Groups in the category of $f$-manifolds

by

Richard S. Millman (Carbondale, Ill.)

Abstract. A structure on a $n$-dimensional differentiable manifold given by a tensor field $f$ of type $(1,1)$ and constant rank $r$ which satisfies $f^2 + f = 0$ is called an $f$-structure. An $f$-map is a map between $f$-manifolds whose differential commutes with the $f$-structure. An $f$-Lie group is a group in the category of $f$-manifolds and $f$-maps.

Theorem A. Every $f$-Lie group is the quotient of the product of a complex Lie group and a Lie group with trivial $f$-structure. An $f$-Lie group is an $f$-contact Lie group if the kernel of the $f$ as a sub-bundle of the tangent bundle is parallelizable by commuting vector fields.

Theorem B. A compact $f$-contact Lie group is isomorphic (as a $f$-group) to a torus.

1. A structure on an $n$-dimensional differentiable manifold given by a tensor field $f$ of type $(1,1)$ and constant rank $r$ which satisfies $f^2 + f = 0$ is called an $f$-structure. This notion has been studied by Xano and Ishihara (among others) [4]. An $f$-structure is integrable if about each point there is a coordinate system in which $f$ has the constant components

$$f = \begin{pmatrix} 0 & -I_p & 0 \\ I_p & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $I_p$ is the $(p \times p)$ identity matrix $(p = \frac{1}{2}n)$. In [1] it is shown that the integrability of $f$ is equivalent to the vanishing of the Nijenhuis tensor of $f$.


where $X$ and $Y$ are vector fields on $M$. We shall write $\chi(M)$ for the set of all vector fields on $M$, $T_m(M)$ for the tangent space of $M$ at $m \in M$ and $T(M)$ for the tangent bundle of $M$. For $m \in M$, let

$$(\text{ker} f)_m = \{X \in T_m M | f_m(X) = 0\}$$

and

$$(\text{im} f)_m = \{X \in T_m M | X = f_m Y \text{ for some } Y \in T_m M\}$$.
The proof of the following corollary is immediate since from Proposition 1 the Niijiihuis torsion of a bi-invariant $f$-structure must vanish at $e$.

**COROLLARY.** A bi-invariant $f$-structure on a Lie group is integrable.

We now prove Theorem A. Let $L_0 = \ker f_0$ and $L_1 = \im f_0$. It is clear from Proposition 1 that both $L_0$ and $L_1$ are Lie subalgebras of $\tilde{G}$. Now if $X = f(Z) \in L_1 \cap L_2$ then $f(Z) = 0$ hence since $f(Z) + f^*(Z) = 0$, $X = f(Z) = 0$ and so $L_1 \cap L_2 = 0$. By dimensions $\tilde{G}$ is therefore the direct sum (as a vector space) of $L_1$ and $L_2$. Furthermore if $X = f(Z) \in L_1$ and $Y \in L_2$ then again applying Proposition 1,

$$[X, Y] = f(Z, Y) = [Z, f(Y)] = 0.$$ 

Thus $\tilde{G} = L_1 \oplus L_2$ as Lie algebras and by standard results of Lie theory we have Theorem A.

3. Before proving Theorem B we need to recall some results of [3].

The kernel of $f, \ker f$, is $\cup (\ker f)_m$ and the image of $f, \im f$, is $\cup (\im f)_m$.

An $f$-manifold is $k$-framed if there are $\xi_1, \ldots, \xi_n \in \chi (M)$ such that $\{\xi_1(m), \ldots, \xi_n(m)\}$ forms a basis for $(\ker f)_m$ for all $m \in M$. We write $n = n - r$. If $M_1$ and $M_2$ are $k$-framed $f$-manifolds then we define an almost complex structure $J$ on $M_1 \times M_2$. We shall denote the $k$-framing on $M_1$ by $\xi_1, \ldots, \xi_k$ and the $f$-structure on $M_1$ by $f_1$. If in addition $(\xi_1, \ldots, \xi_k)$ is $0$ for all $1 \leq k$, then $M_1$ is called an $f$-contact manifold.

The concept of an $f$-contact manifold generalizes the basic features of almost contact structure to $f$-manifold of higher multiplicity (i.e. lower rank). In [3, Lemma 2] we have associated to the framing $(\xi_1, \ldots, \xi_k)$ differential forms $\eta_i$ for $i = 1, 2, j = 1, \ldots, n_0$. We define the almost complex structure $J$ on $M_1 \times M_2$ as follows: if $X_1 \in T_{m_1}M_1, X_2 \in T_{m_2}M_2$ where $p \in M_1, q \in M_2$ then

$$J_{p,q}(X_1, X_2) = \left( f_1(X_1) - \sum \eta_i (X_2) \xi_i(p) \right) + \sum \eta_i (X_1) \xi_i(q) .$$

We also proved the following theorem in [3].

**THEOREM.** Let $M_1$ and $M_2$ be two $k$-framed $f$-manifolds of the same rank. If $f_1$ and $f_2$ are integrable then the almost complex structure $J$ is integrable then the almost complex structure $J$ is integrable if and only if both $M_1$ and $M_2$ are $f$-contact manifolds.

To prove Theorem B we note that if $G$ is an $f$-contact Lie group then $G \times G$ is a complex Lie group. (This is essentially showing that the $\eta_i$ are bi-invariant which follows immediately from the bi-invariance of $f_1$). Hence if $G$ is compact then $G \times G$ is a compact complex Lie group, hence abelian and the result follows. Theorem B is proven in the special case that $f$ defines a structure of an almost contact manifold in [3].
If we let $G = C \times R$ where $C$ is the complex line (considered as a complex manifold) and $R$ is a Lie group with trivial $f$-structure and $D = \{(n+in, n) \mid n \text{ is an integer}\}$ then $G/D$ is an $f$-Lie group which is not the product of a complex Lie group and an $f$-Lie group with trivial $f$-structure. ($G/D$ is of course diffeomorphic to $C \times S^1$, but the $f$-structure on $G/D$ is not the product $f$-structure of $C \times S^1$). This is the example mentioned in the introduction.

References


Reducing hyperarithmetic sequences

by

Hans Georg Carstens (Hannover)

Abstract. Every $a'$-sequence is isomorphic to an $a^+$-sequence. This implies: Every $a'$-theory $T$ with an $a$-language has an $a^+$-model. If $T$ has an infinite normal-model then $T$ has an normal $a^+$-model.

§ 1. Introduction. If you analyse a mathematical construction to evaluate its complexity e.g. in terms of the hyperarithmetic hierarchy, it is not difficult to get $a'$-bounds ($a \in O$, $O$ Kleene's system of ordinal notations, $a' = 2^a$) for you can employ recursive processes to describe the construction. If you try to get $a^+$-bounds (a predicate is $a^+$-bounded if it is a Boolean combination of $\Sigma^a_0$-predicates) you must analyse some tricky constructions often related to wait and see methods.

In this paper we prove a theorem on hyperarithmetic sequences by which in some cases we can avoid this analysis and get an $a^+$-bound by means of $a'$-bound. In § 5 examples regarding models and structures will be discussed.

A model is called normal if its universe is the set of natural numbers and the first predicate is the identity. In [3] Hensel and Putnam have shown that every axiomatized consistent theory based on a finite number of predicates which has an infinite model with "=" interpreted as identity, has a normal model in $B^+(1)$, i.e. all predicates are $1^+$-bounded. Among its consequences the theorem has an analogue to the Hensel-Putnam result for arbitrary hyperarithmetic theories with a recursive language. We can drop the assumption that the theory must be based on a finite number of predicates, and different to Putnam [5] and Hensel-Putnam [3] the result yields a method which solves Mostowski's problem [4, p. 39] simultaneously for theories with and without identity.

§ 2. The hyperarithmetic hierarchy. Let $O$ be Kleene's system of ordinal notations with the ordering $<_\alpha$, $a' = 2^a$ the successor of $a$ in $O$, $A'$ the recursive jump of $A$; we write $A \leq B$ if $A$ is recursive in $B$. $H_a := O$, $H_{a'} := H_a$ for $a \in O$, $H_{a+} := \{ \langle x, y \rangle : y <_{a} 3 \cdot 5^a \& x \in H_y \}$, where $3 \cdot 5^a$ is a notation of a limit ordinal.