On decompositions of hereditarily smooth continua

by

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Abstract. It is proved that a continuum $X$ is hereditarily smooth at $p$ (for the definition see below) if and only if there is an upper semi-continuous monotone decomposition $\omega$ of $X$ such that $Y$ is an arcwise connected continuum which is hereditarily smooth at $\varphi(p)$ and for each subcontinuum $Q$ of $X$ such that $p \in Q$ we have $\varphi^{-1}(Q) = Q$, where $Y$ is the decomposition space of $\omega$ and $\varphi$ is the canonical mapping. This result generalizes a well-known theorem for continua which are hereditarily unicoherent at some point [3].

§ 1. Preliminaries. In this paper we give a characterization of hereditarily smooth continua by their monotone decompositions having an arcwise connected decomposition space. This result generalizes a theorem obtained in [2] by G. R. Gordh and reduces the study of hereditarily smooth continua to the study of hereditarily smooth arcwise connected continua.

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The topological spaces under consideration will be assumed to be metric and compact. If the space under consideration is established, then $ab$ denotes an arbitrary arc with endpoints $a$ and $b$, and $I(a, b)$ denotes an arbitrary irreducible continuum between $a$ and $b$.

The notion of smoothness of continua in a general form has been introduced in [4]. We say that the continuum $X$ is smooth at the point $p \in X$ if, for each convergent sequence $\{x_n\}$ of points of $X$ and for each subcontinuum $K$ of $X$ such that $p, x \in K$, where $x = \lim x_n$, there exists a sequence $\{K_n\}$ of subcontinua of $X$ such that $p, x_n \in K_n$ for each $n = 1, 2, \ldots$ and $\lim K_n = K$ (the topological limit).

We have (see [4], Theorem (3.1)(v), p. 83)

Proposition 1. A continuum $X$ is smooth at the point $p \in X$ if and only if for each subcontinuum $N$ of $X$ and for each open set $V$ of $X$ there exists a continuum $K$ such that $p \in N \cap V$ implies $N \cap K = K \cap V$.

A continuum $X$ is said to be hereditarily smooth at $p$ provided each subcontinuum of $X$ which contains $p$ is smooth at $p$. Recall that a continuum $X$ is
hereditarily unicoherent at $p$ if the intersection of any two subcontinua each of which contains $p$ is connected (see [2], p. 52).

It is well known that for every irreducible continuum $X$ there exists an upper semi-continuous decomposition of $X$ into continua (called layers of $X$; see [3], § 48, IV, p. 199) and the decomposition of $X$ into layers is the finest of all linear upper semi-continuous decompositions of $X$ into continua (see [3], § 48, IV, Theorem 3, p. 200). If each layer of $X$ has a void interior, then $X$ is said to be of type $\lambda$ (see [3], § 48, III, p. 197, footnote). It is well known (see [3], § 48, VII, Theorem 3, p. 216) that an irreducible continuum $X$ is of type $\lambda$ if and only if each indecomposable subcontinuum of $X$ has a void interior.

We have (cf. [1], Proposition 1, p. 46)

PROPOSITION 2. If a continuum $X$ is hereditarily smooth at the point $p$, then any irreducible continuum $I(p, x)$ is of type $\lambda$.

In fact, any irreducible continuum $I(p, x)$ is smooth at $p$ by the hereditary smoothness of $X$ at $p$. Therefore the continuum $I(p, x)$ is hereditarily unicoherent at $p$ (see [4], Theorem 5.3, p. 88); thus $I(p, x)$ is smooth in the sense of Gordan (cf. [2], p. 52). It follows from Corollary 3.3 of [2], p. 55 that every indecomposable subcontinuum of $I(p, x)$ has a void interior in $I(p, x)$, i.e., the continuum $I(p, x)$ is of type $\lambda$.

§ 2. Continua of convergence. Recall that a subcontinuum $K$ of $X$ is called the continuum of convergence (see [3], § 47, VI, p. 245) provided $K$ is the topological limit of the sequence of continua such that

$$K = \lim_{\longrightarrow} K_n$$

and $K \cap K_n = \emptyset$ for each $n = 1, 2, ...$

If $X$ is compact, then we can assume that continua $K_1, K_2, ...$ are mutually disjoint.

We have the following generalization of Theorem 2 of [5].

THEOREM 3. Let a continuum $X$ be hereditarily smooth at the point $p$ and suppose that $L$ is a subcontinuum of $X$ such that $p \in L$. Then, for each continuum $K$ of convergence in $X$, the set $K \cap L$ is a continuum.

Proof. Suppose, on the contrary, that the set $K \cap L$ is not connected. Thus there are closed, nonempty sets $A$ and $B$ such that

(1) $K \cap L = A \cup B$ and $A \cap B = \emptyset$.

Let $I(a, b)$ be an arbitrary subcontinuum of $K$ irreducible between sets $A$ and $B$, where $a \in A$ and $b \in B$. It follows from Theorem 2 of [3], § 48, 1X, p. 223 that

(2) the continuum $I(a, b)$ is irreducible between each point of the set $I(a, b) \cap A$ and each point of the set $I(a, b) \cap B$.

Consider two cases.

1'. The continuum $I(a, b)$ is indecomposable. Let $C$ be a component of the point $a$ in $I(a, b)$ (for the definition of a component see [3], § 48, VI, p. 209). It follows from Theorem 2 of [3], § 48, VI, p. 209, that $C = I(a, b)$. Thus there is a sequence $\{d_n\}$ of points of $C$ such that

(3) $\lim_{n \to \infty} d_n = b$.

Consider the continuum $R = L \cup I(a, b)$. Since $X$ is hereditarily smooth at $p$ and since $p \in L \cup I(a, b)$, we conclude that the continuum $R$ is smooth at $p$. Thus, because $p, b \in L$, it follows from (3) that there is a sequence $\{d_n\}$ of subcontinua of $R$ such that

(4) $p, b \in L_n$ for each $n = 1, 2, ...$

and

(5) $\lim_{n \to \infty} L_n = L$.

Since $p \in L_n \cap L$ and $b \in L_n$ (cf. (4)), we infer that the continuum $L_n$ contains an irreducible continuum $I(b, c_n)$ between $b$ and $L$. Since no proper subcontinuum $S$ of $I(b, c_n)$ such that $b \in S$ intersects $L$, i.e., $S \cap L = \emptyset$, we conclude that the component $C_n$ of the point $b$ in the continuum $I(b, c_n)$ is contained in $I(a, b)$. Therefore $I(b, c_n) = C_n \cap I(a, b)$ (cf. [3], § 48, VI, Theorem 2, p. 209), i.e.,

(6) $I(b, c_n) = I(a, b)$ for each $n = 1, 2, ...$

Moreover, $A \cap I(b, c_n) \neq \emptyset$ for each $n = 1, 2, ...$

In fact, if $I(b, c_n) = I(a, b)$, then obviously (7) holds, because $a \in I(a, b) \cap \cap A \neq \emptyset$. Thus, to show (7), we can assume by (6) that the continuum $I(b, c_n)$ is a proper subcontinuum of $I(a, b)$. Since $b \in I(b, c_n)$ and $b \in C$, we conclude $I(b, c_n) \subset C$. If $C \cap B = \emptyset$, then there is a proper subcontinuum (contained in $C$) of $I(a, b)$ joining sets $A$ and $B$, contrary to the choice of $I(a, b)$. Therefore $C \cap B \neq \emptyset$; thus $I(b, c_n) \subset L \subset C \subset C \subset A$. Since $I(b, c_n) \cap L = \emptyset$, we have that condition (7) holds.

The set $D = L \cup I(b, c_n)$ is a continuum (cf. [3], § 47, II, Theorem 6, p. 171) and $b \in D \subset I(a, b)$ and $D \cap A \neq \emptyset$ by (3), (6) and (7). Thus by the irreducibility of $I(a, b)$ between $A$ and $B$ (cf. (2)) we infer $D = I(a, b)$. Since $I(b, c_n) \subset L$, we conclude by (5) that $I(a, b) = D = L \cup I(b, c_n) \subset L$. Hence we have $I(a, b) \subset K \cap L$, contrary to (1).

2'. The continuum $I(a, b)$ is decomposable. Then there are proper subcontinua $M$ and $N$ of $I(a, b)$ such that $I(a, b) = M \cup N$. It follows from (2) that either $M \cap A = \emptyset$ and $N \cap B = \emptyset$ or inversely

(8) $M \cap B = \emptyset$ and $N \cap A = \emptyset$. 
Without loss of generality we can assume (8).

Since $K$ is a continuum of convergence in $X$, we conclude that there are subcontinua $K_n$ of $X$ such that

$$K = \lim_{n \to \infty} K_n \cap (K \cup K_n) \cap K_n = \emptyset$$

for each $m \neq n$ and $m, n = 1, 2, \ldots$

Let $d \in M \cap N$. Therefore there is a sequence $\{d_n\}$ of points of $X$ such that

$$\lim_{n \to \infty} d_n = d \quad \text{and} \quad d_n \in K_n \quad \text{for each} \ n = 1, 2, \ldots$$

It follows from (1) and (8) by the normality of $X$ that there are open sets $U$ and $V$ such that

$$A \subseteq U, \quad B \subseteq V$$

and

$$(U \cap V) \cup (U \cap N) \cup (V \cap M) = \emptyset.$$  

Then the set $G = X \setminus (K \cup U \cup V)$ is open in $X$. Moreover, conditions (1) and (11) imply $p \in L \cap G$. Since $X$ is smooth at $p$, there is, by Proposition 1, a continuum $Q$ in $X$ such that

$$L \subseteq \text{Int} Q \subseteq Q \cap \emptyset.$$  

Since $\lim_{n \to \infty} K_n \cap L = K \cap L \neq \emptyset$, we can assume by (13) that

$$K_n \cap Q \neq \emptyset \quad \text{for each} \ n = 1, 2, \ldots$$

It follows from (10) and (13) that we can take irreducible subcontinua $I(d_n, a_n)$ of $K_n$ between $d_n$ and $Q$. Consider the set

$$P = Q \cap K \cup \bigcup_{n=1}^{\infty} I(d_n, a_n).$$

Since $Q \cap K \neq \emptyset$ and $I(d_n, a_n) \cap Q \neq \emptyset$ for each $n = 1, 2, \ldots$, we conclude that the set $P$ is connected. Moreover, $\lim_{n \to \infty} (d_n, a_n) = K = (\lim_{n \to \infty} K_n) = K$ (cf. (9)); thus $P$ is closed, i.e.,

$$\text{the set} \ P \ \text{is a continuum.}$$

Furthermore,

$$\text{if} \ F \ \text{is a subcontinuum of} \ P \ \text{such that} \ d_n \in F \ \text{and} \ F \cap Q \neq \emptyset, \ \text{then} \ I(d_n, a_n) \subseteq F.$$  

Indeed, since $d_n \in F$ and $F \cap Q \neq \emptyset$, we infer that the continuum $F$ contains an irreducible continuum $I(d_n, a_n)$ between $d_n$ and $Q$. Therefore no proper sub-continuum $S$ of $I(d_n, a_n)$ such that $d_n \in S$ intersects the continuum $Q$, i.e., $S \cap Q = \emptyset$. Thus

$$S = S \cap P = (S \cap K) \cup \bigcup_{n=1}^{\infty} (S \cap I(d_n, a_n)).$$

Since $S$ is connected and sets $S \cap K, S \cap I(d_n, a_n)$ for $k = 1, 2, \ldots$ are mutually disjoint (cf. (9) and the definition of $I(d_n, a_n)$) and $d_n \in S \cap I(d_n, a_n)$, we conclude that the equality $S = S \cap I(d_n, a_n)$ holds, i.e., $S \subseteq I(d_n, a_n)$. This implies that the component $C$ of the point $d_n$ in the continuum $I(d_n, a_n)$ is contained in $I(d_n, a_n)$. Therefore $I(d_n, a_n) = C \subseteq I(d_n, a_n)$ (cf. [3], § 48, VI, Theorem 2, p. 209).

By the irreducibility of $I(d_n, a_n)$ between $d_n$ and $Q$ we infer $I(d_n, a_n) = I(d_n, a_n)$. Thus $I(d_n, a_n) \subseteq P$ by the choice of $I(d_n, a_n)$, i.e., (16) holds.

The set $W = L \cap I(d_n, a_n)$ is a continuum (cf. [3], § 47, Π, Theorem 6, p. 171).

Moreover, $d \in W \cap K$ and $W \cap Q \neq \emptyset$ by (9), (10) and by the choice of $I(d_n, a_n)$. Let $E \subseteq W \cup Q \subseteq K \cap N$. Since $K \cap N = K \cap G = (K \cap U) \cup (K \cap V)$, it suffices to consider two cases.

a) $e \in K \cap U$. Since $P$ is a continuum (cf. (15)), $p \in L \cap Q \subseteq P$, we infer that $P$ is smooth at $p$ by the hereditary smoothness of $X$ at $p$. Thus, since $p, d \in Q \cup \cup U \cap N \subseteq Q \cap I(a, b) \subseteq Q \cup K \subseteq P$, we infer by (10) that there are continua $F_n$ such that

$$p, d_n \in F_n \ \text{for each} \ n = 1, 2, \ldots$$

and

$$\lim_{n \to \infty} F_n = Q \cup N.$$  

It follows from (16) that $I(d_n, a_n) \subseteq F_n$; thus $W = L \cap I(d_n, a_n) \subseteq L \cap F_n$.

b) $e \in K \cap V$. The continuum $P$ is smooth at $p$; thus, since $p, d \in Q \cup U \subseteq P$, we infer by (10) that there are continua $F_n$ such that

$$p, d_n \in F_n \ \text{for each} \ n = 1, 2, \ldots$$

and

$$\lim_{n \to \infty} F_n = Q \cup M.$$  

It follows from (16) that $I(d_n, a_n) \subseteq F_n$; thus $W \subseteq Q \cup M$. We obtain a contradiction in the same way as in case (a). The proof of Theorem 3 is complete.

**Corollary 4.** Let a continuum $X$ be hereditarily smooth at the point $p$ and suppose that $L$ is a subcontinuum of $X$ such that $p \in L$. Then, for each layer $T$ of an arbitrary irreducible continuum $I(p, x)$, the set $L \cap T$ is connected.
Indeed, by the assumptions, the continuum \( I(p, x) \) is smooth at \( p \). Let \( a \) be an arbitrary point of \( L \cap T \). Therefore, by Lemma 1 of [5], for each \( y \in T \) there is a continuum of convergence \( K_y \) such that \( (a, y) \in K_y \cap T \). Thus \( T = \bigcup \{ K_y \mid y \in T \} \) and \( L \cap T = \bigcup \{ K_y \cap L \mid y \in T \} \). Sits \( K_y \cap L \) for each \( y \in T \) are connected by Theorem 3, and \( a \in K_y \cap L \) for each \( y \in T \). This implies that the set \( L \cap T \) is connected (see [2], § 46, II, Corollary 3 (6), p. 132).

If we transform the proof of Theorem 1 of [5], then we obtain the proof of the following.

**Proposition 5.** Let a continuum \( X \) be hereditarily smooth at the point \( p \) and let \( Q \) be an arbitrary subcontinuum of \( X \). If \( pq \) is an arc in \( X \) which is irreducible between \( p \) and \( q \), then the continuum \( Q \) is hereditarily smooth at \( q \).

We have also

**Proposition 6.** Let a continuum \( X \) be hereditarily smooth at the point \( p \), let \( I(p, c) \) be an arbitrary subcontinuum of \( X \) (irreducible from \( p \) to \( c \)) and let \( T \) be a layer of the point \( c \) in \( I(p, c) \). If there is an arc \( pc \) such that \( pc \cap T = \{ c \} \), then \( T = \{ c \} \).

In fact, one can observe that the assumption of the arcwise connectedness of \( X \) in the proof of Theorem 3 of [5] is used only to conclude that there is an arc \( pc \).

We assume the existence of the arc \( pc \). Now if we transform the proof of Theorem 3 of [5], putting \( d = p \) and using Corollary 4 instead of Corollary 7 of [5] and Proposition 5 instead of Theorem 1 of [5], then we obtain the proof of Proposition 6.

### § 3. Monotone decompositions

Now we prove the following.

**Theorem 7.** Let a continuum \( X \) be hereditarily smooth at the point \( p \). If a monotone mapping \( f \) maps the continuum \( X \) onto \( Y \), then the continuum \( Y \) is hereditarily smooth at \( f(p) \).

**Proof.** Suppose \( Q \) is a subcontinuum of \( Y \) such that \( f(p) \in Q \). Since the mapping \( f \) is monotone, we infer that the set \( f^{-1}(Q) \) is a continuum. Moreover, \( f^{-1}(Q) \) is a continuum of \( X \) at \( p \), we conclude that the continuum \( f^{-1}(Q) \) is smooth at \( p \). Therefore the continuum \( Q = f^{-1}(Q) \) is smooth at \( f(p) \) by Theorem (6.2) of [4], p. 90, i.e., \( Y \) is hereditarily smooth at \( f(p) \).

We have

**Theorem 8.** Let a continuum \( X \) be hereditarily smooth at the point \( p \). If \( T \) and \( T' \) are layers of the point \( c \) in two irreducible continua \( I(p, c) \) and \( I'(p, c) \), respectively, then \( T = T' \).

**Proof.** Consider the continuum \( K \) of the form: \( K = I(p, c) \cup I'(p, c) \). Define a monotone decomposition \( \Phi \) onto \( K \) as follows: if \( x, y \in K \), then \( xy \) if and only if either \( x = y \) or \( x \) and \( y \) belong to the same layer in \( I'(p, c) \). Let \( g \) be the canonical mapping from \( K \) onto \( K \). Obviously \( \Phi(I'(p, c)) \) is an arcwise connected. Hence the mapping \( \Phi(I'(p, c)) \) is monotone, and thus the set \( \Phi(I'(p, c)) \) is an irreducible continuum between \( \phi(p) \) and each point of \( \phi(T) \) by Theorem 3 of [3], § 48, I, p. 192. It follows from Theorem 7 that the continuum \( K \) is hereditarily smooth at the point \( \phi(p) \). Thus Proposition 6 implies that \( \phi(T) = \phi(c) \), and so \( T = \phi^{-1}(T) = \phi^{-1}(c) = T' \), whence \( T = T' \), because the role of \( T \) and \( T' \) is symmetric. The proof of Theorem 8 is complete.

**Corollary 9.** Let a continuum \( X \) be hereditarily smooth at the point \( p \). If \( c \in I(p, x) \cap I(p, y) \) and \( T' \) is a layer of \( c \) in \( I(p, y) \) and \( T \) is a layer of \( c \) in \( I(p, x) \), then \( T' = T \) by Lemma 2 of [5], and similarly: if \( T' = T' \) is a layer of \( c \) in \( I(p, y) \) and \( T' \) is a layer of \( c \) in \( I(p, x) \), then \( T' = T' \) by Lemma 2 of [5]. By Theorem 8 we have \( T = T' \), and thus \( T = T' \).

If \( x \in X \) is an equivalence relation on \( X \), we denote by \( \Phi \) the decomposition of \( X \) induced by \( \Phi \) and we denote by \( \Phi \) the projection mapping from \( X \) onto \( X / \Phi \).

Let \( X \) be an arbitrary continuum and let \( p \in X \). We define a relation \( \Phi \) as follows:

\[
\Phi(p, c) \quad \text{if and only if there are continua} \\
I(p, x) \quad \text{and} \\
I(p, y) \quad \text{such that} \\
I(p, x) = I(p, y) = I(p, c).
\]

**Proposition 10.** The relation \( \Phi \) is reflexive and symmetric.

**Lemma 11.** If a continuum \( X \) is hereditarily smooth at \( p \), then \( \Phi \) is an equivalence relation and the equivalence classes are layers of continua \( I(p, x) \).

**Proof.** By Proposition 10, it suffices to show that the relation \( \Phi \) is transitive. Let \( x \in \Phi y \) and \( y \in \Phi z \). By the definition of \( \Phi \) there exist continua \( I(p, x) \), \( I(p, y) \), \( I(p, z) \), and \( I(p, \Phi) \) such that \( I(p, x) = I(p, y) = I(p, z) \). By Theorem 8 layers \( T \) and \( T' \) of the point \( c \) in \( I(p, y) \) and \( I(p, \Phi) \), respectively, are equal. The point \( c \) is a point of irreducibility of \( I(p, x) \), and thus \( x \in T = T' \).

There are \( \Phi(p, c) \) is a point of irreducibility of \( I(p, y) \), i.e., \( I(p, y) = I(p, c) \); thus \( I(p, c) = I(p, x) \). Hence \( x \in \Phi z \).

Further, if \( T \) is a layer of the point \( x \) of the continuum \( I(p, x) \), then \( T \) is contained in the equivalence class \( [x] \) of \( x \) with respect to the relation \( \Phi \). It follows from the definition of \( \Phi \) and from Corollary 9 that \( [x] = T \).

We have the following generalization of Theorem 5.2 of [2], p. 58.

**Theorem 12.** If a continuum \( X \) is hereditarily smooth at the point \( p \), then the decomposition \( \Phi \) is such that

(i) \( \Phi \) is upper semi-continuous,
(ii) the elements of \( \Phi \) are continua,
(iii) the decomposition space of \( \Phi \) is arcwise connected,
(iv) if \( \delta \) is a decomposition satisfying (i), (ii), and (iii), then \( \Phi \leq \delta \) (i.e., \( \Phi \) refines \( \delta \)),
(v) each element of \( \Phi \) has \( \Phi \) void interior,
(vi) \( X/\Phi \) is hereditarily smooth at \( \Phi(p) \),
(vii) if \( Q \) is a continuum, then \( p \in Q \) implies \( Q = \Phi^{-1}(Q) \).
Proof. (i) In order to prove that $\mathcal{D}_p$ is upper semi-continuous it suffices to show that $q$ is a closed subset of $X \times X$ (see [3], § 43, I, Theorem 4, p. 58). Let $(x_n, y_n): n \in \mathbb{N}$ be a sequence of points of $q$ which converges to $(x, y)$. Then $\lim x_n = x$ and $\lim y_n = y$. Let $I(p, x)$ be an arbitrary subcontinuum of $X$, irreducible $x = x_0$ between $p$ and $x$. By the smoothness of $X$ at $p$ there are continua $I(p, x_n)$ in $X$ such that $\lim I(p, x_n) = I(p, x)$ (cf. [4], Theorem 2.4, p. 81). Since $x_n \in \mathcal{D}_n$, there are continua $I(p, x_n)$ and $I(p, y_n)$ such that $I(p, x_n) = I(p, y_n)$. Thus $y_n$ belongs to a layer $I'(p, x_n)$ in $I(p, x_n)$. We infer $y \in I(p, x)$. Take an irreducible continuum $I(p, y)$ in $I(p, x)$. In a similar way, we obtain $x \in I(p, y)$. But $x \in I(p, y) = I(p, x)$ implies $I(p, x) = I(p, x)$ by the irreducibility of $I(p, x)$. This means $x = y$.

(ii) and (v) The fact that the elements of $\mathcal{D}_p$ are continua, indeed continua with void interiors, follows immediately from Lemma 11 (cf. Proposition 2).

(iii) Let $\phi_p(x)$ denote an arbitrary point of $X \setminus \{\phi_p(y)\}$. Applying Lemma 11 to the arbitrary continuum $I(p, x)$, we find that $\phi_p(I(p, x))$ is an arc containing $\phi_p(x)$ and $\phi_p(y)$. Thus $X \setminus \{\phi_p(x)\}$ is arcwise connected.

(iv) Suppose that there is an equivalence relation $\sim$ such that the arc $\mathcal{D}_\sim = \{\phi_p(x): x \in X \setminus \{\phi_p(y)\}\}$ satisfies (i), (ii) and (iii). If $\mathcal{D}_\sim$ does not refine $\mathcal{D}_p$, then there exist elements $D_1$ and $D_2$ of $\mathcal{D}_\sim$ such that $D_1 \cap D_2 \neq \emptyset$ and $D_1 \cap D_2 \neq \emptyset$.

Since $X \setminus \{\phi_p(x)\}$ is arcwise connected, we may assume that there exists an arc $A$ in $X \setminus \{\phi_p(x)\}$ which contains the points $\phi_p(x)$ and $\phi_p(y)$ but misses $\phi_p(y)$. Now $\phi_p^{-1}(A)$ is a continuum which contains $p$ and intersects $D$ properly. This contradicts the definition of $D$ (cf. Lemma 11); consequently:

(vi) It follows from (i), (ii) and Theorem 7 that $X \setminus \{\phi_p(x)\}$ is hereditarily smooth at $\phi_p(x)$.

(vii) Let $Q$ be an arbitrary continuum such that $p \neq Q \subseteq X$. It is obvious that $Q = \phi_p^{-1}(\phi_p(Q))$. Then there is a point $y \in Q$ such that $x \in y$. It follows from Lemma 11 that points $x$ and $y$ belong to the same layer of any continuum $I(p, y)$. But $p, y \in Q$, and thus $Q$ contains such a continuum. Therefore $x \in \mathcal{D}_p$, i.e., $\phi_p^{-1}(\phi_p(Q)) = Q$. The proof of Theorem 12 is complete.

Theorem 3 of [5] and Theorem 12 (iii), (vii) imply

Corollary 13. If a continuum $X$ is hereditarily smooth at the point $p$, then $X \setminus \{\phi_p(x)\}$ is hereditarily arcwise connected.

We have

Theorem 14. Let $X$ be a continuum and $p \neq X$. If there is an equivalence relation $\sim$ such that

(i) $\mathcal{D}_\sim$ is upper semi-continuous,
(ii) the elements of $\mathcal{D}_\sim$ are continua,
(iii) if $Q$ is a continuum, then $p \notin Q \subseteq X$ implies $Q = \phi_p^{-1}(\phi_p(Q))$, and
(iv) $X \setminus \{\phi_p(x)\}$ is hereditarily smooth at $\phi_p(x)$,

then the continuum $X$ is hereditarily smooth at the point $p$.

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