CHAPTER IV. Derivation of additive functions.

\[ \int f(\xi, \eta) \, dL_\alpha(\xi) \leq A' \] for every \( \eta \in \mathcal{R}_k \). Therefore, on account of \( \mathcal{Q}_k \), we obtain for \( k = 1, 2, \ldots, m \),

\[ \int \int f(\xi, \eta) \, dL_\alpha(\xi) \, dL_\alpha(\eta) \leq A' \cdot L_\alpha(\mathcal{R}_k) = A' \cdot L_\alpha(\mathcal{R}_k), \]

whence it follows by (16.7) and (16.8) that \( \int f(\zeta) \, dL_\alpha(\zeta) \leq A' \cdot L_\alpha(\mathcal{S}_m) \leq A' \).

Making \( m \to \infty \), we obtain in the limit \( M = \int f(\zeta) \, dL_\alpha(\zeta) \leq A' \) and so \( M \leq A \).

By symmetry \( M \geq B \) and, since it is clear that \( A \leq B \), this requires \( A = M = B \) and completes the proof.

CHAPTER V.

Area of a surface \( z = F(x, y) \).

§ 1. Preliminary remarks. We saw (cf. Chap. IV, § 8) that the Lebesgue theory enables us to solve completely the elementary problems concerning the length of a curved line and the expression of this length by an integral. However, similar problems concerning curved surfaces involve difficulties of a much more serious kind. Certain classical treatises on the differential and integral calculus, even in the second half of the XIX-th century, contain an inaccurate definition of the area of a surface. By analogy with the definition of length of a curve, the authors attempted to define the area of a surface as the limit of the areas of polyhedra inscribed in the surface and tending to it. H. A. Schwarz [I, p. 309] (cf. also M. Fréchet [3]) was the first to remark that such a limit may not exist and that it is possible to choose a sequence of inscribed polyhedra whose areas tend to any number not less than the actual area of the surface. About the same time Peano and Hermite subjected the old definition to similar criticisms and proposed new definitions based on quite different ideas. It was H. Lebesgue who first returned in his Thesis [1] to the old method, in a modified form that may be roughly described as follows: the area of a surface is the lower limit of the areas of polyhedra tending uniformly to the surface in question (without, however, being necessarily inscribed in the latter).

Nevertheless, in the more general case in which the surface is given parametrically, this definition requires various additional notions and considerations (cf. T. Rado [I; 1; 4]) and the results obtained are far from being as complete as those available for curves. The difficulties that arise belong to Geometry and
Topology rather than to the Theory of functions of a real variable. (For the special case in which the functions \( x = X(u, v), y = Y(u, v) \) and \( z = Z(u, v) \) which define the surface parametrically fulfill the Lipschitz condition, see T. Radó [4] and H. Rademacher [4]).

We shall therefore restrict ourselves to the case of continuous surfaces of the form \( z = F(x, y) \). The most elegant and the most complete results concerning these surfaces are due to L. Tonelli [5; 6; 7]; they will be given in § 8 and are the principal object of this chapter.

Tonelli's theory is based on the definition of area proposed by Lebesgue. As regards the modern work on area of surfaces based on other definitions, we should mention: W. H. Young [4], J. C. Burkill [5], S. Banach [5], A. Kolmogoroff [5], and J. Schauder [1].

T. Radó [1, pp. 154—169; 2] has developed further the methods of Tonelli by means of older ideas due to de Goeze and with the help of certain functionals introduced by the latter. The principal result of Radó (see Theorem 7.3), applications of which will be discussed below, enables us to define the area of a surface as the limit of certain simple expressions, whereas the Lebesgue definition only enables us to obtain it as a lower limit. Another expression is due to L. C. Young (see below § 8) and constitutes a direct generalization of the classical formula for the area of a surface.

Except where the contrary is expressly stated, the reasoning of this chapter will be formulated for functions of two real variables. The extension to spaces of any number of dimensions offers no difficulty.

§ 2. Area of a surface. By a continuous surface on a plane interval \( I_0 \), we shall mean any equation of the form \( z = F(x, y) \), where \( F \) is a continuous function on \( I_0 \).

A continuous surface \( z = P(x, y) \) on an interval \( I_0 \) is termed polyhedron if there exists a decomposition of \( I_0 \) into a finite number of non-overlapping triangles \( T_1, T_2, ..., T_n \) such that the function \( P \) is linear on each of these triangles, i.e. such that \( P(x, y) = ax + by + c \), for \( (x, y) \in T_n \), where \( n = 1, 2, ..., n \) and \( a, b, c \) are constant coefficients. We shall call, respectively, faces and vertices of the polyhedron \( z = P(x, y) \), the parts and the points of the

graph (cf. Chap. III, § 10) of the function \( P \), which correspond to the triangles \( T_i \), and to the vertices of the \( T_i \). The sum of the areas of the faces in the sense of elementary Geometry, i.e. the number

\[
\sum |T_i| \cdot (a_i^2 + b_i^2 + 1) = \int \int \frac{1}{4} \left( (\partial P/\partial x)^2 + (\partial P/\partial y)^2 + 1 \right) dx dy,
\]

will be called elementary area of the polyhedron \( z = P(x, y) \) on \( I_0 \) and denoted by \( \sum S_0(P; I_0) \).

Given any continuous surface \( z = P(x, y) \) on an interval \( I_0 \), we shall term its area on \( I_0 \) and denote by \( S(F; I_0) \), the lower limit of the elementary areas of polyhedra tending uniformly to this surface, i.e. the lower bound of all the numbers \( s \) for each of which there exists, given any \( \varepsilon > 0 \), a polyhedron \( z = P(x, y) \) on \( I_0 \) such that \( 1^0 |P(x, y) - F(x, y)| < \varepsilon \) at every point \( (x, y) \in I_0 \) and \( 2^0 \sum S_0(P; I_0) \leq s \).

We might verify here that for polyhedrons the elementary area agrees with the area according to the general definition just given. As, however, this is an easy consequence of the theorem given further on (see p. 181), a special proof is unnecessary at this point. It should be remarked that, in accordance with the definition adopted, the area of a surface may be either finite or infinite.

The following theorem is an immediate consequence of the definition.

(2.1) **Theorem.** For any sequence of continuous functions \( \{ F_n \} \) which converges uniformly on an interval \( I_0 \) to a function \( F \), we have

\[
\lim \inf \sum S(F_n; I_0) \geq S(F; I_0).
\]

§ 3. The Burkill integral. Instead of treating the theory of areas of surfaces by itself, it is more convenient to associate it with certain differential properties of functions of an interval. However, the functions of an interval occurring in the theory of area are not in general additive, and in consequence we shall have to complete in some minor points the theory of functions of an interval, developed in the preceding chapters.

We shall begin with some subsidiary definitions. To simplify the wording we shall understand in the sequel by subdvision of a figure \( R_0 \) any finite system of non-overlapping intervals \( I_1, I_2, ..., I_n \), such that \( R_0 = \sum I_i \). Given any function of an interval \( U \) and any finite system of intervals \( S = \{ I_3 \} \), we shall write, for brevity, \( U(S) \) in place of \( \sum U(I_3) \). In particular therefore, \( L(S) \) will denote the sum of the areas of the intervals belonging to the system \( S \).
We call upper and lower integral in the sense of Burkhill of a function of an interval $U(I)$ over a figure $R_0$, and we denote by $\int_U$ and $\bar{\int}_U$ respectively, the upper and the lower limit of the numbers $U(\Sigma)$ for arbitrary subdivisions $\Sigma$ of $R_0$, whose characteristic numbers $d(\Sigma)$ tend to zero (cf. Chap. II, p. 40). When these integrals are equal, their common value is called the Burkhill definite integral (or simply the integral) of the function $U$ over $R_0$ and is denoted by $\int U$. If this integral exists and is finite, the function $U$ is said to be integrable on $R_0$ (in the sense of Burkhill). If the function $U$ is integrable on every figure $R$ in the whole plane or in a figure $R_0$, its integral $\int U$ considered as a function of the figure $R$ is called indefinite integral of $U$ (in the whole plane or on $R_0$).

(3.1) **Theorem.** If $U$ is a function of an interval and $R_1$, $R_2$ are non-overlapping figures, we have

$$\int_{R_1+R_2} U \geq \int_{R_1} U + \int_{R_2} U \quad \text{and} \quad \int_{R_1+R_2} U \leq \int_{R_1} U + \int_{R_2} U,$$

provided that both integrals of $U$ over $R_1 + R_2$ are finite.

Any function of an interval $U$ which is integrable on a figure $R_0$ is equally so on every figure $R \subset R_0$ and its indefinite integral on $R_0$ is an additive function of a figure.

**Proof.** Part 1 of the theorem is a direct consequence of the definition of the Burkhill integrals, and part 2 follows at once from the formulae (3.2) when we subtract the second of these formulae from the first.

If $U$ is a function of an interval on a figure $R$, we shall call variation of $U$ on $R$ a set $D$ the upper limit of $|U(\Sigma)|$ as $d(\Sigma) \to 0$, where $\Sigma$ denotes any finite system of non-overlapping intervals contained in $R$ and possessing common points with $D$. The following analogue of Theorem 4.1, Chap. III, may be noted.

(3.3) **Theorem.** Given on a figure $R_0$ a function of an interval $U$ such that $\int_U^{R_0} U < +\infty$, there can be at most an enumerable infinity of straight lines $D$, which are parallel to the coordinate axes and at which the variation of $U$ on $R_0$ is not zero.

In fact, the number of straight lines which are parallel to the axis of $x$ or of $y$, and at which the variation of $U$ on $R_0$ exceeds a positive number $\epsilon$, cannot be greater than $2\epsilon^{-1}\int_{R_0} |U| < +\infty$.

(3.4) **Lemma.** Given a function of an interval $U$ integrable on a figure $R_0$, there exists, for each $\epsilon > 0$, an $\eta > 0$ such that for every system $\Sigma = (I_1, I_2, \ldots, I_p)$ of non-overlapping intervals situated in $R_0$, the inequality $\Delta(\Sigma) < \eta$ implies the inequality

$$\left| \sum_{k=1}^{p} U(I_k) - \int_U \right| < \epsilon.$$

**Proof.** Let $\eta > 0$ be a number such that, for every subdivision $\Sigma$ of $R_0$, $\Delta(\Sigma) < \eta$ implies $|U(\Sigma) - \int_U| < \epsilon/2$, and let $\Sigma = (I_1, I_2, \ldots, I_p)$ be any finite system of non-overlapping intervals situated in $R_0$ such that $\Delta(\Sigma) < \eta$. Let $R_1 = R_0 - \sum_{k=1}^{p} I_k$. By Theorem 3.1, the function $U$ is integrable on $R_1$. It follows that there exists a subdivision $\Sigma_1$ of $R_1$ such that

$$\Delta(\Sigma_1) < \eta \quad \text{and} \quad |U(\Sigma_1) - \int_{R_1} U| < \epsilon/2.$$

Now $\Sigma + \Sigma_1$ clearly constitutes a subdivision of $R_0$ such that $\Delta(\Sigma + \Sigma_1) < \eta$. We therefore have $|U(\Sigma + \Sigma_1) - \int_R U| < \epsilon/2$, and we need only subtract the second of the relations (3.6) from it to obtain (3.5).

If $R_0$ is a fixed figure, then to any $\eta > 0$ there corresponds a positive integer $p$ such that every interval $I \subset R_0$ may be subdivided in $p$ subintervals of diameter less than $\eta$. Hence applying Lemma 3.4, we obtain at once the following

(3.7) **Theorem.** If a function of an interval $U(I)$ which is integrable on a figure $R_0$, is continuous, then the same is true of its indefinite integral $B(R) = \int_{R} U$.

(3.8) **Theorem.** If $U$ is a function of an interval which is integrable on a figure $R_0$ and if $B$ is its indefinite integral, then $\overline{B}(x,y) = \overline{U}(x,y)$ and $\overline{B}(x,y) = \overline{U}(x,y)$ at almost all points $(x,y) \in R_0$.

In particular therefore, if one of the functions $U$ and $B$ is almost everywhere derivable in $R_0$, the same is true of the other and the derivatives of $U$ and of $B$ are almost everywhere equal.
Proof. Suppose that the set of points \((x, y)\) at which
\[ U(x, y) > B(x, y) \]
has positive measure. We could then determine a set \(E \subseteq R_0\) of positive outer measure and a number \(\alpha > 0\), such that
\[ U(x, y) > \frac{B(x, y)}{\alpha} \]
at each point \((x, y)\) of \(E\). Therefore, on
account of Vitali's Covering Theorem (Chap. IV, Theorem 3.1),
we could determine in \(R_0\) for any \(\eta > 0\), a finite system of non-overlapping intervals \(\Xi = \{L_k\}_{k=1, 2, ..., n}\) such that
\[ d(3) < \eta, \quad L(3) > |E|/2, \]
and
\[ U(L_k) - \frac{B(L_k)}{\alpha} \]
for \(k = 1, 2, ..., n\). Now it follows from the last two relations that
\[ U(3) - B(3) > \frac{1}{\alpha} |E|, \]
which contradicts Lemma 3.1. Hence, \(U(x, y) \leq B(x, y)\) almost everywhere in \(R_0\).
In the same way we prove that the opposite inequality holds also almost everywhere in \(R_0\) and this completes the proof.

(3.9) Theorem. Suppose that \(U\) is a continuous function of an interval on a figure \(R_0\) and that (i) \(\int_{R_0} |U| < \infty\) and (ii) \(U(I) \leq U(\Xi)\)
for every interval \(I \subseteq R_0\) and every subdivision \(\Xi\) of \(I\). Then the function \(U\) is integrable on \(R_0\).

Proof. Given a number \(\varepsilon > 0\), let \(\Xi = \{J_i\}_{i=1, 2, ..., p}\) be a subdivision of \(R_0\) such that
\[ \int_{R_0} |U - \varepsilon| \leq \varepsilon. \]

Let us denote by \(D_1, D_2, ..., D_r\) the sides of the intervals \(\Xi\) which do not belong to the boundary of \(R_0\). By Theorem 3.3 it may be assumed, in view of the continuity of the function \(U\) and of condition (ii), that the variation of \(U\) on \(R_0\) vanishes at each side \(D_i\). It follows that there exists an \(\eta > 0\) such that, given any finite system \(\Xi\) of non-overlapping intervals situated in \(R_0\) and having points in common with the sides \(D_i\) the inequality \(d(\Xi) < \eta\) implies \(|U(\Xi)| < \varepsilon\). We can clearly assume that \(\eta\) does not exceed the length of any side of the intervals \(\Xi\).

This being so, consider an arbitrary subdivision \(\Xi = \{J_1, I_2, ..., I_n\}\)
of \(R_0\) such that \(d(3) < \eta\). By numbering the intervals of \(\Xi\) suitably, we may evidently suppose that \(I_1, I_2, ..., I_n\) are those having points in common with the sides \(D_i\) while the remaining intervals of \(\Xi\) (if any) have none. Finally, let us agree to write \(U(J_i \cap I_k) = 0\) when \(J_i \cap I_k = 0\). Then
\[ \sum_{k=1}^{n} U(I_k) < \varepsilon \]
and
\[ \sum_{k=1}^{n} \sum_{i=1}^{n} U(J_i \cap I_k) < \varepsilon, \]
so that, by (3.10) and by condition (ii) of the theorem, we have
\[ \int_{R_0} U - \varepsilon \leq U(\Xi) - \varepsilon - \sum_{k=1}^{n} U(I_k) + \sum_{k=1}^{n} \sum_{i=1}^{n} U(J_i \cap I_k) < U(3) + 2\varepsilon. \]

It follows that \(\int_{R_0} U \leq \int_{R_0} U + 3\varepsilon\), and so, that \(\int_{R_0} U = \int_{R_0} U\).

In connection with this, see J. C. Burkill [2; 3; 4], R. C. Young [2] and E. Riesz [6; 7].

§ 4. Bounded variation and absolute continuity for functions of two variables. Given a function \(F(x, y)\) continuous on an interval \(I = [a_1, b_1; a_2, b_2]\), let us denote for any value \(x\) subject to \(a_1 < x < b_1\) by \(W_1(F; x; a_2, b_2)\) the absolute variation of the function \(F(x, y)\) with respect to the variable \(y\) on the interval \([a_2, b_2]\), and for any value \(y\) subject to \(a_2 < y < b_2\) by \(W_2(F; y; a_1, b_1)\) that of the function \(F(x, y)\) with respect to \(x\) on \([a_1, b_1]\). Denoting by \(J_1\) and \(J_2\) respectively the linear intervals \([a_1, b_1]\) and \([a_2, b_2]\) we shall also write \(W_1(F; x; J_2)\) for \(W_1(F; x; a_2, b_2)\) and \(W_2(F; y; J_1)\) for \(W_2(F; y; a_1, b_1)\).

By continuity of the function \(F\), the non-negative expressions \(W_1(F; x; J_2)\) and \(W_2(F; y; J_1)\) are, as is easily seen, lower semi-continuous functions of the variables \(x\) and \(y\) respectively. When the integrals \(\int_{a_1}^{b_1} W_1(F; x; J_2) dx\) and \(\int_{a_2}^{b_2} W_2(F; y; J_1) dy\) are both finite, the function \(F\) is said to be of bounded variation in \(I\) in the Tonelli sense. It follows at once that any function of bounded variation of two variables \(x, y\) is of bounded variation with respect to \(x\) for almost every value of \(y\) and with respect to \(y\) for almost every value of \(x\).

A continuous function \(F(x, y)\) will be termed absolutely continuous on an interval \(I = [a_1, b_1; a_2, b_2]\) in the Tonelli sense, if it is of bounded variation on \(I\) and moreover, absolutely continuous with respect to \(x\) for almost every value of \(y\) in \([a_2, b_2]\), and absolutely continuous with respect to \(y\) for almost every value of \(x\) in \([a_1, b_1]\).

We say that a function \(F(x, y)\) fulfills the Lipschitz condition on \(I\), if there exists a finite constant \(N\) such that
\[ |F(x', y') - F(x'', y'')| \leq N(|x' - x''| + |y' - y''|) \]
for every pair of points \((x', y')\) and \((x'', y'')\) of \(I\).

Any function which fulfills the Lipschitz condition on an interval \(I\) is evidently absolutely continuous on \(I\). In particular
polyhedra and also functions of two variables with continuous partial derivatives, are always absolutely continuous functions.

A function \( F \) which is continuous and of bounded variation [absolutely continuous, or subject to the Lipschitz condition] on an interval \( I_0 = [a_0, b_0; a_2, b_2] \) can easily be continued, even so as to be periodic, over the whole plane in such a manner as to remain continuous and of bounded variation [absolutely continuous, or subject to the Lipschitz condition] on every interval. In fact, denoting by \( I_1 \) one of the intervals congruent to \( I_0 \) with a common side parallel to the axis of \( x \), let us continue the function \( F \) from the interval \( I_0 \) on to the interval \( I_1 \) by symmetry relative to the common side of these intervals. Let us further continue similarly the function \( F \) from the interval \( I_0 + I_1 \) on to an interval \( I_2 \) congruent to \( I_0 + I_1 \) which has with the latter a common side parallel to the axis of \( y \). The function \( F \) is then defined on the interval \( I_0 + I_1 + I_2 \) whose sides are respectively of lengths \( 2(b_1 - a_1) \) and \( 2(b_2 - a_2) \). Writing \( u = 2(b_1 - a_1) \) and \( v = 2(b_2 - a_2) \), and continuing the function \( F \) from the interval \( I_0 + I_1 + I_2 \) to the rest of the plane by the periodicity condition \( F(x + u, y) = F(x, y + v) = F(x, y) \), we see easily that the continuation obtained for the function \( F \) has the properties required.

Besides the definition of Tonelli several other definitions have been given of conditions under which a function of two variables is said to be of bounded variation. For a discussion of these definitions see C. R. A. damaging and J. A. Clarkson [1; 2]. Throughout this chapter, use is made of Tonelli’s concept only.

We shall subsequently make use of the following theorem concerning the partial derivatives of any continuous function:

(4.1) Theorem. Given a continuous function \( F(x, y) \), its partial Dini derivatives, \( F_+^x, F_+^y, F_-^x, F_-^y \) and \( F_+^y, F_-^y, F_+^x, F_-^x \), are functions measurable (\( B \)).

Proof. It will suffice to prove this for any one of these derivatives, say \( F_+^x \).

Given an arbitrary real number \( a \), consider the set

\[ E = \{(x, y) : F(x, y) < a\} \]

and denote by \( E_a \) the set of all the points \((x, y)\) such that for every \( h \) the inequality \( 0 < h \leq 1/m \) implies \( |F(x + h, y) - F(x, y)|/h \leq a - 1/m \).

We find that \( E = \bigcup_n E_n \) and, since by continuity of the function \( F \) each of the sets \( E_n \) is closed, \( E \) is a set (\( B \)), so that the derivate \( F_+^x \) is a function measurable (\( B \)).

Theorem 4.1 may be compared with Theorems 4.2 and 4.3 of Chap. IV concerning measurability of the derivatives of functions of one real variable. Nevertheless it is to be remarked that contrary to what occurs for functions of one variable, the partial Dini derivatives of a function measurable (\( B \)) need not in general be measurable (\( B \)), although they are still measurable (\( C \)) (the proof of this requires, however, the theory of analytic sets; see F. Hausdorff [II, p. 274], M. Neubauer [1] and A. E. Currier [1]). On the other hand, a function of two variables may be measurable (\( C \)) without its partial Dini derivatives being so.

§ 5. The expressions of de Gődőe. We shall make correspond to any function \( F(x, y) \) which is continuous on an interval \( I = [a_0, b_0; a_2, b_2] \), the following expressions introduced by Z. de Gődőe [1] into the theory of areas of surfaces:

\[ G_1(F; I) = \int_{a_0}^{b_0} \left| F(x, b_2) - F(x, a_2) \right| dx, \quad G_2(F; I) = \int_{a_0}^{b_0} \left| F(b_2, y) - F(a_1, y) \right| dy, \]

\[ G_3(F; I) = \left( G_1(F; I)^2 + G_2(F; I)^2 + |F|^2 \right)^{1/2}. \]

While studying the fundamental properties of these expressions, we shall often find the following two inequalities useful:

\[ \left( \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 \right)^{1/2} \leq \left( \sum_{i=1}^{n} (x_i^2 + y_i^2 + z_i^2) \right)^{1/2} \]

for any three sequences \((x_i), (y_i), (z_i)\) of real numbers;

\[ \left( \int_{x}^{y} \left( \int_{y}^{z} dx \right)^2 + \int_{y}^{z} dx \right)^{1/2} \leq \left( \int_{x}^{y} \left( \int_{y}^{z} dx \right)^2 + \int_{y}^{z} dx \right)^{1/2} \]

for any measurable set \( E \) in a space \( R_m \) and any three non-negative functions \( x(t), y(t), z(t) \), measurable on \( E \).

The inequality (5.1) is easily deduced by induction from the case \( n = 2 \) which can be verified directly. The inequality (5.2), in the special case in which the functions \( x(t), y(t), z(t) \) are simple, is an obvious consequence of (5.1); and we pass at once to the general case with the help of Theorems 7.4 and 12.6 of Chapter I.
Theorem. The expressions of de Geöcze \( G_1(I) = G_1(F; I), \) \( G_2(I) = G_2(F; I) \) and \( G(I) = G(F; I) \), associated with a continuous function \( F(x, y) \), are continuous functions of the interval \( I \) and their integrals over any interval exist (finite or infinite); these integrals over any interval \( I_0 = [a_1, b_1; a_2, b_2] \) fulfill the following relations:

\[
\int_{a_1}^{b_1} G_1 = \int_{a_1}^{b_1} W_1(F; x; a_2, b_2) \, dx \quad \text{and} \quad \int_{a_1}^{b_1} G_2 = \int_{a_1}^{b_1} W_2(F; y; a_1, b_1) \, dy
\]

(5.4)

\[
G_1(I_0) \leq G_1, \quad G_2(I_0) \leq G_2 \quad \text{and} \quad G(I_0) \leq G
\]

(5.5)

\[
\int_{a}^{b} G_1 \leq \int_{a}^{b} G_1 + \int_{a}^{b} G_2 + |I_0| \quad \text{where} \quad i = 1, 2.
\]

(5.6)

Proof. Given an arbitrary \( \varepsilon > 0 \), let \( \eta < \varepsilon \) be a positive number such that, for any pair of the points \( (x, y) \) and \( (x, y) \) in \( I_0 \),

\[
|y - y| < \eta \quad \text{implies} \quad |F(x, y) - F(a, y)| < \varepsilon.
\]

(5.7)

Let \( M \) denote the upper bound of \( F(x, y) \) on \( I_0 \) and consider in \( I_0 \) an interval \( I = [a_1, b_1; a_2, b_2] \) such that \( |I| < \eta \). We then have either \( b_2 - a_1 < \eta \) or \( b_2 - a_1 < \eta \). In the former case, we find \( G_1(I) \leq (b_2 - a_1), 2M < 2M \eta < 2M \varepsilon \), and in the latter we derive from (5.7), \( G_1(I) \leq (b_2 - a_1) \cdot \varepsilon < (b_2 - a_1) \cdot \varepsilon \), so that in both cases \( G_1(I) \leq (2M + b_2 - a_1) \cdot \varepsilon \). The function \( G_1(I) \) is therefore continuous.

The same is of course true of \( G_2(I) \) and the continuity of these two functions at once implies that of \( G(I) \).

This being so, we shall show that the functions \( G_1 \) and \( G_2 \) are integrable and, at the same time, we shall deduce the formulae (5.4).

Let \( \mathcal{S}_n \) be a sequence of subdivisions of \( I_0 \) such that \( \lim_{n \to \infty} \mathcal{S}_n = 0 \) and \( \lim_{n \to \infty} G_1(\mathcal{S}_n) = \int_{a}^{b} G_1 \); and denote, for any positive integer \( n \) and any \( \xi \in [a_1, b_1] \), by \( V_n(\xi) \) the sum of the absolute increments of the function \( F(\xi, y) \) on the linear intervals cut off on the line \( x = \xi \) by the rectangles of the subdivision \( \mathcal{S}_n \). We then have, on the one hand,

\[
G_1(\mathcal{S}_n) = \int_{a}^{b} V_n(\xi) \, d\xi \quad \text{for} \quad n = 1, 2, \ldots
\]

(5.8)

and on the other hand, on account of continuity of the function \( F \),

\[
\lim_{n \to \infty} V_n(\xi) = W_1(F; \xi; a_2, b_2) \quad \text{for} \quad \xi \in [a_1, b_1].
\]

Therefore, in virtue of Fatou's Lemma (Chap. I, Theorem 12.10) and by (5.8), we obtain

\[
\int_{a}^{b} G_1 \geq \int_{a}^{b} W_1(F; \xi; a_2, b_2) \, d\xi.
\]

But since \( G_1(\mathcal{S}) \leq \int_{a}^{b} W_1(F; \xi; a_2, b_2) \, d\xi \) for every subdivision \( \mathcal{S} \) of \( I_0 \), we have also \( \int_{a}^{b} G_1 \leq \int_{a}^{b} W_1(F; \xi; a_2, b_2) \, d\xi \).

Therefore the function \( G_1 \) has a unique integral over \( I_0 \) and this integral fulfills the first of the relations (5.4). The existence of the integral \( \int_{a}^{b} G_1 \) and the validity of the second of these relations are deduced by symmetry.

Let us pass on now to the function \( G \). We first remark that the integral \( \int_{a}^{b} G \) clearly exists in the case in which one at least of the integrals \( \int_{a}^{b} G_1 \) and \( \int_{a}^{b} G_2 \) is infinite, and is then also infinite on account of the relations

(5.9)

\( G_1(I) \leq G(I) \) and \( G_2(I) \leq G(I) \) for any interval \( I \).

In the remaining case, the two integrals in question being finite, the evident inequality \( G(I) \leq G_1(I) + G_2(I) + |I| \) yields

(5.10)

\[
\int_{a}^{b} G \leq \int_{a}^{b} G_1 + \int_{a}^{b} G_2 + |I| < +\infty;
\]

and on the other hand, for every subdivision \( \mathcal{S} \) of any interval \( I \) the equally obvious relations

(5.11)

\( G_1(I) \leq G_1(\mathcal{S}) \) and \( G_2(I) \leq G_2(\mathcal{S}) \)

lead, in view of the inequality (5.1), to

(5.12)

\( G(I) \leq G(\mathcal{S}). \)

Now, continuity of the function \( G \) being already established, the formulae (5.10) and (5.12) imply, by Theorem 3.9, that this function has over \( I_0 \) a unique integral.

To complete the proof we need only remark that the formulae (5.11) and (5.12) imply at once the formulae (5.5) and finally that formula (5.6) follows directly from (5.9) and (5.10).

As a corollary of Theorem 5.3, and more particularly as a consequence of the formulae (5.4) and (5.6), we have:
(5.13) **Theorem.** In order that the function of an interval \( G(I) = G(F; I) \), associated with a continuous function \( F(x, y) \), be integrable on an interval \( I_0 \) (i.e. in order that \( \int_I G(x, y) \) exists), it is necessary and sufficient that the function \( F(x, y) \) be of bounded variation on \( I_0 \).

§ 6. **Integrals of the expressions of de Geöcze.** Given a continuous function \( F(x, y) \), we shall denote, for any interval \( I_0 \), by \( H_1(F; I_0) \), \( H_2(F; I_0) \) and \( H(F; I_0) \) respectively, the integrals of the functions of an interval \( G_1(I) = G_1(F; I) \), \( G_2(I) = G_2(F; I) \), and \( G(I) = G(F; I) \) over the interval \( I_0 \). All these integrals exist on account of Theorem 5.3 and their importance in the theory of area of surfaces is due to the fact that, as will be shown in the next §, the number \( H(F; I_0) \) coincides with the area of the surface \( z = F(x, y) \) on \( I_0 \).

(6.1) **Theorem.** For any function \( F(x, y) \) which is continuous and of bounded variation, the expressions \( H_1(F; I_0) = H_1(F; I) \), \( H_2(F; I_0) = H_2(F; I) \), and \( H(F; I_0) = H(F; I) \) are additive, continuous, and non-negative functions of the interval \( I \), and we have at almost all points \( (x, y) \) of the plane

\[
\begin{align*}
H(x, y) &= |F'_x(x, y)|, \\
H(x, y) &= |F'_y(x, y)|, \\
H'(x, y) &= [(F'_x(x, y))^2 + (F'_y(x, y))^2 + 1]^{1/2}.
\end{align*}
\]

**Proof.** Additivity and continuity of the functions in question follow at once from Theorems 3.1 and 3.7 on account of Theorem 5.3. We have therefore only to establish the relations (6.2). Now, for any interval \( I = [a_1, b_1; a_2, b_2] \) we have according to Theorem 5.3 and Theorem 7.4 of Chap. IV, the following relation (in which the transformation is effected in accordance with Fubini's Theorem 3.1, Chap. III), rendered applicable to the partial derivatives of the function \( F(x, y) \) by Theorem 4.1):

\[
H_1(F; I) = \int_{a_1}^{b_1} W_1(F; x; a_2, b_2) \, dx \geq \int_{a_1}^{b_1} \left[ \int_{a_2}^{b_2} |F'_y(x, y)| \, dy \right] \, dx = \int_{I} |F'_y(x, y)| \, dx \, dy;
\]

whence

\[
H'(x, y) \geq |F'_y(x, y)| \quad \text{for almost every point} \ (x, y).
\]

Let us now denote by \( \{J_n\} \) the sequence of the linear intervals with rational extremities. In view of Theorem 7.4, Chap. IV, we have for \( n = 1, 2, \ldots \) and for every linear interval \( J \),

\[
\int_{J} W_1(F; x; J_n) \, dx = \int_{J_n} \int_{J} |H'_1(x, y)| \, dx \, dy = \int_{J_n} \left[ \int_{J} |H'(x, y)| \, dx \right] \, dy,
\]

and consequently, for each positive integer \( n \), the inequality

\[
\int_{J} W_1(F; x; J_n) \, dx \geq \int_{J_n} \left[ \int_{J} |H'(x, y)| \, dx \right] \, dy
\]

holds at every point \( x \), except at most those of a set \( E_n \) of linear measure zero. Therefore, writing \( E = \sum_{n} E_n \), we obtain the inequality

\[
\int_{J} W_1(F; x; J) \, dx \geq \int_{J} |H'(x, y)| \, dy
\]

whenever \( J \) has rational extremities and \( x \) lies outside the set \( E \) of linear measure zero. If we now regard the two sides of this inequality, for a given value of \( x \) outside the set \( E \), as functions of the linear interval \( J \), we obtain by derivation with respect to this interval (on account of Theorem 7.9, Chap. IV) for almost all \( y \), the inequality

\[
|F'_y(x, y)| \geq |H'(x, y)|.
\]

Therefore, since the derivatives \( H'_1(x, y) \) and \( F'_y(x, y) \) are measurable (cf. Theorem 4.1), it follows from Fubini's theorem (in the form (8.6), Chap. III) that the set of the points \( (x, y) \) at which the relation (6.4) is not fulfilled, is of plane measure zero. By (6.3) we therefore have almost everywhere the first of the relations (6.2).

The proof of the second relation now follows by symmetry, and that of the third from the remark that if we write \( G_1(I) = G_1(F; I) \), \( G_2(I) = G_2(F; I) \) and \( G(I) = G(F; I) \), we have by Theorem 3.8,

\[
H'(x, y) = G'(x, y) = [G_1(x, y)]^2 + [G_2(x, y)]^2 + 1)^{1/2} = (F'_y(x, y))^2 + [(F'_x(x, y))^2 + 1]^{1/2}
\]

at almost every point \( (x, y) \) of the plane. This completes the proof.

(6.5) **Theorem.** In order that the function of an interval \( H(F; I_0) \), corresponding to a continuous function \( F(x, y) \) of bounded variation on an interval \( I_0 = [a_1, b_1; a_2, b_2] \), be absolutely continuous on this interval, it is necessary and sufficient that the function \( F(x, y) \) itself be absolutely continuous; and when this is the case, we have

\[
H(I_0) = \int_{I_0} \left[ (F'_x(x, y))^2 + (F'_y(x, y))^2 + 1 \right]^{1/2} \, dx \, dy.
\]
Proof. By Theorem 5.3, absolute continuity of the function $H(I)$ is equivalent to absolute continuity of the functions $H_1(I)$ and $H_2(I)$ together.

Therefore if the function $H$ is absolutely continuous on $I_{a}$, we have, by Theorem 6.1, for any interval $I_{a} = [a_{a}, b_{a}]$, where $a_{a} \leq \xi \leq b_{a}$, the relation

$$\int_{a_{a}}^{b_{a}} W_{1}(F; x; a_{a}, b_{a}) \, dx = H_{1}(I_{a}) = \int_{I_{a}} \left[ \int_{a_{a}}^{b_{a}} |F_{x}(x, y)| \, dy \right] dx,$$

and, taking the derivative with respect to $\xi$, we obtain for almost every value of $x$,

$$W_{1}(F; x; a_{a}, b_{a}) = \int_{a_{a}}^{b_{a}} |F_{x}(x, y)| \, dy. \tag{6.7}$$

Now, for any given value of $x$ (for which $F(x, \eta)$ is of bounded variation in $\eta$) the difference $W_{1}(F; x; a_{a}, \eta) - W_{1}(F; x; a_{a}, \eta)$ is a non-negative and non-decreasing function of the variable $\eta$ (cf. Theorem 7.4, Chap. IV). It therefore follows from (6.7) that we have for almost every value of $x$, and for any $\eta \in [a_{a}, b_{a}]$,

$$W_{1}(F; x; a_{a}, \eta) = \int_{a_{a}}^{\eta} |F_{x}(x, y)| \, dy,$$

i.e. that the function $W_{1}(F; x; a_{a}, \eta)$, and consequently also $F(x, \eta)$, is absolutely continuous with respect to $\eta$ on $[a_{a}, b_{a}]$ for almost every value of $x$. By the symmetry of the variables, we conclude also that the function $F(\xi, y)$ is at the same time absolutely continuous with respect to $\xi$ on $[a_{a}, b_{a}]$ for almost every value of $y$. The function $F$, which is by hypothesis of bounded variation on $I_{a}$, is therefore absolutely continuous in the Tonelli sense on this interval.

Conversely, if the function $F$ is absolutely continuous on $I_{a}$, we have by Theorems 7.8 and 7.9, Chap. IV, for every subinterval $I_{a} = [a_{a}, b_{a}]$ of $I_{a}$, the relations:

$$H_{1}(I) = \int_{a_{a}}^{b_{a}} W_{1}(F; x; a_{a}, b_{a}) \, dx = \int_{I_{a}} \left[ \int_{a_{a}}^{b_{a}} |F_{x}(x, y)| \, dy \right] dx,$$

$$H_{2}(I) = \int_{a_{a}}^{b_{a}} W_{2}(F; y; a_{a}, b_{a}) \, dy = \int_{I_{a}} \left[ \int_{a_{a}}^{b_{a}} |F_{y}(x, y)| \, dx \right] dy,$$

so that the two functions of an interval $H_{1}$ and $H_{2}$, and therefore also $H$, are absolutely continuous.

Finally, since the function $H$ is absolutely continuous, the formula (6.6) is a direct consequence of the third of the relations (6.2), the latter being valid almost everywhere on account of Theorem 6.1.

Up to the present we have regarded the expression $H(F; I)$ as a function of an interval $I$. If we treat this expression as a functional depending on the function $F$, we obtain the following theorem, whose geometrical interpretation will appear in § 8 when the theorem appears to be a generalization of Theorem 2.1.

(6.8) **Theorem.** Given any sequence of continuous functions $\{F_{n}\}$ which converges to a continuous function $F$, we have for every interval $I$

$$\lim_{n} \inf_{I} H(F_{n}; I) \geq H(F; I). \tag{6.9}$$

**Proof.** Denoting by $S_{a}$ the subdivision of $I$ into $p^{n}$ equal intervals, similar to $I$, we have by Theorem 5.3 for any pair of integers $p$ and $n$, $H(F_{n}; I) \geq H(F_{p}; S_{p})$, and by Fatou’s Lemma (Chap. I, Theorem 12.10), for every integer $p$, $\lim \inf G(F; S_{p}) \geq G(F; S_{p})$.

We therefore have $\lim \inf H(F_{n}; I) \geq G(F; S_{p})$, and this leads to (6.9) when $p \to \infty$.

§ 7. Radó’s Theorem. Before passing to the proof of the result of Radó, according to which the area of any surface $z=F(x, y)$ on an interval $I$ is equal to $H(F; I)$, we shall prove the following

(7.1) **Theorem.** If a continuous function $F(x, y)$ has on an interval $I_{a} = [a_{a}, b_{a}]$ continuous partial derivatives, there exists a sequence of polyhedra $\{z=P_{n}(x, y)\}$ inscribed in the surface $z=F(x, y)$, such that the sequence converges uniformly to this surface and such that

$$\lim_{n} \int_{I} \left[ [F_{x}(x, y)]^{2} + [F_{y}(x, y)]^{2} + 1 \right] dx dy = H(F; I). \tag{7.2}$$

**Proof.** Let $S_{a} = \{I_{a,1}, I_{a,2}, ..., I_{a,n}\}$ denote the subdivision of $I_{a}$ into $n^{3}$ equal intervals similar to $I_{a}$, and $(x_{n,i}, y_{n,i})$, where $i=1, 2, ..., n$, the lower left-hand vertex of $I_{n,i}$. Let us divide any interval $I_{n,i}$ into two right-angled triangles $T_{n,i}$ and $T_{n,1}$ by a diagonal, in such a way that the vertex $(x_{n,i}, y_{n,i})$ is that of the right angle of $T_{n,i}$. Consider for any $n$ the polyhedron $z=P_{n}(x, y)$ inscribed in the surface $z=F(x, y)$ and corresponding to the net formed on $I_{a}$ by the $2n^{3}$ triangles $T_{n,i}$ and $T_{n,1}$ where $i=1, 2, ..., n$.  

For brevity let \( h_0 = (b_1 - a_1)/m \) and \( k_0 = (b_2 - a_2)/m \); and let \( \mu_0 \) denote the upper bound of the differences \( |F'_x(x', y') - F'_x(x, y')| \) and \( |F'_y(x', y') - F'_y(x, y')| \) for all points \((x', y')\) and \((x, y')\) of \( I_0 \) such that \( |x' - x| \leq h_0 \) and \(|y' - y| \leq k_0 \).

Now if \( s_{n, i} \) and \( s'_{n, i} \) denote respectively the elementary areas of the faces of the polyhedron \( z = F_n(x, y) \) which correspond to the triangles \( T_{x, n} \) and \( T_{y, n} \), we notice at once that the areas of the projections of the former of these faces on the planes \( xy \) and \( yz \) are respectively equal to

\[
\frac{1}{2} h_0 |F(x_n, y_n, y_n) - F(x_n, y_n, y_n)| = \frac{1}{2} k_0 |F(x_n, y_n, x_n) - F(x_n, y_n, x_n)|,
\]

and

\[
\frac{1}{2} h_0 |F(x_n, x_n, y_n) - F(x_n, y_n, y_n)| = \frac{1}{2} k_0 |F(x_n, x_n, y_n) - F(x_n, y_n, y_n)|.
\]

where \( s_{n, i} \leq s'_{n, i} \leq s_{n, i} + h_0 \) and \( y_{n, i} \leq y_{n, i} \leq y_{n, i} + k_0 \).

We therefore have \( s_{n, i} \leq s'_{n, i} \leq s_{n, i} + h_0 \) and \( y_{n, i} \leq y_{n, i} \leq y_{n, i} + k_0 \).

We therefore have \( s_{n, i} = \frac{1}{2} |F(x_n, y_n, y_n) - F(x_n, y_n, y_n)| + |F(x_n, y_n, y_n) - F(x_n, y_n, y_n)| + 1/2 |F(x_n, y_n, y_n)| \), and so, by the inequality (5.1), p. 171,

\[
\sum_{i=1}^{n} s_{n, i} \leq \frac{1}{2} \sum_{i=1}^{n} \left[ |F(x_n, y_n, y_n) - F(x_n, y_n, y_n)|^2 + |F(x_n, y_n, y_n) - F(x_n, y_n, y_n)|^2 + 1/2 |F(x_n, y_n, y_n)| \right],
\]

Since the partial derivatives \( F'_x \) and \( F'_y \) are by hypothesis continuous, it follows by making \( n \to \infty \) that

\[
\lim_{n \to \infty} \sum_{i=1}^{n} s_{n, i} = \frac{1}{2} \int_{I_0} \left[ |F(x, y, y) - F(x, y, y)|^2 + |F(x, y, y) - F(x, y, y)|^2 + 1/2 |F(x, y, y)| \right] dxdy,
\]

and the same limit is clearly obtained for the sum of \( s'_{n, i} \). By addition, together with an appeal to Theorem 6.5, we now derive the formula (7.2) and this completes the proof.

In what follows we shall apply the method of mean value integrals. Given in the plane a summable function \( F(x, y) \), the sequence of functions \( F_n(x, y) = \frac{1}{2} \left[ F(x, y) - F(x, y) \right] + |F(x, y) - F(x, y)| + 1/2 |F(x, y) - F(x, y)| \), where \( n = 1, 2, \ldots \), will be called the sequence of the mean value integrals of the function \( F(x, y) \). It is clear that if the function \( F \) is continuous, (i) the sequence of its mean value integrals \( \{F_n(x, y)\} \) converges to \( F(x, y) \) at every point \((x, y)\) of the plane, and uniformly on any interval, and (ii) the partial derivatives \( \partial F_n/\partial x \) and \( \partial F_n/\partial y \) exist everywhere and are continuous. In fact, at any point \((x, y)\) a direct calculation gives

\[
\begin{align*}
\partial F_n(x, y)/\partial x &= n^2 \int_0^{1/n} \left[ F(x + u, y + v) - F(x, y) \right] du \quad \text{and} \\
\partial F_n(x, y)/\partial y &= n^2 \int_0^{1/n} \left[ F(x + u, y + v) - F(x, y) \right] dv.
\end{align*}
\]

It was T. Radó [2] who first applied in the theory of area of surfaces the method of mean value integrals. The rôle of these mean values is due to the fact that in the case in which the given function \( F \) is continuous, the sequence of areas of the surfaces \( z = F(x, y) \) tends to the area of the surface \( z = F(x, y) \) (vide, below, Theorem 7.3).

In the definition given above, the functions \( F_n \) are defined at each point \((x, y)\) as "mean values" of the function \( F \) over squares of which \((x, y)\) is a vertex; it goes without saying that we could also make use of mean values taken over squares, or circles, having \((x, y)\) as their centres. These mean values over circles are used for instance in potential theory (cf. F. Riesz [4] and G. C. Evans [1]).

(7.3) **Radó's Theorem.** If \( F(x, y) \) is a continuous function and \( \{F_n(x, y)\} \) is the sequence of mean value integrals of \( F(x, y) \), then

\[
\lim_{n \to \infty} S(F_n; I_0) = S(F; I_0)
\]

for every interval \( I_0 \).

**Proof.** Let \( \{z = F_n(x, y)\} \) be a sequence of polyhedra converging uniformly to the surface \( z = F(x, y) \), such that

\[
\lim_{n \to \infty} S_0(P_n; I_0) = S(F; I_0).
\]

Since the functions \( F_n(x, y) \) are absolutely continuous, it follows from Theorem 6.5 (cf. also § 2, p. 165) that \( S_0(P_n; I_0) = H(P_n; I_0) \) for every \( n \). Consequently, since the sequences of \( \{F_n\} \) and \( \{P_n\} \) converge uniformly to the function \( F \), it follows by using successively Theorem 2.1, the formula (7.5) and Theorem 6.8, that

\[
\lim_{n \to \infty} S(F_n; I_0) \geq S(F; I_0) = \lim_{n \to \infty} H(P_n; I_0) \geq H(F; I_0).
\]

Now if the function \( F \) is not of bounded variation on \( I_0 \), it follows from Theorem 5.13 that \( H(F; I_0) = +\infty \) and consequently the formula (7.4) follows at once from (7.6). We may therefore assume that the function \( F \) is of bounded variation on \( I_0 \), and further (cf. § 4, p. 170) that \( F \) is continuous and of bounded variation on each interval of the plane.
Chapter V. Area of a surface $z=F(a, y)$.

Let us agree to denote, for any set $E$ in the plane, by $E^{(u, v)}$ the parallel translation of $E$ by the vector $(u, v)$ (cf. Chap. III, § 11); similarly, for a family of sets $\mathcal{E}$ in the plane, $\mathcal{E}^{(u, v)}$ will denote the family of all the sets obtained from sets $\mathcal{E}$ by subjecting them to this translation. For any subinterval $I=[a_1, b_1; a_2, b_2]$ we then obtain

$$G(F; I) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \left| F(x + u, y + v) - F(x, y) \right| du dv$$

and a similar formula for $G_2$. Hence by the inequality (5.2), p. 171,

$$G(F; I) = \left[ G_1(F; I)^2 + G_2(F; I)^2 + |I|^2 \right]^{1/2} \leq \int_{0}^{1} \left[ \left( \int_{0}^{1} G(F; I^{(u, v)}) du \right)^2 + \left( \int_{0}^{1} G(F; I^{(u, v)}) dv \right)^2 + |I| \right]^{1/2} du dv$$

Denoting by $\mathcal{S}_p$ the subdivision of $I_0$ into $p^2$ equal intervals similar to $I_0$, we obtain therefore, for every $p$,

$$G(F; \mathcal{S}_p) \leq \int_{0}^{1} \int_{0}^{1} G(F; I^{(u, v)}) du dv;$$

and since by Lemma 3.4, $G(F; I^{(u, v)})$ tends to $H(F; I_0^{(u, v)})$ as $p \to \infty$, uniformly in $u$ and $v$, we obtain in the limit

$$H(F; I_0) \leq \int_{0}^{1} \int_{0}^{1} H(F; I_0^{(u, v)}) du dv.$$  \hspace{1cm} (7.7)

Finally the areas of the figures $I_0^{(u, v)} \cap I_0$ and $I_0 \cap I_0^{(u, v)}$ tend to 0 with $u$ and $v$ and each of these figures is a sum of two intervals. Hence since the expression $H(F; I)$ is by Theorem 6.1 a continuous function of the interval $I$, we have $\lim_{n \to \infty} H(F; I_0^{(u, v)}) = H(F; I_0)$. On the other hand, since the functions $F(x, y)$ have continuous partial derivatives, we have, by Theorem 7.1, $S(F; I_0^{(u, v)}) = H(F; I_0)$ for each $n$. Therefore making $n \to \infty$ in (7.7), we find $\lim_{n \to \infty} S(F; I_0) = \leq H(F; I_0)$, which in conjunction with (7.6) gives the required relation (7.4).

§ 8. Tonelli's Theorem. The theorems of Radô just established, enables us to replace in all the theorems of this chapter the expression $H(F; I)$ by the surface area $S(F; I)$.

Thus for instance, Theorem 6.5 (formula (6.6)) expresses the fact that the elementary area of a polyhedron coincides with its area according to the general definition of area of a surface.

Theorem 6.8 contains a generalization of Theorem 2.1; it enables us to replace in its statement uniform convergence by ordinary convergence: we thus obtain a theorem similar to Lemma of Fatou (Chap. I, Theorem 12.10). It follows that the uniform convergence of the inscribed polygons, required in the definition of area, may be replaced by the ordinary convergence, so that the area of a continuous surface $z=F(x, y)$ is the lower limit of the areas of polyhedra tending to this surface. Further, by Theorem 7.1, if a function $F(x, y)$ has continuous partial derivatives, there exists a sequence of polyhedra inscribed in the surface $z=F(x, y)$, tending uniformly to the latter and having areas which converge to the area of this surface. (For further generalizations see S. Kempisty [1]. Cf. also on this subject H. Rademacher [2], W. H. Young [3], M. Fréchet [4] and T. Radô [5].) Finally, we obtain the following theorem, which sums up the most essential considerations of this chapter:

(8.1) **Tonelli's Theorem.** a) In order that a continuous surface $z=F(x, y)$ have a finite area on an interval $I_0$, it is necessary and sufficient that the function $F(x, y)$ be of bounded variation on $I_0$.

b) When this is the case, we have

$$S(F; I_0) = \int_{\mathcal{I}} \left[ \left( \frac{\partial F}{\partial x} \right)^2 + \left( \frac{\partial F}{\partial y} \right)^2 + 1 \right] dx dy;$$

the expression $S(I) = S(F; I)$ is then an additive continuous function of the interval $I \subset I_0$ and we have for almost every point $(x, y) \in I_0$

$$S(x, y) = \left[ \left( \frac{\partial F}{\partial x} \right)^2 + \left( \frac{\partial F}{\partial y} \right)^2 + 1 \right]^{1/2}. \hspace{1cm} (8.2)$$

c) In order that we should have

$$S(F; I_0) = \int_{\mathcal{I}} \left[ \left( \frac{\partial F}{\partial x} \right)^2 + \left( \frac{\partial F}{\partial y} \right)^2 + 1 \right] dx dy,$$

it is necessary and sufficient that the function $F(x, y)$ be absolutely continuous on $I_0$; and in order that this be the case it is necessary and sufficient that the area $S(F; I)$ be an absolutely continuous function of the interval $I \subset I_0$. 

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Proof. The assertion a) follows directly from Theorem 5.13; b) and c) follows from Theorems 6.1 and 6.5 on account of Theorem 7.4, Chap. IV.

With regard to Theorem 8.1 see L. Tonelli [5; 6; 7]. The necessity of condition a) was established a little earlier by G. Lampariello [1].

According to Tonelli's theorem, the relation of equality (8.2) can hold for a continuous surface \( z = F(x, y) \) only in the case in which the function \( F \) is absolutely continuous. Nevertheless, as proved by L. C. Young, this relation will remain valid for arbitrary continuous surfaces, as soon as we replace on the right-hand side the partial derivatives by ratios of finite differences and transpose the passage to the limit outside the integral sign. In fact:

(8.3) Theorem. For any continuous surface \( z = F(x, y) \) and any interval \( I_0 \) we have

\[
\begin{align*}
S(F; I_0) &= \lim_{n, \beta \to 0} \int_{I_0} \left[ \frac{F(x + a, y) - F(x, y)}{\alpha} \right]^2 + \\
&\quad + \left[ \frac{F(x, y + \beta) - F(x, y)}{\beta} \right]^2 + 1 \right]^{1/2} dx dy,
\end{align*}
\]

and in order that the function \( F \) be of bounded variation on \( I_0 \), it is necessary and sufficient that

\[
\limsup_{a, \beta \to 0} \frac{1}{|a| + |\beta|} \int_{I_0} |F(x + a, y + \beta) - F(x, y)| dx dy < +\infty.
\]

Proof. Let \( \{F_n\} \) be the sequence of mean value integrals (cf. § 7, p. 178) of the function \( F \). Denote, for brevity, by \( R(x, y; a, \beta) \) the expression under the integral sign on the right-hand side of (8.4), and by \( R_n(x, y; a, \beta) \), for each positive integer \( n \), the expression obtained from \( R(x, y; a, \beta) \) by replacing \( F \) by \( F_n \). Finally let us write for \( n = 1, 2, ... 

\[
R_n(x, y; a, \beta) = \lim_{a, \beta \to 0} R_n(x, y; a, \beta) = ((\partial F_n(x, y)/\partial x)^2 + (\partial F_n(x, y)/\partial y)^2 + 1)^{1/2}.
\]

In order to establish the identity (8.4), it evidently suffices to show that

(8.6) \( S(F; I_0) \leq \liminf_{a, \beta \to 0} \int_{I_0} R(x, y; a, \beta) dx dy \)

and

(8.7) \( S(F; I_0) \geq \limsup_{a, \beta \to 0} \int_{I_0} R(x, y; a, \beta) dx dy \).

For this purpose, let \( I \) be any interval in the interior of \( I_0 \). By means of the inequality (5.2), p. 171, we easily find that

\[
R_n(x, y; a, \beta) \leq \frac{n}{\epsilon} \int_{I_0} R(x + u, y + v; a, \beta) du dv.
\]

Now let \( n \) be a positive integer, sufficiently large in order that \( (x, y) \in I \), \( |x| < 1/n \) and \( |y| < 1/n \) should imply \( (x + u, y + v) \in I_0 \). We then have

\[
\int_{I_0} R(x + u, y + v; a, \beta) du dv \leq \int_{I_0} R(x, y; a, \beta) dx dy
\]

and consequently, by (8.8),

\[
\int_{I_0} R_n(x, y; a, \beta) dx dy \leq \int_{I_0} R(x, y; a, \beta) dx dy.
\]

Making \( a \to 0 \) and \( \beta \to 0 \), we obtain in the limit \( S(F; I) \leq \lim_{a, \beta \to 0} \int_{I_0} R(x, y; a, \beta) dx dy \). This relation being thus established for each interval \( I \), we may replace, on its left-hand side, \( I \) by \( I_0 \) and making still \( n \to \infty \) we obtain the relation (8.6).

In order to prove the relation (8.7), let's first observe that the latter is obvious in the case in which \( S(F; I_0) = +\infty \). We may therefore assume that the function \( F(x, y) \) is of bounded variation on \( I_0 \) and moreover (cf. § 4, p. 170) of bounded variation on every interval in the plane and periodic with respect to each variable. We can therefore determine an interval \( I_0 = [a_1, b_1; a_2, b_2] \) containing \( I_0 \) in its interior, such that its sides \( b_1 - a_1 \) and \( b_2 - a_2 \) are the periods of \( F(x, y) \) with respect to \( x \) and \( y \) respectively.

This being so, we find easily, on account of the inequality (5.1), p. 171, that, for any pair of positive integers \( n \) and \( k \),

\[
R_n(x, y; a, \beta) \leq \frac{1}{k} \sum_{j=0}^{k-1} R_n(x + ja/k, y + j\beta/k; a/k, \beta/k).
\]

By integrating the two sides of this inequality over \( J_0 \) and taking account of the periodicity of the function \( F \), we obtain

\[
\int_{J_0} R_n(x, y; a, \beta) dx dy \leq \int_{J_0} R_n(x, y; a/k, \beta/k) dx dy;
\]

and hence, passing to the limit, making first \( k \to \infty \), and then \( n \to \infty \), we find by Radô's Theorem 7.3,
\[ \lim_{n \to \infty} \int_{J_0} R(x, y; a, \beta) \, dx \, dy = \lim_{n \to \infty} \int_{J_0} R_n(x, y) \, dx \, dy = S(F; J_0) = S(F; J_0) , \]

and so

\[ (8.9) \quad \lim_{n \to \infty} \sup_{\alpha, \beta} \int_{J_0} R(x, y; a, \beta) \, dx \, dy \leq S(F; J_0) . \]

Now, by the result already established in the inequality (8.6), we have \( \lim_{\alpha, \beta \to 0} \int_{I} R(x, y; a, \beta) \, dx \, dy \geq S(F; I) \) for every interval \( I \).

It follows at once that (8.9) remains valid when we replace the interval \( J_0 \) by any subinterval of \( J_0 \), and in particular by the interval \( I \). We thus obtain the relation (8.7).

Finally, let us remark that upon modifying the relation (3.4), in order that the area of the surface \( z = F(x, y) \) on \( I_0 \) be finite, it is necessary and sufficient that

\[ \lim_{x \to 0} \frac{1}{|\partial|} \int_{I} |F(x, y) - F(x, y_0)| \, dx \, dy < +\infty \]

and

\[ \lim_{y \to 0} \frac{1}{|\partial|} \int_{I} |F(x, y) - F(x, y_0)| \, dx \, dy < +\infty . \]

Now this pair of relations is easily seen to be equivalent to the relation (8.5) which therefore expresses a condition necessary and sufficient in order that the function \( F \) should be of bounded variation on \( I_0 \). This completes the proof.

A statement analogous to Theorem 3.3 can be made for curves (cf. Chap. IV, § 8). If \( C \) is a continuous curve defined by the equations \( x = X(t), y = Y(t) \), its length on any interval \( I_0 = [a, b] \) is given by the formula

\[ (8.10) \quad S(C; I_0) = \lim_{h \to 0} \int_{a}^{b} \left[ \frac{|X(t+h) - X(t)|}{h} \right]^2 + \left[ \frac{|Y(t+h) - Y(t)|}{h} \right]^2 \, dt . \]

In particular therefore, in order that a continuous functions \( G(t) \) be of bounded variation on an interval \([a, b]\), it is necessary and sufficient that

\[ (8.11) \quad \lim_{h \to 0} \sup_{t} \frac{1}{|I|} \int_{a}^{b} |G(t+h) - G(t)| \, dt < +\infty . \]