Subdirect decomposition of distributive quasilattices

by

J. A. Kalman (Auckland)

Following Plonka [3] we define a quasilattice to be a nonempty set with binary operations \( \land \) and \( \lor \) which are idempotent, commutative, and associative, and a distributive quasilattice to be one which obeys the laws

\[
\begin{align*}
(x \land y) \lor z &= (x \land z) \lor (y \land z), \\
(x \lor y) \land z &= (x \lor z) \land (y \lor z).
\end{align*}
\]

It is easily checked that the tables

\[
\begin{array}{c|ccc|cccc}
\land & 0 & 1 & \infty & 0 & 1 & \infty & 1 \\
\hline
0 & 0 & 0 & \infty & 0 & 1 & \infty & 1 \\
1 & 0 & 1 & \infty & 1 & 1 & \infty & 1 \\
\infty & 0 & 1 & \infty & 0 & 1 & \infty & 1 \\
\end{array}
\]

define a distributive quasilattice, \( X \) say. Let \( \mathcal{Q} \) and \( \mathcal{S} \) be the sub-quasilattices of \( X \) with underlying sets \( \{0, 1\} \) and \( \{0, \infty\} \) respectively; \( \mathcal{Q} \) is a lattice, and \( \mathcal{S} \) is essentially a semilattice (it obeys the law \( x \lor y = y \land x \)).

The object of this paper is to prove the following

**Theorem.** A distributive quasilattice with more than one element is isomorphic to a subdirect product of copies of \( X, \mathcal{Q}, \) and \( \mathcal{S} \).

This extends Birkhoff's subdirect decomposition theorem for distributive lattices ([1], p. 193, Theorem 15, Corollary 1), and also contains a similar theorem for semilattices.

In any quasilattice an identity element for \( \land \) (resp. \( \lor \)) if it exists, is unique, and will be denoted by \( I \) (resp. \( 0 \)) (cf. [2]), p. 63, ex. 7, and (2), but note that the free distributive quasilattice with \( O, I \), and one generator has five, not seven, elements.

**Lemma.** Let \( Q \) be a distributive quasilattice with \( O \) and \( I \). Then, for all \( x \) and \( y \) in \( Q \),

\[
\begin{align*}
(i) & \quad x \land O = O \text{ if and only if } x \lor I = I; \\
(ii) & \quad x \lor y = I \text{ if and only if } x = y = I; \text{ and} \\
(iii) & \quad x \land y \lor O = O \text{ if and only if } x \lor O = y \land O = 0.
\end{align*}
\]
A. J. Kalman

Also, we may define a congruence relation $B$ on $Q$ by setting $x\sim y$ if and only if $x = y$ or $x \land O \neq O$ and $y \land O \neq O$.

Proof. (i) If $x \land O = 0$ then
\[ I = 0 \lor I = (x \land y) \land (y \lor I) = (x \lor I) \land I = x \lor I, \]
and dually.

(ii) If $x \land y = I$ then $x = x \land I = x \land (x \land y) = x \land y = I$, and similarly $y = I$. The converse is trivial.

(iii) If $x \land y \land O = 0$, then, by (i), $I = (x \land y) \lor I = (x \lor I) \lor (y \lor I)$, hence $x \lor I = y \lor I = I$ by (ii), and hence $x \land O = y \land O = O$ by (i). The converse is trivial.

$B$ is obviously an equivalence relation, and is selfdual by (i). If $x \land y$, then, by (ii) and its dual, $(x \land z) \land (y \land z)$ and $(x \lor z) \lor (y \lor z)$ for all $z$. This completes the proof.

**Lemma 2.** Let $Q$ be a distributive quasi-lattice, and let $a \in Q$. Then

(i) we may define congruence relations $C_a$, $D_a$ on $Q$ by setting $x \sim y$ if and only if $x \land a = y \land a$, and $x \lor D_a = y$ if and only if $x \lor a = y \lor a$;

(ii) $x \land (x \lor a) = y \land (y \lor a)$ if and only if $x \land a = y \land a$;

(iii) $x \lor (x \land a) = y \lor (y \land a)$ if and only if $x \lor a = y \lor a$.

Proof. (i) is easily verified. To prove (ii), we note first that if $x \land (x \lor a) = x \land (y \lor a)$ then
\[ x \land (x \lor a) = x \land (y \lor a) = y \land (y \lor a), \]
whence $x \lor a = y \lor a$, moreover the condition $x \land (x \lor a) = x \lor (y \lor a)$ is equivalent to its dual, hence, by duality, $x \lor (x \lor a) = y \lor (y \lor a) = y \lor a$.

Conversely, if $x \land (y \lor a) = y \land (y \lor a)$, then
\[ x \lor a = (x \lor a) \land (x \lor a) = (x \land (x \lor a)) \lor a = (y \land (y \lor a)) \lor a = y \lor a, \]
whence $x \land (x \lor a) = y \land (y \lor a)$ is equivalent to its dual, hence, by duality, $x \lor (x \lor a) = y \lor (y \lor a)$.

**Theorem 1.** Let $Q$ be a subdirectly irreducible distributive quasi-lattice. Then

(i) $Q$ possesses elements $O$ and $I$ (not necessarily distinct); and

(ii) $a \land O = 0$ if and only if $a \sim O$ or $a \sim I$.

**Proof.** (i) Let $C = \{a \in Q : a \land O \neq O\}$. Then if $x \land y$ we have $x \land y$, i.e., $x \land y = y \land x$ and $x \land y = y \land x$, hence $x = y$. Thus $C = O$. Since $Q$ is subdirectly irreducible it follows that $C_a = O$ for some $a$ in $Q$, and $a = I$ by Lemma 2 (iii). Dually, $Q$ has an $O$.

(ii) If $a \land O = 0$ then $C_a \lor D_a = 0$; for
\[ x = x \lor O = x \lor (a \land O) = (x \land a) \lor (x \lor O) = x \lor (x \lor a) \]
for all $x$, and hence, by Lemma 2 (ii), if $x \land (x \lor a) = y \land (y \lor a)$, then $x = y$. Since $Q$ is subdirectly irreducible, it follows that $C_a = O$ or $D_a = O$ and hence, by Lemma 2 (iii), that $a = O$ or $a = I$. The converse of (ii) is trivial.

**Lemma 3.** A subdirectly irreducible distributive quasi-lattice $Q$ with more than one element is isomorphic to $K_2$ or $\langle \rangle$.

**Proof.** Let $P = (x \in Q : x \sim O \neq O) = Q \setminus \{0, I\}$ (cf. Lemma 3 (ii)). If $P = 0$ then $Q \cong \langle \rangle$. Suppose therefore that $P \neq 0$ and let $B = B \cap (\{0, I\} \lor \{a \land a, a \lor a\})$, where $B$ is defined as in Lemma 1. We show that $B = O$.

We wish to prove that if $x \lor y$ then $x = y$, and we may assume that $x \in P$ and $y \in P$ for otherwise $x = y$ since $SYB$. But then $x \land (x \lor D_x) = y \land (y \lor D_y)$, and hence, by Lemma 1 (ii), $x \land (x \lor D_x) = y \land (y \lor D_y)$, i.e., $x = y \land (y \lor D_y)$, whence $y \land x = x$. Similarly, by Lemma 1 (ii), $x \land (x \lor D_x) = y \land (y \lor D_y)$, and thus $x = y$. This proves that $B = O$, and, since $C_a \neq O$ and $D_a \neq O$ for all $a$ in $P$ by Lemma 3 (iii), it follows, since $Q$ is subdirectly irreducible, that $B = O$. Hence $P$ has just one element, and $Q \cong \langle \rangle$ or $Q \cong \langle \rangle$ according as $O = I$ or $O \neq I$.

The theorem stated in the first paragraph follows from Lemma 4 and Birkhoff's general subdirect decomposition theorem ([1], p. 195, Theorem 13).