Closed retraction of Euclidean spaces

by

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Abstract. The problem of the characterization of the images of the Euclidean spaces under closed retraction is studied. The c-retract of the space $X$ is defined as the image of $X$ under some closed retraction. The following theorems are proved:

**Theorem 1.** Every compact c-retract $E$ of $E^n$ is the $c$-retract of $E^n$.

**Theorem 2.** For every non-compact c-retract $R$ of $E^n$, $R^n(a,b) = 0$ for $m = 1, \ldots, n-1$ and if $R \neq E^n$ then $R^n(a,b) = 0$. ($R^n(a,b)$ denotes the $m$-th Čech cohomology group of $X$ and $aX$ denotes the one-point (Alexandrov) compactification of $X$).

**Theorem 3.** The retract $R$ of the Euclidean plane $R^2$ is the $c$-retract of $R^2$ if and only if it does not disconnect $R^2$.

The main purpose of this paper is to apply some methods investigated in [6] to the study of closed retraction. The paper gives some results about closed retractions of Euclidean spaces, particularly a complete characterization of all subsets $R$ of the Euclidean plane $R^2$ for which there exists a closed retraction $r: R^2 \to R$.

All notions and notations which are not defined here are taken from [1] and [2].

**Definition.** The $c$-retract of the space $X$ is the subset of $X$ which is the image of $X$ under some closed retraction.

**Proposition.** Let $R$ be a compact retract of $E^n$ for some $n$. Then $R$ is the $c$-retract of $E^n$.

**Proof.** The set $R$ is an absolute retract in the sense of [1], Sec. V. 1. On the other hand, $R \subset E^n(0,r)$ for some positive $r$. (We denote by $O$ the element $(0, \ldots, 0)$ of $E^n$). We denote by $R(r)$ the sphere obtained by matching to a point the set $E^n(0,r)$. It is clear that $R(r)$ is a compact metric space and the quotient mapping $\pi: E^n \to R$ is closed (see [6]).

Proposition 1. Simultaneously there exists a retraction $r: R(\infty) \to \bar{R}$ which is closed since $R(r)$ is compact. The composition $r = r_0 \circ \pi$ is the desired closed retraction.

We denote by $R_1^n(\infty)$ the $n$th Čech cohomology group with integer coefficients of the space $X$. We can now prove the first main result of this paper.

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THEOREM. For every non-compact c-retract of $E^n$, $H^n(\alpha R) = 0$ for $i < n - 1$ and if $R \neq F^n$ then $H^n(\alpha R) = 0$.

Proof. If $n = 1$, there is nothing to prove. Suppose now that $n > 1$, then, by (6), Corollary to Theorem 7, we have $\gamma F^n = \alpha F^n \neq S^n$. The closed retraction $\tau: F^n \rightarrow R$ can be extended to $\gamma: F^n \rightarrow \gamma R$. (It is possible by (5), Theorem 4.) The mapping $\gamma$ is an epimorphism and hence $\gamma F^n \rightarrow \gamma R$, which means that $\gamma F^n = \alpha F^n$. We can therefore obtain a mapping $\tau: S^n \rightarrow \alpha F^n$ as the composition of the homeomorphism $\tau: S^n \rightarrow \alpha F^n$ and the mapping $\gamma$. Let $j: \alpha R \rightarrow S^n$ be the extension of the identity map of $R$ into $F^n$. Since $\gamma j = \alpha F^n$, the mapping $j^{*+1}n$ = $\alpha F^n$ is the identity on $H^n(\alpha R)$ for every $m$. So $m: H^n(\alpha R) \rightarrow H^n(S^n)$ is a monomorphism, and since $H^n(S^n) = 0$ for $0 < m < n - 1$, the groups $H^n(\alpha R)$ must vanish for $m > n - 1$.

On the other hand, assuming that $R \neq F^n$, we can easily check that $\alpha R$ is a proper subset of $S^n$ and hence $H^n(\alpha R) = 0$.

Corollary. If $1 < k < n$, then $F^n$ cannot be a c-retract of $E^n$.

Proof. $\alpha F^n = S^n$, hence $H^n(\alpha F^n) = 0$ and it remains to apply Theorem 1.

In the case $n = 2$ we prove the following

LEMMA 1. If $R$ is a c-retract of $F^2$, then $R$ does not disconnect $F^2$.

Proof. If $R$ is compact, then our lemma follows from (1), Theorem V.13.1. Suppose now that $R$ is not compact. The closure of the region in $S^2 = \alpha F^2$ is homeomorphic to $\alpha R$ and $S^2 - \alpha R = F^2 - R$. This means that if $F^2 - R$ is not connected then so is $S^2 - \alpha R$. Applying to the pair $(S^2, \alpha R)$ the Borsuk Theorem (11, Theorem X.13.1), we obtain an essential mapping $f: S^2 \rightarrow S^2$. This means that $f$ is not homotopic to a constant map. Therefore we infer that the Brunkhouski group $\pi_1(\alpha R)$ is not trivial. It remains now to observe that $\pi_1(\alpha R) = H^1(\alpha R)$ (see [3], Theorem II.1.1), hence $H^1(\alpha R) \neq 0$, which is impossible in connection with Theorem 1.

This contradiction ends the proof.

We can now formulate the main theorem of this paper.

THEOREM 2. The retract $E$ of the Euclidean plane $E^2$ is the c-retract of $E^2$ if and only if it does not disconnect $E^2$.

Proof. The necessity of this condition is a consequence of Lemma 1 and, if $R$ is compact, then the condition is sufficient by Proposition 1.

So it remains to prove that if $E$ is a non-compact retract of $E^2$ and $E^2 - R$ is connected, then $E$ is a c-retract of the plane. To prove this, we formulate and prove four lemmas.

LEMMA 2. Let $J: X \rightarrow Y$ be a continuous mapping and let $Y$ be a $T_n$-space. If there exists such a covering $A = \{A_i\} \times \alpha I$ of $X$ by compact sets that $f(A) = (f(A_i)_{a\in A}$ is a locally finite collection, then the mapping $f$ is closed.

Proof. Let $D$ be a closed subset of $X$. The sets $D_i = D \cap \alpha S_i$ are compact for every $\alpha S_i$ and hence the sets $f(D_i)$ are closed in $Y$. So $f(D) = \bigcup f(D_i)$ is the sum of a locally finite family of closed sets and hence $f(D)$ is closed.

It is easy to check that

(i) $f(\alpha S_i) = \alpha F_i$ for any $\alpha S_i$,

(ii) if $A = \alpha S_i$ then $f(A) = \alpha F_i$,

(iii) if $A \subset F_i$ then $f(A) \subset F_i$,

(iv) if $\alpha S_i \subset F_i$, then $f(A)$ is the sum of $\alpha S_i$ and all the bounded components of the set $F_i - A$.

LEMMA 3. If $Z$ is a compact subset of the plane not disconnecting $F^2$, then for every $e > 0$ there exists an $\eta[Z, (e)] > 0$ such that $\eta[F(Z, \eta[Z, e])] \subset F(\eta[Z, e])$.

The proof is an easy modification of the proof of (11, Lemma V.3.2) and will be omitted.

LEMMA 4. If $Z$ is a retract of $F^2$, then for every $e$ such that $0 < e < s$ only finitely many components $W_1, \ldots, W_n$ of $F^2 - R$ intersect simultaneously $S_t$ and $S_r$.

Proof. Let us fix one of such components, $W_2$. It contains an arc $l_2$ joining $S_t$ and $S_r$. Now if $W_2 \cap W_3$ then $l_2 \cup W_3$ disconnects $F^2$ and the components $V_1$ and $V_2$ of $F^2 - (l_2 \cup W_3)$ both contain points of $R$. Denoting by $A_2$ the set $W_2 \cap S_{r+e}$, we can observe that if $r: E^2 - R$ is the retraction, then $r(A_2) \cap F^2 = \emptyset$. Int fact, if $r(A_2) \subset F^2$, then, since $A_2 \cap V_1 \neq \emptyset$, there exists some arc joining $V_2$ and $V_3$, contained in $R \cap F^2$, which contradicts the definition of $V_1$ and $V_2$. So there exists a point $a_2 \in (W_2 - r^{-1}(F^2 - P) \cap S_{r+e})$.

Assume now that there exist infinitely many components $W_n$ of $F^2$ joining $S_t$ and $S_r$. We can easily repeat the construction of the point $a_n$ and denoting by $a$ the accumulation point of the set $\{a_1, a_2, \ldots\}$ we obtain:

(i) $a \in R$, since $S_{r+e} - R$ is open in $S_{r+e}$ and the points $a_2$ are from disjoint components of the set $S_{r+e} - R$.

(ii) $r(a) \in Int F^2$, since $r(a) \subset r(A)$ and $r(A) \cap F^2 = \emptyset$.

The contradiction between (i) and (ii) establishes our lemma.
LEMMA 5. Let $R$ be a non-compact retract of $E^3$ which does not disconnect the plane and let $(s) = s_1, s_2, \ldots$ be a strictly decreasing sequence of positive real numbers such that $s_i < 1/16$. Then there exists a sequence $(\eta) = \eta_1, \eta_2, \ldots$ of positive reals such that $\mathcal{F}_i(K(R(\eta_i))) \subset K(R(s))$.

Proof. We define inductively the sequence of sets $X_1 \subset X_2 \subset \ldots \subset E^3$ and the sequence of positive real numbers $(\eta) = \eta_1, \eta_2, \ldots$ satisfying the following conditions:

(i) $E^3 \cup K(R(\eta)) \subset X \subset K(R(s)), s_i$;  
(ii) $X = \mathcal{F}(X_0)$;  
(iii) $X_0 \cap X_1 \subset \text{Int}P_{s_1}^{1/16}$;  
(iv) $X_1 \setminus X_0 \subset K(R(\eta_1), \eta_1)$;  
(v) $X_i \setminus X_{i+1} \subset K(R(\eta_i+1/16))$

for $n = 1, 2, \ldots$

In the first step we construct the set $X_1$. Let $W_1, \ldots, W_s$ be the components of $P_{s_1}^{1/16} \cap R$ intersecting both $S_{s_1}$ and $S_{s_2}$. We select for $i = 1, \ldots, s$ a point $w_i \in W_i$ and we put $\xi_i = \min \{d(w_i, R)\}$. Now, let $V_i$ be the component of $E^3 \setminus (X_{\xi_i} \cup R)$ containing $W_i$. All bounded sets of the family $V_1, \ldots, V_s$ are contained in some $K_N$ ($N > 3$). We put $\eta_i = \eta_i(w_i, \xi_i)$, where $\eta_i$ is defined as in Lemma 3. The set $X = X_0 \setminus K(R(\eta), \eta) \cup R$ satisfies the conditions (i), (ii), (iv) and (v) given above, and if we put $X_0 = \emptyset$ then condition (iii) is also satisfied (see Figure).

To verify this, we notice first that $\eta_i < \xi_i < 1/16$. Hence

$K(R(\eta)) \subset K_{s_1} \subset K_{s_1} \cup R$.

on the other hand, if $A$ is a bounded component of $E^3 \setminus K(R(\eta)) \cup R$ then $A$ cannot intersect both $S_{s_1}$ and $S_{s_2}$, which follows from the definition of $\eta_i$. Thus conditions (ii'), (iv) and (v) are satisfied. Condition (ii) is satisfied by the property (i) of the operation $\mathcal{F}$. Conditions (i) and (v) are also easy to check.

We now assume that we have defined the sets $X_1, \ldots, X_{n-1}$ and the numbers $\eta_1, \ldots, \eta_{n-1}$ satisfying conditions (i)-(v). We define $X_n$ and $\eta_n$ as follows:

Let $W_1, \ldots, W_s$ be the components of $P_{s_n}^{1/16} \cap R$ joining $S_{s_n}$ and $S_{s_{n+1}}$. As above, we select $w_i \in W_i$ and we put $\xi_i = \min \{d(w_i, X_{n-1})\}$. Now, let $N > s_n + 2$ be such a number that all bounded components of $E^3 \setminus (K(R(\eta_n)) \cup R)$ are contained in $K_N$. Observe now that only a finite number of the components $A_i$ of $P_{s_n}^{1/16} \setminus X_{n-1}$ intersect both $S_{s_n}$ and $S_{s_{n+1}}$. Using similar arguments as in the proof of Lemma 4 we can check that the number of such components is not greater than the number of the components of $P_{s_{n+1}}^{1/16} \setminus X_{n-1}$ joining $S_{s_{n+1}}$ and $S_{s_{n+2}}$. But every such component contains at least one of the points of $R$, and so it contains some ball of diameter $\eta_{n+1}$, and those balls are mutually disjoint. We select the points $g_i \in A_i$ for every $i$ and we put $\xi_i = \min \{d(g_i, X_{n-1})\}$. We define $\eta_n = \eta(X_{n-1} \setminus K_N, \min(\xi_1, \xi_2, \xi_3))$ and we denote $X_n = X_{n-1} \setminus K_{s_n} \cup R \setminus \mathcal{F}(X_{n-1} \setminus K_{s_n} \cup R)$.

We now prove that the set $X_n$ satisfies conditions (i)-(v). The first part of (i) and (ii) follow immediately from the definition of $X_n$. Notice now that all of the bounded components of $E^3 \setminus (X_{n-1} \setminus K_{s_n})$ are contained in $P_{s_{n+1}}^{1/16}$ which follows from the definition of $N$, $\xi_1$, and $\xi_2$. Thus conditions (iii) and (iv) are satisfied. Condition (v) and the second inclusion of (i) are now easy to verify. This finishes the description of the inductive step.

We define $X = \bigcup_{n=1}^s X_n$. The set $X$ satisfies the following conditions:

(iii) $E^3 \setminus K(R(\eta)) \subset \text{Int}P^{1/16}_N$;  
(iv) $\mathcal{F}(X) = X = X$.

Condition (vi) follows from (i). The set $X$ is closed as the sum of the locally finite family of closed sets $X_i \cap P^{1/18}_i$. Now, let $U$ be a bounded component of the set $E^3 \setminus X$. So $U \subset K_N$ for some $N$ and since $X \setminus K_N = X_{n+1} \setminus K_N$, we have $U \subset \mathcal{F}(X_{n+1})$, which is impossible as $X_{n+1} = \mathcal{F}(X_{n+1})$.

This finishes the proof of Lemma 5.

We can now return to the proof of Theorem 2.

Let $v: E^3 \to E^3$ be some retraction. If $R = E^3$, then there is nothing to prove; hence we can assume that $R \neq E^3$. We prove that there exists a closed set $P$ satisfying the following conditions:

(i) $P \subset P$;  
(ii) $v(P)$ is closed;  
(iii) there exists a homeomorphism $h: E^3 \to E^3$ such that $h(P)$ is a closed half-plane.
Let \( \alpha = \alpha_1, \alpha_2, \ldots \) be a sequence of positive numbers such that

(i) \( \alpha_1 < 1/16 \),

(ii) \( a_3 < a_{n+1} \) if \( x, y \in K_{n+1} \) and \( g(x, y) < a_3 \) then \( g(r(x), r(y)) < 1 \).

Such a sequence exists because of the uniform continuity of the mapping \( r \) on every closed ball. The sequence satisfies the assumptions of Lemma 5 and hence there exists a sequence \( \eta_1, \eta_2, \ldots \) such that

\[ F(K(R, \eta)) \subseteq K(R, \eta) \]

Let \( \mathcal{C} \) be such a locally finite triangulation of the plane that

(vi) if \( \varphi \in \mathcal{C} \) and \( \varphi \cap \mathcal{F}^l \subseteq K_{n-1} \), then \( \dim \varphi < \eta_3 \),

(vii) if \( \sigma \in \mathcal{C} \) is a two-dimensional closed simplex and \( \sigma \cap R \neq \emptyset \), then \( \inf \sigma \cap R \neq \emptyset \).

We put \( P \) as the sum of all two-dimensional closed simplexes of \( \mathcal{C} \) intersecting \( R \) and let \( P = P(F(P)) \). It follows from the definition of \( \eta_1 \) and \( \mathcal{C} \) that \( P \cap K(R, \eta) \). Let us observe now that \( F(P) \) is a simple closed line. In fact, let us assume that \( a \) is a point of self-intersection of \( F(P) \). This means that \( a \) is the centroid of at least two two-dimensional simplexes of \( \mathcal{C} \) and \( (P \cap st(a, \mathcal{C})) \} \} \) is not connected. It can easily be checked, by using \( \eta_2 \) that both components of \( P \cap st(a, \mathcal{C}) \) must contain points from \( R \). Let \( p, q \) be nearest to \( a \) such points. It is clear that \( p \neq q \neq q \). Let \( k = \sigma, p, q, \) and \( l = k \cup r(k) \cap C \). It is clear that the set \( l \) disconnects the plane and that one of the components of \( \mathbb{F}^l \) is bounded and contains some non-void component of \( \mathbb{F}^l \), which is impossible, since \( P = F(P) \).

We now prove that \( P \) does not disconnect the plane. Assuming the contrary, we denote by \( U, V \), the components of \( \mathbb{F}^l \). Since \( \mathbb{F}^l \) is connected, there exists an arc \( \gamma \) joining \( U \) and \( V \), disjoint with \( R \). Let \( \varepsilon > 0 \) be such a number that \( K([0, \varepsilon]) \cap R = \emptyset \). We can easily check, using similar arguments as above, that \( l \) disconnects \( P \) and both components of \( P \) \( \cap \gamma \) are unbounded. Let \( K \) be such an integer that \( \eta_2 < \eta_2 \) and \( l \subseteq K_{\gamma} \). Denoting by \( K, L \), the two unbounded components of \( P \) \( \cap \gamma \) we obtain that \( K_{\gamma} \cap L \neq \emptyset \). Moreover, \( P \cap (K_{\gamma} \cap L) 
eq 0 \). Let \( x \) be an element of \( P \), \( y \in (K_{\gamma} \cap L) \). Let \( x \in K_{\gamma} \cap L \) and \( y \in C \) be the definition of \( P \), that there exist \( x, y \in K_{\gamma} \cap L \) such that \( g(x, y) < \eta_2 \), \( p, x \in C \). Hence \( R \cap K \neq \emptyset \) \( \cap L \) and, since \( R \cap l = \emptyset \), \( R \) is not connected. This contradiction finishes the proof of the fact that \( \mathbb{F}^l \) is connected.

We now fix a one-dimensional simplex \( \alpha_1 = \alpha_2 \in \mathcal{C} \) and we define a homeomorphical embedding \( f: \mathbb{E}^{l} \to F(P) \) as follows:

Let \( f([0, 1]) \) be a linear mapping onto \( \alpha_1 \). Assume now that we have defined the mapping \( f \) on the segment \([k, 1]\) and let \( a_k = f(k) \) and \( a_1 = f(1) \) be the endpoints of the broken line \( f([k, 1]) \). It is clear that there exist two one-dimensional simplexes \( \alpha_1 \) and \( \alpha_2 \) from \( F(P) \( \cap \) (k, 1) \) such that \( \alpha_2 \in \alpha_1 \), \( \alpha_1 \in \alpha_1 \). We can extend \( f \) over \([k, 1] \), \( l(1, l+1) \), putting on the segments \([k-1, k] \) and \([l, l+1] \) the uniquely defined linear maps onto \( \alpha_1 \) and \( \alpha_2 \), respectively. Since the triangulation \( \mathcal{C} \) is locally finite, the mapping \( f \) is closed on \([k, 1] \), \( l(1, l+1) \) and hence it can be extended to \( \mathcal{C} \). \( \gamma \mathbb{F}^l \to \gamma \mathbb{F}^l = \alpha_1 \mathbb{F}^l \) (see [8], Theorem 4). It is easy to check that \( \gamma \mathcal{C} = \alpha_2 \mathbb{F}^l \), for \( \mathcal{C} = \gamma \mathbb{F}^l \), \( \gamma \mathcal{C} = \alpha_2 \mathbb{F}^l \), which is an extension of \( f \).

We now prove that \( F(\mathbb{E}) = F(P) \). In fact, if \( f(\mathbb{E}) \) is not then we can repeat this construction for some \( \mathcal{C} \subseteq F(P) \) obtaining the embedding \( f: \mathbb{E} \to F(P) \). It is then easy to verify that \( F(P) \) disconnects the plane into at least three components, which is impossible, since both \( P \) and \( \mathbb{F}^l \) are connected. We can now apply ([5], Theorem 9, 9.1) to obtain a homeomorphism \( \gamma: \mathbb{F}^l \to \mathbb{F}^l \) such that \( G(\mathbb{E}) = \mathcal{C} \) and \( G(\mathbb{F}) = \mathbb{F}^l \) is the equator of the sphere. It is clear that \( G(\mathbb{F}) = \mathbb{F}^l \) is the closed half-plane \( \mathbb{E} \times [0, \infty] \). We now denote by \( s \) the retraction of \( \mathbb{E} \) on \( P \) obtained by the composition \( G \circ s \circ G \), where \( s: \mathbb{E} \to \mathbb{E} \times [0, \infty] \) is a closed retraction defined by \( s(x, y) = (s, |y|) \). We now prove that \( P = s \) is the desired closed retraction. It follows from the definition of \( s \), \( s(\mathbb{E} \cap P_{\infty}) \subseteq P_{\infty} \) and hence the family \( s \) and the mapping \( r \) satisfy the assumptions of Lemma 2 and thus the mapping \( r \) \( \mathcal{C} \) is closed. \( P \) is a closed retraction as the superposition of two closed retractions and the proof of Theorem 2 is finished.

References


Reçu par le Redaction le 10. 7. 1972