5.5. Remark. Since the spaces satisfying the hypotheses of Theorem 5.3 form a hereditary class, we see that a generalized ordered space with a \( G\)-Souslin diagonal must be hereditarily paracompact. Furthermore, (5.4) shows that a generalized ordered space is \( c\)-semistratifiable (and hence paracompact) if it has a quasi-\( G\) diagonal (i.e., if it admits a countable collection \( \mathcal{F} \), as in the proof of (5.4)).

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Homogeneity, universality and saturatedness of limit reduced powers (II)

by

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Abstract. We give some necessary conditions on the pair \( (\mathcal{B}, \mathcal{G}) \), where \( \mathcal{B} \) is an ultrafilter on \( I \) and \( \mathcal{G} \) is a filter on \( I \times I \), which imply that for every structure \( \mathfrak{A} \), the limit ultrapower \( \mathfrak{A}^{\mathcal{B}}/\mathcal{G} \) is \( \lambda \)-universal (or \( \kappa \)-saturated).

The paper is a continuation of [1]. In § 1, we prove Embedding Theorem which says that every limit ultrapower \( \mathfrak{A}^{\mathcal{B}}/\mathcal{G} \) contains a lot of elementary submodels which are isomorphic to ultrapowers of \( \mathfrak{A} \) reduced by ultrafilters which are obtained in a natural way from \( \mathcal{B} \). The idea of Embedding Theorem (in fact contained in the proof of Theorem 4 in [4]) was suggested to the author by the proof of Wierzejewski's Theorem 1 in [5].

In § 2, we apply Embedding Theorem to give some necessary combinatorial conditions on the pair \( (\mathcal{B}, \mathcal{G}) \) which imply that for every structure \( \mathfrak{A} \), the limit ultrapower \( \mathfrak{A}^{\mathcal{B}}/\mathcal{G} \) is \( \kappa \)-universal (or \( \kappa \)-saturated).

The author is deeply indebted to L. Pacholski and J. Wierzejewski for a lot of very stimulating discussions which helped to formulate and prove the results presented below.

§ 1. Embedding Theorem. Let \( \mathcal{B} \) be a filter on \( I \) and \( \mathcal{G} \) an equivalence relation on \( I \). Let \( I/\mathcal{G} = \{ I_J : J \in \mathcal{G} \} \). For every \( \mathcal{F} \in \mathcal{P}(J) \) defined by: \( X \in \mathcal{F} \) if and only if \( \bigcup J \in \mathcal{F} \) called the \( \mathcal{F} \)-image of \( \mathcal{B} \) and is denoted by \( \mathcal{B}/\mathcal{F} \). It is easy to see that if \( J \in \mathcal{G} \) is a filter on \( I \), we can say that \( X \in \mathcal{G} \)-composable if there is \( Y \in \mathcal{J} \) such that \( X = \bigcup J \). Let \( \mathcal{F} \) be a filter on \( I \times I \), then the family of all \( \mathcal{G} \)-composable sets for \( \mathcal{F} \in \mathcal{G} \) we call the family of \( \mathcal{G} \)-composable sets. This family coincides with \( 2 \mathcal{G} \).
EMBEDDING THEOREM. Let $\mathcal{B}$ be a filter on $I$ and let $\mathcal{F}$ be a filter on $I \times I$. Assume that $\mathcal{F} = \mathcal{B}$ is an equivalence relation on $I$. Put $I|q = \{j \in I : j \leq q\}$ and $\mathcal{E} = \mathcal{B}|q$.

Then:

(i) there is an isomorphism $F: \mathcal{B}|q \to \mathcal{B}|q'$,

(ii) if $f : A^0|q \cong eq(q) \geq 0$, then $\bigcap I|q \in Rng(F)$,

(iii) if $\mathcal{F}$ is an ultrafilter then $F$ is an elementary embedding of $\mathcal{B}|q \to \mathcal{B}|q'$.

Proof. Let $g \in A^0$. Let us define a function $F_0 : A^0 \to A^0|q'$ by $F_0(g)(i) = q$ and $g(j) = q$ for all $i \leq j$. Set $X = \{i \in I : g(i) = g(j)\}$. Then $X = \bigcap I|q \in Rng(F)$. But then $F_0(g)(i) = F_0(g)(j) \in \mathcal{B}|q'$, consequently we have $F_0(g) = F_0(q) \geq 0$. Thus, we can define a function $F$ from $A^0 \to A^0|q'$ by the condition: $F[g] = \bigcap I|q \in Rng(F)$.

Let $\phi = \phi(x_1, \ldots, x_j)$ be an atomic formula. Then the following statements are pairwise equivalent:

$\mathcal{B}|q \models \forall[j] \phi[j_1, \ldots, j_x]$,

$\bigcap I|q \in Rng(F)$,

$\mathcal{B}|q \models \forall[j] \phi[j_1, \ldots, j_x] \Rightarrow F_0(q) \models \forall[j] \phi[j_1, \ldots, j_x]$.  

So, $F$ is an isomorphism, which proves (i).

To check (ii), it suffices to notice that if $\mathcal{F}$ is an ultrafilter, then $\mathcal{E}$ is also an ultrafilter and the statements from (i) are equivalent for arbitrary formula $\phi$.

It remains to prove (iii). Let $f : A^0|q \cong eq(q) \geq 0$. Then, for each $i \in I$, the function $f$ is constant on $I|q$. Consequently, we can define a function $g : A^0 \to A^0|q$ by $g(j) = f(i)$ for all $i \leq j$. But then we have $F_0(g) = f$, so $\bigcap I|q \in Rng(F)$. Q.E.D.

EXAMPLE 1. The assumptions of maximality of $\mathcal{F}$ in clause (iii) cannot be removed. Indeed, let $\mathcal{F}$ be the two-elements Boolean algebra, let $\mathcal{B}$ be the Fréchet filter on $\omega$ and $\mathcal{E}$ be the filter on $\omega \times \omega$ generated by all the equivalence relations $\equiv$ on $\omega$ such that $\omega / \equiv$ is finite. Then for any equivalence relation $\equiv$ in $\mathcal{E}$, the embedding $F$ from (i) of Embedding Theorem is not elementary.

COROLLARY 1. Let $\mathcal{B}$ be a filter on $I$ and let $\mathcal{F}$ be an $x$-complete filter on $I \times I$. Let $\bigcap I|q \in A^0|q$, for all $q \leq \alpha$. Then there is an equivalence relation $\equiv$ in $\mathcal{B}$ such that there is an isomorphism $F : \mathcal{B}|q \cong \mathcal{B}|q'$, where $\bigcap I|q = \{i \in I : j \leq q\}$ and $\bigcap \mathcal{B}|q \cong \bigcap \mathcal{B}|q'$, for all $q \leq \alpha$. Moreover, $\mathcal{F}$ is an ultrafilter if $F$ is an elementary embedding.

Proof. Let $\exists q (q \equiv x)$, for all $q \leq \alpha$. Since $\mathcal{F}$ is $x$-complete, we have $q = \bigcap q \in \mathcal{F}$. Consequently, by Embedding Theorem, for $\bigcap I|q = \{i \in I : j \leq q\}$ and $\bigcap \mathcal{B}|q$, we have an isomorphism $F : \mathcal{B}|q \cong \mathcal{B}|q'$, which is an elementary embedding when $\mathcal{E}$ is an ultrafilter. Finally, by (ii), we have $F_0(q) \in Rng(F)$, because of $\mathcal{B}|q \models F_0(q)$, for all $q \leq \alpha$. $\square$

Remark. The condition of $x$-completeness of $\mathcal{E}$ in Corollary 1 is not necessary (see Example 2). To define a weaker condition which gives the thesis of Corollary 1, we need some auxiliary notions.

DEFINITION. Let $\mathcal{B}$ be a filter on $I$ and let $\mathcal{F}$ be a filter on $I \times I$. Let $(\mathcal{B}^n)_{n\in\mathbb{N}}$ be a sequence of equivalence relations from $\mathcal{F}$. Then an equivalence relation $q$ on $I \times I$ is a $\beta$-lower bound of $(\mathcal{B}^n)_{n\in\mathbb{N}}$ if and only if there is a sequence $(\mathcal{B}^n)_{n\in\mathbb{N}}$ of $\mathcal{F}$-composable elements of $\mathcal{B}$ such that for each $\xi < \kappa$ we have

$\mathcal{B} \cap (X_1 \times X_2) \subseteq X_1 \times X_2$.  

If for every sequence $(\mathcal{B}^n)_{n\in\mathbb{N}}$ of elements of $\mathcal{F}$ there is a $\beta$-lower bound of $(\mathcal{B}^n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ then we say that the pair $(\mathcal{B}, \mathcal{F})$ is $x$-closed.

THEOREM 1. Let $\mathcal{B}$ be a filter on $I$ and let $\mathcal{F}$ be a filter on $I \times I$ such that $\mathcal{B}$ is $x$-closed. Then if $(\mathcal{B}^n)_{n\in\mathbb{N}}$ is a sequence of elements of $\mathcal{F}$, then there exists an equivalence relation $q \in \mathcal{B}$ such that $\mathcal{B}^n \cap (X_1 \times X_2) \subseteq X_1 \times X_2$, for all $n \in \mathbb{N}$. Moreover, if $\mathcal{F}$ is an ultrafilter then $F$ is an elementary embedding.

Proof. Let $q_0 = eq(q_0)$, for $\xi < \kappa$. Consider the sequence $(\mathcal{B}^n)_{n\in\mathbb{N}}$ of elements of $\mathcal{F}$. By our assumptions there is a $\beta$-lower bound of $(\mathcal{B}^n)_{n\in\mathbb{N}}$ in $\mathcal{F}$, say $q$. Thus there is a sequence $(\mathcal{B}^n)_{n\in\mathbb{N}}$ of $\mathcal{F}$-composable elements of $\mathcal{B}$ such that $\mathcal{B}^n \cap (X_1 \times X_2) \subseteq X_1 \times X_2$, for all $n \in \mathbb{N}$. Take $\bigcap \mathcal{B}^n = \{i \in I : j \leq q\}$ and $\mathcal{F} = \mathcal{B}|q$. Then by embedding theorem there is an isomorphism $F : \mathcal{B}|q \cong \mathcal{B}|q'$ which is an elementary embedding when $\mathcal{E}$ is an ultrafilter. It remains to prove that $\bigcap \mathcal{B}^n \in Rng(F)$, for all $n \in \mathbb{N}$. For every $\xi < \kappa$, let $\phi$ be a function defined in such a way that $\phi_1 : X_1 \to X_1$ and $\phi_2 : X_2 \to X_2$. Of course, by the construction, we have $\phi_1 = f_1$ (mod $\mathcal{E}$). Since $\phi_2 \in \mathcal{B}|q$ by Embedding Theorem, we have $\phi_2 \in Rng(F)$, for all $\xi < \kappa$, and consequently $\bigcap \mathcal{B}^n \in Rng(F)$, for all $n \in \mathbb{N}$. Q.E.D.

EXAMPLE 2. Let $I$ be the set of all positive rationals and let $\mathcal{B}$ be a filter on $I$ such that for each $\alpha \in I$, the set $\{x \in I : x \leq \alpha\}$ is in $\mathcal{B}$. For each strictly increasing sequence $\psi = (\psi_n)_{n\in\mathbb{N}}$ of rationals without any accumulation point such that $\psi_0 = 0$, define $q_\psi = \bigcap \mathcal{B}|\psi$, if and only if there is some $n \in \mathbb{N}$ such that $\psi_n \leq \alpha < \psi_{n+1}$. Let $G$ be the filter on $I \times I$ generated by all $q_\psi$. Then $G$ is not $\alpha$-complete.

On the other hand, for each sequence $(\mathcal{B}^n)_{n\in\mathbb{N}}$ of elements of $\mathcal{F}$ there is a $\beta$-lower bound of $(\mathcal{B}^n)_{n\in\mathbb{N}}$ in $\mathcal{F}$. Thus the pair $(\mathcal{B}, \mathcal{F})$ is $\alpha$-closed. Consequently the assumptions of Theorem 1, even in the countable case are weaker than those in Corollary 1.

We have also the following converse theorem.

THEOREM 2. Let $\mathcal{B}$ be a filter on $I$ and let $\mathcal{F}$ be a filter on $I \times I$ such that for each structure $\mathcal{B}$ and for each sequence $(\mathcal{B}^n)_{n\in\mathbb{N}}$ of elements of $\mathcal{F}$ there is an equivalence relation $q \in \mathcal{B}$ such that if $\bigcap \mathcal{B}^n = \{i \in I : j \leq q\}$ and $\mathcal{F} = \mathcal{B}|q$, then there is an isomorphism $F : \mathcal{B}|q \cong \mathcal{B}|q'$ with $\bigcap \mathcal{B}^n \in Rng(F)$, for all $n \in \mathbb{N}$. Then the pair $(\mathcal{B}, \mathcal{F})$ is $x$-closed.
Proof. Let \( \langle q_x \rangle_{x<\alpha} \) be a sequence of elements of \( B \). If \( \vert \alpha \vert = \vert \beta \vert \) then there are functions \( f_x \in A^B \) such that \( (q_x)^B = q_x \), for all \( x<\alpha \). Take \( q \in B \) such that if \( q_x = \langle i_j \rangle_{j<\alpha} \) and \( \delta = [q] \) then there is an isomorphism \( F : \mathcal{W}^A \to \mathcal{W}^B \) with \( \mathcal{L}(q) \cong \mathcal{L}(F(q)) \), for all \( x<\alpha \). Then there are functions \( q_x \in A^B \) such that putting \( h_x = F(q) \) we have \( h_x = f_x(\text{mod} \delta) \). Let \( X_x = \{ i \in \{ i : h(i) = f(i) \} \). Of course \( X_x \) is a \( \mathcal{G} \)-compositional element of \( B \). Moreover, if \( (i, j) \in q \) and \( (i, j) \in X_x \times X_x \) then \( (i, j) \in \text{eq}(h) \) because of \( q \approx \text{eq}(h) \). Since \( h_x = F(q) \) and \( X_x \in X_x \), we have \( (i, j) \in q_x \). Whence \( q \approx \langle X_x \times X_x \rangle = q_x \cap (X_x \times X_x) \), for all \( x<\alpha \) which shows that \( q \) is a \( B \)-lower bound of \( \langle q_x \rangle_{x<\alpha} \). Q.E.D.

§ 2. Applications to the universality and saturatedness. To use Embedding Theorem to the universality and saturatedness of limit ultrapowers we need the following facts:

**FACT I** (Keisler [2], Theorem 1.4). An ultrafilter \( D \) on \( I \) is \( \kappa^* \)-good if and only if for every structure \( A \), the ultrapower \( A^D \) is \( \kappa^* \)-saturated.

**FACT II** (Keisler [2], Theorem 1.5). An ultrafilter \( D \) on \( I \) is \( (\kappa, \alpha, \delta) \)-regular if and only if for every structure \( A \), the ultrapower \( A^D \) is \( \kappa^* \)-universal.

**FACT III.** The following three conditions for an ultrafilter \( D \) on \( I \) are equivalent:

(a) \( D \) is \( \alpha \)-complete.
(b) \( D \) is \( (\alpha, \alpha, \delta) \)-regular.
(c) \( D \) is \( \alpha \)-incomplete.

**FACT IV.** If for every \( \lambda \leq \kappa \) and every sequence \( \langle i_x \rangle_{x<\lambda} \) of elements of \( B \) there is an \( \gamma \)-saturated model \( \mathcal{A} \) and an elementary embedding \( F : \mathcal{A} \to \mathcal{M} \), with \( \mathcal{L} \), then \( \mathcal{M}^D \) is \( \kappa^* \)-universal.

Now these facts together with Embedding Theorem yield the following theorems:

**THEOREM A.** Let \( D \) be an ultrafilter on \( I \) such that for some \( \mathcal{Q} \in \mathcal{F} \), the \( \mathcal{Q} \)-image of \( B \) is \( (\alpha, \alpha, \delta) \)-universal. Then the ultrapower \( A^D \) is \( \alpha \)-saturated.

Proof. Let us remark that if \( [\mathcal{F}]^D \) is \( (\alpha, \alpha, \delta) \)-regular then for every \( q \in B \) there is \( q \approx [\mathcal{F}]^D \) such that \( q \in B \) and \( \mathcal{F}^D \) is also \( (\alpha, \alpha, \delta) \)-regular. In fact, we can take \( q = [\mathcal{F}]^D \) and \( \mathcal{F}^D \).

Now put \( B = [\mathcal{Q}]^D \). It is well known (see [1], Theorem 3.1) that there is a set \( I \) and an ultrafilter \( D \) on \( I \) such that for some filter \( \mathcal{F} \) on \( I \times I \), we have \( B = \mathcal{F}^D \). On the other hand it is easy to see that \( B \) is not \( \alpha \)-saturated, for, there is a countable increasing sequence of elements of \( B \) of which \( \mathcal{Q} \) is cofinal in \( B \). Thus, in Theorem A, we cannot replace \( \alpha \) by \( \alpha \).

**THEOREM B.** Let \( D \) be an ultrafilter on \( I \) and let \( \mathcal{Q} \) be a filter on \( I \times I \). Let \( \mathcal{F} \) be a structure such that \( \mathcal{F} \not= \mathcal{W}^D \). Then \( \mathcal{F} \) is not \( \alpha \)-saturated.

(i) If \( \mathcal{F} \) is \( \alpha \)-saturated then \( \mathcal{W}^D \) is \( \alpha \)-saturated.

(ii) If \( \mathcal{F} \) is \( \alpha \)-saturated then \( \mathcal{W}^D \) is \( \alpha \)-saturated without any assumption on \( \mathcal{F} \).

Proof. Since \( \mathcal{F} \not= \mathcal{W}^D \), there is a function \( f \in A^D \) which is not constant on any set from \( D \). Take \( q = \text{eq}(f) \). Then for each \( q \approx q \), the filter \( [\mathcal{F}]^D \) is nonprincipal.

If \( [\mathcal{F}]^D \) is not \( \alpha \)-complete then \( [\mathcal{F}]^D \) is \( (\alpha, \alpha, \delta) \)-regular by Fact III and we can get the theses of Theorem B from Theorem A.

So, suppose that for no \( q \approx q \), \( q \approx [\mathcal{F}]^D \) the filter \( [\mathcal{F}]^D \) is \( (\alpha, \alpha, \delta) \)-regular. Then both \( [\mathcal{F}]^D \) and \( [\mathcal{F}]^D \) must be measurable and we need the assumption of (0).

Let \( \langle \mathcal{F}_n \rangle_{n<\omega} \) be a finite sequence of elements of \( A^D \). Let \( q^* = \mathcal{Q} \times q \cap \ldots \cap q_{n-1} \) where \( q_n = \text{eq}(f_n) \), \( n = 0, \ldots, m-1 \). Then \( [\mathcal{F}]^D \) is \( (\alpha, \alpha, \delta) \)-complete ultrafilter on \( I \). By Ultrafilter Theorem for \( \alpha \)-complete ultrafilters \( A^D \) is \( \alpha \)-saturated if and only if \( B \) is \( \alpha \)-saturated. Consequently, by Embedding Theorem we have an elementary embedding \( F : \mathcal{W}^D \to \mathcal{W}^D \), with \( \mathcal{L}(q) \cong \mathcal{L}(F) \), for all \( n<\kappa \). Thus, by Fact IV, we see that \( \mathcal{W}^D \) is \( \alpha \)-saturated. Q.E.D.

Remark. Theorem B is closely related to a theorem of Wierzejewski ([5], Theorem 2) that if \( B \) is an \( \alpha \)-homogeneous then \( \mathcal{W}^D \) is \( \alpha \)-homogeneous but in Theorem B, for the nonmeasurable case, we have a stronger thesis without any assumption on \( \mathcal{F} \).

**EXAMPLE.** Let \( D \) be filters from Example 1, and let \( \mathcal{F} \) be the ring of integers. Then it is easy to check that \( \mathcal{W}^D \) is not \( \alpha \)-saturated. Consequently, in Theorem A and B we cannot omit the assumption that \( D \) is maximal.

**THEOREM C.** Let \( D \) be an ultrafilter on \( I \) and let \( \mathcal{Q} \) be a filter on \( I \times I \). Then there exists \( q \in B \) such that \( \mathcal{Q} \approx [\mathcal{Q}]^D \) if and only if for every structure \( A \), the ultrapower \( A^D \) is \( \kappa^* \)-universal.

Proof. Suppose there is \( q \in B \) such that \( \mathcal{Q} \approx [\mathcal{Q}]^D \). Then, by Fact II, the ultrapower \( A^D \) is \( \kappa^* \)-universal. By Embedding Theorem there is an elementary embedding \( F : A^D \to A^D \). Consequently \( A^D \) is \( \kappa^* \)-universal as an elementary extension of \( A^D \).

The converse implication follows in the same way as in Keisler's proof of Fact II (see [2]).
Remark. After we had the result that the existence of $g \in \mathcal{G}$ such that $\mathcal{G}/g$ is $(\alpha, \alpha)$-regular implies the $\pi'$-universality of $\mathbb{W}_0^\alpha\mathcal{G}$, L. Pacholski has drawn our attention that the condition above is also sufficient for the $\pi'$-universality of $\mathbb{W}_0^\alpha\mathcal{G}$ and that the Keisler's proof from [2] works also in our case.

**Theorem D.** Suppose $\mathcal{G}$ is an ultrafilter on $[1, \infty)$ and $\mathcal{F}$ a filter on $[1, \infty)$ such that the pair $(\mathcal{G}, \mathcal{F})$ is $x'$-closed. Suppose that for every $\xi \in \mathcal{G}$ there is $\eta \leq \xi$ such that $\mathcal{G}/\eta$ is $x'$-good. Then for every structure $\mathcal{M}$, the limit ultrapower $\mathbb{W}_0^\alpha\mathcal{G}$ is $x'$-saturated.

**Proof.** Let $(\bigcup_{\xi < \eta} G, G)$ be a sequence of elements of $\mathbb{W}_0^\alpha\mathcal{G}$. From Theorem 1, it follows that there is a relation $g \in \mathcal{G}$ such that if $\xi \in \mathcal{G}/g = \{ \gamma \in \mathcal{F} : \gamma \leq \xi \}$ then there is an elementary embedding $\mathcal{M} \rightarrow \mathbb{W}_0^\alpha\mathcal{G}$ with $\bigcup_{\xi < \eta} G = \text{Rag}(\mathcal{F})$, for all $\xi < \eta$. From our hypotheses we can additionally assume that $\mathcal{G}/g$ is $x'$-good. Then, by Fact I, $\mathbb{W}_0^\alpha\mathcal{G}$ is $x'$-saturated. Thus the result follows from Fact IV.

Remark. L. Pacholski has informed me that he has a combinatorial condition on a pair $(\mathcal{G}, \mathcal{F})$ which is equivalent to the statement: "for every $\mathcal{M}$ the limit ultrapower $\mathbb{W}_0^\alpha\mathcal{G}$ is $x'$-saturated". For more informations see [3].

**References**


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**The irreducibility of continua which are the inverse limit of a collection of Hausdorff arcs**

by

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**Abstract.** Consider the space which is the inverse limit of a collection of generalized (non metric) arcs over a linearly ordered index set. Such a space is a hereditarily unicoherent ariodic Hausdorff continuum. It is shown that every indecomposable subcontinuum of the space is irreducible between some two points. A necessary and sufficient condition in order for a subcontinuum of the space to be indecomposable is stated. Further it is shown that the space must be a generalized arc if it is not the inverse limit over a countable subset of the index set. Thus it follows that the space must be an irreducible continuum.

**Introduction.** In this work a continuum is a closed connected subset of a Hausdorff space and an arc is a compact continuum which has only two non-cut points. It is known that if $M$ is a nondegenerate compact ariodic hereditarily unicoherent continuum and every nondegenerate indecomposable subcontinuum of $M$ is irreducible between some two points then $M$ is irreducible between some two points. (See M. H. Profitt [4] for a stronger result.) Suppose $S$ is the inverse limit of a collection of Hausdorff arcs over a linearly ordered index set. Then $S$ is a compact ariodic hereditarily unicoherent continuum. In this paper we show that every nondegenerate indecomposable subcontinuum of $S$ is irreducible between some two points. Further we show that if $S$ is not an arc then it must be the inverse limit of a collection of arcs over a countable index set (this result has also been independently discovered by G. R. Gordh and S. Mardesić). Also a necessary and sufficient condition in order for a subcontinuum of $S$ to be indecomposable is stated.

Following are some definitions used in this paper. For theorems concerning inverse limits the reader should consult Ellenberg and Sieradski [1], and for theorems concerning arcs the reader should consult Hocking and Young [2], and R. L. Moore [3].

**Definition.** Suppose $M$ is an arc and $0$ and $1$ are the two non-cut points of $M$. Then the statement that $M$ is ordered from $0$ to $1$ means that if $x$ and $y$ are two points of $M$ then $x < y$ (or $x$ precedes $y$) if and only if $x \neq y$ and it is true that $y = 1$ or $M - y$ is the sum of two mutually separated sets, one containing $0$ and $x$ and the other $y$.