Chapter V. Properties of some special series.

9. Let $5.4(1)$ be a $\mathcal{E}[f]$ and let $n_{k+1}/n_k \rightarrow 1$. Then we have the inequalities $m_{\lambda} \mathcal{W}_k[f] \leq \mathcal{W}_k[f] \leq M_{\lambda} \mathcal{W}_k[f]$, where $\gamma_0^2 = a^2 + b^2$, and the constants $m$ and $M$ depend only on $\alpha$ and $\lambda$.

It is sufficient to prove, for sums of series, a theorem analogous to Theorem 5.5(1). The proof is similar if, for fixed $\alpha$, $\lambda$ is sufficiently large. In the general case we split up the series considered into a finite number of series for each of which the number $\lambda$ is large.

10. If the series $5.4(1)$, with $n_{k+1}/n_k > \lambda > 0$, converges in an interval $(a, b)$, then the series converges absolutely. Fatou [1].

[Let $a_k \cos n_k x + b_k \sin n_k x = \rho_k \cos (n_k x + \phi_k)$. It is easily seen geometrically that there is a point $x^*_{\lambda}$ in $(a, b)$ such that $\cos (n_k x^*_{\lambda} + \phi_k) > 0$ for $k$ sufficiently large. The theorem holds for $\lambda > 1$. See Zygmund [6]].

11. The points of convergence and those of divergence for the series $\Sigma (\sin^{10} x)/n!$ are everywhere dense in the interval $(0, 2\pi)$.

12. Let $0 < \alpha < 1$ and $0 < \beta$. The series $\Sigma_{n=1}^{\infty} n^{-\beta} \rho^n e^{\alpha x} e^{iu x}$ converges everywhere if $\alpha + \beta > 1$; the convergence is uniform if $\alpha + \beta > 1$. H. Hardy [1].

13. If $1 \leq \alpha + \beta \leq 2$, the sum of the previous series belongs to $\text{Lip}(\varphi = \beta - 1)$. H. Hardy [1], Zygmund [7].

[Apply van der Corput's lemma and an argument similar to that of § 5, 23].

14. The function $F(x) = -x + \lim_{m \to \infty} \frac{1}{\pi} \int \frac{m}{\pi} (1 + \cos 2\pi t) dt$ is a continuous function of bounded variation with Fourier coefficients $O(1/m)$. F. Riesz [5].

The product $p_m = (1 + \cos 2\pi) \ldots (1 + \cos 4\pi t)$ is a trigonometrical polynomial of order $a_m = 4^m - 4^{m-1} + \ldots + 4$. Since the lowest term of the polynomial $p_{m+1} - p_m = p_m \cos 4\pi t$ is of order $p_{m+1} = 4^{m+1} - 4^m + \ldots + 4 > p_m$, $p_m$ is a partial sum of $p_{m+1}$. I.e. $\{p_m\}$ is a subsequence of the polynomial sums of a trigonometrical series $f(x) = a_1 \cos x + a_2 \cos 2x + \ldots$. Let $P_m(x)$ be the integral of $p_m$ over the interval $(0, x)$, and let $\gamma_m$ be the number of non-vanishing terms in $p_m$. It is easy to see that $\gamma_{m+1} = \gamma_m - 1$, i.e.

$\gamma_{m+1} - \gamma_m = 3(\gamma_m - \gamma_{m-1})$. Since $p_{m+1} - p_m$ consists of $3^m$ terms each of which does not exceed $1$ in absolute value, we have $|p_{m+1} - p_m| < 3^m/3 = O(3^m/\gamma_m)$ and so the function $F(x) = \lim_{m \to \infty} P_m(x) = P(x) + \lim_{m \to \infty} (P_m - P_{m-1}) + \ldots$ is continuous. $P_m(x)$ is non-decreasing and so is its limit. It follows that the function $F(x) = -x + P(x)$ is continuous and of bounded variation. To obtain $\mathcal{E}[f]$ we reject the linear term from the series $f(x)$ integrated term by term. Since $a_m = 1$, the coefficients of $\mathcal{E}[f]$ are not $O(1/m)$.

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CHAPTER VI.

The absolute convergence of trigonometrical series.

6.1. The Lusin-Denjoy theorem. The convergence of the series

$$\sum_{n=1}^{\infty} |a_n + b_n|$$

implies the absolute convergence of the series

$$\sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

The series (2) may be absolutely convergent at an infinite set of points without (1) being convergent. A simple example is given by the series $\sum \sin mx$, whose terms vanish from some place onwards for every $x$ commensurable with $\pi$.

If the series (2) converges absolutely in a set $E$ of positive measure, the series (1) converges.

Suppose, for simplicity, that $a_0 = 0$, and let $a_n \cos nx + b_n \sin nx = \rho_n \cos (nx + \phi_n)$, where $\rho_n > 0$, $\rho_n^2 = a_n^2 + b_n^2$. The function

$$\alpha(x) = \sum_{n=1}^{\infty} \rho_n \cos (nx + \phi_n)$$

is finite at every point of $E$. Hence there exists a set $E \subset E$, $|E| > 0$, such that $\alpha(x)$ is bounded on $E$, $\alpha(x) < M$ say. Since the partial sums $\alpha_n(x)$ of (3) are uniformly bounded on $E$, the series may be integrated formally over $E$.

$$\sum_{n=1}^{\infty} \rho_n \int_{E} \cos (nx + \phi_n) \, dx \leq M |E|.$$
To prove the convergence of $p_1 + p_2 + \ldots$, which is equivalent to our theorem, it is sufficient to show that the integrals $I_n$ on the left in (4) all exceed an $s > 0$. Let $I_n$ be the integral analogous to $I_n$ with $|\cos(nx + x_0)|$ replaced by $\cos^2(nx + x_0)$. Since $I_n > s$, it is sufficient to prove that $I_n > s$. For this purpose we use the formula $2\cos^2(nx + x_0) = 1 + \cos 2nx \cos 2x_0 - \sin 2nx \sin 2x_0$. Since the Fourier coefficients of the characteristic function of the set $E$ tend to 0, we obtain that $I_n > \frac{1}{2} |E|$, which completes the proof, all $I_n$ being positive.

The set $E$ in the theorem which we have established is of positive measure. This property, while sufficient for the convergence of (1), is not necessary. The problem of necessary and sufficient conditions seems to be unsolved.

6.11. We shall supplement the previous theorem by a few results of the same character. Suppose that, for the series 6.1(2), we have $p_1 + p_2 + \ldots = \infty$, and let $E$ be the set of points where $\alpha(x) < \infty$. The complementary set $H$, where the upper limit of the sequence $\{\alpha_n(x)\}$ of continuous functions is equal to $\infty$, is a product of a sequence of open sets; for if $G_n$ denotes the open set of points where at least one of the functions $\alpha_n(x)$ exceeds $N$, we have $H = G_1 G_2 \ldots$. It follows that $E$ is the sum of a sequence of closed sets. None of these closed sets contains an interval; for otherwise we should have $p_1 + p_2 + \ldots < \infty$. It follows that all of them are non-empty, $E$ is of the first category, and therefore, if 6.1(2) converges absolutely in a set of the second category, even if it is of measure 0, the series 6.1(1) converges$^1$.

6.12. There exist trigonometrical series absolutely convergent in a perfect set but not everywhere (§ 6.6.1). On the other hand, as we shall prove, there exist perfect sets $P$ of measure 0, which, as regards the absolute convergence of trigonometrical series, resemble sets of positive measure: every trigonometrical series absolutely convergent in $P$ is absolutely convergent everywhere. In particular Cantor's well-known set has this property.

A point-set $B$ will be called a basis, if every real $x$ can be represented in the form $a_1 x_1 + a_2 x_2 + \ldots + a_n x_n$, where $a_i, a_n \ldots$ are integers, and $x_1, x_2 \ldots$ belong to $B$. We may also write $x = \varepsilon_1 x_1 + \ldots + \varepsilon_n x_n$, where $\varepsilon_1 = \pm 1$ and the $x_i$ are not necessarily different. We require the following lemma.

Let $B$ be a basis, and let $B^* = B^*$ denote the set $B$ translated by a number $u$. There exists a set $S$ of the second category such that, for every $y \in S$, we have $y = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n$, with $x_i$ integral and $x_i \in B^*$). To prove this, we observe that for every $x$ we have $x = a_1 (x_1 - u) + a_2 (x_2 - u) + \ldots$, i.e. $x + ku = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n$, where $k = k_n$ is an integer. Let $E_n = \infty < n < \infty$, denote the set of $x$ for which $k_n = n$. For any $x$ may exist several $k_n$; we choose one of them. At least one of these sets, say $E_n$, is not of the first category, and we may take for $S$ the set $E_n$ translated by $n u$. We may say that $B^*$ is a basis for $S$.

If $B$ is a basis, every trigonometrical series absolutely convergent in $B$ is absolutely convergent everywhere$^2$). Suppose first that the trigonometrical series considered contains only sine terms. We prove by induction that $|\sin a (\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n)| \leq \frac{1}{1 + n x_1 + \ldots + n x_n}$ if $\varepsilon_1 = \pm 1$, and the result follows. In the general case let $u$ be any point of $B$, and let $x = y + u$. We have $a_n \cos nx + b_n \sin nx = a_n (x) \cos ny + b_n (x) \sin ny$, where $a_n (x) = a_n \cos nu + b_n \sin nu$, $b_n (x) = b_n \cos nx - a_n \sin nu$. The absolute convergence of the series at the point $y = 0$ implies $|a_n (x)| + |a_n (u)| + \ldots < \infty$, and therefore the series $b_n (u) \sin ny + b_n (x) \sin 2ny + \ldots$ converges absolutely in a set $B^*$ obtained from $B$ by translating it by $-u$. In virtue of the lemma, $B^*$ is a basis for a set $S$ of the second category. The argument which we applied to sine series shows that $b_n (u) \sin y + b_n (x) \sin 2ny + \ldots$ is absolutely convergent in $S$, and consequently, by Theorem 6.11, everywhere. The same may be said of the series with terms $a_n (u) \cos ny + \ldots + b_n (x) \sin nx = a_n \cos u x + b_n \sin x$, and the theorem is established.

6.13. To give an example, we shall show that the Cantor ternary set $C$ constructed on $(0,1)$ (or on any other interval) is a basis. More precisely, we will show that the set of all possible sums $x + y$, with $x \in C$, $y \in C$, fills up the whole interval $(0,1)$. This could be deduced from the fact that the ternary development of any $x \in C$ can be written in the form not containing the digit 1.

1) Thence it is not difficult to deduce that $B^*$ is itself a basis (§ 6.6.2), but this is not necessary for our purposes.
2) See Niemytzki [1], for the case of sine series.
3) Steinhaus [4]. More general results will be found in Denjoy [2], Mirimanoff [1].
but a geometrical proof is more illuminating. Consider the set \( F \) of points \((x,y)\) of the plane such that \( x \in G, y \in C \). The set \( F \) may be obtained by the following procedure: Divide the square \( Q_0 \) with opposite corners at \((0,0)\) and \((1,1)\) into nine equal parts, and, removing the interior of the five squares forming a cross, consider the sum \( Q_1 \) of the remaining four corner squares. For any of these corner squares we repeat our procedure, and let \( Q_0 \) be the sum of the new corner squares, and so on. Plainly \( F = Q_0 \cap Q_1 \cap \ldots \). The projection of any \( Q_k \) on the diagonal joining the points \((0,0)\) and \((1,1)\) fills up this diagonal. In other words, any straight line \( L_n \) with the equation \( x + y = h \), \( 0 < h < 1 \), meets every \( Q_k \) at one point at least. Since the \( Q_k \) are closed and form a decreasing sequence, \( F \cap L_n = \emptyset \) for \( 0 < h < 1 \), and this is just what we wanted to prove.

### 6.2. Fatou's theorems

The problem of the absolute convergence for sine or cosine series has a very simple solution in the case when the moduli of the coefficients form a decreasing sequence.

If the series \( a_1 \cos x + a_2 \cos 2x + \ldots + a_r \cos ax + \ldots \), is absolutely convergent at a point \( x_0 \), then \( |a_1| + |a_2| + \ldots < \infty \). The same is true for the series \( a_1 \sin x + a_2 \sin 2x + \ldots, \) provided that \( x_0 \equiv 0 \) (mod \( \pi \)). To prove the first part of the theorem we may simply suppose that \( 0 < x_0 < \pi \). From the hypothesis it follows that \( |a_1| \cos^2 x_0 + |a_2| \cos^2 2x_0 + \ldots < \infty \). Since \( 2 \cos^2 x_0 = 1 + \cos 2x_0 \), and since the series \( |a_1| \cos y_0 + |a_2| \cos 2y_0 + \ldots \) converges (§1.28), the result follows. The second part is obtained by a similar argument.

### 6.21. The set of points where a trigonometrical series converges absolutely.

The set of points where a trigonometrical series 6.1(2) converges absolutely, possesses curious properties. Let \( A \) denote the set of points of absolute convergence for the series conjugate to 6.1(2), and let \( B \) and \( C \) be the sets of points where the series 6.1(2) and its conjugate converge, not necessarily absolutely. It will be convenient to place all these sets on the circumference of the unit circle.

Every point of \( A \) is a point of symmetry for the sets \( A, A', B, \overline{B} \).

The proof follows from the formula

\[
a_0(x+h)+a_0(x-h)=2a_0(x)\cos nh, \quad b_0(x+h)=b_0(x-h)=-2a_0(x)\sin nh,
\]

where the notation is that of §6.12. From the first of them we deduce that, if \( |a_0(x)| + |a_0(x)| + \ldots < \infty \), and if the series \( a_0(x-h)+a_0(x-h)+\ldots \) converges, or converges absolutely, so does the series \( a_0(x+h)+a_0(x+h)+\ldots \).

The theorem remains true if we consider the points of summability, the arcs of uniform convergence, etc.

#### 6.22. If \( A \) is infinite, then \( B \), and similarly \( C \), is either of measure 0 or \( 2\pi \).

If \( x \in A \), \( x + h \in A \), then all the points \( x + h, x + 2h, x + 3h, \ldots \) belong to \( A \). Since \( A \) is infinite, \( h \) may be arbitrarily small, and so \( A \) is everywhere dense. Suppose that \( B \) and its complement \( C \) are both of positive measure, and let \( x_1 \) and \( x_2 \) be points of density 1 for \( B \) and \( C \) respectively. There exists an \( \epsilon > 0 \) such that, if any interval \( I \), \( |I| < 2\delta \), contains \( x_1 \), we have \( |IB| > \frac{1}{3} |I| \), and if any interval \( I \), \( |I| < 2\delta \), contains \( x_2 \), then \( |IC| > \frac{1}{3} |I| \). Let \( I = (x_1 - \epsilon, x_1 + \epsilon) \), and take an \( x \) belonging to \( B \) and distant by less than \( \frac{1}{3} \epsilon \) from the middle-point of the arc \((x_1, x_3)\). The set \( B \) reflected in \( x_0 \) goes into itself, and \( I \) into an interval \( I' \), \( |I'| = 2\beta \), containing \( x_2 \). Since the inequalities \( |IB| > \frac{1}{3} |I'| \), \( |IC| > \frac{1}{3} |I'| \) are incompatible, we have a contradiction.

#### 6.23. The absolute convergence of Fourier series.

We begin by the following theorem due to S. Bernstein.

If \( f \in \text{Lip } \alpha, \alpha > \frac{1}{2}, \) then \( \sum f \) converges absolutely. For \( \alpha = \frac{1}{2} \) this is no longer true.

Suppose that 6.1(2) is \( \sum f \). Then

\[
(1) \quad f(x+h) - f(x-h) \sim 2 \sum b_0(x) \sin nh,
\]

\[
\frac{1}{\pi} \int \left| f(x+h) - f(x-h) \right|^2 \, dx = 4 \sum \rho_n^2 \sin^2 nh,
\]

where \( \rho_n = a_n^2 + b_n^2 \). The left-hand side of the last formula is \( \leq C h^{2\alpha} \), where \( C, a_1, \ldots \) denote constants. On setting \( h = \pi/2N \) we obtain two inequalities

\[
(2) \quad \sum_{n=1}^{N} \rho_n^2 \sin^2 \frac{\pi n}{2N} \leq C N^{-2\alpha}, \quad \sum_{n=1}^{N} \rho_n^2 \sin^2 \frac{\pi n}{2N} \leq C N^{-2\alpha}.
\]
Let us now assume that \( N = 2^r \), \( r = 1, 2, \ldots \). Taking into account only the terms with indices \( n \) exceeding \( \frac{1}{2} N \), we obtain from the last inequality

\[
(3) \quad \sum_{n=2^{r-1}+1}^{2^r} \rho_n < 2C 2^{-n \alpha}.
\]

Thence, by Schwarz’s inequality,

\[
(4) \quad \sum_{n=2^{r-1}+1}^{2^r} \rho_n < \left( \sum_{n=2^{r-1}+1}^{2^r} \rho_n^2 \right)^{1/2} \left( \sum_{n=2^{r-1}+1}^{2^r} 1 \right)^{1/2} < C_1 2^{(1/4 - \alpha)} \leq \frac{\alpha}{2} \left( \sum_{n=2^{r-1}+1}^{2^r} \rho_n^2 \right)^{1/2} \left( \sum_{n=2^{r-1}+1}^{2^r} 1 \right)^{1/2} < C_1 2^{(1/4 - \alpha)},
\]

and finally

\[
(5) \quad \sum_{n=2}^{2^r} \rho_n \leq C_1 \sum_{n=2}^{2^r} 2^{(1/4 - \alpha)} < \infty.
\]

The last series is convergent since \( \alpha > \frac{1}{2} \). The proof of the second part of the theorem we postpone to § 6.33.

6.31. If \( f(x) \) is of bounded variation and belongs to Lip \( \alpha \) for any positive \( \alpha \), \( f \) converges absolutely.\(^1\) That the second condition imposed on \( f \) is not superfluous is seen from the example of the series

\[
(1) \quad \sum_{n=2}^{2^r} \sin nx/n \log n
\]

which, being the Fourier series of a function of bounded variation, indeed of an absolutely continuous function (§ 5.12), diverges absolutely (§ 6.2).

Let \( \omega(\beta) \) be the modulus of continuity of \( f \), and \( V \) the total variation of \( f \) over \((0, 2\pi)\). We start from the inequality

\[
\sum_{k=1}^{2N} \left| f \left( x + \frac{k\pi}{N} \right) - f \left( x + (k-1) \frac{\pi}{N} \right) \right|^2 \leq \omega \left( \frac{\pi}{N} \right) \sum_{k=1}^{2N} \left| f \left( x + \frac{k\pi}{N} \right) - f \left( x + (k-1) \frac{\pi}{N} \right) \right| \leq V \omega \left( \frac{\pi}{N} \right)
\]

which we integrate over \((0, 2\pi)\). On account of the periodicity,

\[
\int_0^{2\pi} \left| f \left( x + \xi \right) - f \left( x - \frac{\pi}{2N} \right) \right|^2 \leq 2\pi V \omega \left( \frac{\pi}{N} \right),
\]

\[
\sum_{n=1}^{2N} \rho_n^2 \leq C_2 2^{-\gamma n},
\]

\[
\sum_{n=2}^{2N} \rho_n^2 \leq C_3 2^{-\gamma n},
\]

\[
\sum_{n=1}^{2N} \rho_n^2 \leq C_4 2^{-\gamma n},
\]

\[
\sum_{n=1}^{2N} \rho_n^2 \leq C_4 2^{-\gamma n} < \infty.
\]

6.32. The problem of the absolute convergence of trigonometrical series may be generalized as follows. Given a series 6.1(2), we ask about the values of the exponent \( \beta \) which makes

\[
(1) \quad \sum_{n=1}^{\infty} \left\{ |a_n|^\beta + |b_n|^\beta \right\}
\]

convergent. Theorem 6.3 is special a case of the following theorem; it is, in fact, the most important case of it.

If \( f \in \text{Lip} \alpha, 0 < \alpha \leq 1 \), the series (1) converges for every \( \beta > 2/(2\alpha + 1) \), but not necessarily for \( \beta = 2/(2\alpha + 1) \).\(^2\)

The proof of the first part resembles the proof of the first part of Theorem 6.3. Let \( \gamma = 2/(2\alpha + 1) \). Since \( 0 < \gamma < 2 \), we may also assume that \( 0 < \beta < 2 \). Starting with 6.3(3), and applying Hölder’s inequality, we obtain

\[
\sum_{n=1}^{2^r} \rho_n^\beta \leq \left( \sum_{n=1}^{2^r} \rho_n^2 \right)^{\gamma/2} \left( \sum_{n=1}^{2^r} 1 \right)^{1-\gamma/2} \leq C_4 2^{-\gamma n} < \infty.
\]

Here \( 1-\gamma/\gamma < 0 \), and an argument similar to 6.3(5) yields the convergence of \( \rho_n^\beta + \rho_{n+1}^\beta + \ldots \) or, what is the same thing, of the series (1). This gives the first part of the theorem.

6.33. The second part of Theorem 6.32, and of Theorem 6.3, is a simple corollary of the results obtained in § 5.3. It was

\[\text{Zygmund [8].}\]

\[\text{Szász [2].}\]
proved there that the real and imaginary components of the first of the series

\[ \sum_{n=1}^{\infty} \frac{e^{\pi n \log n}}{n^{1+\alpha}} \sin x, \quad 0 < \alpha < 1, \quad \sum_{n=1}^{\infty} \frac{e^{\pi n \log n}}{(n \log n)^{\frac{1}{\alpha}}} \cos x, \]

belong to \( \text{Lip} \alpha \), and it is easy to see that, for these components, the series with terms \( \frac{1}{n^{1+\alpha}} \) diverge. The components of the second series in (1) belong to \( \text{Lip} 1 \) (§ 5.33), and the series with terms \( \frac{1}{n^{1+\alpha}} \) diverges.

6.34. If \( f \) is of bounded variation and also \( f \in \text{Lip} \alpha, 0 < \alpha < 1 \), the series 6.32(1) converges for \( \beta > 2/(2 + \alpha) \).

The proof, which is analogous to that of Theorems 6.31 and 6.32, may be left to the reader (see also § 6.67).

6.35. Let \( F(x) \) be an absolutely continuous and periodic function whose derivative \( F'(x) = f(x) \) belongs to \( L^p \).

If \( a_n, b_n \) are the Fourier coefficients of \( f \), those of \( F \) will be \( -b_n/n, a_n/n \). From the inequalities

\[ \frac{|a_n|}{n} \leq \frac{1}{2} \left( \frac{a_n^2 + 1}{n^2} \right), \quad \frac{|b_n|}{n} \leq \frac{1}{2} \left( \frac{b_n^2 + 1}{n^2} \right), \]

we see that \( \mathcal{S}[F] \) converges absolutely. More generally, if \( F \) is absolutely continuous and \( F' \in L^p, p > 1 \), then \( \mathcal{S}[F] \) converges absolutely.\(^2\) The proof remains essentially the same as in the case \( p = 2 \), if, instead Bessel’s inequality, we use a more general inequality, due to Young, which will be established in Chapter IX. It is however much simpler to deduce the theorem from Theorem 6.31, observing that, if \( F' \in L^p, p > 1 \), then \( F \) satisfies a Lipschitz condition of positive order (§ 4.7.8).

The result which we have established is, in turn, contained in the following theorem

6.36. (i) If \( F(x) \) is absolutely continuous, \( F'(x) = f(x) \), and \( |f| \log^+ |f| \) is integrable, then \( \mathcal{S}[F] \) converges absolutely.\(^2\) It will be convenient to postpone the proof of (i) to Chapter VII, where we shall obtain this theorem as a corollary of the following important result due to Hardy and Littlewood:

(ii) If \( \mathcal{S}[F] \) and \( \mathcal{S}[F'] \) are both Fourier series of functions of bounded variation, \( \mathcal{S}[F] \) converges absolutely.

Here we only observe that the integrability of \( |f| |\log^+ |f||^{1-\varepsilon} \), \( \varepsilon > 0 \), would not be sufficient for the truth of (i). For if we take for \( \mathcal{S}[F] \) the series 6.31(1), which converges absolutely only at the points \( x = 0 \mod \pi \), we have \( f(x) \sim 1, x \log^3 x \) as \( x \to 0, + \infty \), (§ 5.221), so that \( |f| |\log^+ |f||^{1-\varepsilon} \) is integrable for every \( \varepsilon > 0 \).

6.4. Szidon’s theorem on lacunary series. The following theorem on the absolute convergence of Fourier series bears a different character.

If a lacunary trigonometrical series

\[ \sum_{n=1}^{\infty} (a_n \cos n_0 x + b_n \sin n_0 x), \quad n_{0+k+1}/n_0 > \lambda > 1, \]

is the Fourier series of a bounded function \( f(x), |f| \leq M \), the series converges absolutely\(^3\).

Taking, instead of \( f(x) \), the functions \( f(x) \pm f(-x) \), we may restrict ourselves to purely cosine or purely sine series, e. g. to the former. The idea of the proof consists in considering the non-negative polynomials

\[ P_1(x) = \prod_{k=1}^{l} (1 + \epsilon_k \cos m_k x), \]

where \( \epsilon_k = \pm 1 \) and the positive integers \( m_k \) satisfy a condition \( m_{k+1}/m_k > \eta > 0 \). Multiplying out the product \( P \), we see that it consists of the constant term 1, and of terms \( A_0 \cos vu \), where \( v = \pm m_k \pm m_{k+1} \pm \cdots \pm m_q > 0, m_k < m_{k+1} < \cdots < m_q < m_0 \). From the last equation we see that \( v \) is contained between \( m_q(1-\nu^{-1}-\nu^{-2} - \cdots) \) and \( m_q(1+\nu^{-1}+\nu^{-2} + \cdots) \), i. e. between \( m_q(\nu^{-2})(\nu^{-1}) \) and \( m_q(\nu^{-2})(\nu^{-1}) \). Therefore, since \( \nu > 0 \), the numbers \( \pm m_k \pm \cdots \pm m_q \) corresponding to various sequences \( k \) are all different, and, if \( m \) is large enough, \( m > m_0(\epsilon) \), the indices \( v \) corresponding to \( A_0 \neq 0 \) concentrate in the neighbourhoods \( (m_1 - \epsilon), m_1 + 1, \cdots) \) of the numbers \( m_0 \) where \( \epsilon > 0 \) is arbitrary.

Returning to the series (1), take \( \epsilon \) so small that the intervals \( (n_0(\epsilon), n_0(\epsilon+1)), k = 1, 2, \cdots \), do not overlap, and an integer \( r \) such that \( \lambda^r > m_q(\epsilon) \). Put \( m_0^2 = m_0^{k_r-1}, \quad k = 1, 2, \cdots, 0 \leq s \leq r-1, \)

\(^1\) Szidon [2]; for a generalization see Zygmund [6].
and let \( P^0(x) \) denote the polynomial (2) formed with \( m^0(x) \), \( 1 \leq k \leq l \), and \( s_k = \text{sign}\, a_{2k+1} \). Since \( m^0(x)/m^0(x) > 1 \), we have

\[
(3) \quad \sum_{k=1}^{l} a_{2k+1} = \frac{1}{\pi} \int f(x) P^0(x) \, dx \leq \frac{M}{\pi} \int P^0(x) \, dx = 2M,
\]

since the constant term of \( P^0(x) \) is equal to 1. Making \( l \to \infty \), we find that each of the \( r \) partial sums into which we have decomposed the series \( |a_r| + |a_{r+1}| + |a_{r+2}| + \ldots \) converges. This completes the proof.

If (1) is a pure sine series, we consider, instead of (2), analogous polynomials, with cosines replaced by sines.

### 6.5. Wiener's theorem.

It is obvious that the absolute convergence of \( \mathcal{Z} [f] \) at a point \( x_0 \) is not a local property but depends on the behaviour of \( f(x) \) in the whole interval \((0, 2\pi)\). However, if to every point \( x_0 \) corresponds a neighbourhood \( \delta_0 \) of \( x_0 \) and a function \( g(x) = g_0(x) \) such that (i) \( \mathcal{Z} [g] \) converges absolutely, and (ii) \( g(x) = f(x) \) in \( \delta_0 \), then \( \mathcal{Z} [f] \) converges absolutely.

By the Heine-Borel theorem we can find a finite number of points \( x_1, x_2, \ldots, x_n \) such that the intervals \( \delta_0, \delta_0, \ldots, \delta_0 \) overlap and cover the whole interval \( 0 < x < 2\pi \). Let \( \delta_0 = (u_k, v_k) \). Without loss of generality we may suppose that \( u_k < v_{k-1} < u_{k+1} < v_k \), \( k = 1, 2, \ldots, n \), where \((u_{m+1}, v_{m+1}) = (u_1, v_1)\). Let \( \lambda(x) \) be the periodic and continuous function equal to 1 in \((u_{m-1}, u_{m+1})\), vanishing outside \((u_m, v_m)\) and linear in the intervals \((u_m, u_{m-1})\) and \((u_{m+1}, v_m)\). It will be readily seen that \( \lambda(x) + \lambda(x) + \ldots + \lambda(x) = 1 \). Since \( \lambda \) has a derivative of bounded variation, the Fourier coefficients of \( \lambda \) are \( O(n^{-\gamma}) \), so that \( \mathcal{Z} [\lambda_0] \) converges absolutely.

Since \( \mathcal{Z} [f\lambda_0] = \mathcal{Z} [g_0\lambda_0] = \mathcal{Z} [g_0\lambda_0] \mathcal{Z} [\lambda_0] \), we obtain that \( \mathcal{Z} [f\lambda_0] \) converges absolutely (\S 4.451). To prove the theorem it is sufficient to observe that \( \mathcal{Z} [f] = \mathcal{Z} [f' + \ldots + \lambda] = \mathcal{Z} [f' \lambda_1 + \ldots + \mathcal{Z} [f' \lambda_1] \ldots] \), where the sum of the last series is \( \ll |a_0| + |a_1| + \ldots \). Hence \( \mathcal{Z} [f] \) converges for \( |z| < 1 \).

### 6.51.

Let \( f(t) = \sum_{n=0}^{+\infty} c_n e^{int} \), \( f^*(t) = \sum_{n=-\infty}^{+\infty} c_n^{(0)} e^{int} \), \( k = 0, 1, 2, \ldots \). Since \( \mathcal{Z} [f] \) is obtained from \( \mathcal{Z} [f] \) by formal multiplicity, it is easy to see that if \( \ldots + |c_1| + |c_0| + |c_1| + \ldots = M \), then \( \ldots + |c_1^{(0)}| + |c_0^{(0)}| + \ldots \leq M^2 \). Suppose that the series \( \sum_{n=0}^{+\infty} c_n z^n \) converges for \( |z| < r \). In the case \( M < r \) the theorem is fairly simple; for the series \( \sum_{n=0}^{+\infty} c_n z^n \) converges uniformly, and, if \( \gamma_n \) are the complex Fourier coefficients of \( \{ f \} \), then

\[
\gamma_n = \sum_{m=0}^{+\infty} a_m c_{m+n}^{(0)}, \quad \sum_{n=-\infty}^{+\infty} |\gamma_n| \leq \sum_{m=0}^{+\infty} |a_m| c_{m+n}^{(0)} = \sum_{m=0}^{+\infty} a_m \sum_{n=-\infty}^{+\infty} c_{m+n}^{(0)},
\]

where the sum of the last series is \( \ll |a_0| + |a_1| + \ldots \). Hence \( \mathcal{Z} [f] \) converges absolutely.

Let \( t_n \) be an arbitrary point of the interval \( 0 \leq t \leq 2\pi \). To prove the theorem in the general case it is sufficient to show that there is a function \( g(t) \) such that \( \mathcal{Z} [\gamma_{\infty}] \) converges absolutely and that \( g(t) = f(t) \) for some point \( t_0 \) in \((0, 2\pi)\). Suppose, for simplicity, that \( t_0 = 0 \) and \( f(0) = 0 \). Without real loss of generality we may suppose that \( u_0 = 0 \), for otherwise we have \( \mathcal{Z} [f(t)] = \mathcal{Z} [f(t) - u_0 + u] = \mathcal{Z} [f(t)] \), where \( f(t) = f(t) - u_0 + u \), and we may consider the functions \( f(t), \gamma_{\infty} \) instead of \( f, \gamma_{\infty} \).

Let \( \omega = a_0 + a_z + \ldots \) be convergent for \( |z| < r \). In virtue of the special case already dealt with, it is sufficient to construct a function \( g(t) \) with Fourier coefficients \( c_n \) such that \( g(t) = f(t) \) in \((-\pi, \pi)\) and that \( \ldots + |c_1| + |c_0| + |c_1| + \ldots = M^2 < r \); for then \( \mathcal{Z} [g(t)] \) will be absolutely convergent.

Let \( \lambda(t) = \lambda_0(x) \) be a continuous periodic function such that

(i) \( \lambda(t) = 1 \) for \( 0 \leq t \leq \rho \),
(ii) \( \lambda(t) = 0 \) for \( 2\rho \leq t \leq \pi \),
(iii) \( \lambda(t) \) is linear in the interval \((\rho, 2\rho)\),
(iv) \( \lambda(t) \) is even. If \( I_n = \rho/2\pi \), \( I_n = (2 \sin \lambda/2 \rho \cos \lambda/2 \rho n) \pi \rho n^2 \), \( n \neq 0 \). We shall require the following relations

\[
(1) \quad \sum_{n=-\infty}^{+\infty} |I_n| = A, \quad (2) \quad \sum_{n=-\infty}^{+\infty} |I_n - I_{n-1}| = 0 \quad \text{as} \quad \rho \to 0,
\]

where \( A, B, \ldots \) denote constants independent of \( \rho \). To prove (1) we observe that from the inequalities \( |\sin u| \leq 1 \), \( |\sin u| \leq |u| \),

we obtain \( |I_n| \leq 2/\pi \rho n^2 \), \( |I_n| \leq 2 \rho /2\pi \), and so, if \( N = [1/\rho] + 1 \), the sum in (1) is less than

\[
3\rho + 2 \sum_{n=1}^{N} 3\rho - 2 \sum_{n=N+1}^{+\infty} 2/\pi n^2 < \lambda N + 4/\pi \rho N < A.
\]
Now $\nu_n - \nu_{n-1}$ is the complex Fourier coefficient of the function $\lambda_\nu(t)$ $(1 - e^{it})$. Considering the real and imaginary parts of the derivative $\lambda_\nu(t)$ $(1 - e^{it}) - e^{it}\lambda_\nu(t)$ of this function, we easily find that the total variation of this derivative over $(-\pi, \pi)$ is uniformly bounded, and so, in virtue of the results obtained in § 2.118, we have $|\nu_n - \nu_{n-1}| \leq B/n^2$. If $\nu$ is a positive integer, the series in (2) is equal to

$$
\sum_{n=0}^{\infty} \left( \sum_{n=0}^{\omega-1} + \sum_{n=\omega}^{\infty} \right) \leq \sum_{n=0}^{\infty} |\nu_n - \nu_{n-1}| + 2B \sum_{n=1}^{\infty} \frac{1}{n^2} = P + Q.
$$

Taking $\nu$ large enough we have $Q < \frac{1}{2} \varepsilon$. If $p > 0$, then $\|p_n\| \to 0$, and so $\nu_n < 0$ for every $n$. Hence, for fixed $P$, $P < \frac{1}{2} \varepsilon$, $P + Q < \varepsilon$ if $p$ is small enough, and this proves (2).

Let $q > 0$ be an integer which we shall define in a moment, and let $d_\nu = u_p + v_p$, where $u_p = c_p$, $v_p = 0$ for $\|p\| < q$, and $u_p = 0$, $v_p = c_p$ for $\|p\| > q$. Since $f(0) = \sum c_p = 0$, $\sum |c_p| < \infty$ we have

$$
\sum_{p=0}^{\infty} |u_p| < r/3A, \quad \sum_{p=0}^{\infty} |v_p| < r/3A
$$

if $q$ is large enough. Denoting by $\nu_n$ the Fourier coefficients of the function $f(t)$ $\lambda_\nu(t)$, we have

$$
d_\nu = \sum_{\mu=0}^{\infty} c_\mu \nu_n - \nu_{n-1}
$$

$$
\sum_{n=0}^{\infty} |d_\nu|^2 \leq \sum_{n=0}^{\infty} \sum_{\mu=0}^{\infty} |u_\mu| |\nu_n - \nu_{n-1}| + \sum_{n=0}^{\infty} \sum_{\mu=0}^{\infty} |v_\mu| |\nu_n - \nu_{n-1}| = S + T,
$$

$$
T = \sum_{n=0}^{\infty} \sum_{\mu=0}^{\infty} |v_\mu| |\nu_n - \nu_{n-1}| = \sum_{n=0}^{\infty} \sum_{\mu=0}^{\infty} |v_\mu| |\nu_n - \nu_{n-1}| < r \cdot A = \frac{r}{3A},
$$

$$
S = \sum_{n=0}^{\infty} \sum_{\mu=0}^{q} c_\mu (l_{n+1} - l_n) \leq \sum_{n=0}^{\infty} \sum_{\mu=0}^{q} c_\mu (l_{n+1} - l_n) + \sum_{n=0}^{\infty} \sum_{\mu=0}^{q} c_\mu |l_n| = S_1 + S_2.
$$

It is plain that $S_2 < \frac{1}{4} r$. Since $|l_{n+1} - l_n| \leq |l_{n+1} - l_{n-1}| + \ldots + |l_1 - l_0|$, for $p > 0$, $|l_{n+1} - l_n| \leq |l_{n+1} - l_{n-1}| + \ldots + |l_1 - l_0|$, for $p < 0$, $S_1$ is less than a multiple of the series (2) and so tends to 0 with $p$. If $p = p_0$ is small enough, then $S_1 < \frac{1}{4} r$.

As a corollary we obtain that, if $\bar{f}$ converges absolutely and $f(x) \neq 0$, then $\bar{f}$ converges absolutely.

### 6.6. Miscellaneous theorems and examples.

1. The set of points where the series $\sum x^{-1}$ converges absolutely contains a perfect subset.

[Consider the graph of the curves $y = \sin n!$ x.]

2. (i) Every measurable set of positive measure is a basis; (ii) every set of the second category is a basis.

[Let $E$ be an arbitrary set of positive measure, and $x \in E$, $y \in E$. To prove (i) it is sufficient to show that the set of the differences $x - y$ contains an interval. To show this let $E_n$ denote the set $E$ translated by $h$. Considering the neighbourhood of a point of density 1 for the set $E_n$, it is easy to show that $E_n \in E$ if $h$ is sufficiently small. This theorem is due to Steinhaus [5]. The proof of (ii) is similar.]

3. A necessary and sufficient condition that the Fourier series of a function $f(x)$ should converge absolutely is that there should exist two functions $f$ and $g$ of the class $L^2$ such that $h(x) = \frac{1}{2\pi} \int_0^\infty f(x+\gamma)g(\gamma) d\gamma$. 

(M. Riesz; see Hardy and Littlewood [8].)

[That the condition is sufficient follows from § 2.11. Let $c_\nu$ be the complex Fourier coefficients of $h$; to prove that the condition is necessary consider the functions with Fourier coefficients $|c_\nu|$ and $|c_\nu|$ and $\text{sign} c_\nu$.

4. The conditions of Theorems 6.3 and 6.3.2 are unnecessarily stringent. Thus Theorems 6.3 and 6.3.2 remain true, and the proofs unchanged, if we assume that $f \in \text{Lip} (\alpha, 2)$. In Theorem 6.3.1 we may assume that the function $f$ is of bounded variation and belongs to $\text{Lip} (\alpha, 1)$.

5. Let $0 < \alpha < 1$, $0 < p < 2$. If $a_n, b_n$ are the Fourier coefficients of an $f \in \text{Lip} (\alpha, p)$, then $\sum (|a_n|^2 + |b_n|^2) < \infty$ for every $\delta < p < (1 + \alpha - 1)$. Szász [5].

[The proof is similar to that of Theorem 6.3.2 if, instead of Parseval's relation, we use the inequality of Hausdorff-Young which will be established in Chapter IX.]

6. (i) If $f \in \text{Lip} (\alpha, 0 < \alpha < 1)$, then $\sum a_n^{p-1} \left| a_n + b_n \right| < \infty$ for every $\beta < \alpha$.

(Hardy [4].) (ii) If $f$ is, in addition, of bounded variation then $\sum a_n^{p-1} \left| a_n + b_n \right| < \infty$.

(iii) If $f \in \text{Lip} (\alpha, p)$, $0 < \alpha < 1$, $1 < p < 2$, then $\sum a_n^{p-1} \left| a_n + b_n \right| < \infty$ for every $\gamma < a - 1/p$.

[To prove the first part of the theorem consider the inequality 6.3(4).]
7. Let \( f(x) = \sum_{n=1}^{\infty} \frac{\epsilon^{|a_n|^2}}{(\log n)^\beta} \) \( \epsilon^{\cos nx} \) where \( 0 < \alpha < 1 \), and \( \beta - 1 \) is positive and sufficiently small. Then the real and imaginary parts of \( f \) are bounded in variation, belong to Lip \( \alpha \), and yet \( \sum_{n=1}^{\infty} \) for \( k = 2/(2 - \alpha) \). It follows that Theorem 0.34 cannot be improved. For the proof see Zygmund [7]. See also § 5.7.13.

8. Let \( a_n, b_n \) be the Fourier coefficients of a function \( f(x) \) and let \( t_n = t_n(f) = \frac{1}{2} \sum_{n=1}^{2n} \epsilon^{\cos nx} = \frac{1}{2} \sum_{n=1}^{2n} \epsilon^{\cos nx} \) where \( \gamma_n \geq 0 \), \( \gamma_n^2 = a_n^2 + b_n^2 \).

(i) If \( |f(x)| \leq 1 \), then \( t_n \leq (2n + 1)^{1/2} \). (ii) For every \( n \) there is a function \( f_n(x) = f(x) \) such that \( t_n \geq A n^{1/4} \) where \( A \) is a positive absolute constant.

See Bernstein [3], where a little more is proved, viz. that for \( f \) we may take a trigonometrical polynomial of order \( n \).

(iii) follows from the inequalities of Bessel and Schwarz. To obtain (ii) let \( g_n(x) = g_n(t) \cos x + \ldots ... + \cos nx \), where \( \gamma_1, \gamma_2 \ldots \) are Riemann’s functions. Then

\[
\int_0^{2\pi} \left| g_n(x) \right| dx \leq \int_0^{2\pi} \left| g_n(x) \right| dx \leq \int_0^{2\pi} \cos^m x + \ldots \cos nx dx =
\]

\[
\leq \frac{m}{2} \int_0^{2\pi} \cos^{m-1} x + \ldots \cos nx dx \leq \frac{m}{2} \int_0^{2\pi} \cos^{m-1} x + \ldots \cos nx dx \leq \frac{\pi}{\sin \pi/m} \int_0^{2\pi} \cos^{m-1} x + \ldots \cos nx dx = \pi n^{1/m}
\]

§§ 5.7.3, 4.13(3). Let \( t_n \) be a value of \( t \) such that the integral of \( |g_n(x)| \) over \( (0, 2\pi) \) exceeds \( \pi n^{1/4} \), and let \( a_n, b_n \) be the Fourier coefficients of the function \( f(x) = \text{sign} g_n(x) \). Then

\[
\sum_{k=1}^{N} \left| \frac{\epsilon_{a_k} + b_k} {\epsilon^{\cos x}} \right| = \left| \frac{\epsilon_{a_k} + b_k} {\epsilon^{\cos x}} \right| \leq \frac{1}{\pi} \int_0^{2\pi} f(x) g_n(x) dx
\]

\[
\leq \frac{1}{\pi} \int_0^{2\pi} |g_n(x)| dx \leq \pi n^{1/4}
\]

The idea of the proof is taken from Paley [2], where it is applied to another problem. The result may be used to prove the second part of Theorem 6.3.

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CHAPTER VII.

Conjugate series and complex methods in the theory of Fourier series.

7.1. Summability of conjugate series 1). In Chapter III we proved some results on the summability \((C, r)\) of Fourier series. As regards the conjugate series our results were less complete. The obstacle was that we knew nothing about the existence of the integral

\[
(1) \quad \frac{1}{\pi} \int_0^{\pi} \frac{f(x+t)-f(x-t)}{2\sin \frac{t}{2}} dt = \lim_{h \to 0} \left[ \frac{1}{\pi} \int_0^{\pi} \frac{f(x+t)-f(x-t)}{2\sin \frac{t}{2}} dt \right].
\]

We proved that, almost everywhere, the existence of (1) was equivalent to the summability \( A \) of \( f \). We now intend to prove the latter fact using complex methods, independently of the behaviour of (1). This will just enable us to prove the existence of (1) for almost every \( x \). The proof will be based on the following lemma.

Let \( G(z) = a_0 + a_1 z + a_2 z^2 + \ldots, z = re^{i\theta} \), be a function which is regular, bounded, and non-vanishing in the circle \( |z| < R \). The function \( l(z) = \log |G(re^{i\theta})| + i \arg G(re^{i\theta}) \). Since \( G(z) \neq 0 \) for \( |z| < 1 \), \( \log G(z) \) is regular

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1) Privaloff [2], Plessner [2], Hardy and Littlewood [1], Zygmund [3].