \( \zeta = m\omega + n\omega' \) and the two sides intersecting at this vertex, but without the end-points \( \zeta + \omega, \zeta + \omega' \). Let us denote by \( R_{m,n} \) the figure obtained, which we shall call a period-parallelogram. The period-parallelograms \( R_{m,n} \) do not have points in common and cover the entire open plane. The parallelogram \( R_{0,n} \) with vertices at the points \( 0, \omega, \omega + \omega', \omega' \), is called the fundamental parallelogram.

Generalizing the definition of congruence, introduced in § 9, Chapter I, we shall say that \( z_2 \) is congruent to \( z_1 \), modulo \( \omega, \omega' \), and write

\[
z_2 \equiv z_1 \pmod{\omega, \omega'},
\]

if the difference \( z_2 - z_1 \) is of the form \( m\omega + n\omega' \), where \( m \) and \( n \) are integers. If the numbers \( \omega \) and \( \omega' \) are fixed, then we shall simply write \( z_2 \equiv z_1 \) and say that \( z_2 \) is congruent to \( z_1 \).
§ 2. Expansion of a periodic function in a Fourier series.

If the function \( F(x) \) is meromorphic in the open plane, then the function \( F(\zeta) = F(2\pi n \zeta) \) has period 1. Therefore without loss of generality we may assume from the start that the function \( F(\zeta) \) has period 1 (we do not assume, however, that this period is primitive; \( F(z) \) may even be doubly periodic).

Let us suppose that \( F(z) \) is a function meromorphic in the strip \( R \) defined by the inequality \( b < \Im z < B \). The function
\[
\zeta = e^{2\pi iz}
\]
transforms this strip (of course not in a one-to-one manner) into the annulus \( 1 < \Im \zeta < \infty \). We shall prove that the function
\[
G(\zeta) = F(z) = F\left(\frac{1}{2\pi i} \log \zeta\right)
\]
is a function meromorphic in \( P \). Let us notice first of all that formula (2.2) defines \( G(\zeta) \) in \( P \) uniquely. In fact, although the expression \( \frac{1}{2\pi i} \log \zeta \) has infinitely many values, they differ by integers.

Because of the fact that 1 is a period of the function \( F(z) \), from formula (2.2) we obtain the same value for \( G(\zeta) \) in all cases. Now, let \( \zeta_0 \) be an arbitrary point of the annulus \( P \) and let \( \zeta_0 = e^{2\pi i z_0} \).

In the neighbourhood of the point \( \zeta_0 \) there exists a holomorphic branch \( L(\zeta) \) of the function \( \frac{1}{2\pi i} \log \zeta \). Since \( G(\zeta) = F(L(\zeta)) \) in this neighbourhood, we see that, if the function \( F(z) \) is meromorphic at the point \( z_0 \), then \( G(\zeta) \) is holomorphic at the point \( \zeta_0 \). If \( F(x) \) has a \( k \)-tuple pole at \( z_0 \), since \( L'(\zeta_0) \neq 0 \), the function \( G(\zeta) \) has a \( k \)-tuple pole at \( \zeta_0 \) (cf. Chapter III, theorem 8.3). Consequently, the function \( G(\zeta) \) is, in fact, meromorphic in \( P \).

Similarly, if the function \( F(z) \) is holomorphic in the strip \( R \), then, obviously, the function \( G(\zeta) \) is holomorphic in the annulus \( P \).
CHAPTER VIII. Elliptic functions.

Let us consider, for example, the function $F(z) = \cot \pi z$, having the period 1. Taking $\zeta = e^{\pi i}$, we find that $F(z) = i(1 + \zeta^{-1})(1 + \zeta)$. The right side of this equation has the point $\zeta = 1$ as the only singular point, and it's Taylor series in the circles $K(0, 1)$ and $K(\infty, 1)$ are, respectively,

$-i(1 + 2\zeta + 2\zeta + \ldots), \quad i(1 + 2\zeta^{-1} + 2\zeta^{-2} + \ldots)$.

From this we obtain the following two Fourier expansions for the function $\cot \pi z$,

$\cot \pi z = -i \left(1 + 2 \sum_{n=1}^{\infty} e^{-\pi n \omega}\right), \quad \cot \pi z = i \left(1 + 2 \sum_{n=1}^{\infty} e^{-\pi n \omega}\right),$

for $\omega > 0$, and the second for $\omega < 0$.

Making use of the equation $e^{2\pi n \omega} = \cos 2\pi n \omega + i \sin 2\pi n \omega$ we may write (2.3) in the form:

$F(z) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi n \omega + b_n \sin 2\pi n \omega\right),$

where

$\begin{align*}
 a_n &= 2a_0, \quad a_n = a_n + c_{-n}, \quad b_n = i(c_n - c_{-n}) \quad \text{for} \quad n = 1, 2, \ldots
\end{align*}$

Series (2.6) is the trigonometric form of a Fourier series.

Now let $H(z)$ be a function with period $\omega \neq 0$, holomorphic in a strip $b < \mathcal{S}(\omega) < B$, parallel to the straight line $\omega$. The function $F(z) = H(\zeta)$ has period 1 and is holomorphic in the strip $\omega < \mathcal{S} < B$, and hence we have formulae (2.3) or (2.5), or what amounts to the same thing, formula (2.6). Consequently, for $b < \mathcal{S}(\omega) < B$,

$H(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi n \omega} = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi n \omega + b_n \sin 2\pi n \omega\right),$

where the last two series differ only in appearance; we can easily pass from one to the other by using formula (2.7).

EXERCISES. 1. Let $F(z)$ be a function of period 1, holomorphic and bounded for $\omega > b$. Show that $F(z)$ tends to a finite limit, as $z$ tends to $\infty$ in such a way that $\omega \to \infty$.

2. An entire function $F(z)$ with period 1, bounded in the strip $0 < \Re z < 1$, is constant.

3. We shall say that a holomorphic function $F(z)$ with period 1 belongs to the class $\mathcal{C}$ if the function $G(\zeta)$, given by formula (2.2), is a rational function of the variable $\zeta$. Show that necessary and sufficient condition that a function $F(z)$, meromorphic and having period 1, belong to the class $\mathcal{C}$, is that it tend to a limit, finite or infinite, as $z$ tends to $\infty$ in the strip $0 < \Re z < 1$. (The limits for $\omega \to \infty$ and for $\omega \to -\infty$ need not be the same.)

§ 3. General theorems on elliptic functions.

4. If a function $F(z)$ of class $\mathcal{C}$ does not tend either to 0 or to $\infty$, when $0 < \Re z < 1$ and $\omega \to \pm \infty$, then in the strip $0 < \Re z < 1$ the function $F(z)$ has exactly as many roots as poles (taking into account the multiplicities of the roots and the poles).

5. Let $F(z)$ satisfy the hypotheses of exercise 4 and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_n$, respectively, be all the roots and all the poles of the function, in the strip $0 < \Re z < 1$. Then

$F(z) = O \left(\frac{\sum_{n=1}^{\infty} (z - e^{2\pi n \omega})}{\sum_{n=1}^{\infty} (z - e^{2\pi n \omega})}\right),$

where $O$ is a constant.

§ 3. General theorems on elliptic functions. Among the doubly periodic meromorphic functions a particularly important role is played by those functions whose region of meromorphism is the entire open plane. They are called elliptic functions. In discussing doubly periodic functions we shall limit ourselves almost exclusively to elliptic functions.

The name "elliptic functions" is, perhaps, not the most suitable one and expresses a rather accidental property of these functions. Historically, it arose because of the fact that the length of an arc of an ellipse is expressible in terms of certain integrals intimately connected with elliptic functions, the so-called elliptic integrals (see § 14). However, the arc length of many other curves, for example, the lemniscate, is expressible in terms of elliptic integrals.

Let $F(z)$ be an elliptic function, and $\omega, \omega'$, a pair of its primitive periods. Changing possibly the order of the periods we may always assume that $\mathcal{S}(\omega') > 0$, or $0 < \arg(\omega/\omega') < \pi$.

The parallelogram formed from the segments $[0, \omega], [\omega, \omega + \omega'], [\omega + \omega', \omega'],$ and $[\omega', 0]$, is therefore positively oriented (Chapter IV, § 11, p. 209).

We shall begin with the proof of the theorem that

(3.1) The only entire elliptic function is a constant.

Proof. A doubly periodic entire function $F(z)$ is bounded in the fundamental parallelogram. Since the function $F(z)$ assumes the same values in all the period-parallelograms as in the fundamental parallelogram, it is therefore bounded in the entire open plane, and consequently a constant.

We see, therefore, that every elliptic function which is not a constant must have at least one finite pole and therefore at least one pole in every period-parallelogram.
§ 3 General theorems on elliptic functions.

Let \( z_0 \) be an arbitrary complex number. Let us consider the parallelogram \( R \) with vertices \( z_0, z_0 + \omega, z_0 + \omega + \omega', \) and \( z_0 + \omega' \), obtained from the fundamental parallelogram by a translation through \( z_0 \) (in particular, \( R \) may be a period-parallelogram).

(3.4) If an elliptic function \( F(z) \) is holomorphic on the perimeter of the parallelogram \( R \), then its integral along this perimeter is zero.

In fact, this integral is equal to

\[
\int_{z_0}^{z_0 + \omega} F(z) \, dz + \int_{z_0 + \omega}^{z_0 + \omega + \omega'} F(z) \, dz + \int_{z_0 + \omega + \omega'}^{z_0 + \omega} F(z) \, dz + \int_{z_0 + \omega}^{z_0} F(z) \, dz,
\]

where we integrate along rectilinear segments. The sum of the first and third integrals is, as one easily sees,

\[
\int_{z_0}^{z_0 + \omega} F(z) \, dz - \int_{z_0}^{z_0 + \omega} F(z + \omega') \, dz = \int_{z_0}^{z_0 + \omega} F(z) \, dz + \int_{z_0}^{z_0} F(z) \, dz,
\]

and is therefore zero.

From (3.4) we have the following theorem:

(3.5) The sum of the residues of an arbitrary elliptic function \( F(z) \), corresponding to all the poles belonging to any period-parallelogram \( R \), is equal to zero.

Proof. Let us suppose, first, that the function \( F(z) \) is holomorphic on the perimeter of the parallelogram \( R \) under consideration. The sum of the residues, multiplied by \( 2\pi i \), is then equal to the integral of the function \( F(z) \), taken along the perimeter of the parallelogram \( R \) in the positive sense, and hence, by theorem 3.4, is zero.

Obviously, it would be sufficient to assume merely that \( R \) is a translation of a period-parallelogram.

If \( F(z) \) has poles on the perimeter of the parallelogram \( R \), let us consider the parallelogram \( R' \) formed from \( R \) by a translation through \( c \), where \( c \) is a complex constant with a sufficiently small absolute value. The function \( F(z) \) has a finite number of poles in \( R \). Therefore, if \( c \) is suitably chosen, then the function \( F(z) \) will have the same poles in \( R' \) as in \( R \), and will be holomorphic on the perimeter of the parallelogram \( R' \). By virtue of the case already considered, the sum of its residues, corresponding to the poles lying in \( R' \), is equal to zero. Consequently, the sum of the
residues of the function $F(z)$, for poles lying in $R$, is also equal to zero and the theorem is proved.

(3.6) Every non-constant elliptic function is of order $\geq 2$.

Proof. An elliptic function of order 1 would have exactly one pole $z_0$ in the fundamental parallelogram, with principal part $c/(z-z_0).$ Since $c \neq 0$, we should have a contradiction to theorem 3.5.

Let us note now that together with $F(z)$ the function $F'(z)/F(z)$ is also elliptic and the sum of its residues in the period-parallelogram is equal to the difference between the number of roots and the number of poles of the function $F(z)$ in this parallelogram.

Applying theorem 3.5 we deduce from this that

(3.7) The number of roots of an elliptic function $F(z)$ in an arbitrary period-parallelogram is equal to the number of its poles in this parallelogram.

Replacing the function $F(z)$ by $F(z)-c$ we obtain that

(3.8) An elliptic function of order $r>0$ assumes every value $c$ exactly $r$ times in the period-parallelogram.

In view of theorem 3.7 the number of roots of an elliptic function in the period parallelogram depends on the number of poles lying in it. We shall now show that also the position of the roots depends on the position of the poles. Namely:

(3.9) Let $F(z)$ be an elliptic function of order $r>0$, and let $a_1, a_2, \ldots , a_r$ and $b_1, b_2, \ldots , b_r$ be the roots and poles, respectively, of the function $F(z)$ in the period-parallelogram $R$, where every root and every pole is counted as many times as its multiplicity indicates. Then

\[ a_1 + a_2 + \ldots + a_r \equiv b_1 + b_2 + \ldots + b_r \pmod{2m\pi \omega}. \]

Proof. Let us suppose, first, that $F(z)$ does not have any roots or poles on the perimeter of the parallelogram $R$ under consideration. Let us denote its vertices by $z_0, z_0+\omega, z_0+\omega+\omega'$ and $z_0+\omega'.$

The difference between the left and right sides of formula (3.10), multiplied by $2\pi i$, is equal to the integral of the function $zF'(z)/F(z)$ (see Chapter IV, theorem 7.5), taken along the perimeter of $R$, i.e.

\[ \frac{z_0}{z_0} F'(z) \frac{z_0}{z_0} + \frac{z_0+\omega}{z_0+\omega} F'(z) \frac{z_0+\omega}{z_0+\omega} + \frac{z_0+\omega'}{z_0+\omega'} F'(z) \frac{z_0+\omega'}{z_0+\omega'} + \frac{z_0}{z_0} F'(z) \frac{z_0}{z_0}. \]

\[ \int z_0 \frac{F'(z)}{F(z)} dz + \int \frac{z_0+\omega}{z_0+\omega} \frac{F'(z)}{F(z)} dz + \int \frac{z_0+\omega'}{z_0+\omega'} \frac{F'(z)}{F(z)} dz + \int z_0 \frac{F'(z)}{F(z)} dz. \]

The sum of the first and third integrals is

\[ \int z_0 \frac{F'(z)}{F(z)} dz + \int \frac{z_0+\omega}{z_0+\omega} \frac{F'(z+\omega)}{F(z+\omega)} dz + \int \frac{z_0+\omega'}{z_0+\omega'} \frac{F'(z+\omega')}{F(z+\omega')} dz + \int z_0 \frac{F'(z)}{F(z)} dz. \]

\[ = \int \frac{F'(z)}{F(z)} dz + \int \frac{z_0+\omega}{z_0+\omega} \frac{F'(z+\omega)}{F(z+\omega)} dz + \int \frac{z_0+\omega'}{z_0+\omega'} \frac{F'(z+\omega')}{F(z+\omega')} dz + \int z_0 \frac{F'(z)}{F(z)} dz. \]

Since, by hypothesis, the function $F(z)$ is holomorphic and different from zero on the side $[z_0, z_0+\omega]$, the integral of the function $F'(z)/F(z)$ along this side is equal to the increment of $\log F(z)$ (Chapter IV, § 5, p. 185), which, in view of $F(z_0)=F(z_0+\omega)$, is equal to $2n\pi i$, where $n$ is an integer.

Consequently, the sum of the first and third integrals in (3.11) is $-2n\pi i$. Similarly, the sum of the second and fourth integrals is equal to $2n\pi i$, where $m$ is an integer. Therefore the difference $a_1+a_2+\ldots+a_r-(b_1+b_2+\ldots+b_r)$ is equal to $ma_0-\omega a'$, and the formula (3.10) is proved. In the above reasoning it was sufficient to assume that $R$ is only a translation of a period-parallelogram.

In the case when $R$ contains roots or poles of the function $F(z)$ on the perimeter, we proceed as in the proof of theorem 3.5.

(3.12) If $\omega$ and $\omega'$ are arbitrary complex numbers, finite or not, and $a_1, a_2, \ldots , a_r$ and $a_1', a_2', \ldots , a_r'$, the roots of the equations $F(z)=-\omega'$ and $F(z)=\omega'$, respectively, lying in a period-parallelogram of the elliptic function $F(z)$ of order $r>0$, then

\[ a_1 + a_2 + \ldots + a_r \equiv a_1' + a_2' + \ldots + a_r' \pmod{2m\pi \omega, \omega'}. \]

Proof. If $\omega'=\infty$, for example, and $\omega$ is a finite number, then (3.13) is a consequence of theorem 3.9 applied to the function $F(z)-\omega'$. On the other hand, if $\omega$ and $\omega'$ are finite, then by theorem 3.9, applied successively to the functions $F(z)-\omega'$ and $F(z)-\omega''$, the left as well as the right sides of formula (3.13) are congruent to $b_1+b_2+\ldots+b_r \pmod{2m\pi \omega, \omega'}$, where $b_1, b_2, \ldots , b_r$ are the poles of the function $F(z)$ in the parallelogram under consideration. Therefore also in this case formula (3.13) is true.

We shall introduce still another notion. Let $a_1, a_2, \ldots , a_r$ be the set of all roots of the function $F(z)$ in the fundamental parallelogram, where the multiple roots are counted with the corresponding multiplicity. The set of numbers $a_1, a_2, \ldots , a_r$ will be called a complete system of roots of the function $F(z)$, if
$a_i \equiv a_i \pmod{\omega, \omega'}$ for $i = 1, 2, \ldots, r$.

The system $a_1, a_2, \ldots, a_r$ obviously represents the roots of the function $F(z)$ in the entire plane equally as well as the system $a_1, a_2, \ldots, a_r$ does, since by translating the points $a_1, a_2, \ldots, a_r$ through $m\omega + n\omega'$, where $m$ and $n$ assume all integral values, we obtain all the roots of the function $F(z)$. We similarly define a complete system of poles of the function $F(z)$, as well as a complete system of roots of the equation $F(z) = c = 0$.

Formula (3.10) remains valid if by $a_1, a_2, \ldots, a_r$ and $b_1, b_2, \ldots, b_r$ we mean arbitrary complete systems of roots and poles of the function $F(z)$. Similarly, in (3.13), $a_1, a_2, \ldots, a_r$ and $a'_1, a'_2, \ldots, a'_r$ may respectively denote complete systems of roots of the equations $F(z) = c' = 0$ and $F(z) = c'' = 0$.

§ 4. The function $\wp(z)$. In § 3 we proved a series of theorems on elliptic functions; so far, however, we have not given any example of such a function (different from a constant). We shall now show that the meromorphic function $\wp(z) = \wp(z; \omega, \omega')$ (see § 5, Chapter VII) is elliptic.

Let $\omega, \omega'$ be a pair of complex numbers different from 0 and such that $\mathbb{D}(\omega, \omega') > 0$. Let us consider the set $\Omega$ of points $w = m\omega + n\omega'$ in the plane, where $m, n = 0, \pm 1, \pm 2, \ldots$ Let us arrange all these points in an infinite sequence

$w_0 = 0, w_1, w_2, \ldots, w_k, \ldots$

Then

$$\wp(z; \omega, \omega') = \frac{1}{z^2} + \sum_{k=1}^{\infty} \frac{1}{(z - w_k)^2} - \frac{1}{w_k^2}.$$  \hspace{1cm} (4.1)

The function $\wp$ is holomorphic in the entire open plane, with the exception of the points $w_k$, where it has double poles. At points different from the points $w_k$, the series (4.1) is absolutely convergent, and hence its sum does not depend on the order of the terms. In addition, in every circle of finite radius the series (4.1) is uniformly convergent after discarding a sufficient number of initial terms. We can therefore differentiate term by term, which gives the formula

$$\wp'(z) = -\frac{2}{z^3} - \sum_{k=1}^{\infty} \frac{2}{(z - w_k)^3} = -2 \sum_{k=1}^{\infty} \frac{1}{(z - w_k)^3}. \hspace{1cm} (4.2)$$

If $z$ is not a period, then the series (4.2) is absolutely convergent, as a consequence of the convergence of the series $\sum_{k=1}^{\infty} \frac{1}{|w_k|}$, proved in § 5, Chapter VII.

We shall show, first, that the function $\wp'(z)$ has periods $\omega$ and $\omega'$. Now, from formula (4.2) it follows that

$$\wp'(z + \omega) = -2 \sum_{k=1}^{\infty} \frac{1}{(z - (w_k + \omega))^3} = -2 \sum_{k=1}^{\infty} \frac{1}{(z - w_k)^3} = \wp'(z),$$

because like $w_k$, the expression $w_k + \omega$ also assumes all the values of the set $\Omega$. In view of the symmetric role played by $\omega$ and $\omega'$, we also have $\wp'(z + \omega') = \wp'(z)$.

Integrating the equation $\wp'(z + \omega') = 0$ we get the formula

$$\wp(z + \omega) - \wp(z) = C,$$

where $C$ is a constant. In order to find its value, let us note that $\wp(z)$ is an even function of the variable $z$, because formula (4.1) gives

$$\wp(-z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} \frac{1}{(z + w_k)^2} - \frac{1}{w_k^2} = \frac{1}{z^2} + \sum_{k=1}^{\infty} \frac{1}{(z - (w_k + \omega))^2} + \frac{1}{(z - w_k)^2},$$

and $-w_k$ assumes, like $w_k$, all the values of the set $\Omega$. Putting $z = -\omega/2$ in the equation defining the constant $C$, we find

$$C = \wp(-\omega/2) - \wp(-\omega/2) - \wp(-\omega/2) = 0.$$

Consequently, $\wp(z + \omega) = \wp(z)$. Similarly, $\wp(z + \omega') = \wp(z)$. The function $\wp(z)$, being doubly periodic and meromorphic, is therefore elliptic.

We have shown that the numbers $w_k = m\omega + n\omega'$ are periods of the function $\wp(z)$. This function has no other periods. This follows from the fact that $z = 0$ is a pole of $\wp(z)$; hence, if there were periods different from the numbers $w_k$, then the function $\wp(z)$ would have poles at points different from $w_k$, which is not true. Consequently $\omega$ and $\omega'$ form a pair of primitive periods for the function $\wp(z; \omega, \omega')$. Since $\wp(z)$ has one double pole in every period-parallelogram, $\wp(z)$ is an elliptic function of order 2. The derivative $\wp'(z)$ is an elliptic function of order 3, and $\omega, \omega'$ is again a pair of primitive periods.

Elliptic functions form only a special, although the most important, class of doubly periodic functions. An example of a non-elliptic doubly periodic function is the function $\exp{\wp(z)}$. It is meromorphic and doubly periodic in

S. Saks and A. Zygmund, Analytic Functions.
the open plane minus the points \( w_0 \), at which it has essential singularities. For, in the neighbourhood of the point \( z = 0 \) we have

\[
\exp \phi(z) = \exp \left( z + G(z) \right) = H(z) \exp z^2,
\]

where \( G(z) \) and \( H(z) \) are functions holomorphic at the point 0, and hence \( z = 0 \) is an essential singularity of the function \( \exp \phi(z) \). Not being meromorphic in the entire open plane, the function \( \exp \phi(z) \) is not elliptic.

By theorem 3.6, the order of an elliptic function (different from a constant) is at least 2. Functions of order 2 are therefore the simplest elliptic functions. In the theory of doubly periodic functions they play the same fundamental role as the exponential function in the theory of periodic functions.

Let us now consider the derivative \( \phi'(z) \). As is easily seen from formula (4.2), it is an odd function. If \( w \) is its period, then we have the equations

\[
\phi'(w/2) = -\phi'(w/2), \quad \phi'(-w/2) = \phi'(w/2),
\]

of which the first is a consequence of the oddness, and the second of the periodicity, of the function \( \phi'(z) \). We get from this

\[
\phi'(w/2) = 0.
\]

In this argument we assumed implicitly that the number \( w/2 \) is not itself a period, because otherwise both sides of the formulæ (4.3) would be infinite for \( z = w/2 \). Limiting ourselves to the fundamental parallelogram, let us consider the points belonging to it which are half-periods, but not periods. There are three such points, namely:

\[
\frac{1}{2} \omega, \quad \frac{1}{2} \omega', \quad \frac{1}{2} (\omega + \omega').
\]

By formula (4.4) these are roots of the function \( \phi'(z) \). Since \( \phi'(z) \) is an elliptic function of order 3, it has exactly three roots in the fundamental parallelogram. Hence, the numbers (4.5) are the only roots of \( \phi'(z) \) in the fundamental parallelogram.

It follows from this that, except for \( z = 0 \), only the values assumed by the function \( \phi(z) \) at the points (4.5) are multiple; the values assumed by \( \phi(z) \) at the remaining points of the fundamental parallelogram are simple.

Let

\[
\phi(\omega/2) = \epsilon_1, \quad \phi(\omega'/2) = \epsilon_3, \quad \phi(\omega/2 + \omega'/2) = \epsilon_2.
\]

§ 5. Differential equation of the function \( \phi(z) \). The properties proved for the function \( \phi'(z) \) enable us to derive the differential equation of the function \( \phi(z) \). To that end, let us note that \( \phi'(z) \) has a triple pole at \( z = 0 \) and simple roots at the points (4.5). Consequently, \( \phi''(z) \) has a sextuple pole at \( z = 0 \), and double roots at the points (4.5). Let us consider the product

\[
[\phi(z) - \epsilon_1][\phi(z) - \epsilon_2][\phi(z) - \epsilon_3].
\]

The point \( z = 0 \) is a sextuple pole of this expression, and the points (4.5) are double roots. In view of theorem 3.3, \( \phi''(z) \) differs from the product (5.1) only by a constant factor. In order to find this factor it is sufficient to compare the principal parts of these functions at the point \( z = 0 \). From formulæ (4.1) and (4.2) we find that the principal part of the function \( \phi''(z) \) is \( 4z^{-1} + \ldots \), and the principal part of the product (5.1) is \( z^{-1} + \ldots \). Therefore the desired factor is 4, and consequently,

\[
\phi''(z) = 4[\phi(z) - \epsilon_1][\phi(z) - \epsilon_2][\phi(z) - \epsilon_3].
\]

This is precisely the differential equation of the function \( \phi(z) \) which we were seeking.
We shall give still another form of this equation. Let us put,
for \( n = 3, 4, 5, \ldots, \)
\[
(5.3) \quad g_n = \sum_{k=1}^{\infty} \frac{1}{w_k^n},
\]
where the sum extends over all the periods \( w_k \) different from 0.
For \( n \geq 3 \) the series \( (5.3) \) are absolutely convergent (see p. 316),
and, if \( n \) is odd, then \( s_n = 0 \), since the terms corresponding to
periods differing in sign cancel each other.

Let us now note that
\[
\frac{1}{(z - w_k)^3} = \frac{1}{w_k^3 \left( \frac{z}{w_k} \right)^3} = \frac{1}{w_k^3} \left( \frac{z}{w_k} \right)^3 + \frac{3}{w_k^3} \frac{z^2}{w_k^2} + \ldots \quad \text{for} \quad |z| < |w_k|.
\]

Let us apply this formula to each of the terms on the right side
of equation (4.1). By theorem 5.9, Chapter III, for \( z \) sufficiently
close to the point 0, we obtain the expansion
\[
(5.4) \quad \varphi(z) = z^2 + 3s_4z^4 + 35s_6z^6 + \ldots,
\]
whence
\[
\varphi'(z) = -2z^3 + 6s_4z + 20s_6z^3 + \ldots, \quad \varphi''(z) = 4z^4 - 24s_4z^2 - 80s_6z + \ldots, \quad \varphi'''(z) = z^5 - 10s_4z^3 + 15s_6z + \ldots.
\]
From the last two formulae we see that
\[
\varphi''(z) - 4\varphi'(z) = -60s_6z^2 - 140s_6z + \ldots,
\]
and therefore
\[
(5.5) \quad \varphi''(z) - 4\varphi'(z) + 60s_4\varphi(z) = -140s_6z + \ldots.
\]

The left side, which is an elliptic function, can have poles
only at the points \( w_k \). As is seen from the last equation, the function
under consideration is holomorphic at the point \( z = 0 \). It is therefore
holomorphic everywhere, and consequently a constant.

The value of this constant, as also follows from (5.5), is \(-140s_6\).
The function \( \varphi(z) \) therefore satisfies the differential equation
\[
(5.6) \quad \varphi''(z) - 4\varphi'(z) - g_3\varphi(z) = g_3,
\]
where, following Weierstrass, we use the notation
\[
(5.7) \quad g_4 = 60s_4 = 60 \sum_{k=1}^{\infty} \frac{1}{w_k^4}, \quad g_6 = 140s_6 = 140 \sum_{k=1}^{\infty} \frac{1}{w_k^6}.
\]

The numbers \( g_4 \) and \( g_6 \), which play an important role in the
theory of elliptic functions, are known by the name of \textit{invariants}.
(The reason for this name will appear in \( \S \text{11} \), p. 391.)
relations among the quantities \( s_n \) by comparing coefficients. In order to simplify the calculation, let us first differentiate the equation (5.6). Dividing by \( 2P'(z) \) we see that

\[
P'(z) = 6P(z) - \frac{1}{z} \frac{g_4}{z^2}.
\]

Let us introduce one more simplification by putting \( (2n-1)s_n = c_n \) for \( n = 2, 3, \ldots \). Formula (5.4) now has the form \( \varphi(z) = z^2 + \sum_{n=3}^\infty c_n z^{2n-2} \).

Since \( g_4 = 60c_4 = 20c_4 \), we can also rewrite (5.11) in the form

\[
6s^{-4} + \sum_{n=1}^\infty (2n-2)(2n-3)c_n z^{2n-4} = -10c_4 + 6 \frac{1}{z^2} + \sum_{n=3}^\infty c_n z^{2n-2}.
\]

Comparing the coefficients of \( z^{2n-4} \) we have the formula

\[
(n-3)(2n+1)c_n = 3(c_n c_{n-4} + c_{n-4} c_{n-8} + \ldots + c_4 c_0) \quad \text{for} \quad n = 4, 5, \ldots
\]

In particular:

\[
c_0 = \frac{1}{3} c_4^2, \quad c_4 = \frac{3}{11} c_{11} c_4, \quad c_6 = \frac{1}{13} (2c_4 c_6 + c_4 c_{12}) = \frac{1}{2} \left( \frac{2}{3} c_4^2 + c_6^2 \right), \quad \text{etc.}
\]

All the numbers \( c_n \) are therefore polynomials in \( c_4 \) and \( c_6 \) with rational coefficients. In other words, we have the following interesting theorem:

(5.12) The quantities \( s_1, s_3, s_5, \ldots \), defined by formula (5.3), are expressible as polynomials in the invariants \( g_2 \) and \( g_4 \) (and hence also in \( s_4 \) and \( s_6 \)) with rational coefficients.

We recall that the quantities \( s_4, s_6, s_8, \ldots \) are equal to zero.

Theorem 5.12 is the analogue of the more elementary theorem concerning the numbers \( S_n = \sum_{k=1}^n \frac{1}{k^2} \), where the summation is extended over all the integers \( k \neq 0 \). From formula (5.7), Chapter VII, it follows that \( S_n \) is a rational multiple of the number \( \pi^2 \) and hence also a rational multiple of \( S_n^\prime \).

Thus far we have treated \( \varphi(z) \) exclusively as a function of the variable \( z \), having a fixed pair of primitive periods \( \omega, \omega' \). In certain problems, however, it is also necessary to consider the dependence of the function \( \varphi \) on the periods \( \omega \) and \( \omega' \). Let us note that the function \( \varphi(z; \omega, \omega') \), treated as a function of all three variables, is homogeneous of degree \( -2 \), i.e., that for an arbitrary \( \lambda \neq 0 \) we have

\[
\varphi(\lambda z; \lambda \omega, \lambda \omega') = \lambda^{-2} \varphi(z; \omega, \omega').
\]

This is immediately evident if we replace \( z \) in formula (4.1) by \( \lambda z \) and \( w_k = \lambda \omega + \lambda \omega' \) by \( \lambda w_k \).

The numbers \( s_1, s_3, s_5, \ldots \), defined by formulae (4.6), are functions of \( \omega \) and \( \omega' \), i.e., \( s_i = s_i(\omega, \omega') \) for \( i = 1, 2, 3 \). From the preceding observations it follows that the numbers \( s_i \) are homogeneous functions of degree \( -2 \) of the variables \( \omega \) and \( \omega' \).

\section*{§ 6.
EXERCISE. Show that

\[
\varphi' (z) \varphi' (z + \omega / 2) \varphi' (z + \omega' / 2) \varphi' (z + \omega + \omega' / 2) = g_4^2 - 27g_6^2.
\]

[Hint. Having verified that the left side is constant, investigate its value when \( z = 0 \). To that end apply formula (5.11).]

\section*{§ 6. The functions \( \zeta(z) \) and \( \sigma(z) \).}

The functions \( \sigma(z) \) and \( \zeta(z) \), with which we have met earlier (cf. Chapter VII, § 5), also play an important role in the theory of elliptic functions. The first one of these is an entire function defined by the absolutely convergent product

\[
\sigma(z) = \exp \left( \sum_{k=1}^\infty \frac{1}{e^{\pi k^2 z} - 1} \right),
\]

extended over all the points \( w_k = m \omega + n \omega' \) different from 0. The function \( \sigma(z) \) has (simple) roots at the points \( w_k \) and only at these points. The function \( \zeta(z) \) is the logarithmic derivative of the function \( \sigma(z) \):

\[
\zeta(z) = \frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{k=1}^\infty \left( \frac{1}{z - w_k} + \frac{1}{w_k + w_k^2} \right)
\]

and has simple poles at the points \( w_k \). All of its residues are equal to 1.

The functions \( \sigma(z) \) and \( \zeta(z) \) are related to \( \varphi(z) \) by the equations

\[
\varphi(z) = -\frac{d}{dz} \zeta(z) = -\frac{d}{dz} \log \sigma(z).
\]

From formula (6.1) we see that the function \( \sigma(z) \) changes its sign when we replace \( z \) by \( -z \) (since at the same time we may change \( w_k \) to \( -w_k \)). Consequently, \( \sigma(z) \) is an odd function of the variable \( z \).

Similarly, replacing the variable \( z \) by \( -z \) in the series (6.2) defining the function \( \zeta(z) \), and \( w_k \) by \( -w_k \), we deduce that \( \zeta(z) \) is an odd function of the variable \( z \).

From the equation

\[
\frac{d}{dz} (\zeta(z + \omega) - \zeta(z)) = -[\varphi(z + \omega) - \varphi(z)] = 0
\]

and from an analogous equation obtained by replacing \( \omega \) by \( \omega' \), we get

\[
\frac{d}{dz} (\zeta(z + \omega') - \zeta(z)) = -[\varphi(z + \omega') - \varphi(z)] = 0
\]
(6.5) \[ \zeta(z + \omega') - \zeta(z) = \eta', \quad \zeta(z + \omega) - \zeta(z) = \eta, \]
where \( \eta \) and \( \eta' \) are certain constants. Applying (6.5) repeatedly, we obtain the general formula
\begin{equation}
(6.6) \quad \zeta(z + m\omega + n\omega') = \zeta(z) + m\eta + n\eta'.
\end{equation}

We see, therefore, that the function \( \zeta(z) \) has a certain “pseudo-periodicity”: by increasing the independent variable \( z \) by the quantity \( \omega_n \), we change the function only by an additive constant. It may happen that one of the numbers \( \eta, \eta' \) is equal to 0, but both cannot vanish simultaneously, because the function \( \zeta(z) \) has only one (simple) pole in the fundamental parallelogram, and hence it would then be an elliptic function of order 1 (cf. theorem 3.6).

Let us put \( z = -\omega/2 \) in the first of the formulae (6.5) and \( z = -\omega'/2 \) in the second. Taking the oddness of \( \zeta(z) \) into account, we find:
\[ \eta = 2z \left( \frac{1}{2} \omega \right), \quad \eta' = 2z \left( \frac{1}{2} \omega' \right). \]

The quantities \( \eta, \eta', \omega, \omega' \) are connected by a certain relation. In order to find it, let us consider the integral of the function \( \zeta(z) \) along the perimeter of the parallelogram \( \mathcal{P} \) with successive vertices \( z_0, z_0 + \omega, z_0 + \omega + \omega' \) and \( z_0 + \omega' \), where \( z_0 \) is an arbitrary number, not a pole of the function \( \zeta(z) \). This integral may be written in the form
\[ \int_{z_0}^{z_0 + \omega} [\zeta(z + \omega) - \zeta(z)] dz - \int_{z_0}^{z_0 + \omega'} [\zeta(z + \omega') - \zeta(z)] dz = \omega'\eta - \omega\eta'. \]

Since \( \mathcal{P} \) contains only one pole of the function \( \zeta(z) \), with residue 1, we have
\begin{equation}
(6.7) \quad \omega'\eta - \omega\eta' = 2\pi i.
\end{equation}

This is precisely the relation which we were seeking. It is known as Legendre’s equation. We recall that the periods \( \omega \) and \( \omega' \) were chosen so that \( \mathcal{S}(\omega'/\omega) > 0 \).

Let us take:
\[ \omega = m\omega + n\omega', \quad \eta = m\eta + n\eta'. \]

Writing the formula (6.6) in the form
\[ \frac{\sigma(z + \omega)}{\sigma(z + \omega')} = \frac{\sigma(z)}{\sigma(z)} + \frac{\eta}{\omega'}, \]
and integrating, we get the equation
\begin{equation}
(6.8) \quad \sigma(z + w) = C \exp(\eta w/2),
\end{equation}
where \( C \) is a constant. Let us suppose, first, that \( \sigma(w/2) \neq 0 \) and let us set \( z = -w/2 \) in the last equation. Taking the oddness of \( \sigma(z) \) into account, we obtain \( C = -\exp(\eta w/2) \).

The preceding reasoning fails when \( \sigma(w/2) = 0 \), because in that case, after substituting \( z = -w/2 \), both sides of formula (6.8) become equal to zero. But let us note that since \( \sigma(z) \) is an odd function, \( \sigma'(z) \) is an even function. Let us now differentiate equation (6.8) and take \( z = -w/2 \). If \( \sigma(w/2) = 0 \), then \( \sigma'(w/2) = 0 \), because the roots of the function \( \sigma(z) \) are simple. Let us divide both sides of the equation by \( \sigma'(w/2) \) and let us take the evenness of the function \( \sigma'(z) \) into account. In the case under consideration we get \( C = \exp(\eta w/2) \), and therefore an expression differing in sign from the one in the preceding case.

Let us note now that \( w/2 \) is a root of the function \( \sigma \) if and only if \( m \) and \( n \) are even simultaneously. Since \( m \) and \( n \) are both even if and only if the expression \( mn + m + n \) is even, we get the general formula
\[ \sigma(z + \omega) = -\sigma(z + \omega + \omega', \eta) \sigma(z), \quad \eta = m\eta + n\eta'. \]

In particular, putting \( m = 1, n = 0 \), and \( m = 0, n = 1 \), we have:
\begin{equation}
(6.9) \quad \sigma(z + \omega) = -\sigma(z + \omega', \eta) \sigma(z), \quad \sigma(z + \omega') = -\sigma(z + \omega, \eta) \sigma(z).
\end{equation}

Consequently, the function \( \sigma(z) \) is also “pseudo-periodic”, although in another sense than \( \zeta(z) \); by increasing the variable \( z \) by \( \omega \) or \( \omega' \) the function is multiplied by an exponential factor.

**EXERCISES.** 1. For \( \lambda \neq 0 \) we have:
\[ \sigma(z; \omega; \lambda) = \lambda^s \sigma(z; \omega, \omega'), \quad \zeta(z; \omega; \lambda) = \lambda^{-1} \zeta(z; \omega, \omega'). \]

2. Prove that the function \( \sigma(nz)/[\sigma(z)]^n \) is elliptic. Show that \( \sigma(2z)/\sigma(z) = -\sigma(z'/\omega) \).

3. Let \( w/\omega = \pi \) and \( w/\omega' = \varphi \). Show that:
\begin{align*}
\sigma(z; \omega, \omega') &= \frac{1}{\sqrt{\pi}} \exp\left( \frac{\omega}{\omega'} \right) \sin \frac{\pi}{\omega'} \left( 1 - \frac{\sin \pi}{\sin \pi} \right), \\
\zeta(z; \omega, \omega') &= \pi \left( \cot \pi \omega + \sum_{n=1}^{\infty} \left( \cot \pi (\omega + \omega') + \cot \pi (\omega - \omega') \right) \right), \\
\varphi(z; \omega, \omega') &= -\pi \omega + \frac{1}{\omega} \sum_{n=1}^{\infty} \frac{\sin \pi (\omega + \omega')}{\sin \pi (\omega - \omega')},
\end{align*}
where \( \pi = 2\zeta(\omega/2; \omega, \omega') \) (Weierstrass).
§ 7. Construction of elliptic functions by means of the function \( \sigma(z) \). Let \( F(z) \) be an elliptic function of order \( r \), and \( \sigma_1, \sigma_2, \ldots, \sigma_r \) and \( \beta_1, \beta_2, \ldots, \beta_r \), complete systems (p. 367 and 368) of roots and poles. By theorem 3.9 and the remark at the end of § 3, the difference between the sum of the numbers \( \sigma_i \) and the sum of the numbers \( \beta_i \) is of the form \( m \omega + n \omega' \), where \( m \) and \( n \) are integers. Replacing \( \beta_r \), for example, by \( \beta_r + m \omega + n \omega' \), we may assume that

\[
\alpha_1 + \alpha_2 + \ldots + \alpha_r = \beta_1 + \beta_2 + \ldots + \beta_r.
\]

Let us now consider the meromorphic function

\[
\Phi(z) = \frac{\sigma(z - \alpha_1) \sigma(z - \alpha_2) \ldots \sigma(z - \alpha_r)}{\sigma(z - \beta_1) \sigma(z - \beta_2) \ldots \sigma(z - \beta_r)}.
\]

We shall show that \( \Phi(z) \) has the periods \( \omega \) and \( \omega' \). For let \( a \) and \( \beta \) denote two arbitrary constants and \( G(z) = \sigma(z - a) / \sigma(z - \beta) \). From the first formula of (6.9) it follows that \( G(z + \omega) = e^{i \rho (\omega - \omega')} G(z) \). Let us apply this to the right side of (7.2). Taking condition (7.1) into account, we see that

\[
\Phi(z + \omega) = e^{i \rho (-\omega - \omega' - \beta_1 + \ldots + \beta_r)} \Phi(z) = \Phi(z).
\]

Consequently \( \omega \) and similarly \( \omega' \), is a period of the function \( \Phi(z) \). The latter has the same roots and poles as \( F(z) \), and hence, by theorem 3.3, it differs from \( F(z) \) by a constant factor. Whence:

\[
(7.3) \quad \text{Every elliptic function } F(z) \text{ of order } r \text{ can be represented in the form}
\]

\[
F(z) = C \frac{\sigma(z - \alpha_1) \sigma(z - \alpha_2) \ldots \sigma(z - \alpha_r)}{\sigma(z - \beta_1) \sigma(z - \beta_2) \ldots \sigma(z - \beta_r)},
\]

where \( C \) is a constant, and \( \alpha_1, \alpha_2, \ldots, \alpha_r \) and \( \beta_1, \beta_2, \ldots, \beta_r \) are, respectively, complete systems of roots and poles of the function \( F(z) \), satisfying condition (7.1). Conversely, every function of the form (7.4),

\[
\text{where } a_1, a_2, \ldots, a_r \text{ and } \beta_1, \beta_2, \ldots, \beta_r \text{ are arbitrary numbers satisfying equation (7.1), is an elliptic function.}
\]

EXAMPLES. 1. Express \( \varphi'(z) \) in terms of the function \( \sigma(z) \). The function \( \varphi'(z) \) has a triple pole at \( z = 0 \) and the roots \( \omega/2 \), \( (\omega + \omega')/2 \) and \( \omega'/2 \) in the fundamental parallelogram. Therefore, if we set:

\[
a_1 = \frac{1}{2} \omega, \quad a_2 = -\frac{1}{2} (\omega + \omega'), \quad a_3 = -\frac{1}{2} \omega', \quad \beta_1 = \beta_2 = \beta_3 = 0,
\]

then the condition \( a_1 + a_2 + a_3 = 1 + \beta_1 + \beta_2 + \beta_3 \) will be satisfied, and by virtue of theorem 7.3,

\[
\varphi'(z) = \frac{C}{\sigma(z - \omega/2) \sigma(z + \omega/2 + \omega'/2) \sigma(z - \omega'/2)} \sigma(z) \quad \quad (7.5)
\]

In order to determine the constant \( C \), let us consider the coefficients of \( z^3 \) in the expansion of both sides in a Laurent series at the point \( z = 0 \). We get

\[
C = \frac{2}{\sigma(\omega/2) \sigma(\omega/2 + \omega'/2) \sigma(\omega'/2)}.
\]

2. Express the function \( F(z) = \varphi(z) - \varphi(u) \) in terms of \( \sigma(z) \), where \( u \) is a constant, not a period of the function \( \varphi \).

Let us suppose that \( u \) is not a half-period. The function \( F(z) \) has a double pole at \( z = 0 \) and two simple non-congruent roots \( u \) and \( -u \). We may therefore put \( a_1 = u, a_2 = -u, \) and \( \beta_1 = \beta_2 = 0 \) in formula (7.4), which gives

\[
F(z) = \frac{C \sigma(z - u) \sigma(z + u)}{\sigma^2(z)}.
\]

Considering the principal parts of both sides at \( z = 0 \), we get

\[
C = -1/\sigma^2(u), \text{ i.e.}
\]

\[
\varphi(z) - \varphi(u) = -\frac{\sigma(z - u) \sigma(z + u)}{\sigma^2(z) \sigma^2(u)}.
\]

By continuity, we verify this formula also for the case when \( u \) is a half-period.

EXERCISES. 1. In some cases it is convenient to write \( a_1 \) instead of \( a_1 \) and \( a_0 \) instead of \( a_0 \), and to consider an auxiliary period \( a_0 \) defined by the condition \( \omega_1 + \omega_2 + \omega_3 = 0 \). Let \( e_i = \varphi(\omega_i/2) \) for \( i = 1, 2, 3 \). Prove that the func-
The function $G(x)$ has the periods $\omega$ and $\omega'$, for if we increase $x$ by $\omega$, for example, then $G(x)$ will increase by the number $\eta_1{(\omega^1+\omega^2+\ldots+\omega^r)}$, which is equal to zero in view of theorem 3.5. Being meromorphic, $G(z)$ is therefore an elliptic function. It has the same poles as $F(x)$ and the same residues. By theorem 3.3, $F(x)$ differs from $G(x)$ by an additive constant. Therefore:

(8.2) If an elliptic function $F(x)$ has only simple poles, then

(8.3) $F(x) = C + C^1(x-\beta_1) + C^2(x-\beta_2) + \ldots + C^r(x-\beta_{r})$

where $\beta_1, \beta_2, \ldots, \beta_r$ is a system of all the poles of the function belonging to an arbitrary period-parallellogram, the numbers $C^1, C^2, \ldots, C^r$ are the corresponding residues, and $C$ is a constant.

Proceeding to the general case, let us consider an arbitrary pole with multiplicity $k \geq 1$ of an elliptic function $F(x)$. The principal part of the function $F(x)$ at the point $\beta$ can be written in the form

(8.4) $\frac{C_1}{x-\beta} + \frac{1}{(x-\beta)^2} + \ldots + \frac{(-1)^{k-2}(k-1)!}{(x-\beta)^k} C_k$

Since the principal part of the function $\zeta(x-\beta)$ at the point $\beta$ is equal to $1/(x-\beta)$, the function

(8.5) $C_1 \zeta(x-\beta) + C_2 \zeta^2(x-\beta) + \ldots + C_k \zeta^{k-1}(x-\beta)$

will have the principal part (8.4) at the point $\beta$. Let $\beta_1, \beta_2, \ldots, \beta_s$ be a system of all the distinct poles of the function $F(x)$, lying in a period-parallellogram, with the respective multiplicities $k_1, k_2, \ldots, k_s$. Let us write the principal part of the function $F(x)$ at each of these poles in the form (8.4), replacing $\beta$ by $\beta_k$, $k$ by $k_i$, and $C_1, C_2, \ldots, C_k$ by $C_1^{(k_1)}, C_2^{(k_2)}, \ldots, \ldots, C_s^{(k_s)}$, $i = 1, 2, \ldots, s$. Therefore, if we put

$G(x) = \sum_{i=1}^{s} (C_1^{(k_1)} + C_2^{(k_2)} + \ldots + C_s^{(k_s-1)})(x-\beta_i)$

then the difference $F(x) - G(x)$ will be an entire function. We assert that it is doubly periodic. In fact, the functions $\zeta' = -\varphi, \zeta'' = -\varphi', \ldots$ are all doubly periodic; consequently, if we increase $x$ by $\omega$, the expression (8.5) increases by $C_1 \eta_1$, and hence $G(x)$ increases by the quantity $C_1^{(k_1)} + C_2^{(k_2)} + \ldots + C_s^{(k_s-1)} \eta$, which is equal to zero in view of theorem 3.5. Consequently, the function $G(x)$ has the period $\omega$, and similarly the period $\omega'$. The same can be said of the function $F(x) - G(x)$. This function, being doubly periodic and entire, must be a constant. From this we obtain the following theorem:
(8.6) Every elliptic function \( F(z) \) can be represented in the form
\[
A + \sum \gamma C_i(z - \beta_i) + C_1'(z - \beta_1) + \ldots + C_n'(z - \beta_n),
\]
where \( A \) is a constant, and the sum \( \sum \) is extended over all the distinct poles of \( F(z) \) in an arbitrary period-parallelogram. The numbers \( C_1, C_2, \ldots, C_n \) are the coefficients of the principal part of \( F(z) \) at the point \( \beta_i \), written in the form (8.4).

Conversely, every function of the form (8.7) is elliptic, provided that \( \sum C_i(z) = 0 \).

Obviously, we may replace \( \zeta, \zeta', \zeta'', \ldots \) in formula (8.7) by \(-\zeta, -\zeta', -\zeta'', \ldots\) respectively.

We shall now consider the problem of expressing an elliptic function in terms of the function \( \varphi(z) \), and we shall begin by making some observations.

(a) If an elliptic function \( G(z) \) is odd and is holomorphic at a point \( z_0 = w/2 \), which is a half-period, then \( G(z_0) = 0 \). Indeed, on the one hand, in view of the periodicity, we have \( G(-z_0) = -G(z_0) \), and on the other hand, in view of the oddness of the function \( G(z) \), \( G(-z_0) = -G(z_0) \). Consequently, \( G(z_0) = -G(z_0) \) and since the value \( G(z_0) \) is finite, it is equal to zero.

(b) If \( F(z) \) is an even function, then the derivative \( F'(z) \) is odd; similarly, the oddness of the function \( F(z) \) implies the evenness of \( F'(z) \). From this it follows, generally, that if \( F(z) \) is an even function, then the function \( F^{(2k)}(z) \) is even, and \( F^{(2k+1)}(z) \) odd; if \( F(z) \) is an odd function, then the function \( F^{(2k)}(z) \) is odd, and \( F^{(2k+1)}(z) \) even \((k = 0, 1, 2, \ldots)\).

(c) If an even elliptic function \( F(z) \) has a root at a point \( z_0 = w/2 \), which is a half-period, then the multiplicity of this root is even. This follows from the fact that, in view of (b) and (a), all the derivatives of odd order vanish at the point \( z_0 \).

Since a pole of a function \( F(z) \) is a root of the function \( 1/F(z) \), we see that, if \( z_0 = w/2 \) is a pole of an even elliptic function \( F(z) \), then the multiplicity of the pole is even.

Let us now consider an arbitrary even elliptic function \( F(z) \) and let \( a \) be a root of \( F(z) \). In view of the evenness of the function \( F(z) \), the number \(-a\) will also be a root, not convergent to \( a \), provided that \( a \) is not a half-period. In the latter case, \( a \) will be a root of even multiplicity. In any case, therefore, there exist numbers \( a_1, a_2, \ldots, a_n \), which together with \(-a_1, -a_2, \ldots, -a_n \) form a complete system of roots of the function \( F(z) \). (It follows from this, in particular, that the even elliptic functions are of even order.) Similarly, we can find numbers \( \beta_1, \beta_2, \ldots, \beta_n \) forming together with \(-\beta_1, -\beta_2, \ldots, -\beta_n \) a complete system of poles of the function \( F(z) \).

Let us assume for the moment that the periods \( w \) are neither roots nor poles of the function \( F(z) \). Since \( \varphi(z) \) is an even function, it is easy to see that the function \( F(z) \) has the same roots and poles as the function
\[
\varphi(z) - \varphi(a), \ldots, \varphi(z) - \varphi(a_n),
\]

By virtue of theorem 3.3, \( F(z) \) differs from the quotient (8.8) by a constant factor, and therefore

(8.9) Every even elliptic function \( F(z) \) can be expressed as a rational function of the function \( \varphi(z) \).

We have proved this under the assumption that \( z = 0 \) is neither a root nor a pole of the function \( F(z) \). If the point \( z = 0 \) is a root or a pole, then in any case it is one of even multiplicity, in view of the evenness of the function \( F(z) \). Therefore, there exists an integer \( k \) such that \( F(z)^k \varphi(z)^k \) is a function holomorphic and different from zero at \( z = 0 \). Applying the result obtained to the last function we see that theorem 8.9 is true in the general case.

If \( F(z) \) is an odd elliptic function, then the quotient \( F(z)/\varphi(z) \) is an even elliptic function, and, therefore, a rational function of \( \varphi(z) \). In other words, an odd elliptic function is the product of \( \varphi'(z) \) by a rational function of the function \( \varphi(z) \). Moreover, since for every function \( F(z) \) we have
\[
F(z) = \frac{1}{2} [F(z) + F(-z)] + \frac{1}{2} [F(z) - F(-z)],
\]
where the first term on the right side is an even function and the second odd,

(8.10) Every elliptic function \( F(z) \) can be written in the form
\[
R(\varphi) + \varphi\cdot R_1(\varphi),
\]
where \( R(\varphi) \) and \( R_1(\varphi) \) are rational functions of the variable \( u \). Conversely, every function of the form (8.11) is elliptic.
Let \( W(x, y) \) be an arbitrary rational function of the variables \( x \) and \( y \). The function \( W(p, p') \) is elliptic. On the other hand, every elliptic function has the form (8.11), and therefore is a rational function of \( p \) and \( p' \). Consequently,

(8.12) The class of elliptic functions is identical with the class of functions of the form \( W(p, p') \), where \( W(x, y) \) is an arbitrary rational function of two variables.

Theorem 8.12 is equivalent to theorem 8.10, since the expression \( W(p, p') \) is only in appearance more general than the expression (8.11). For, it is sufficient to note that, in view of equation (5.8), every even power of \( p' \) can be expressed rationally in \( p \), and hence \( W(p, p') \) has the form (8.11).

Exercises. 1. Let \( \beta_1 \) and \( \beta_2 \) be two distinct points of the fundamental parallelogram. Show that the most general elliptic function of the second order, having poles at the points \( \beta_1 \) and \( \beta_2 \), can be written in each of the two forms:

\[
A\left[ \frac{\zeta(\beta_1 - \beta)}{\zeta(\beta_1 - \beta_2)} + B, \frac{1}{\zeta(\beta_1 - \beta_2)} \right] + B,
\]

where \( A \) and \( B \) are constants, and \( \beta \) is the mid-point of the segment \([\beta_1, \beta_2]\).

2. Every elliptic function of the second order, having a double pole \( \beta \) in the fundamental parallelogram, has the form \( A\Phi(x - \beta) + B \), where \( A \) and \( B \) are constants.

3. Prove that

\[
\Phi(x - u) - \Phi(x + u) = \frac{\Phi'(x)\Phi'(u)}{[\Phi(x) - \Phi(u)]^3},
\]

where \( \Phi \) is a meromorphic function.

4. Prove that

\[
\frac{1}{\Phi'(x)} = \frac{1}{\Phi(x)} - \sum \frac{1}{3\Phi'(\beta_{mn})} \left[ \zeta(\beta_{mn}) + \zeta(\beta_{mn}^* \right],
\]

where \( \beta_{mn} = (m\omega + n\omega^*)/3 \), and \( m \) and \( n \) assume the values 0, 1, 2, but are not simultaneously zero.

§ 9. Algebraic addition theorem for the function \( \Phi(x) \). We say that a meromorphic function \( F(x) \) satisfies an algebraic addition theorem if there exists an algebraic relation among the quantities \( F(x), F(y), F(x + y) \), i.e. if

\[
W(F(x), F(y), F(x + y)) = 0
\]

identically, where \( W(u, v, w) \) is a polynomial in the variables \( u, v, w \), not identically equal to zero. For example, an algebraic addition

 theorem is satisfied by the functions \( e^x, \cos x, \tan x \). We shall show that

(9.1) The function \( \Phi(x) \) satisfies an algebraic addition theorem.

Proof. Let us write (7.6) in the form

\[
\Phi(x) - \Phi(y) = -\frac{\sigma(x + y)}{\sigma^2(x)\sigma^2(y)} \cdot \Phi(x) \cdot \Phi(y).
\]

Taking the logarithmic derivatives of both sides, first with respect to \( x \), and then with respect to \( y \), we obtain the equations:

\[
\frac{\Phi'(x)}{\Phi(x) - \Phi(y)} = \zeta(x + y) + \zeta(x - y) - 2\zeta(x),
\]

\[
\frac{\Phi'(y)}{\Phi(x - \Phi(y)} = \zeta(x + y) - \zeta(x - y) - 2\zeta(y),
\]

whence, by addition,

\[
\frac{\Phi'(x) - \Phi'(y)}{2[\Phi(x) - \Phi(y)]} = \zeta(x + y) - \zeta(x - y).
\]

Let us differentiate this formula e. g. with respect to \( x \). We get

(9.2)

\[
\Phi(x + y) = \Phi(x) - \frac{1}{2} \frac{\Phi'(x) - \Phi'(y)}{[\Phi(x) - \Phi(y)]^2}.
\]

If we now carry out the differentiation and apply the equation \( \alpha^2 = 6\Phi^2 - g_2/2 \), we obtain a formula expressing \( \Phi(x + y) \) in terms of \( \Phi(x), \Phi(y), \Phi'(x), \Phi'(y) \) (just as \( e^x \) is expressed in terms of \( \sin x, \sin y \)). Hence, \( \Phi(x) \) and \( \Phi'(x) \) (up to a power of \( \Phi(x) \)), which could then be removed by raising to a power.

The addition theorem for the function \( \Phi(x) \) can be given a more symmetrical form. Let us write (9.2) in the form

(9.3)

\[
\Phi(x + y) = \Phi(x) - \frac{1}{2} \left\{ \frac{\Phi'(x)}{\Phi(x) - \Phi(y)} - \frac{\Phi'(y)}{[\Phi(x) - \Phi(y)]^2} \right\}.
\]

Let us interchange here \( x \) and \( y \) and add the formula so obtained to (9.3). We get

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\[ 2\varphi(x+y) = \varphi(x) + \varphi(y) - \frac{1}{2} \left( \varphi''(x) - \varphi''(y) \right) + \frac{1}{2} \left( \varphi'(x) - \varphi'(y) \right)^2. \]

Let us apply the equation \( \varphi'' = 6 \varphi^3 - g_2/2 \) (cf. (5.11)). We get the formula

\[ \varphi(x+y) + \varphi(x) + \varphi(y) - \frac{1}{4} \left( \varphi'(x) - \varphi'(y) \right)^2. \]

This is precisely the form of the addition theorem which we were seeking.

EXERCISES. 1. Prove that

\[ \varphi(x + \frac{1}{2} \omega) = e_i + \frac{(\epsilon_i - 1) (\epsilon_j - 1)}{\varphi(x) - e_i}, \]

where the number triple \( i, j, k \) is a permutation of the triple 1, 2, 3, and the numbers \( \omega_1, \omega_2, \omega_3 \) are defined as in exercise 1, § 7.

2. Show that

\[ \varphi(2x) = \varphi^3 + \frac{g_2}{4} \varphi^2 + \frac{g_1}{2} \varphi + \frac{1}{2} \varphi^2, \]

where \( \varphi = \varphi(x). \)

§ 10. Algebraic relations between elliptic functions.

(10.1) Between every two co-periodic elliptic functions \( F_1(x) \) and \( F_2(x) \) there exists an algebraic relation, i.e. a relation of the form

\[ W(F_1(x), F_2(x)) = 0, \]

where \( W(u, v) \) is a polynomial in the variables \( u, v \), not identically equal to zero.

Proof. Let \( \omega, \omega' \) be a common pair of periods of the functions \( F_1(x) \) and \( F_2(x) \), and \( x_1, x_2, \ldots, x_r \) the system of all the distinct points which are poles of at least one of the functions \( F_1(x), F_2(x) \), and which lie in the parallelogram with vertices \( 0, \omega, \omega + \omega', \omega'. \) Let \( k_i \) and \( k'_i \), respectively, be the multiplicities of the poles of \( F_1(x) \) and \( F_2(x) \) at the point \( x = x_i \), and let \( k = \text{Max}(k_i, k'_i) \). Finally, let \( N \) be a positive integer such that \( N + 3 > 2(k_1 + k_2 + \ldots + k_r) \). The general form of a polynomial \( V(u, v) \) in the two variables \( u, v \), of degree \( N \), is

\[ V(u, v) = c_0 + \sum_{0 < \ell_1 + \ell_2 < N} c_{\ell_1 \ell_2} u^{\ell_1} v^{\ell_2}. \]

§ 11. The modular function \( J(v) \).

Let us consider a pair of primitive periods \( \omega, \omega' \) of the function \( \varphi(x; \omega, \omega') \), where, as always, we assume that \( \Im(\omega'/\omega) > 0 \).

Let the numbers \( g_2 = g_2(\omega, \omega') \) and \( g_3 = g_3(\omega, \omega') \) be the invariants (cf. § 5, p. 372), i.e. let

\[ g_2(\omega, \omega') = 60 \sum_{m,n=-\infty}^{\infty} \frac{1}{(m\omega + n\omega')^4}, \]

\[ g_3(\omega, \omega') = 140 \sum_{m,n=-\infty}^{\infty} \frac{1}{(m\omega + n\omega')^6}, \]

where the sign ‘indicates that the term corresponding to the indices \( m = n = 0 \) is omitted in the summation. In § 5 we proved
that the discriminant \( g_2^2 - 27g_3^3 \) is different from zero. The following problem is important for the theory of elliptic functions: Two arbitrary numbers \( a \) and \( b \), satisfying the condition \( a^3 - 27b^2 \neq 0 \), are given. Does there exist a pair of periods \( \omega, \omega' \), such that \( g_2(\omega, \omega') = a \), \( g_3(\omega, \omega') = b \)? The solution of this problem will be given in § 13. It will be based on the properties of a certain special function, the so-called modular function \( J(\tau) \), with which we shall be concerned presently.

We shall first introduce certain general definitions. A function \( F \) meromorphic in a region \( G \) is said to be automorphic with respect to a given group \( \Sigma \) of transformations of the region \( G \) into itself (see Introduction, § 7, p. 15) if for every transformation \( T \in \Sigma \) we have \( F(T(z)) = F(z) \), identically in the region \( G \). For example, if \( \omega \neq 0 \) is an arbitrary complex number, and \( G \) denotes the entire plane or strip bounded by two parallels to the straight line \( \omega_0 \), then the transformations of the form \( z = x + m\omega \), where \( m = 0, \pm 1, \pm 2, \ldots \), form a group of transformations of the region \( G \), and in order that the meromorphic function \( F \) in \( G \) be automorphic with respect to this group it is necessary and sufficient that the function \( F \) be periodic with period \( \omega \). Similarly, if \( \omega \) and \( \omega' \) denote complex numbers different from zero, having a non-real quotient, then the transformations of the form \( z = x + m\omega + n\omega' \), where \( m \) and \( n \) are arbitrary integers, form a group of transformations of the open plane, and in order that the meromorphic function \( F \) be automorphic with respect to this group, it is necessary and sufficient that it be doubly periodic with periods \( \omega \) and \( \omega' \).

As we verify immediately (cf. Chapter I, § 14, p. 82 and 83), the homographic transformations which can be written in the form

\[
3 = \frac{ax + \beta}{\gamma x + \delta},
\]

where \( a, \beta, \gamma, \delta \) are real integers, and \( a\delta - \beta\gamma = 1 \), form a group of transformations of the closed plane. This group is known as the modular group and plays a particularly important role in the theory of functions. Two points, one of which can be carried into the other by means of any transformation of this group, are said to be congruent with respect to the modular group.

1) An introduction to the theory of such functions can be found in the monograph of E. L. Ford, Automorphic Functions, New York 1929.

\* \* \*

Every transformation belonging to the modular group maps the upper half-plane \( \Im z > 0 \) into itself. Since the coefficients of the transformations (11.2) are real, the real axis is mapped into itself under these transformations, and it is sufficient to show that under transformations (11.2) at least one point of the upper half-plane goes into a point of the same half-plane; now, substituting \( e. y. z = i \) in (11.2), we find in fact that

\[
\Im \beta = \frac{a\delta - \beta\gamma}{\gamma^2 + \delta^2} > 0.
\]

From this it follows at the same time that the modular group constitutes not only a group of transformations of the entire closed plane, but also a group of transformations of the open upper half-plane.

A function \( F(z) \), meromorphic in the upper half-plane, is said to be a modular elliptic function if it is automorphic with respect to the modular group, or at least with respect to a subgroup of the modular group (i.e., to a family of transformations contained in the modular group and forming a group) not reducible to the identity transformation only. Before we give examples of modular functions, we shall prove the following lemma:

(11.3) Let \( \omega, \omega' \) be a pair of primitive periods of any elliptic function, and \( \omega, \omega' \) a pair of periods given by the formula:

\[
\omega' = \alpha \omega + \beta \omega', \quad \omega = \gamma \omega + \delta \omega,
\]

where \( \alpha, \beta, \gamma, \delta \) are integers. Then a necessary and sufficient condition that the pair \( \omega, \omega' \) also be a pair of primitive periods is the relation

\[
a\delta - \beta\gamma = \pm 1;
\]

and in order that the numbers \( \Im (\omega'/\omega) \) and \( \Im (\omega'/\omega) \) be of the same sign, a necessary and sufficient condition is the relation \( a\delta - \beta\gamma = 1 \).

Proof. Let us put \( A = a\delta - \beta\gamma \). Solving the system (11.4) for \( \omega \) and \( \omega' \), we obtain:

\[
\omega' = \frac{\beta \omega - \alpha \omega'}{A}, \quad \omega = \frac{\gamma \omega + \delta \omega'}{A}.
\]

If \( A = \pm 1 \), then \( \omega \) and \( \omega' \) are the sums of integral multiples of the numbers \( \omega \) and \( \omega' \). Therefore every period of the elliptic function under consideration, as the sum of integral multiples of the numbers \( \omega \) and \( \omega' \), is the sum of integral multiples of the
numbers \( w, w' \). The pair of periods \( w, w' \) is, consequently, primitive and the sufficiency of the condition (11.5) is proved. Proceeding to the proof of the necessity of the condition, let us assume that the pair \( w, w' \), defined by the formulae (11.4), is a primitive pair. Then \( \Delta \neq 0 \), and from (11.6) it follows that all four of the coefficients:

\[
a = \frac{\beta}{\Delta}, \quad b = \frac{\gamma}{\Delta}, \quad c = -\frac{\gamma}{\Delta}, \quad d = \frac{\alpha}{\Delta}
\]

must be integers. Let us put \( D = ad - bc \). It is easy to verify that \( D = (\alpha \gamma - \beta \gamma)/\Delta \). Moreover, since \( D \) and \( \Delta \) are integers, \( \Delta = \pm 1 \).

In order to prove the second part of the lemma let us note that if we take \( w'/w = z \) and \( w'/w = z' \), then from (11.4) it follows that

\[
z' = \frac{\alpha z + \beta}{\gamma z + \delta},
\]

and the point \( z' \) lies in the same half-plane (upper or lower) as the point \( z \) if and only if \( \alpha \delta - \beta \gamma > 0 \), i.e. if \( \Delta = 1 \). Lemma 11.3 is therefore proved.

Returning to the invariants \( g_2 \) and \( g_3 \), let us note first of all that they are homogeneous functions of the periods \( \omega \) and \( \omega' \), of degrees \(-4\) and \(-6\), respectively. If we put \( \omega'/\omega = \tau \), then

\[
g_2(\omega, \omega') = -g_2(1, \tau), \quad g_3(\omega, \omega') = -g_3(1, \tau).
\]

The discriminant \( \Delta(\omega, \omega') = g_2^3 - 27g_3^2 \) is therefore a homogeneous function of degree \(-12\). The ratio \( g_2/\Delta \), as a homogeneous function of degree \(0\) of the variables \( \omega, \omega' \), depends therefore only on the ratio \( \tau = \omega'/\omega \). The ratio \( g_2/\Delta \) will be denoted by \( J(\tau) \). Consequently,

\[
J(\tau) = \frac{g_2(1, \tau)}{\Delta(1, \tau)}, \quad J(\tau) - 1 = \frac{27g_3^2(1, \tau)}{\Delta(1, \tau)}.
\]

(11.7) The function \( J(\tau) \) is holomorphic in the half-plane \( \Im(\tau) > 0 \) and automorphic with respect to the modular group, i.e.

\[
J(\tau) = J\left(\frac{\alpha \tau + \beta}{\gamma \tau + \delta}\right) \quad \text{when} \quad \Im(\tau) > 0,
\]

for every system of integers \( \alpha, \beta, \gamma, \delta \) for which \( \alpha \delta - \beta \gamma = 1 \).

Proof. First, we shall show that the function

\[
g_2(1, \tau) = 60 \sum_{m,n=-\infty}^{\infty} \frac{1}{(m+\tau)^4}
\]

is holomorphic for \( \Im(\tau) > 0 \). Since every term of this series is holomorphic in the upper half-plane, it is sufficient to prove that the series is uniformly convergent in every half-strip \( S \) defined by the inequalities

\[
a \leq \Re(\tau) \leq b, \quad \Im(\tau) > c, \quad a, b, c > 0.
\]

If \( \tau \) belongs to \( S \), then the altitude \( h \) is bounded from below by some number \( \varepsilon > 0 \), and as follows from the considerations on p. 316 — the absolute values of the terms of the series (11.10) do not exceed in \( S \) the terms of a certain convergent numerical series. This proves that the series (11.10) is uniformly convergent in \( S \). Consequently \( g_2(1, \tau) \) is indeed a holomorphic function in the upper half-plane.

The same result is obtained for the function \( g_3(1, \tau) \). It follows from this that \( \Delta(1, \tau) \) also is a holomorphic function in the upper half-plane, and since \( \Delta(1, \tau) = 0 \), the function \( J(\tau) \) is also holomorphic there.

For the proof of the remaining part of the theorem let us consider the transformation (11.4), where \( \alpha, \beta, \gamma, \delta \) are integers and \( \alpha \delta - \beta \gamma = 1 \). If \( \alpha \omega, \omega' \) is a pair of primitive periods of the function \( p(z) \), then \( w, w' \) is also a pair of primitive periods; this means that when \( m \) and \( n \) assume all integral values, the set of periods \( m\omega + n\omega' \) is identical with the set of periods \( m\omega + n\omega' \). By the same token, as follows from (11.1),

\[
g_2(\omega, \omega') = g_2(w, w'), \quad g_3(\omega, \omega') = g_3(w, w')
\]

(whence the name invariants of the numbers \( g_2 \) and \( g_3 \)). Therefore \( A(\omega, \omega') = A(w, w') \) also, and we can write

\[
J(\tau) = J\left(\frac{\alpha \tau + \beta}{\gamma \tau + \delta}\right)
\]

since \( w'/w = (\alpha \tau + \beta)/(\gamma \tau + \delta) \) in view of (11.4). Theorem 11.8 is therefore proved.

Particular transformations of the modular group are the transformations:

\[
\tau' = \tau + 1 = \frac{1}{\tau + 1}, \quad \tau' = -\frac{1}{\tau} = \frac{0 \cdot \tau + 1}{-1 \cdot \tau + 0}
\]

Consequently,

\[
J(\tau + 1) = J(\tau), \quad J(-1/\tau) = J(\tau), \quad \text{for} \quad \Im(\tau) > 0.
\]
EXERCISES. 1. Given are: a family $\mathcal{R}$ of transformations belonging to the modular group (although not necessarily forming a group) and a function $F(z)$, meromorphic in the half-plane $\Im z > 0$ and such that $F(T(z))=F(z)$, when $\Im z > 0$ and $T \in \mathcal{R}$. Then there exists a subgroup $\mathcal{E}$ of the modular group, containing $\mathcal{R}$ and such that the function $F(z)$ is automorphic with respect to the group $\mathcal{E}$.

2. The transformations $z'=(\alpha z+\beta)/(\gamma z+\delta)$ of the modular group, where $\beta$ and $\gamma$ are divisible by the integer $n$, form a group (and hence a subgroup of the modular group). In the case $n=2$ we obtain the so-called even subgroup of the modular group (the numbers $\alpha$ and $\delta$ are then odd).

3. Let $\lambda(\tau)=(\varepsilon_1-\varepsilon_2)/(\varepsilon_3-\varepsilon_4)$, where the numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are defined by the formulae (4.6). If $\tau'$ is congruent to $\tau$ with respect to the modular group, then $\lambda(\tau')$ assumes one of the 6 values:

\[ \lambda(\tau), \ 1-\lambda(\tau), \ \frac{1}{1-\lambda(\tau)}, \ \frac{\lambda(\tau)}{1-\lambda(\tau)}, \ \frac{\lambda(\tau)}{\lambda(\tau)-1}, \ 1-\frac{1}{\lambda(\tau)}. \]

Investigate, when $\lambda(\tau')=\lambda(\tau)$.

[Hint. If the pairs of primitive periods $\omega, \omega'$ and $\omega, \omega'$ are related by formula (11.4), then the number triples $\varphi(\omega, \omega'), \varphi(\omega', \omega), \varphi(\omega, \omega'), \varphi(\omega', \omega)$ differ in order at most.]

4. The function $\lambda(\tau)$ in exercise 3 is holomorphic in the half-plane $\Re \tau > 0$, does not assume the values 0, 1, and is automorphic with respect to the even subgroup of the modular group. (Next to $\lambda(\tau)$, the function $\lambda(\tau)$ is the most important modular elliptic function.)

5. Let $\lambda=\lambda(\tau)$. The function

\[ F(\tau)=(\lambda+1)(1-\lambda+1)\left(\frac{1}{1-\lambda+1}\right)^2\left(\frac{2}{\lambda-1+1}\right) \]

is a modular elliptic function, automorphic with respect to the modular group. (If we simply took the product of the functions $\lambda$ in exercise 3 instead of $F(\tau)$, we should obtain 1.)

6. Show that the function $F(\tau)$ in exercise 5 satisfies the equation

\[ F(\tau)=27(1-J(\tau)), \]

and hence that

\[ J(\tau)=\frac{4}{27}\left(1-\lambda(\tau)^{3}\right). \]

[Hint. Express $\lambda$ in formula (**) in terms of the quantities $\varepsilon_1, \varepsilon_2, \varepsilon_3$, and apply the formulae (5.8), (5.10), (11.7).]

§ 12. Further properties of the function $J(\tau)$. The first of the formulae (13.11) indicates that the function $J(\tau)$ has the period 1. Because of this (see § 2), $J(\tau)$ is a holomorphic func-

\[ \text{tion of the variable } \zeta=e^{2\pi i \tau} \text{ in the annulus } P(0;0,1). \text{ We shall prove that} \]

\[ \text{(12.1) } J(\tau) \text{ as a function of the variable } \zeta=\exp 2\pi i \tau \text{ has a simple pole at the point } \zeta=0. \]

Proof. To that end, we shall proceed from the expansion of the function $\cot \pi \tau$ into simple fractions (see Chapter VII, formula (5.2)):

\[ \pi \cot \pi \tau=\frac{1}{\tau}+\sum_{n=-\infty}^{\infty}\frac{1}{\tau-n+\varepsilon}, \]

and we shall make use of the following formula, true for $\Re \tau > 0$:

\[ \pi \cot \pi \tau=-\pi i(1+2\zeta^2+2\zeta^4+\ldots), \text{ where } \zeta=e^{2\pi i \tau}. \]

(cf. (2.5)). In this formula, let us replace $\pi \cot \pi \tau$ by the expansion (12.2) and let us differentiate the equation obtained three times and five times with respect to $\tau$, remembering that $d\zeta/d\tau=2\pi i \zeta$.

We get the equations:

\[ -6\sum_{m=-\infty}^{\infty}\frac{1}{(m+\tau)^4}=-16\pi^4(1+2\zeta^2+\ldots), \]

\[ -120\sum_{m=-\infty}^{\infty}\frac{1}{(m+\tau)^6}=64\pi^6(1+2\zeta^2+\ldots), \]

which we shall make use of when investigating the invariants $g_2$ and $g_3$. To that end we shall also need the equations:

\[ \sum_{m=-\infty}^{\infty}\frac{1}{m+\tau}^4=\frac{\pi^4}{45}, \sum_{m=-\infty}^{\infty}\frac{1}{m+\tau}^6=\frac{2\pi^6}{945}, \]

which we can obtain e. g. by differentiating the equation (12.2) three and five times, respectively, and putting $\tau=0$ (cf. also Chapter VII, formulae (5.4) and (5.6)).

Let us now replace $\tau$ by $m\tau$ in the first of the equations, and hence $\zeta$ by $e^{2\pi i m \tau}$, where $m=1, 2, \ldots$. Since $(m+\tau)^4=-(m+\tau)^4$,

\[ g_4(1, \tau)=60\sum_{m=-\infty}^{\infty}\frac{1}{m+\tau}^4+2\sum_{m=-\infty}^{\infty}\frac{1}{m+\tau}^6=60\frac{\pi^4}{45}+16\pi^4 \sum_{n=1}^{\infty}(-1)^{n+1}(\zeta^2+8\zeta^2+\ldots). \]

Let us denote by $G_4(\zeta)$ the sum of the power series $\zeta^2+8\zeta^2+\ldots$, standing under the last summation sign. The series $G_4(\zeta)+G_4(\zeta)+\ldots$
is almost uniformly convergent in the circle \( K=K(0;1) \) to the sum \( G(\zeta) \) holomorphic in \( K \). Therefore we obtain the expansion of the function \( G(\zeta) \) in a power series by adding formally the power series defining the functions \( G_1(\zeta), G_2(\zeta), \ldots \) (Chapter III, theorem 5.9). It follows from this that
\[
g_1(1, \tau) = \pi^4 \left( \frac{1}{3} + \frac{320}{3} \zeta + \ldots \right).
\]
Reasoning similarly we obtain
\[
g_1(1, \tau) = 140 \left( \sum_{m=1}^{\infty} \frac{1}{m^4} + 2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(m+n)^4} \right)
\]
\[
= 140 \left( \frac{2\pi^4}{945} \sum_{n=1}^{\infty} \left( \frac{1}{n^3} + 32 \zeta + \ldots \right) \right)
\]
\[
= \pi^4 \left( \frac{8}{27} + \frac{448}{3} \zeta + \ldots \right).
\]
From the formulae for \( g_1(1, \tau) \) and \( g_4(1, \tau) \) it follows that
\[
A(1, \tau) = g_1^2(1, \tau) - 27g_1^2(1, \tau) = \pi^4 \left( 4096 \zeta + \ldots \right),
\]
which gives for \( J(\tau) = g_1^2(1, \tau)/A(1, \tau) \) the expression
\[
\left( \frac{4}{3} + \frac{320}{3} \zeta + \ldots \right)^3
\]
\[
= \frac{1}{1728} + c_0 + c_1 \zeta + \ldots.
\]
Theorem 12.1 is therefore proved. From it there follows, in particular, that \( J(\tau) \rightarrow \infty \) when \( \Im \tau \rightarrow \infty \). Because of this, we can extend the definition of the function \( J \) to the point \( \infty \), taking \( J(\infty) = \infty \).

Let us consider an arbitrary pair of numbers \( \omega, \omega' \), different from zero and with a non-real quotient. If the integers \( \alpha, \beta, \gamma, \delta \) satisfy the condition \( \alpha \delta - \beta \gamma = 1 \), then the pair of numbers \( \omega, \omega' \), defined by formula (11.4) will be called a pair equivalent to the pair \( \omega, \omega' \). The pairs \( \omega, \omega' \) and \( \omega, \omega' \) obviously play a symmetric role in this definition. We shall prove the following lemma:

(12.5) Given an arbitrary pair of numbers \( \omega, \omega' \), different from zero and with a non-real quotient, one can choose an equivalent pair \( w, w' \) such that
\[
|w'| \geq |w|, \quad |w' \pm w| \geq |w'|.
\]

Proof. Let us consider the set \( \Omega \) of points \( m\omega + n\omega' \), where \( m \) and \( n \) are arbitrary integers. This set can here be considered as the set of periods of a doubly periodic function \( g(\zeta) \). Let \( \omega \) be an arbitrary element of the set \( \Omega \) different from \( \theta \) and having the smallest absolute value. Let us consider the subset \( \Omega_1 \), consisting of all the elements of the set \( \Omega \) not lying on the line \( \omega \), and let us take for \( \omega' \) that point of the set \( \Omega_1 \) which has the smallest absolute value. Changing the sign of \( \omega' \), if necessary (because, like \( \omega' \), \( -\omega' \) also belongs to \( \Omega_1 \)), we may assume that the numbers \( \mathcal{S}(\omega'/\omega) \) and \( \mathcal{S}(\omega'/\omega) \) are of the same sign. We obviously have \( |\omega'| \geq |\omega| \).

From the definition of the number \( \omega' \) it follows that the numbers \( \omega' \pm \omega \) do not lie on the line \( \omega \), and hence the second of the conditions (12.6) is also satisfied. Reasoning such as that on pp. 357-358 indicates that the pair \( \omega, \omega' \) is a pair of primitive periods of the function \( g(\zeta) \), and therefore, by virtue of theorem 11.3, is equivalent to the pair \( \omega, \omega' \). Hence lemma 12.5 is proved.

Let \( G_0 \) denote the set of all those points \( \tau \) of the upper half-plane for which
\[
-\frac{1}{2} \leq \Re \tau < 0, \quad |\tau| \geq 1 \quad \text{or} \quad 0 \leq \Re \tau < \frac{1}{2}, \quad |\tau| > 1.
\]
The point \( \infty \) is also included in \( G_0 \). Consequently, \( G_0 \) is a curvilinear triangle bounded by two half-lines parallel to the imaginary axis and by an arc of the circumference \( C(0;1) \) (see Fig. 33 on p. 396). Of the boundary points, however, we include in \( G_0 \) only the side \( A \infty \) and the arc \( A \beta \), and exclude the side \( A' \infty \) as well as the interior of the arc \( A'B \). We shall call the set \( G_0 \) the fundamental region, and the points \( A, B, \infty \), the vertices of the fundamental region.

(12.7) For every point \( \tau \) of the upper half-plane there exists a point congruent to it with respect to the modular group and lying in the fundamental region.

Proof. Let us consider the pair of numbers \( 1, \tau \). By virtue of lemma 12.5, we can find an equivalent pair \( w, w' \) satisfying condition (12.6). We therefore have \( \omega' = \alpha \omega + \beta, \omega = \gamma \omega + \delta \), where the numbers \( \alpha, \beta, \gamma, \delta \) are integers and \( \alpha \delta - \beta \gamma = 1 \). If we put \( w'/w = \tau^* \), then \( \tau^* = (\alpha \omega + \beta)/(\gamma \omega + \delta) \), and from (12.6) it follows that \( |\tau^*| \geq 1 \) and \( |\tau^* + 1| \geq |\tau^*| \). Consequently, \( \tau^* \) lies either in \( G_0 \) or on the side \( A' \infty \), or else is an interior point of the arc \( A'B \). In the first case the lemma is proved. Therefore we may assume that one of the two remaining cases holds.

Now, the side \( A \infty \) is obtained from side \( A' \infty \) by means of the transformation \( \tau = -1/\tau \). Similarly, the arc \( AB \) is obtained from the arc \( A'B \) by means of the transformation \( \tau = -1/\tau \). Both
of these transformations belong to the modular group. Applying
an appropriate one of these to the point \( \tau \), we obtain a point
congruent to \( \tau \) with respect to the modular group and lying in \( \mathcal{G}_0 \).

From theorems 11.8 and 12.7 it follows that in order to in-
vestigate what values the function \( J(\tau) \) assumes for \( \Im \tau > 0 \) it is
sufficient to limit ourselves to the fundamental region.

\[ (12.9) \quad J(\tau) = e^c, \]

where \( c \) is an arbitrary number (finite or infinite), has exactly one
solution in the fundamental region \( \mathcal{G}_0 \).

Proof. Since \( J(\tau) = \infty \) at the vertex \( \infty \) of the fundamental
region \( \mathcal{G}_0 \) (cf. p. 394), the theorem is true for \( c = \infty \). We may
therefore assume that \( c \) is a finite number.

Let \( s \) be a positive number so large that \( |J(\tau)| > |c| \) for \( \Im \tau > s \).
Therefore, if we denote by \( H_1 \) that part of the
set \( \mathcal{G}_0 \) where \( \Im \tau < s \), then all the possible
roots of the equation \( (12.9) \) belonging to \( \mathcal{G}_0 \) will have to lie in \( H_1 \).

Let us denote the points \(-1/2+i\kappa\) and \(1/2+i\kappa\) by \( D \) and \( D' \), respectively,
and by \( \Gamma_1 \) the closed curve \( AB A'D'D'A \), bounding the set \( H_1 \) (see Fig. 33).

1° Let us first consider the case in which equation
\((12.9)\) does not have
roots on \( \Gamma_1 \). By virtue of theorems 7.5 and 5.4, Chapter IV, the number
\( N \) of roots of this equation in the
interior of \( \Gamma_1 \) (counting every root as
many times as its multiplicity in-
dicates), multiplied by \( 2\pi \), is equal to the increment of the argument of the function \( J(\tau) - c \) along the curve \( \Gamma_1 \). This increment is
equal to the sum of the increments along the arcs \( DA, AB, BA', A'D', D'D' \). But, in view of the relation \( J(\tau + 1) = J(\tau) \), the
increments along the segments \( DA \) and \( D'A' \) are equal, i.e., the
sum of the increments of the arguments along the segments \( DA \) and \( A'D' \) is equal to zero. Similarly, since \( J(-1/2+i\kappa) = J(\tau) \), the sum of the increments of the arguments along the arcs \( A'H \) and \( BA' \)
is equal to zero. Consequently, \( 2\pi N \) is equal to the increment of \( \arg \{J(\tau) - c\} \) along the segment \( D'D' = [1/2+i\kappa, -1/2+i\kappa] \).

But when the point \( \tau \) describes this segment, the point \( \zeta = e^{2\pi i \tau} \)
describes the circumference \( C_\tau = C(0; e^{-2\pi i \tau}) \) oriented negatively. By
virtue of theorems 7.5 and 5.4, Chapter IV, \( 2\pi N \) is therefore equal to
\( 2\pi \) times the difference between the number of poles and the number
of roots of the expression \( J(\tau) - c \), considered as a func-
tion of the variable \( \zeta \), inside the circumference \( C_\tau \). From the
definition of the segment \( DD' \) it follows that \( J(\tau) - c \) does not have
any roots inside \( C_\tau \), while by theorem 12.1 it has exactly one pole
inside \( C_\tau \), namely, the simple pole at the point \( \zeta = 0 \). It follows
from this that \( N = 1 \) and the theorem is proved in the particular
case considered.

2° Let us next consider the case in which the equation
\((12.9)\) also has roots on the curve \( \Gamma_1 \), but at points different
from \( A \) and \( B \) (and hence also from \( A' \)), namely, at the points
\( \lambda_1, \lambda_2, \ldots, \lambda_k \) lying in the interior of the segment \( AD \), and at the
points \( \mu_1, \mu_2, \ldots, \mu_l \) in the interior of the arc \( AB \) (of course we
can have \( k = 0 \) or \( l = 0 \). On Fig. 33 (p. 396) we have taken for
simplicity \( k = l = 1 \)). Because of this, the equation \( (12.9) \) will also
have roots at the points \( \lambda'_1, \lambda'_2, \ldots, \mu'_1, \mu'_2, \ldots \), symmetric,
respectively, to the preceding ones with respect to the imaginary axis. Let the number \( \eta \), positive and smaller than \( 1/2 \), be so small that the circles
\( K_p = K(\lambda_p, \eta) \) and \( K_q = K(\mu_q, \eta) \) do not contain the points \( D, A, B \)
or any roots of the equation \( (12.9) \) other than the centres \( \lambda_p \) and
\( \mu_q \). Let the transformation \( \tau' = \tau + 1 \) carry the circle \( K_p \) into
the circle \( K'_p \), and let the transformation \( \tau' = -1/\tau \) carry the
circle \( K_q \) into the circle \( K'_q \) (the centre of the circle \( K'_q \) need not go into
the centre of the circle \( K'_q \)). Let us consider the set
\( H_1 = H_1 + \sum_{\tau} K_p + \sum_{\tau'} K_q - \sum_{\tau'} K'_p - \sum_{\tau'} K'_q \).

Let us denote by \( \Gamma_1 \) the positively oriented curve which is
the boundary of the set \( H_1 \). On the curve \( \Gamma_1 \) the equation \( (12.9) \)
does not have any roots, and inside \( \Gamma_1 \) the number \( N \) of these roots
is the same as in \( \mathcal{G}_0 \). The increments of the arguments on the arcs
of the curve \( \Gamma_1 \), corresponding to each other by means of the transfor-
mations \( \tau' = \tau + 1 \) or \( \tau' = -1/\tau \), are equal, and — reasoning as
before — we find that \( 2\pi N \) is equal to the increment of \( \arg \{J(\tau) - c\} \) along the segment \( D'D' \). This again gives \( N = 1 \).
3. There still remains for us to consider the case in which the equation (12.9) has roots \( a = e^{\frac{N\pi i}{m}} \) and \( b \). These points have the respective coordinates \( a = e^{\frac{N\pi i}{m}} \) and \( b \). Let us note now that, in view of the identity \( m \equiv \infty \equiv (n + im) \), we have

\[
\frac{1}{140} g_{z}(1, i) = \sum_{m,n=\infty}^{\infty} \frac{1}{(m-in)^{2}} = \sum_{m,n=\infty}^{\infty} \frac{1}{(n+im)^{2}} = -\frac{1}{140} g_{z}(1, i).
\]

Consequently, \( g_{z}(1, i) = 0 \). Similarly, if we take into account that \( \beta^{2} = 1 \) and that \( \beta^{2} + \beta + 1 = 0 \), we obtain

\[
g_{z}(1, \beta) = \frac{\sum_{m,n=\infty}^{\infty} \frac{1}{(m+\beta n)^{2}}}{\sum_{m,n=\infty}^{\infty} \frac{1}{(\beta m + n)^{2}}} = \frac{\sum_{m,n=\infty}^{\infty} \frac{1}{(m-n) + n\beta}}{\sum_{m,n=\infty}^{\infty} \frac{1}{m + n\beta}}.
\]

When the system \( m, n \) runs through all possible pairs of integers, with the exception of the pair \( 0, 0 \), then the system \( m-n, m \) does the same thing. Consequently, the last expression equals \( g_{z}(1, \beta) / 60 \), which gives \( g_{z}(1, \beta) = 0 \). Collecting the equalities obtained and taking into consideration formulae (11.7), we have

\[
(12.10) \quad g_{z}(1, \beta) = 0, \quad g_{z}(1, i) = 0, \quad J(\beta) = 0, \quad J(i) = 1.
\]

There have thus remained only the cases \( c = 1 \) and \( c = 0 \). In considering these cases we shall make use of the following simple observation. Let \( F(\tau) \) be a function holomorphic in the circle \( K = K(\tau; \varepsilon) \) with an \( m \)-tuple root at the centre of this circle. Let \( L_{m} \) denote an arbitrary arc of the circumference \( C(\tau; \varepsilon) \), and \( L_{m} \), the angular measure of the arc \( L_{m} \), where \( \varepsilon < R \). If, as \( \epsilon \to 0 \), we have \( L_{m} \to 0 \), then the increment of the argument of the function \( F(\tau) \) along the arc \( L_{m} \), tends to \( ma \). To prove this let us note that if \( \tau_{0} = 0 \), for example, then \( F(\tau) = e^{\tau G(\tau)} \), where \( G(\tau) \) is a function holomorphic in \( K \), and \( G(0) \neq 0 \). As \( \epsilon \to 0 \), the increment of \( \arg G(\tau) \) along the arc \( L_{m} \), tends to zero and the increment of \( \arg \tau^{m} \) to \( ma \). Let us suppose now that \( c = 1 \). In addition to the roots \( \lambda_{1}, \lambda_{2}, \ldots, \lambda_{m}, \mu_{1}, \mu_{2}, \ldots \) considered in case 2, the root \( i \) of the equation \( J(\tau) = 1 = 0 \) now enters in. Let \( m \) be the multiplicity of this root. Let \( H_{1} = H_{2} - K(i; \varepsilon) \), where \( H_{1} \) is the set considered in case 2 (with \( c = 1 \)), and \( \varepsilon \) is sufficiently small. Finally, let \( L_{m} \) denote that arc of the circumference \( C(i; \varepsilon) \) which lies in the closure of the set \( H_{2} \), and \( N \) the number of roots of the equation \( J(\tau) = 1 = 0 \) in \( H_{2} \). Reasoning as before, we find that \( 2\pi N \) is equal to the sum of the increments of \( \arg[J(\tau) - 1] \) along the segment \( D' \) and along the arc \( -L_{m} \). Applying the property of the increment, mentioned above, we find, when \( \epsilon \to 0 \), that \( N = 1 = -m/2 \). The number \( m \) is integral and positive, and the number \( N \) integral and negative. From this it follows that \( m = 2 \) and \( N = 0 \). In other words, the equation \( J(\tau) - 1 = 0 \) has in \( G_{0} \) exactly one root (double) at the point \( \tau = \tau_{0} = 0 \). The case of the equation \( J(\tau) = 0 \) is resolved similarly by considering the increment of \( \arg J(\tau) \) along the positively oriented boundary of the set

\[
H_{2} = K(\varepsilon; \tau; \varepsilon) - K(\varepsilon; \tau; \varepsilon),
\]

where \( \varepsilon \to 0 \). Since the arcs \( AB \) and \( AD \) form the angle \( \pi/3 \) at the point \( \varepsilon = e^{\pi i/2} \), we find that the equation \( J(\tau) = 0 \) has exactly one root (triple) in the fundamental region \( G_{0} \), namely, \( \tau = q \). Theorem 12.8 is therefore entirely proved. Let \( G_{0} \) and \( G_{0}' \) respectively, denote the parts of the fundamental region \( G_{0} \) where \( \Re \tau < 0 \) and \( \Re \tau > 0 \). Completing theorem 12.8, we shall show that

\[
(12.11) \quad \text{The function } w = J(\tau) \text{ maps the set } G_{0} \text{ in a one-to-one manner into the half-plane } \Re w > 0, \text{ and the closure of the set } G_{0}' \text{ in a one-to-one manner into the half-plane } \Re w < 0.
\]

Proof. We shall first show that \( J(\tau) \) assumes conjugate values at points situated symmetrically with respect to the imaginary axis, i.e., that \( J(-\tau) = J(\tau) \). In fact

\[
g_{z}(1, -\tau) = 60 \sum_{m,n=\infty}^{\infty} \frac{1}{(m+n\tau)^{2}} = 60 \sum_{m,n=\infty}^{\infty} \frac{1}{(m-n\tau)^{2}} = g_{z}(1, \tau),
\]

and similarly we prove the formula \( g_{z}(1, -\tau) = g_{z}(1, \tau) \). From this we obtain \( J(-\tau) = J(\tau) \). From this equation it follows that \( J(\tau) \) assumes real values on the imaginary axis. If the point \( \tau \) is on the boundary of the fundamental region \( G_{0} \), then in view of the formulae (11.11) we have \( J(-\tau) = J(\tau) \). From this and the preceding equation it follows that \( J(\tau) \) is real. Consequently, the function \( J(\tau) \) assumes real values on the boundaries of the sets \( G_{0} \) and \( G_{0}' \).

When the point \( \tau \) describes the boundary of the region \( G_{0} \), namely, the half-line \( [\infty, 0] \), the arc \([0, i] \) of the circle \( C(0; 1) \), and the half-line \([i, \infty] \), the point \( w = J(\tau) \) describes the real axis from \( -\infty \) through the points 0 and 1 to the point \( +\infty \).
In view of theorem 12.8 and of the continuity of the function \( J(\tau) \), this function assumes on the boundary of the set \( G'_2 \) every real value \( w \) at exactly one point. It follows from this that the function \( J(\tau) \) does not assume real values in the interior of \( G'_2 \). We shall show that in the interior of \( G'_2 \) the imaginary part of the function \( J(\tau) \) is either always positive or always negative. For if there were two points inside \( G'_2 \) where the imaginary part of the function \( J \) has different signs, then at some point of any arc joining these two points inside \( G'_2 \) the function \( J(\tau) \) would be real, which is impossible. Applying the theorem on the preservation of angles (Chapter I, theorem 15.8), we easily see that for \( \tau \) belonging to the interior of \( G'_2 \) we have \( \Im J(\tau) > 0 \). Consequently, the function \( w = J(\tau) \) transforms the set \( G'_2 \) into the closed half-plane \( \Im w \geq 0 \) in a one-to-one manner. From this and from the equation \( J(-\tau) = J(\tau) \) it follows that this function transforms the closure of the set \( G'_2 \) into the half-plane \( \Im w \leq 0 \) in a one-to-one manner.

Let \( T_0, T_1, T_2, \ldots \) be a sequence of all the transformations of the modular group, and let \( T_a \) denote the identity transformation. Let \( G_{k} = T_k(G_0) \) for \( k = 0, 1, \ldots \). The points of the set \( G_k \) into which the transformation \( T_k \) carries the vertices of the set \( G_0 \) we shall call the vertices of the set \( G_k \). The sets \( G_k \) are curvilinear triangles lying in the half-plane \( \Im \tau > 0 \) (some of the vertices lie on the real axis), and, in view of theorem 12.8, the function \( J(\tau) \) assumes in \( G_k \) every value exactly at one point. From theorem 12.7 it follows that the set \( G_0 + G_1 + \ldots \) covers the entire half-plane \( \Im \tau > 0 \). We shall now show that

(12.12) Two different sets \( G_k \) and \( G_l \) can have at most one vertex as a common point.

First we shall prove the following lemma:

(12.13) In the fundamental region \( G_0 \) there exist exactly three points which can be carried into points belonging to \( G_k \) by some transformations of the modular group other than the identity. These are the vertices \( \infty, g, i \), and each of them can be carried only into itself.

Proof. Let us suppose that \( \tau \) and \( \tau' \) are two points of the region \( G_0 \) and that the transformation \( \tau' = (\alpha \tau + \beta) / (\gamma \tau + \delta) \), where \( \alpha, \beta, \gamma, \delta \) are integers satisfying the condition \( \alpha \delta - \beta \gamma = 1 \), is not the identity transformation. If one of the points \( \tau, \tau' \) lies at infinity, then the same can be said of the other, because the transforma-

Fig. 34 shows the distribution of a certain number of the triangles of \( G_k \). Shaded are those parts of the triangles where the imaginary part of the function \( J(\tau) \) is positive.
EXERCISES. 1. In an arbitrary neighbourhood of any point \( \tau_0 \) of the real axis there are contained infinitely many sets \( G_\tau \).

([Hint] It is sufficient to consider the case when \( \tau_0 = i/\gamma \), where \( \alpha \) and \( \gamma \) are relatively prime integers. As is known, we can then find integers \( \beta \) and \( \delta \) such that \( \alpha \delta - \beta \gamma = 1 \). Let \( T(\tau) = (\alpha \tau + \beta)/(|\gamma \tau + \delta|) \) and let \( G_\tau \) denote the set \( G_\tau \) translated through \( \tau \). Consider the sets \( T(G_\tau) \).

2. Every closed and bounded set contained in the half-plane \( \Im \tau > 0 \) is covered by a finite number of sets \( G_\tau \).

3. The half-plane \( \Re \tau > 0 \) is a natural region for \( J(\tau) \) (cf. Chapter VI, § 4).

4. The function \( J(\tau) \) tends to \( \infty \) as \( \tau \to \infty \), remaining in the angle \( \varepsilon \leq \text{Arg} \tau < \varepsilon \pi /2 \), whatever \( \varepsilon > 0 \) is.

5. The inverse function \( J^{-1}(z) \) of the modular function \( J \) is an infinitely valued analytic function having critical points at the points \( 0, 1, \infty \) only (Chapter VII, § 11); all its values belong to the half-plane \( \Re \tau > 0 \). If \( F(z) \) is an entire function not assuming the values \( 0 \) or \( 1 \), then each of the functions \( J^{-1} F \) (cf. Chapter VI, §§ 5, 9) reduces to a constant. Deduce from this the 'small theorem of Picard' (Chapter VII, theorem 12.1).

([Hint] Cf. theorem 12.8 and Chapter VI, theorem 11.1. By the monodromy theorem (Chapter VI, theorem 6.3), the function \( J^{-1} F \) is single-valued and therefore holomorphic; apply the theorem of exercise 6, Chapter II, § 5.)

6. In addition to the properties mentioned in exercise 5, the function \( J^{-1} \) has the following property: if \( P \) is a closed and bounded set contained in the half-plane \( \Re \tau > 0 \), then the set of those points of the plane at which the function \( J^{-1} \) assumes at least one value \( \tau \in P \) is also bounded.

([Hint] Cf. theorem 2.)

7. Using the properties of the function \( J^{-1} \) prove Montel's theorem: If \( [F_\tau(z)] \) is a sequence of functions holomorphic in a region \( G \) and none of the functions of this sequence assumes the values \( 0 \) or \( 1 \) in \( G \), then the sequence \( [F_\tau(z)] \) is normal (Chapter VII, theorem 13.12); this theorem implies 'the great theorem of Picard'.

([Hint] One may assume that \( G \) is a circle and it is sufficient to show that the sequence \( [F_\tau(z)] \) contains a subsequence almost uniformly convergent in \( G \), or almost uniformly divergent in \( G \) to \( \infty \). Let us consider that this is not so. Hence, there exists in \( G \) a sequence of points \( \{z_\nu\} \) such that \( z_\nu \to \tau_0 \) in \( G \), \( F_\tau(z_\nu) \to \alpha \), where \( \alpha \neq 0, 1, \infty \). Let \( \tau_0 \to \infty \), \( \tau_\nu \to 0 \) (see theorem 12.8). Let us now fix, for every \( \nu \), one of the functions \( J^{-1} F_\nu \) (Chapter VI, § 5, 9), denoting it by \( \Phi_\nu \). By virtue of the monodromy theorem for the circle (Chapter VI, theorem 6.2), the functions \( \Phi_\nu \) are holomorphic in \( G \), and they can be so chosen that \( \Phi_\nu(z_\nu) \to \alpha \). Finally, from the sequence \( \{\Phi_\nu\} \) (cf. Chapter III, theorem 11.4) one can choose a subsequence \( \{\Phi_{\nu_\nu}\} \), almost uniformly convergent in \( G \) to a function \( \Phi \). If the function \( \Phi \) is not a constant, then the region \( \Phi(G) \) is contained, together with the regions \( \Phi_\nu(G) \) (cf. Chapter III, theorem 11.2), in the open half-

§ 13 Solution of the system of equations \( g_2(\omega, \omega') = a \), \( g_3(\omega, \omega') = b \). From theorem 12.8 there follows a positive solution of the problem formulated at the beginning of § 11, namely:

(13.1) If \( a, b \) are arbitrary finite numbers satisfying the condition \( a^2 - 27b^2 \neq 0 \), then there always exists a pair of periods \( \omega, \omega' \), with a non-real quotient, such that

(13.2) \( g_2(\omega, \omega') = a \), \( g_3(\omega, \omega') = b \).

Let us suppose, first, that \( a \neq 0 \) and \( b \neq 0 \). If the equations (13.2) are satisfied, then we have

(13.3) \( g_3(\omega, \omega') = \frac{a^2}{g_2^3(\omega, \omega') - 27g_2^2(\omega, \omega')} = \frac{a}{g_2^2(\omega, \omega') - 27b^2} = \frac{1}{b} \),

conversely, if the expressions \( g_2 \) and \( g_3 \) satisfy equations (13.3), then the equations (13.2) also hold. The systems (13.2) and (13.3) are therefore equivalent. Setting \( \omega' = 1/\tau \), we write the first one of the equations (13.3) in the form \( J(\tau) = \omega^2(\omega^2 - 27b^2) \). By theorem 12.8, this equation always has a solution in the upper half-plane. If \( \tau \) is already known, then the second one of the equations (13.3), which can be written in the form

\[ \omega^2g_3(1, \tau) = \frac{a}{g_2(1, \tau)} \]

gives us \( \omega \), and therefore also \( \omega' = 1/\omega \).

Next, let us suppose that \( a = 0 \), for example. The system of equations (13.2) is therefore equivalent to the system \( g_3^2(1, \tau) = 0 \), \( g_3 = b \), or — what amounts to the same thing — to the system \( J(\tau) = 0 \), \( \omega' = g_3(1, \tau) = b \). By virtue of the third one of the equations (12.10) we may therefore take \( \tau = 0 \), and from the equation
$o^{-q} g_0(1, q) = b$ determine $a$. We consider the case of $b = 0$ in a similar manner. Theorem 13.1 is therefore entirely proved 1).

EXERCISE. Given are three distinct numbers $e_1, e_2, e_3$, satisfying the condition $e_1 + e_2 + e_3 = 0$. Then there exists a pair of periods $o, o'$ such that:

$$
\varphi(o + 2o; o) = e_1, \quad \varphi(o + o' + o + o'; o) = e_2, \quad \varphi(o; o) = e_3.
$$

§ 14. Elliptic integrals. In the present section we shall denote a complex variable by $x$. Let $P(x)$ be an arbitrary polynomial without multiple roots. Let us consider the equation

$$(14.1) \quad y^2 = P(x).$$

This equation defines $y$ as a double-valued function of the variable $x$. If the circle $K = K(x_0, r)$ with a finite centre does not contain any root of the polynomial $P$, then there exist in $K$ two holomorphic branches of the function $y$, differing from each other in sign. By fixing the value of $\sqrt{P(x)}$ at one of the points of the circle $K$, we choose a definite branch of the function $y$. This branch is continuous along every curve whose initial point is $x_0$ and which does not pass through the point $\infty$ or through any of the roots of the polynomial $P$.

By a definite elliptic integral we mean a curvilinear integral of the form

$$(14.2) \quad \int_{L(x_0, x_1)} W(x, y) \, dx,$$

where $W(x, y)$ is a rational function of two variables, $y$ is a function of the variable $x$, defined by equation (14.1), in which $P(x)$ is a polynomial of degree $3$ or $4$ without multiple roots, and $L(x_0, x_1)$ is a regular curve with initial point $x_0$ and terminal point $x_1$, not passing through any of the roots of the function $P(x)$. In addition, we assume that the integrand $W(x, y)$ does not reduce to a rational function in $x$ only, and does not assume the value $\infty$ on $L$.

(The conditions concerning the curve $L(x_0, x_1)$ and the values assumed by $W(x, y)$ along $L$, are not necessary and can be disregarded in certain cases. We shall then have to deal with improper elliptic integrals. We shall not consider them, however, in order not to introduce unessential difficulties.)

1) The above proof of the solvability of the system $g_2 = a, g_3 = b$ is taken from Hurwitz-Courant, Funktionentheorie, 3rd edition, Berlin 1929, p. 227.

Obviously, the integral (14.2) becomes defined only when we fix the value of the function $y$ at some point of the curve $L(x_0, x_1)$, for example, at the point $x_0$. Then the values of $y$ are determined along the entire curve $L(x_0, x_1)$.

If we fix the point $x_0$ and vary the point $x_1 = x$, then the integral (14.2) will depend on $x$ and on the curve $L$. Such an integral, to which we add an arbitrary constant, will be called an indefinite elliptic integral. We shall write it in the form

$$(14.3) \quad \int_{L(x_0, x)} W(x, y) \, dx.$$
CHAPTER VIII. Elliptic functions.

The last condition states that \( g_{1}^{2} - 27g_{3}^{3} \neq 0 \). By virtue of theorem 13.1, we may therefore consider the numbers \( g_{2} \) and \( g_{3} \) as the invariants \( g_{2}(\omega, \omega') \) and \( g_{3}(\omega, \omega') \) for a certain pair of periods \( \omega, \omega' \) with a non-real quotient.

(14.5) If \( g_{2} = g_{2}(\omega, \omega') \), \( g_{3} = g_{3}(\omega, \omega') \), and \( \varphi(u) = \varphi(u; \omega, \omega') \), then to every pair of numbers \( x, y \) satisfying equation (14.4), there corresponds in every period-parallelogram \( R \) of the function \( \varphi(u) \) exactly one point \( \gamma \) such that \( x = \varphi(\gamma) \) and \( y = \varphi'(\gamma) \).

Proof. The parallelogram \( R \) certainly contains one or at most two points \( u \) such that \( \varphi(u) = x \) (see theorem 3.8). Let us denote them by \( u_{1} \) and \( u_{2} \), taking \( u_{1} = u_{2} \) if there exists only one point \( u \) satisfying this equation. We have \( u_{1} = -u_{2} \), and therefore \( \varphi'(u_{1}) = -\varphi'(u_{2}) \) (this also holds when \( u_{1} = u_{2} \), since in that case \( \varphi'(u_{1}) = \varphi'(u_{2}) = 0 \)). On the other hand (cf. (5.6),

\[ (\varphi'(u_{k}))^{3} = 4[\varphi(u_{k})]^{3} - g_{2} \varphi(u_{k}) - g_{3} = 4x^{3} - g_{2} x - g_{3} = y^{3}, \]

where \( k = 1, 2 \). It follows from this that \( y = \varphi'(u_{k}) \) for one of the values \( k = 1, 2 \). Denoting the corresponding point \( u_{k} \) by \( \gamma \), we have \( x = \varphi(\gamma) \), \( y = \varphi'(\gamma) \).

(14.6) Let \( g_{2} = g_{2}(\omega, \omega') \), \( g_{3} = g_{3}(\omega, \omega') \), and let \( x = x(t) \), where \( a \leq t \leq b \), be a regular curve not passing through any one of the roots of the right side of equation (14.4). Furthermore, let \( y = y(t) \) be a continuous function in \([a, b]\), related to \( x = x(t) \) by means of the equation (14.4).

Then there exists a regular curve \( u = u(t) \), where \( a \leq t \leq b \), such that:

\[ x(t) = \varphi[u(t)], \quad y(t) = \varphi'[u(t)], \quad \text{for } a \leq t \leq b. \]

Proof. Let \( a \leq t_{0} \leq b \), \( x_{0} = x(t_{0}) \), and \( y_{0} = y(t_{0}) \). Then \( y_{0} = 4x_{0}^{3} - g_{2} x_{0} - g_{3} = 0 \), and by lemma 14.5 there exists a point \( u_{0} \) such that \( \varphi(u_{0}) = x_{0} \), \( \varphi'(u_{0}) = y_{0} = 0 \). Therefore (Chapter III, theorem 12.4) the function \( \varphi(u) \) is uniquely invertible in the neighbourhood of the point \( u_{0} \) and its inverse transforms the arc of the curve \( x = x(t) \) in a sufficiently small interval \([t_{0} - h, t_{0} + h]\) into a regular arc \( u = u(t) \) in the same interval (if \( t_{0} = a \) or \( t_{0} = b \), we consider, of course, only the interval \([t_{0} - h, t_{0} + h]\) or \([t_{0} - h, t_{0} + h]\)). For \( t_{0} - h \leq t \leq t_{0} + h \) we shall therefore have \( \varphi[u(t)] = x(t) \) and \( \varphi'[u(t)]^{3} = y(t)^{3} \), and because \( \varphi'(u_{0}) = y_{0} \) and \( y(t) \neq 0 \), we have precisely \( \varphi'[u(t)] = y(t) \).

Applying now e.g. the Borel-Lebesgue theorem (Introduction, theorem 6.4), we can divide the interval \([a, b]\) into a finite number of subintervals \([t_{1}, t_{2}], [t_{2}, t_{3}], \ldots, [t_{n-1}, t_{n}]\), where \( a = t_{1} \), \( b = t_{n} \), in such a way that it is possible to define in them the corresponding regular curves \( C_{k} \) given by equations \( u = u_{k}(t) \), satisfying the conditions \( \varphi[u_{k}(t)] = x(t), \quad \varphi'[u_{k}(t)] = y(t), \) for \( k \leq t \leq k_{k+1} \), where \( k_{1} = 1, 2, \ldots, n - 1 \). In view of the periodicity of the functions \( \varphi \) and \( \varphi' \), we may assume that for \( k = 2, 3, \ldots, n - 1 \) the initial point of the curve \( C_{k} \) and the terminal point of the curve \( C_{k-1} \) belong to the same period-parallelogram. Then, in view of lemma 14.3, the initial point of the curve \( C_{k} \) coincides with the terminal point of the curve \( C_{k-1} \). We can therefore define the function \( u(t) \) in \([a, b]\), taking \( u(t) = u_{k}(t) \) for \( t_{k} \leq t \leq t_{k+1} \), \( k = 1, 2, \ldots, n - 1 \). The function \( u(t) \) satisfies the condition (14.7).

Let us denote by \( L \) and \( l \), respectively, the curves \( x = x(t) \) and \( u = u(t) \) appearing in theorem 14.6, and let us assume that the function \( W(x, y) \) is finite along \( L \). We then have

\[ \int_{a}^{b} \int_{a}^{b} W(x, y) \, dx \, dy = \int_{a}^{b} \int_{a}^{b} \left[ W[\varphi(u), \varphi'(u)] \right] \, du \, dt; \]

for the left side is here equal to

\[ \int_{a}^{b} \int_{a}^{b} W(x, y) \, dx \, dy = \int_{a}^{b} \int_{a}^{b} \left[ W[\varphi(u), \varphi'(u)] \right] \varphi'[u(t)] \, du \, dt \]

(cf. (14.7)) and the last integral is identical to the right side of formula (14.8).

Let us note now that the function \( F(u) = W[\varphi(u), \varphi'(u)] \) is elliptic, and hence by theorem 8.6 it can be expressed in the form of a sum

\[ A + \sum_{\nu} C_{\nu}^{\prime} \zeta(u - \beta_{\nu}) + C_{\nu} \zeta^{2}(u - \beta_{\nu}) + \ldots + C_{\nu}^{n} \zeta^{n-1}(u - \beta_{\nu}), \]

where the constants have the same meaning as in theorem 8.6. Let us make use of formula (14.8) and let us replace the integral on the left side by an indefinite integral. We obtain the equation

\[ \int_{a}^{b} \int_{a}^{b} W(x, y) \, dx \, dy = C + A u + \sum_{\nu} C_{\nu} \zeta^{\nu}(u - \beta_{\nu}) + \ldots + C_{\nu}^{n} \zeta^{n-1}(u - \beta_{\nu}) \]

where in the integrals on the right side we do not display the curve of integration, because it depends on the curve of integration on the left side. In other words,

\[ \int_{a}^{b} \int_{a}^{b} W(x, y) \, dx \, dy = C + A u + \sum_{\nu} C_{\nu} \log |u - \beta_{\nu}| + \sum_{\nu} C_{\nu}^{\prime} \zeta^{\nu}(u - \beta_{\nu}) \]

\[ - \sum_{\nu} C_{\nu} \varphi(u - \beta_{\nu}) + \ldots + C_{\nu}^{n} \varphi^{n-3}(u - \beta_{\nu}). \]
The last sum is an elliptic function, and hence, in view of theorem 8.10, it is a rational function in \( \varphi(u) \) and \( \varphi'(u) \). In addition \( \zeta(u-\beta_i)=\zeta(u)+\zeta(u-\beta_i)-\zeta(u) \), where the difference \( \zeta(u-\beta_i)-\zeta(u) \) has the periods \( \alpha \) and \( \omega' \) (cf. formula (6.6)), and hence is elliptic. Consequently,

\[
\int W(x,y) \, dx = C + A u + A' \zeta(u) + \sum_i C_i \log (u-\beta_i) + R[\varphi(u), \varphi'(u)],
\]

where \( C_i, A, A', \) \( C_i \) are constants, and \( R[\varphi(u), \varphi'(u)] \) is a rational function in \( \varphi(u) \) and \( \varphi'(u) \). We may therefore state the following theorem:

**(14.9)** Every elliptic integral \( \int W(x,y) \, dx \), where \( y \) is defined by formula (14.4), can, by means of a suitable substitution \( x=\varphi(u) \), \( y=\varphi'(u) \), be expressed as the sum of rational functions in \( \varphi(u) \) and \( \varphi'(u) \), and a linear expression in \( u, \zeta(u) \) and a finite number of functions \( \log (u-\beta_i) \).

The definite elliptic integral (14.2) depends not only on the end-points \( x_0, x_1 \) of the curve of integration, but also on the curve itself. By means of formula (14.8) one can investigate this dependence. For simplicity let us limit ourselves to the integral

\[
\int \frac{dx}{y} = \int \frac{dx}{y} \sqrt{4\alpha x^3 - g_2 x - g_3},
\]

called Weierstrass’s elliptic integral of the first kind. A change of variables, \( x=\varphi(u) \), \( y=\varphi'(u) \) (cf. theorem 14.6), gives the formula

\[
\int \frac{dx}{y} = \int \frac{du}{y} = \int \frac{dx}{u_2 - u_1},
\]

where \( u_2 \) and \( u_1 \) denote the initial point and the terminal point of the curve \( I \).

Let \( x_0 \) and \( x_1 \) denote the initial point and the terminal point of the curve \( L \), and \( y_0 \) and \( y_1 \) the values of the function \( y \) at the points \( x_0 \) and \( x_1 \). Consequently,

(a) \( \varphi(u_0)=x_0, \varphi'(u_0)=y_0 \), (b) \( \varphi(u_1)=x_1, \varphi'(u_1)=y_1 \).

Let us now replace \( L \) by a curve \( \hat{L} \), having the same initial point and the same terminal point as \( L \), and let \( J \) denote the value of the integral under consideration along \( \hat{L} \). In addition, we assume that in integrating along \( \hat{L} \), we go from the same value \( y_0 \) of the function \( y \) at the point \( x_0 \).

§ 14 Elliptic integrals.

Let \( \hat{u}_0 \) and \( \hat{u}_1 \) denote the initial point and the terminal point, respectively, of the curve \( \hat{I} \) corresponding to the curve \( I \). We may assume that \( \hat{u}_0 = u_0 \). If \( \hat{y}_1 \) denotes the value of the function \( y \) at the point \( \hat{x}_1 \), after the continuation along \( \hat{L} \), then either \( \hat{y}_1 = y_1 \), or \( \hat{y}_1 = -y_1 \). In the first case, from equation (b) and from the analogous equations \( \varphi(\hat{u}_1)=x_1, \varphi'(\hat{u}_1)=y_1 \), we obtain \( \hat{u}_1 = u_1 + m\alpha + n\omega' \), where \( m \) and \( n \) are integers (see theorem 14.5). In the second case we have \( \hat{u}_1 = -u_1 + m\alpha + n\omega' \). From equations \( J = u_1 - u_0 \) and \( J = \hat{u}_1 - \hat{u}_0 \) it follows, therefore, that:

**(14.11)** If in the integral (14.10) we replace the curve \( L \) by a curve \( \hat{L} \), having the same initial point \( u_0 \) and the same terminal point as \( I \), and if in both cases we give the function \( y \) the same value \( y_0 \) at the initial point of the curve \( L \), then the new integral \( \hat{J} \) will be related to \( J \) by means of one of the two formulae:

(a) \( \hat{J} = J + m\alpha + n\omega' \), \( \hat{J} = -J - 2u_0 + m\omega + n\omega' \),

where the numbers \( m \) and \( n \) \( \omega' \) have a non-real quotient, the coefficients \( m \) and \( n \) are integers, and \( \varphi(u_0) = x_0 \).

The first one of the formulae (14.12) holds in the case when the function \( y \) assumes the same values as the terminal points of the curves \( L \) and \( \hat{L} \), and the second one when these values differ in sign.

Of course \( m \) and \( n \) can be arbitrary integers. For example, in order to realize the first one of the formulae (14.12) it is sufficient to choose two arbitrary curves of integration \( I \) and \( \hat{I} \), having a common initial point \( u_0 \), and terminal points differing by \( m\alpha + n\omega' \), and to take \( \int L \) and, respectively, \( \int \hat{L} \) the curves obtained from \( L \) and \( \hat{L} \) by the transformation \( x=\varphi(u) \).

If \( x_0 \) is a finite number, and the circle \( K = K(x_0; R) \) does not contain any one of the roots \( \xi, \epsilon_1, \epsilon_2, \epsilon_3 \) of the right side of equation (14.4), where \( \beta - 2\beta Y \neq 0 \), then the function

\[
F(\xi) = \int \frac{dx}{y},
\]

where \( y = x^2 - g_2 x - g_3 \), \( \xi \in K \), and we integrate along the segment \( [x_0, \xi] \), is holomorphic in the circle \( K \). This function is continuous along every curve not passing through any one of the points \( \epsilon_1, \epsilon_2, \epsilon_3, \infty \). It is easy to see that all the values which the function \( F \) assumes at a point \( \xi \) are given by integrals of the form \( \int_{L(\xi; L)} \frac{dx}{y} \), where \( L(\xi; L) \) is an arbitrary curve not passing through the points \( \epsilon_1, \epsilon_2, \epsilon_3 \), with its initial point at the point \( x_0 \) and with
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Its terminal point at the point \( z \). The analytic function \( F(z) \) which we obtain in this way is also called Weierstrass's elliptic integral of the first kind. If \( J \) is one of the values of the function \( F \) at the point \( z \), then all of its values are given by formula (14.12). Consequently, \( F(z) \) is an infinitely-valued function, having at most the critical points \( c_1, c_2, c_3, \infty \).

Expanding the function \( 1/y \) in the neighbourhood of the points \( c_i \) in a Laurent series in \((z-e_i)^{1/2}\), we easily verify that these points are algebraic critical points of the function \( F(z) \), with order of ramification 1. In an analogous manner we verify the same thing for the point \( \infty \). Applying a suitable substitution \( z = \Phi(w) \), \( y = \Phi'(w) \), for \( w \in K \), we see without difficulty that the function \( F(z) \) is the inverse (cf. Chapter VI, § 5) of the function \( \Phi(w+u_0) \), where \( u_0 \) is a constant.

In this way, obtaining elliptic integrals, we arrive in a natural manner at elliptic functions. It was precisely this road which Abel and Jacobi took to introduce elliptic functions for the first time. The theory of elliptic functions based on the notion of double periodicity and developed in §§ 3-10 of the present chapter is historically more recent, and is due principally to Liouville and Weierstrass.

**EXERCISES.** 1. Let \( L \) and \( \bar{L} \) denote two curves with initial point \( x_0 \) and terminal point \( x_1 \), not passing through the point \( \infty \) or through any one of the distinct roots \( e_1, e_2, \ldots, e_n \) of a given polynomial \( P(z) \) of degree \( n \). If the analytic function \( y \) defined by the formula \( y^n = P(z) \), assumes the value \( y_1 \) at the point \( x_0 \), and at the point \( x_1 \), after a continuation along the curves \( L \) and \( \bar{L} \), the values \( y_1 \) and \( \bar{y}_1 \), respectively, then a necessary and sufficient condition that \( y_1 = \bar{y}_1 \) is that the number

\[
\sum_{i=1}^{n} \text{ind} e_i, \quad \text{where} \quad C = L + (-\bar{L}),
\]

be even.

2. If \( F(z) \) is a polynomial of the fourth degree with simple roots, and the circle \( K = K(x_0; R) \) does not contain roots of the polynomial \( F \), then the function

\[
F(z) = \left( \frac{dz}{y} \right),
\]

where \( y^4 = P(z) \), and we integrate along the segment \([x_0, z]\), is holomorphic in the circle \( K \). This function is continuous along every curve not passing through any one of the roots of the polynomial \( P \). Prove that, with a suitable choice of the periods \( \omega, \omega' \), the function \( F \) is the inverse of the function

\[
a\Phi(u + u_0; \omega, \omega') + b \quad \text{and} \quad c\Phi(u + u_0; \omega, \omega') + d.
\]

where \( a, b, c, d, u_0 \) are constants.

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CHAPTER IX

THE FUNCTIONS \( \Gamma(s) \) AND \( \zeta(s) \). DIRICHLET SERIES

§ 1. The function \( \Gamma(s) \). In Chapter VII, § 5, we introduced the meromorphic function \( \Gamma \). At present we shall study somewhat in detail the properties of this function.

Let us consider the integral

\[
\Gamma(s) = \int_0^\infty u^{s-1} e^{-u} du,
\]

where \( s = \sigma + it \) is a complex variable and \( e^{\gamma} = \exp((s-1)\log u) \). Integral (1.1) is known as Euler's integral of the second kind.

Let us note that \( |u^{s-1}e^{-u}| = |u^{\sigma-1}e^{-u} | \) and that the function \( u^{\sigma-1}e^{-u} \) is integrable over the interval \( 0 < u < 1 \), provided that \( \sigma > 0 \). On the other hand, for every \( \sigma \) we have the inequality \( u^{\sigma-2}e^{-u} < e^{-u} \) if \( u \) is sufficiently large, and hence the function \( u^{\sigma-1}e^{-u} \) is integrable over the interval \( 1 < u < +\infty \). Consequently, the integral (1.1) is convergent, and even absolutely convergent, if \( \sigma > 0 \).

Integral (1.1) is improper, because the interval of integration is infinite and, in addition, if \( 0 < R < 1 \), the integrand is unbounded in the neighbourhood of the point \( u = 0 \).

Let us denote by \( F(s) \) the value of the integral (1.1). We shall prove that the integral under consideration is almost uniformly convergent in the half-plane \( \Re s > 0 \), i.e., that if we take \( F_{\alpha, R}(s) = \int_{\alpha}^{R} u^{s-1}e^{-u} du \), then \( F_{\alpha, R}(s) \) tends almost uniformly to the limit \( F(s) \) in the half-plane \( \Re s > 0 \), as \( \alpha \to 0+ \) and \( R \to +\infty \). It is sufficient to prove that the function \( F_{\alpha, R}(s) \) tends uniformly to \( F(s) \) in every strip \( \alpha < \Re s < \beta \), where \( 0 < \alpha < b < +\infty \). We may assume that \( b < R \). Then, if \( s \) belongs to the strip mentioned, we have