On $\omega_1$-categorical theories of fields

by

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0. Introduction. In this paper we prove that the only $\omega_1$-categorical first-order theories of infinite fields are the theories of algebraically closed fields.

We adopt the usual formalization of the elementary theory of fields, in terms of an applied first-order predicate logic with equality, individual constants 0 and 1, and binary operation-symbols $+$ and $\cdot$. All these symbols get the usual interpretation. By considering any of the usual axioms for fields, we see easily that the class of fields is an E0 class. We will be dealing more generally with $E_\kappa$ classes of fields.

For $\kappa$ a prime, or 0, let $ACF_\kappa$ be the class of algebraically closed fields of characteristic $\kappa$. It is well-known that $ACF_\kappa$ is an $E_\kappa$ class. A basic property of $ACF_\kappa$, due to Steinitz [19], is: If $\kappa$ is an uncountable cardinal, and $L_1$ and $L_2$ are members of $ACF_\kappa$ of cardinality $\kappa$, then $L_1 \equiv L_2$.

From this one concludes by Vaught’s Test that any two members of $ACF_\kappa$ are elementarily equivalent, i.e. satisfy exactly the same sentences of the language of field-theory. Let $Th(ACF_\kappa)$ be the set of all sentences of field-theory that hold in an arbitrary member of $ACF_\kappa$. Then Steinitz’s result may be formulated in model-theoretic terms as: $Th(ACF_\kappa)$ is $\kappa$-categorical, for each uncountable $\kappa$.

It is known, from Morley’s proof of the Loj Conjecture [14] that, for countable first-order logics, the $\omega_1$-categoricity of a theory $T$ implies the $\kappa$-categoricity of $T$ for each uncountable $\kappa$.

With this background we investigated the possibility of extending the above result of Steinitz to other classes of fields. We have proved the following, which is the principal result of our paper:

Theorem. If $K$ is an $E_\kappa$ class of fields such that $Th(K)$ is $\omega_1$-categorical, and $K$ has no finite members, then $K$ is one of the classes $ACF_\kappa$.

The above theorem is a corollary of a theorem about totally transcendental theories of fields. This theorem says that the only totally transcendental complete theories of infinite fields are the theories $Th(ACF_\kappa)$.
Though this is a stronger model-theoretic result than the theorem stated above, it has apparently no more algebraic interest than its corollary.

Let us consider briefly the $E_0$ classes of fields which are best understood at the present time.

A. Real-closed fields. One normally discusses the class of real-closed fields as a class of ordered fields, and in this formulation the class of real-closed fields is an $E_0$ class. But in a real-closed field the order $\succ$ is definable in terms of the field structure thus:

$$x \succ y \iff (\exists z)(x = y + z^2 \land z \neq 0).$$

Thus the class of real-closed fields is an $E_0$ class.

B. Algebrically complete fields with valuation. Ax and Kochen ([4], [5], [6]) and Ersov [11] have given complete sets of axioms for various classes of fields with valuation. Ax [3] and Ersov [9] have shown that in certain of these classes we can define the valuation-rings in terms of the field structure. It follows that we can interpret in terms of the field structure all statements about the residue-class field and all statements about the ordered value-group. In this way many classes of Henselian valued fields can be construed as $E_0$ classes of fields.

Pseudo-finite fields. Ax [3] has classified all complete theories of pseudo-finite fields. More generally, he has discussed $S$-pseudo-finite fields, where $S$ is a set of primes. There is an overlap with Ersov's paper [10], where absolutely algebraic fields of prime characteristic are discussed, for the latter fields are $S$-pseudo-finite for suitable $S$.

Separably closed fields. Ersov [10] has classified all complete theories of separably closed fields. This of course includes the case of algebraically closed fields.

If $K$ is an $E_0$ class of fields of any of the above types, and $K$ is not one of the classes $ACF_0$, then very little is known about the isomorphism types of members of $K$.

For real-closed fields there is the result of Erdős, Gillman and Hendriksen [8] that if $a > 0$ then any two $a$-real-closed fields of cardinality $\kappa$ are isomorphic, but this is vacuous unless $\kappa = \sum_{n \geq 0} 2^n$. In model-theoretic terms this result identifies the saturated uncountable real-closed fields, assuming the generalized continuum hypothesis. See, for example, [15]. Without using the generalized continuum hypothesis, one can prove the existence of special [15] real-closed fields of certain cardinalities, and using [15] deduce some isomorphism theorems.


Using a condition of Ehrenfeucht [7], and results of Morley [14], one can show that for any uncountable cardinal $\kappa$ there exist non-isomorphic real-closed fields of cardinality $\kappa$. This is because we can define a linear order in each real-closed field, and the Ehrenfeucht condition prohibits this for $\omega_1$-categorical theories. The situation is similar for many classes of algebraically complete fields with valuation, since we can often interpret the theory of the ordered value-group in terms of the basic field-structure.

For the complete theories of pseudo-finite fields, and the complete theories of separably closed, but not algebraically closed, fields, we will point out, in the course of the proof of the main theorem, why these theories are not $\omega_1$-categorical.

A consequence of the main theorem is that the $ACF_0$, $ACF_0^*$ classes of infinite fields allowing elimination of quantifiers. Of course, the theory of real-closed fields allows elimination of quantifiers, but only when $\succ$ is taken as a primitive notion. See [18]. The situation is analogous for certain valued fields. See [6].

In this paper we presuppose acquaintance with our paper [13] on the corresponding problems for abelian groups. We refer to that paper for explanation of notation and basic ideas.

The main idea taken from [13] is that of definable filtration. This idea is applied twice, firstly to the multiplicative group of a field, and secondly to the additive group of a field of finite characteristic.

A model-theoretic result which we did not use in [13], but which is very useful now, is Ehrenfeucht's condition [7].

As well as facts on abelian groups already utilized in [13], we use some results on field-extensions, to be found in [1], [12].

1. Outline of proof. We indicate the main steps of the proof.

(a) We prove that if $K$ is a field with $Char(K)$ totally transcendental, and $E_0$ is a finite algebraic extension of $K$, then $Char(E_0)$ is totally transcendental.

(b) We prove that if $Char(K)$ is totally transcendental then $K^*$ (the multiplicative group of non-zero elements of $K$) is the direct sum of a divisible group and a finite group. The proof uses a filtration on $K^*$.

(c) Using Ehrenfeucht’s condition we refine (b) to prove that if $K$ is infinite and $Char(K)$ is totally transcendental then $K^*$ is divisible.

(d) From (a) and (c) we conclude that if $K$ is infinite and $Char(K)$ is totally transcendental then $(K^*)^*$ is divisible for each finite algebraic extension $K_i$ of $K$. We then prove, by Galois theory, that if $(K^*)^*$ is divisible for every finite extension $K_i$ of a field $K$ of characteristic 0, then $K$ is algebraically closed. We conclude that if $K$ is an infinite field
of characteristic 0, and $Th(K)$ is totally transcendental, then $K$ is algebraically closed.

(e) The characteristic $p$ case is more involved. In this case we define a filtration of the additive group of $K$, using the endomorphism $a \mapsto x^p - x$. Using results from [13], we then show that if $Th(K)$ is totally transcendental, then either

(i) for each finite extension $K_i$ of $K$, $K_i$ has no cyclic extension of dimension $p$, or

(ii) for each finite extension $K_i$ of $K$, $K_i$ has exactly one cyclic extension of dimension $p$.

In case (i), using (a) and (c) and some Galois theory, we show that $K$ is algebraically closed.

The case (ii) includes the $(p)$-pseudo-finite fields of $Ax$. A special argument is used to prove that if $K$ is infinite and satisfies (ii) then $Th(K)$ is not totally transcendental.

2. Model-theoretic preliminaries. For the various facts assumed in this paper, one should consult Section 1 of [13], where there are references to the literature.

We will be working with first-order predicate logics $L$, with connectives $\neg$ and $\land$, quantifiers $\forall$ and $\exists$, identity symbol $=$, finitary relation-symbols and operation-symbols, and variables $v_0, v_1, \ldots, v_n, \ldots$.

The basic semantic notions are assumed known. It $\mathcal{M}$ will be the underlying set of $\mathcal{M}$.

If $a$ is an ordinal, we form a logic $L(a)$ by adding to $L$ distinct new individual constants $a_\eta$ for $\eta < a$. If $\mathcal{N}$ is an $L$-structure and $a \in [\mathcal{N}]^a$, then $(\mathcal{N}, a)$ is the obvious $L(a)$-structure where $a(\eta)$ corresponds to $a_\eta$ for each $\eta < a$.

If $L$ is an $L$-theory and $a$ is a cardinal, $L$ is $\kappa$-categorical if any two models of $L$ of cardinality $\kappa$ are isomorphic.

Suppose $L$ is countable, and $L$ is an $L$-theory. Then $L$ is totally transcendental if, for every model $M$ of $L$ and every $a \in [\mathcal{M}]^a$, $Th((\mathcal{M}, a))$ has at most $a$ complete extensions in $L(a+1)$.

Morley [14] proved that if $L$ is an $\omega_1$-categorical theory in a countable logic $L$, then $L$ is totally transcendental.

For proving that a theory is not totally transcendental, Ehrenfeucht's Condition is very useful. Suppose $\mathcal{M}$ is an $L$-structure, $X \subseteq [\mathcal{M}]$, and $\varphi(v_0, \ldots, v_n)$ is an $L$-formula whose free variables occur in the list $v_0, \ldots, v_n$. $\varphi$ is said to be connected over $X$ (relative to $\mathcal{M}$) if, for all distinct elements $a_0, \ldots, a_n$ of $X$, there is a permutation $\pi$ of $\{0, 1, \ldots, n\}$ such that $\langle a_{\pi(0)}, \ldots, a_{\pi(n)} \rangle$ satisfies $\varphi(v_0, \ldots, v_n)$ in $\mathcal{M}$. Then Theorem (Ehrenfeucht's Condition): Suppose $L$ is countable, $\mathcal{M}$ is an $L$-structure, $X$ is an infinite subset of $[\mathcal{M}]$ and $\varphi(v_0, \ldots, v_n)$ is an $L$-formula. Suppose both $\varphi$ and $\neg \varphi$ are connected over $X$. Then $Th(\mathcal{M})$ is not totally transcendental.

For a proof, see [14].

3. Let $L$ be the logic for field-theory, as described in the introduction. We construct fields as $L$-structures. If $K$ is a field, $Th(K)$ is the set of all $L$-sentences that hold in $K$.

We come now to the first, and most tedious, step of the proof.

We have to prove that if $K$ and $K_1$ are fields, with $Th(K)$ totally transcendental, and $K_1$, a finite algebraic extension of $K$, then $Th(K_1)$ is totally transcendental.

The basic idea is simple. Let $m$ be the dimension of $K_1$ over $K$. Let $x_0, \ldots, x_{m-1}$ be a basis for $K_1$ over $K$, where, without loss of generality, $x_0 = 1$. Each element of $K_1$ is uniquely of the form $\lambda_0 x_0 + \cdots + \lambda_{m-1} x_{m-1}$ where $\lambda_0, \ldots, \lambda_{m-1} \in K$. We define a map $\pi: K_1 \to K^m$ by $\pi(\lambda_0 x_0 + \cdots + \lambda_{m-1} x_{m-1}) = (\lambda_0, \ldots, \lambda_{m-1})$. $\pi$ is 1-1 and onto. Under $\pi$, addition and multiplication on $K_1$ induce operations $+ \otimes$ and $\oplus$ on $K^m$, as follows:

$$u \otimes v = \pi(\varphi(u, v)),$$

and

$$u \oplus v = \pi(\varphi(u, v)).$$

for all $u, v \in K^m$.

Clearly $\langle \lambda_0, \ldots, \lambda_{m-1} \rangle \otimes \langle \mu_0, \ldots, \mu_{m-1} \rangle = \langle \lambda_0 + \mu_0, \ldots, \lambda_{m-1} + \mu_{m-1} \rangle$. To give a corresponding definition for $\otimes$, we first define elements $\tau_{i,j}$ $(0 \leq i, j, k \leq m-1)$ of $K$ by:

$$\tau_{i,k} = \sum_{i=0}^{m-1} \rho_{k, i}. $$

Then $\langle \lambda_0, \ldots, \lambda_{m-1} \rangle \otimes \langle \mu_0, \ldots, \mu_{m-1} \rangle = \langle \lambda_0 \mu_0 + \cdots + \lambda_{m-1} \mu_{m-1} \rangle$, where

$$\delta_{k} = \sum_{i,j \leq m-1} \lambda_i \mu_j \tau_{i,j},$$

for $0 \leq k \leq m-1$.

In this way we can interpret every sentence about elements of $K_1$ as a sentence about $m$-tuples of elements of $K$.

We now give this a more precise metamathematical formulation. We preserve the notation of the preceding paragraph. Let $f$ be some fixed map of the set of all ordered triples $\langle i, j, k \rangle$, where $0 \leq i, j, k \leq m-1, 1, 1$ onto the set $\{0, 1, \ldots, m^2-1\}$. We are going to define, by induction, maps $\alpha_0, \ldots, \alpha_{m^2-1}$ from the set of terms of $L(m+1)$ to the set of terms of $L(m+1)$.\n
First we extend the $\Sigma$-notation to terms of $\mathcal{L}(a)$ for any $a$. Suppose $\gamma_0, \ldots, \gamma_{n+1}$ are terms of $\mathcal{L}(a)$. We define
$$\sum_{\gamma_0} \gamma_1 = \gamma_0$$
and
$$\sum_{\gamma_0} \gamma_1 = \left( \sum_{\gamma_0} \gamma_1 \right) + \gamma_{n+1}.$$
We need also a $\Sigma$-notation over pairs of subscripts. Thus, let
$$\gamma_{ij} (0 \leq i, j \leq a)$$
be terms of $\mathcal{L}(a)$. We define
$$\sum_{\gamma_{ij}} \gamma_{ij} = \sum_{\gamma_{ij}} \sum_{\gamma_{ik}} \gamma_{ij}.$$
Now we define $\pi_0, \ldots, \pi_{n-1}$ as follows:

(i) $\pi_k(0) = 0$ for $0 \leq k \leq m - 1$;

(ii) $\pi_k(1) = 1$, $\pi_k(1) = 0$ for $1 \leq k \leq m - 1$;

(iii) $\pi_k(\alpha) = \pi_{k+1}$ for $0 \leq k \leq m - 1$ and $\alpha < \omega$;

(iv) $\pi_k(\alpha + \beta) = \pi_{k+1}(\alpha) + \pi_{k+1}(\beta)$ for $0 \leq k \leq m - 1$ and $\alpha < \omega$;

(v) $\pi_k(\alpha) = \pi_{k+1}$ for $0 \leq k \leq m - 1$;

(vi) $\pi_k(\gamma_{ij}) = \pi_k(\gamma_1) + \pi_k(\gamma_2)$ for $0 \leq k \leq m - 1$;

(vii) $\pi_k(\gamma_{ij}) = \sum_{\gamma_{ij}} \pi_k(\gamma_{ij}) \cdot \pi_{k+1}(\gamma_{ij}).$

We are using the constants $\pi_k(\gamma_{ij})$ to correspond to the field elements $\pi_k(\alpha)$. The map $\gamma_{ij} \mapsto \pi_k(\gamma_{ij})$, $\cdots, \pi_{n-1}(\gamma_{ij})$ should be thought of as a formal counterpart of the map $\pi$ described earlier.

Next we define a map $\varphi \mapsto \varphi$ from formulas of $\mathcal{L}(a)$, to formulas of $\mathcal{L}(a+1)$, by the following induction:

(i) If $\varphi$ is $\gamma_1 \varphi$, then $\varphi$ is $\gamma_1 \varphi_{i_1} \cdots \varphi_{i_n} \varphi_{i_n}$;

(ii) If $\varphi$ is respectively $\neg \varphi$, $\varphi \varphi$, then $\varphi$ is respectively $\neg \varphi$, $\varphi$;

(iii) If $\varphi$ is respectively $\varphi \varphi$, $\varphi \varphi$, then $\varphi$ is respectively $\varphi \varphi$, $\varphi \varphi$;

(iv) If $\varphi$ is respectively $\varphi \varphi$, $\varphi \varphi$, then $\varphi$ is respectively $\varphi \varphi$, $\varphi \varphi$.

If $\tau$ is a set of formulas of $\mathcal{L}(a)$ then we define $T \vdash \varphi \in \tau$.

Next we define a map $f \mapsto f$ from $|K|^a$ to $|K'|^a$, by:

(i) If $f = t(x_1, \ldots, x_a)$ then $f(n) = \varphi(n)$;

(ii) If $f = t(a^n n + m + k)$, then $m$ and $k$ are uniquely determined, and $f(n) = \varphi(\varphi(\varphi(\varphi(\varphi(n)))))$.

Finally, we define a map $f \mapsto f$ from $|K|^a$ to $|K'|^a$, using conditions (i) and (ii) of the preceding paragraph, and in addition

(iii) If $f = t(a^n + m + k)$, then $f(n + k) = \varphi(n) + k$.

The following basic lemma can now be proved by induction, and we omit the proof.

**Lemma 1.** Let $K$, $K'$ be as above, and let $\varphi$ be a sentence of $\mathcal{L}(a)$. If $s \in |K|^a \cup |K'|^a$ and $|K|, s \models \varphi$, then $|K|, s \models \varphi$.

**Lemma 2.** Suppose $K$, $K'$ are fields, with $K$ a finite algebraic extension of $K'$, of dimension $m$. Suppose that $s_1 \in |K'|^m$ and $T(K, s_1)$ has uncountably many complete extensions in $\mathcal{L}(a+1)$. Then there exists $s \in |K'|^m$ such that $T(K, s_1)$ has uncountably many complete extensions in $\mathcal{L}(a+1)$.

**Proof.** We adopt the notation of the discussion preceding Lemma 1. Let $T_0 \vdash \varphi$ be a well-ordering of the complete extensions of $T(K, s_1)$ in $\mathcal{L}(a+1)$, where $\lambda$ is an uncountable cardinal.

Fix $\eta < \lambda$. Let

$$\{\eta(\alpha_0), \ldots, \eta(\alpha_n)\}$$
be a finite subset of $\mathcal{T}_0$. Since $T_0$ extends $T(K, s_1)$, it follows that

$$\langle K | \eta \rangle \models \langle \eta(\alpha_0), \ldots, \eta(\alpha_n) \rangle,$$

and therefore there exists $s_1 \in |K'|^m$, extending $s_1$, such that

$$\langle K | s_1 \rangle \models \eta(\alpha_0) \land \eta(\alpha_1) \land \ldots \land \eta(\alpha_n).$$

Therefore, by Lemma 1,

$$\langle K | s_1 \rangle \models \eta(\alpha_0) \land \eta(\alpha_1) \land \ldots \land \eta(\alpha_n) \land \ldots \land \eta(\alpha_n).$$

Therefore every finite subset of $\mathcal{T}_0$ is consistent with $T(K, s_1)$, since $s_1$ extends $s_1$. Thus $T_0$ can be embedded in a complete extension of $T(K, s_1)$ in $\mathcal{L}(a+1)$.

Suppose $\eta \neq \lambda$ and $\eta \neq \lambda$. Then there exists $\varphi$ such that $\varphi \in \mathcal{T}_0$ and $\varphi \notin \mathcal{T}_0$. Thus $\varphi \notin \mathcal{T}_0$ and $\varphi \notin \mathcal{T}_0$, so $\mathcal{T}_0$ and $\mathcal{T}_0$ have no common complete extension.

We conclude that $T(K, s_1)$ has at least $\lambda$ complete extensions in $\mathcal{L}(a+1)$. Put $s = s_1$ and the lemma is proved.

The reason for the next lemma is that, in order to show that $T(K, s_1)$ is totally transcendental, we have to look at arbitrary structures $(K| s_1)$ where $K = K$ and $s_1 \in |K'|^m$.

**Lemma 3.** Suppose $K$, $K'$ are fields, with $K$ a finite algebraic extension of $K'$, of dimension $m$. Suppose $K = K$ and $s_1 \in |K'|^m$.

Then there exists $K''$, $K''$ and $s_1$ such that

(i) $K'' = K$, $K'' = K'$, and $s_1 \in |K''|^m$;

(ii) $|K'| \models (K_0, \ldots, K_0)$.

**Proof.** Select a basis $a_0, \ldots, a_{m-1}$ for $K$, over $K$, with $a_0 = 1$.

We augment the logic $\mathcal{L}$ by adjoining individual constants $b_0, \ldots, b_{m-1}$, and a 1-ary predicate-symbol $L$. Let $\mathcal{L}_1$ be the resulting logic. We construe $\mathcal{L}_1$-structures as $\mathcal{L}$-structures with distinguished elements corresponding to $b_0, \ldots, b_{m-1}$, and with a distinguished subset corresponding to $L$. We
are particularly interested in those $L_2$-structures where the underlying $L$-structure is a field, the distinguished set forms a subfield, and the distinguished elements are a basis for the field over the subfield. It is obvious that this class of $L_2$-structures is an $E_0$. Our canonical example of an $E_2$-structure has $K_i$ as its underlying field, $[K]$ corresponding to $L$, and $z_i (0 < i < m - 1)$ corresponding to $b_i$ $(0 < i < m - 1)$. We denote this structure by $(K_1, K, z_1, \ldots, z_{n-1})$.

Let $A$ be the following set of $L(\omega)$-sentences:

$Th([K_1, z_1]) = Th([K_1, K, z_1, z_2, \ldots, z_{n-1}])$.

We claim $A$ is satisfiable. By the Compactness Theorem it suffices to prove that every finite subset of $A$ is satisfiable. In fact we show that if $A_0$ is a finite subset of $Th([K_1, z_1])$ then there exists $a_0 \in [K_1]^\omega$ such that $(K_1, a_0) \models A_0$. From this it follows that every finite subset of $A$ is satisfiable, since $(K_1, K, z_1, z_2, \ldots, z_{n-1}) \models Th([K_1, K, z_1, z_2, \ldots, z_{n-1}])$.

So, let $A_0$ be a finite subset of $Th([K_1, z_1])$. Select $r < \omega$ such that if $z_i$ occurs in a member of $A_0$ then $k < r$. Write $A_0$ as $(\{\phi_i(a_0, \ldots, a_r) \mid i \leq r\}) = Th([K_1, z_1, z_2, \ldots, z_{n-1}])$. Then $(K_1, z_1) \models \phi_i(a_0, \ldots, a_r) \models \phi_i(a_0, \ldots, a_r)$, so $(K_1, z_1) \models Th([K_1, z_1, z_2, \ldots, z_{n-1}])$.

But $K_1'' = K_1$, so $K_1'' \models \exists y (\phi_0 y \ldots, \ldots, \phi_r y \ldots)$.

Therefore there exists a function $s$ from $\{0, \ldots, r\}$ to $K_1$ such that $(K_1, z_1) \models \phi_i(a_0, \ldots, a_r) \models \phi_i(a_0, \ldots, a_r)$, i.e., $(K_1, z_1) \models A_0$.

Now let $A_0$ be any extension of $A_0$ to an element of $[K_1]^\omega$, and clearly $(K_1, A_0) \models A_0$.

We conclude that $A$ is satisfiable. Let $\mathcal{A}$ be an $L(\omega)$-structure satisfying $A$. Then $\mathcal{A}$ is of the form $(N, s')$, where $N$ is an $L_2'$-structure and $s' \in [N]^\omega$.

$s' \in [K_1]^\omega$ such that $K_1'' = K_1$, $K_1' = K_1$ $(K_1', s' \in [K_1]^\omega)$, and $K_1$ is a finite algebraic extension of $K_1'$, of dimension $n$. Thus $Th([K_1, z_1]) = Th([K_1, z_1])$, so $(K_1', z_1')$ has uncountably many complete extensions in $L(\omega+1)$. Now we apply Lemma 2 to the data $E_1, K_1', z_1'$. We conclude that there exists $s'' \in [K_1]^\omega$ such that $Th([K_1', s''])$ has uncountably many complete extensions in $L(\omega+1)$. This proves the lemma.

The next lemma is probably well-known (cf. [14], proof of 5.7).

**Lemma 5.** Suppose $\mathcal{U}$ is an arbitrary countable logic, $\mathcal{V}$ an $L$-theory, and $1 < m < n$. Then $\mathcal{V}$ is totally transcendental if and only if for every model $\mathcal{A}$ of $\mathcal{V}$ and every $s \in [\mathcal{A}]^\omega$, $Th([\mathcal{A}, s])$ has at most $\omega$ complete extensions in $L(\omega+m)$.

**Proof.** Sufficiency is clear, since distinct complete extensions of $Th([\mathcal{A}, s])$ in $L(\omega+1)$ extend to distinct complete extensions of $Th([\mathcal{A}, s])$ in $L(\omega+m)$.

Necessity is proved by induction on $m$.

The result is trivial for $m = 1$. Suppose we have the result for $m < k$. Now suppose $\mathcal{A}$ is a model of $\mathcal{V}$ and $s \in [\mathcal{A}]^\omega$, and $Th([\mathcal{A}, s])$ has uncountably many complete extensions in $L(\omega+k+1)$. Since $\mathcal{V}$ is totally transcendental, our induction hypothesis tells us that $Th([\mathcal{A}, s])$ has at most $\omega$ complete extensions in $L(\omega+k)$. It follows that there exists $\mathcal{A}_1$, a complete extension of $Th([\mathcal{A}, s])$ in $L(\omega+k)$, such that $\mathcal{A}_1$ has uncountably many complete extensions in $L(\omega+k+1)$. Let $\mathcal{A}_0$ be a model of $\mathcal{V}_1$, and for $n < \omega + k$ let $\mathcal{A}_n$ be the element of $[\mathcal{A}_0]$ corresponding to $n$. Let $\mathcal{A}_0$ be the $\mathcal{L}$-structure got from $\mathcal{A}_0$ by forgetting the structure corresponding to the constants $e_i$. Then $[\mathcal{A}_0] = [\mathcal{A}_0]_0$. Since $\mathcal{A}_1$ extends $Th([\mathcal{A}_0, s])$ it is clear that $\mathcal{A}_0 = \mathcal{A}_0$. We define $s_\omega = [\mathcal{A}_0]_\omega$ by:

$s_\omega(n) = s_\omega + n$ for $0 \leq n < \omega$

and $s_\omega(\omega + n) = s_n$ for $n > 0$.

Since $\mathcal{V}_1$ is complete, and has uncountably many complete extensions in $L(\omega+k+1)$, it follows that $Th([\mathcal{A}_0, s])$ has uncountably many complete extensions in $L(\omega+k+1)$, contradicting the assumption that $\mathcal{V}_1$ is totally transcendental. We conclude that $Th([\mathcal{V}_1, s])$ has at most $\omega$ complete extensions in $L(\omega+m)$.

This completes the inductive step and the proof.

**Corollary.** Suppose $\mathcal{U}$ is countable, $\mathcal{V}$ is a totally transcendental $L$-theory, $n < \alpha$, and $\mathcal{A}_1$ is an extension of $\mathcal{U}$ in $L(n)$. Then $\mathcal{V}_1$ is totally transcendental.

**Proof.** Assume the hypotheses. Let $\mathcal{V}_1 = L(n)$. If $\mathcal{V}_1$ has uncountably many complete extensions in $L(n+1)$, then $\mathcal{V}_1$ has uncountably many complete extensions in $L(n+\omega+1)$, contradicting Lemma 5.
The following lemma, the goal of this section is of basic importance for us because it enables us to use Galois theory on our problem.

**Lemma 6.** Suppose \( K, K' \) are fields, with \( K \) a finite algebraic extension of \( K \). Then if \( \text{Th}(K) \) is totally transcendental, \( \text{Th}(K') \) is totally transcendental.

**Proof.** Let \( m \) be the dimension of \( K \) over \( K \). Suppose \( \text{Th}(K) \) is not totally transcendental. Then, by Lemma 4, there exists \( K'' = K \) and \( s'' \in \text{Gal}(K') \) such that \( \text{Th}(K'', s'') \) has uncountably many complete extensions in \( \mathcal{L}(\omega + m) \). Then, by Lemma 5, \( \text{Th}(K) \) is not totally transcendental. This proves the lemma.

**Remark.** The following is an example of fields \( K \) and \( K' \), with \( K' \) a finite algebraic extension of \( K \), \( \text{Th}(K) \) totally transcendental and \( \text{Th}(K') \) not totally transcendental. Take \( K \) as the field of real numbers and \( K' \) as the field of complex numbers. See [14] for proofs that \( \text{Th}(K) \) is totally transcendental and \( \text{Th}(K) \) is not totally transcendental. By Theorem 1 of this paper, and the celebrated theorem of Artin–Schreier that the only fields of finite codimension in their algebraic closure are real-closed or algebraically closed, the above is the only possible example (up to elementary equivalence).

4. The second step of the proof uses the notion of definable filtration, as in [13]. Our only modification of the treatment in [13] is that we use multiplicative notation rather than additive notation.

If \( K \) is a field, let \( K^* \) be the group of non-zero elements of \( K \) under multiplication. Then \( K^* \) is abelian. If \( n \) is a positive integer, let \( (K^*)^n \) be the subgroup of \( K^* \) consisting of \( n \)-th powers of elements of \( K^* \). (Since we have no further use for the cardinality of the statement, no confusion should arise.) Clearly, when \( n \) divides \( m \), \( (K^*)^m \subset (K^*)^n \). Thus \( (K^*)^m \) is a definable filtration of \( K \). Let \( (K^*)_m = \bigcap_{n \leq m} (K^*)^n \). Suppose \( K \) is \( \omega \)-saturated. Then \( K^* \) is \( \omega \)-saturated, and, by 3.5 of [13], \( (K^*)_m \) is divisible, and so \( (K^*)_m \) is a direct summand of \( K^* \). Select a subgroup \( H \) of \( K^* \) such that \( K^* = (K^*)_m \oplus H \). Then \( (K^*)_m \subset H \subset \mathcal{L}(\omega) \). Now suppose \( \text{Th}(K) \) is totally transcendental. By Lemma 3 of [13], there exists an integer \( m \) such that \( H \subset (K^*)_m = H \subset (K^*)_n \). It follows that

\[
H \cap (K^*)_m = H \cap (K^*)_n = \mathcal{L}(\omega).
\]

Therefore for every \( a \) in \( H \), \( a^m = 1 \), so \( H \) is a group of \( m \)-th roots of unity, and so is finite, since \( K \) is a field.

We have proved that if \( K \) is an \( \omega \)-saturated field and \( \text{Th}(K) \) is totally transcendental then \( K^* \) is of the form \( D \oplus H \) where \( D \) is divisible and \( H \) is finite. Since any field is elementarily equivalent to an \( \omega \)-saturated field, we may apply Lemma 2 of [13] to conclude:

**Corollary.** Suppose \( K \) is a field and \( \text{Th}(K) \) is totally transcendental. Then \( K^* \) is of the form \( D \oplus H \) where \( D \) is divisible and \( H \) is finite.

We now use the Ehrenfeucht Condition to get a refinement of Lemma 7.

**Lemma 8.** Suppose \( K \) is a field with \( \text{Th}(K) \) totally transcendental. Suppose \( K^* = D \oplus H \) where \( D \) is divisible and \( H \) is finite. Then \( H = \{0\} \) or \( D = \{0\} \).

**Proof.** Assume the hypotheses of the lemma, and assume \( H \neq \{0\} \) and \( D \neq \{0\} \). Then \( D \) is infinite, so \( K \) is infinite.

Since \( H \) is a finite subgroup of \( K^* \), \( H \) is cyclic [12]. Suppose \( H \) has \( n \) elements, and is generated by \( \zeta \). Then \( n > 1 \) and \( \zeta^2 = 1 \) for all \( \zeta \) in \( H \). Since \( D \) is divisible, \( D \subset (K^*)^n \). Since \( K^* = D \oplus H \) and \( \zeta^2 = 1 \) for all \( \zeta \) in \( H \), we conclude that \( (K^*)^n \subset D \), so \( (K^*)^n = D \). Thus \( K^* = (K^*)^n \oplus H \). Therefore if \( y \in K^* \), there exists \( k \) with \( 0 < k < n-1 \) such that \( y^{-1} \zeta (K^*)^n \). On the other hand \( \zeta \notin (K^*)^n \).

Now we define an \( n \)-ary relation \( R \) on \( K \) by:

\[
\langle x_0, \ldots, x_{n-1} \rangle \in R \text{ if and only if either}
\]

\[
x_0 + x_1 \zeta + \ldots + x_{n-1} \zeta^{n-1} \in (K^*)^n
\]

or

\[
x_0 + x_1 \zeta + \ldots + x_{n-1} \zeta^{n-1} = 0.
\]

Suppose \( x_0, \ldots, x_{n-1} \) are distinct elements of \( K \). We look at two cases.

**Case 1.** \( \langle x_0, \ldots, x_{n-1} \rangle \notin R \). Then \( x_0 + x_1 \zeta + \ldots + x_{n-1} \zeta^{n-1} \neq 0 \) so

\[
x_0 + x_1 \zeta + \ldots + x_{n-1} \zeta^{n-1} \in (K^*)^n.
\]

Therefore there exists \( k \) with \( 0 < k < n-1 \) such that

\[
\zeta^{-k}(x_0 + x_1 \zeta + \ldots + x_{n-1} \zeta^{n-1}) \in (K^*)^n.
\]

Therefore

\[
x_0 + x_1 \zeta^{k+1} + \ldots + x_{n-1} \zeta^{n-1} + x_{n-1} \zeta^{k+1} + \ldots + x_0 \zeta^{n-1} \in (K^*)^n.
\]

Therefore

\[
\langle x_0, x_1, x_{n-1}, x_0, x_1, \ldots, x_{n-1} \rangle \in R.
\]

Therefore there exists a permutation \( \pi \) of \( \{0, \ldots, n-1\} \) such that

\[
\langle x_{\pi(0)}, \ldots, x_{\pi(n-1)} \rangle \in R.
\]

**Case 2.** \( \langle x_0, \ldots, x_{n-1} \rangle \in R \). We have two subcases.

**Subcase 1.** \( x_0 \neq x_1 \zeta \). We have two subcases.

Therefore

\[
\langle x_0, x_1, \ldots, x_{n-1} \rangle \notin R.
\]

Therefore the element \( \xi \) is an \( (n+1) \)-ary relation on \( K \) which is elementarily equivalent to \( (K^*)^n \).
Also, since $\zeta \notin (K^*)^n$, 
\[ \zeta \cdot (a_0 + a_1 \zeta + \ldots + a_{n-1} \zeta^{n-1}) \in (K^*)^n. \]

Now 
\[ \zeta \cdot (a_0 + a_1 \zeta + \ldots + a_{n-1} \zeta^{n-1}) = a_{n-1} + a_0 \zeta + \ldots + a_{n-1} \zeta^{n-1}. \]

Therefore 
\[ a_{n-1} + a_0 \zeta + \ldots + a_{n-1} \zeta^{n-1} \neq 0 \quad \text{and} \quad a_{n-1} + a_0 \zeta + \ldots + a_{n-1} \zeta^{n-1} \notin (K^*)^n. \]

Therefore 
\[ \langle a_{n-1}, a_0, \ldots, a_{n-1} \rangle \notin R. \]

Therefore there exists a permutation $\pi$ of $\{0, \ldots, n-1\}$ such that 
\[ \langle a_{n-1}, a_n, \ldots, a_{n-1} \rangle \notin R. \]

Subcase 2. $a_0 + a_1 \zeta + \ldots + a_{n-1} \zeta^{n-1} = 0$. Now $a_0, \ldots, a_{n-1}$ are distinct, and $\zeta \neq 1$, so 
\[ a_0 + a_1 \zeta + \ldots + a_{n-1} \zeta^{n-1} \neq 0. \]

Therefore 
\[ a_0 + a_1 \zeta + a_2 \zeta^2 + \ldots + a_{n-1} \zeta^{n-1} \neq 0. \]

Now, if $a_0 + a_1 \zeta + a_2 \zeta^2 + \ldots + a_{n-1} \zeta^{n-1} \notin (K^*)^n$, then 
\[ \langle a_0, a_1, a_2, \ldots, a_{n-1} \rangle \notin R. \]

On the other hand, if $a_0 + a_1 \zeta + a_2 \zeta^2 + \ldots + a_{n-1} \zeta^{n-1} \notin (K^*)^n$, then 
\[ \langle a_0, a_1, a_2, \ldots, a_{n-1} \rangle \notin R, \]

and the argument of Subcase 1 proves that 
\[ \langle a_{n-1}, a_0, a_1, \ldots, a_{n-1} \rangle \notin R. \]

In both cases there exists a permutation $\pi$ of $\{0, \ldots, n-1\}$ such that 
\[ \langle a_{n-1}, a_0, \ldots, a_{n-1} \rangle \notin R. \]

It is clear that $R$ is first-order definable using the constants $\zeta^k$ ($0 \leq k \leq n-1$). We define a function $\sigma$ from $\{0, \ldots, n-1\}$ to $[K]$ by: 
\[ \sigma(k) = \zeta^k \quad \text{for} \quad 0 \leq k \leq n-1. \]

Then, by the corollary to Lemma 5, $\operatorname{Th}(K, \sigma)$ is totally transcendental.

Let $\varphi(a_0, \ldots, a_{n-1})$ be a formula of $\mathcal{L}(n)$ defining the relation $R$ in the structure $([K], \sigma)$. It is clear how to write down such a formula.

By what we have proved above about $R$ it is clear that $\varphi$ and $\neg \varphi$ are connected over $[K]$. But $[K]$ is infinite, since $D \neq \{1\}$. Now Ehrenfeucht's Condition implies that $\operatorname{Th}(K, \sigma)$ is not totally transcendental. This gives a contradiction.

It follows that either $D = \{1\}$ or $H = \{1\}$.

Corollary. Let $K$ be a field and suppose $\operatorname{Th}(K)$ is totally transcendental. Then either $K$ is finite or $K^*$ is divisible.
THEOREM 1 (Characteristic 0 case). Suppose \( K \) is a field of characteristic 0. Then \( Th(K) \) is totally transcendental if and only if \( K \) is algebraically closed.

Proof. Sufficiency is proved in [14]. Suppose \( K \) is a field of characteristic 0 and \( Th(K) \) is totally transcendental. Then \( K \) is infinite. By Lemma 6, \( Th(K) \) is totally transcendental, for each finite extension \( K_0 \) of \( K \). Applying the Corollary to Lemma 8, we conclude that \( (K_0)^* \) is divisible for every finite extension \( K_0 \) of \( K \). By Lemma 9, \( K \) is algebraically closed. This proves the theorem.

6. Before getting into the characteristic \( p \) case, we pause to show that some important theories of fields are not totally transcendental. The proofs use the Corollary to Lemma 8, but not Lemma 6, and so are significantly simpler than the proof of Theorem 1.

Separably closed fields. Let \( K \) be a field which is separably closed but not algebraically closed. Then \( K \) is of characteristic \( p \) for some prime \( p \). Also, \( K \) is not perfect, so \( K \) is infinite and \( K^p \neq (K^p)^p \). Thus \( K^p \) is not divisible, so \( Th(K) \) is not totally transcendental.

Quasi-finite fields. Let \( K \) be a quasi-finite field, i.e. a perfect field with exactly one extension of each degree. (Actually, all we use is that \( K \) has an extension of degree 2.) We will assume that the characteristic of \( K \) is not 2.

Let \( K_0 \) be an extension of \( K \) of degree 2. Then obviously (or by Fact 2) there exist \( \xi, a \) such that \( K_1 = K(\xi), \xi^2 = \xi = a \in K_1 \), and \( a \) has no square root in \( K \). Thus \( K^p \neq (K^p)^p \), so \( K^p \) is not divisible.

Therefore if \( K \) is infinite \( Th(K) \) is not totally transcendental.

Remark. The case when the characteristic of \( K \) is 2 will have to wait till we prove the general case of Theorem 1.

7. For the characteristic \( p \) case of Theorem 1, the argument resembles that for the characteristic 0 case, but with some extra details. We use the Kummer theory as before, but also the Artin-Schreier theory. In particular we need:

Fact 3. Suppose \( K \) is a field of prime characteristic \( p \), and suppose \( K_1 \) is a Galois extension of \( K \) with Galois group \( \mathbb{Z}(p) \). Then there exist \( \xi, a \) such that \( K_1 = K(\xi), \xi^p = \xi = a \in K_1 \), and there is no \( \beta \) in \( K \) such that \( \beta^p = \beta = a \).

LEMMA 10. Suppose \( K \) is an infinite field of prime characteristic \( p \), such that \( Th(K) \) is totally transcendental. Then for each \( y \) in \( K \) there exists an \( a \) in \( K \) such that \( x^p = x = y \).

The proof of Lemma 10 takes some time, and we postpone it. Right now we show how to complete the proof of Theorem 1, modulo Lemma 10.

LEMMA 11. Suppose \( K \) is a field of prime characteristic \( p \) such that for every finite algebraic extension \( K_0 \) of \( K \) we have:

(i) \( (K_0)^* \) is divisible;

(ii) For each \( y \) in \( K_0 \) there exists an \( x \) in \( K_0 \) such that \( x^p = x = y \).

Then \( K \) is algebraically closed.

Proof. We prove first that for each prime \( q \neq p \) \( K \) has \( q \) distinct \( q \)-th roots of unity. Suppose not, and let \( r \) be the least prime \( \neq p \) such that \( K \) has fewer than \( r \)-th roots of unity. Let \( K_0 \) be a splitting field for \( \mathbb{F} \) over \( K \). Then \( K_0 \neq K \). Let \( G \) be the Galois group of \( K_0 \) over \( K \). Then by Fact 1 \( G \) is cyclic and of order \( \leq r-1 \). Let \( q \) be a prime dividing the order of \( G \). Then \( q < r \). By the converse of Lagrange’s Theorem, \( G \) has a subgroup \( G_q \) isomorphic to \( \mathbb{Z}(q) \). Let \( K_0 \) be the fixed field of \( G_q \). Then the Galois group of \( K_0 \) over \( K \) is \( K_0 \) by the Fundamental Theorem of Galois Theory.

Suppose first \( q = p \). Then by Fact 3 there exist \( \xi, a \) such that \( K_1 = K(\xi), \xi^p = \xi = a \in K_1 \), and there is no \( \beta \) in \( K_0 \) such that \( \beta^p = \beta = a \). But this contradicts assumption (i). Therefore \( q \neq p \).

Thus \( q \neq p \) and \( q < r \). By the minimality of \( r \), \( K \) has \( q \) distinct \( q \)-th roots of unity. Therefore by Fact 2 there exist \( \xi, a \) such that \( K_1 = K(\xi), \xi^p = \xi = a \in K_1 \), and \( a \) has no \( q \)-th root in \( K_0 \). Thus \( (K_0)^* \neq (K^p)^p \), so \( (K_0)^* \) is divisible, contradicting assumption (i).

We conclude that for each prime \( q \neq p \) \( K \) has \( q \) distinct \( q \)-th roots of unity.

Now suppose \( K \) is not algebraically closed. Then, by the same argument as in the proof of Lemma 9, there exist a prime \( q \), and finite extensions \( K_1, K_2 \) of \( K \) such that \( K_1 \) is a Galois extension of \( K_2 \) with Galois group \( \mathbb{Z}(q) \).

Suppose \( q \neq p \). Then by Fact 3 there exist \( \xi, a \) such that \( K_1 = K(\xi), \xi^p = \xi = a \in K_1 \), and there is no \( \beta \) in \( K_3 \) such that \( \beta^p = \beta = a \). This contradicts assumption (i).

Suppose \( q \neq p \). Then \( K \) has \( q \) distinct \( q \)-th roots of unity. Then by Fact 3 there exist \( \xi, a \) such that \( K_1 = K(\xi), \xi^p = \xi = a \in K_1 \), and \( a \) has no \( q \)-th root in \( K_2 \). Thus \( (K_2)^* \neq (K^p)^p \), so \( (K_2)^* \) is not divisible, contradicting assumption (i).

We conclude that \( K \) is algebraically closed.

THEOREM 1 (Characteristic \( p \) case). Suppose \( K \) is a field of prime characteristic \( p \). Then \( Th(K) \) is totally transcendental if and only if \( K \) is finite or algebraically closed.

Proof. Sufficiency for algebraically closed \( K \) is proved in [14], and is trivial for finite \( K \).
Necessity. Suppose $K$ is infinite and $Tb(K)$ is totally transcendental. Then by Lemma 6 $Tb(K_n)$ is totally transcendental for each finite extension $K_n$ of $K$. By the Corollary to Lemma 8, $(K_n)^*$ is divisible for each finite extension $K_n$ of $K$. Furthermore, by Lemma 10, for each $y$ in $K_n$, there is an $x$ in $K_n$ such that $x^n - x = y$. By Lemma 11, $K$ is algebraically closed. This completes the proof.

We put together the parts of Theorem 1 to get:

**Theorem 1.** Suppose $K$ is a field. Then $Tb(K)$ is totally transcendental if and only if $K$ is finite or algebraically closed.

8. We have now to prove Lemma 10. We use the technique of definable filtrations from [13]. In the present application, we work with additive notation.

In this section all fields are of prime characteristic $p$. Let $F_p$ be the finite field of $p$ elements. Any field of characteristic $p$ can be construed as a vector-space over $F_p$. If $V$ is any vector-space over $F_p$, we write $dim V$ for the dimension of $V$.

If $K$ is a field of characteristic $p$, let $\text{Abs}(K)$ be the field of absolute numbers of $K$, i.e. the algebraic closure in $K$ of $F_p$.

We define $\text{Add}(K)$ as the underlying additive group of $K$. We will define a filtration of $K$ consisting of subgroups of $\text{Add}(K)$.

We define a map $\tau: \text{Add}(K) \to \text{Add}(K)$ by:

$$\tau(x) = x^n - x \quad \text{for } x \in \text{Add}(K).$$

**Fact 4.** $\tau$ is a homomorphism with kernel $F_p$.

For a proof, see [12].

We define subgroups $H_m$, $(m \leq n)$, of $\text{Add}(K)$ by:

$$H_0 = \text{Add}(K)$$

$$H_{m+1} = \tau[H_m] \quad \text{for } m \geq 0.$$

Clearly $H_n \subseteq H_m$, whence by induction $H_{m+n} \subseteq H_m$ for all $m, n$, so $\langle H_{m+n} \rangle_{m+n}$ is a filtration of $K$. It is clear that each $H_m$ is definable, so $\langle H_m \rangle_{m+n}$ is a definable filtration. We note also that if $H_{m+1} = H_m$, for some $m$, then $H_n = H_m$ for all $n \geq m$.

There are three possibilities:

(A) $H_n = H_{n+1} = \ldots = H_m = \ldots$

(B) There exists a $k \geq 0$ such that $H_n \neq H_{n+1} \neq \ldots \neq H_{n+k} = H_{n+k+1} = \ldots$

(C) For all $m, n$ with $m \neq n$, $H_m \neq H_n$.

We now analyze these possibilities, and show that if $Tb(K)$ is totally transcendental then (B) and (C) cannot occur.

**Case C.** Suppose $Tb(K)$ is totally transcendental and Case C holds.

Then we have the strictly descending chain

$$H_n \supset H_{n+1} \supset \ldots \supset H_m \supset H_{m+1} \supset \ldots$$

of subgroups of $\text{Add}(K)$. We can strum all subgroups of $\text{Add}(K)$ as vector-spaces over $F_p$. Define $H_m$ as $\cap H_n$, then $H_m$ is a subspace of $\text{Add}(K)$.

Choose a subspace $A$ complementary to $H_m$ in $\text{Add}(K)$. Then $\text{Add}(K) = H_m \oplus A$. Since $Tb(K)$ is totally transcendental it follows by the corollary to Lemma 3 of [13] that there exists a such that

$$H_m \cap A = H_m \quad \text{for all } m \geq n.$$ 

Then $H_n \cap A = H_m \cap A = (0)$. It follows that $\text{Add}(K) = H_m \oplus A$, since $H_n \subseteq H_m$.

We claim that $H_n = H_m$. Suppose not. Then there exists an element $x$ which is in $H_n$ but not in $H_m$. Since $\text{Add}(K) = H_m \oplus A$, there exist $y, z$ such that $y \in H_m, z \in A, y + z \in H_n$. Then $y \in H_n \cap A = (0)$. But $x - y = x - y \in A$. Since $H_n \cap A = (0), x = y$, contrary to assumption. Therefore $H_m = H_n$.

Therefore $H_m = H_n$ for all $m \geq n$. But this contradicts our assumption that $H_m \neq H_n$ if $m \neq n$.

Thus, if $Tb(K)$ is totally transcendental, Case C cannot occur.

**Case B.** Suppose $Tb(K)$ is totally transcendental and Case B holds. Then there exists $k$ such that

$$H_n \neq H_{n+1} \neq \ldots \neq H_{n+k} = H_{n+k+1} = \ldots$$

First some notation. For $n \geq 1$ we define maps $\tau_n: \text{Add}(K) \to \text{Add}(K)$ by:

$$\tau_n(x) = x^n - x \quad \text{for all } x;$$

$$\tau_{n+1}(x) = \tau(\tau_n(x)) \quad \text{for all } x, \text{ and all } m \geq 1.$$ 

Then clearly $\tau_n$ is a homomorphism of $\text{Add}(K)$ to $\text{Add}(K)$, and $\tau_n[H_m] = H_n$.

For each integer $n \geq 1$ let $F_p^n$ be the finite field of cardinality $p^n$. We know that $F_p \subseteq K$, but we cannot determine $F_p^n \cap K$ immediately, when $n > 1$.

Consider the equation (over $F_p$): $\tau_n(x) = 0$.

This equation has degree $p^{n+1}$. It is a simple consequence of the basic theory of finite fields [12] that the equation has $p^n$ roots in $F_{p^{n+1}}$, i.e. that $\tau_{n+1}(x)$ splits into linear factors over $F_{p^{n+1}}$. Moreover, if $F_p^n$ contains a root of the above equation, that root is a member of $F_{p^{n+1}}$.

It follows that

$$\langle x \rangle \cap \tau_{n+1}(x) = (0) \subseteq K \cap F_{p^{n+1}}.$$ 

We define the field $K_n$ by $K_n = K \cap F_{p^{n+1}}$. Then $K_n$ is a finite subfield of $K$.
$K$, is of course closed under $\tau$. We consider the quotient vector-space (over $F_p$) $K_\theta / [K_\theta]$. We claim that $\dim_{F_p} K_\theta / [K_\theta] = 1$.

Firstly, since $K_\theta$ is a finite field it has a unique cyclic extension of degree $p$.

Next we have the following important fact, established in [1], pp. 203-4.

Fact 5. Suppose $F$ is any field of characteristic $p$, and $\tau: F \rightarrow F$ is given by $\tau(x) = x^p - x$. Then the number of cyclic extensions of $F$ of degree $p$ is equal to $\dim_{F_p} F / [F_\tau]$.

From Fact 5 and the preceding paragraph we conclude that $\dim_{F_p} K_\theta / [K_\theta] = 1$.

Now we claim $\dim_{F_p} K / [K] = 1$.

Firstly, since $H_\theta \neq H_\theta$ and $K \neq [K]$ so $\dim_{F_p} K / [K] \geq 1$.

Next, let $\lambda$ be an arbitrary member of $K$. Then $\tau_{\theta}(\lambda) \in H_{p+1} = H_{p+1} = \tau_{\theta}(H_{p+1})$. Therefore there exists $y$ in $K$ such that $\tau_{\theta}(\lambda) = \tau_{\theta}(y)$. Therefore $\tau_{\theta}(\lambda - \tau_{\theta}(y)) = 0$. Therefore $\lambda - \tau_{\theta}(y)$ is a root of the equation $\tau_{\theta}(\lambda) = 0$.

Therefore $\lambda - \tau_{\theta}(y) \in K \cap F_{p+1} = K_\theta$. Since $\dim_{F_p} K_\theta / [K_\theta] = 1$, we can select a such that $a \in K_\theta$, $a \neq \tau_{\theta}(K_\theta)$, and for all $u$ in $K$, there exists $t$ with $0 < t < p$ such that $\lambda = u - \tau_{\theta}(y)$ is a root of $\tau_{\theta}(K_\theta)$. Therefore there exists $\tau$ with $0 < \tau < p - 1$ such that $\lambda - \tau_{\theta}(y) = \tau_{\theta}(a)$. Therefore $\lambda - \tau_{\theta}(y)$ is a root of $\tau_{\theta}(K_\theta)$.

Since $\lambda$ was arbitrary it follows that $\dim_{F_p} K / [K] \leq 1$. Therefore $\dim_{F_p} K / [K] = 1$. In addition, $a \not\in [K]$, for otherwise $K = [K]$.

By Fact 5, $K$ has a unique cyclic extension of degree $p$. Furthermore we know that this extension is generated by a root of the equation $\tau(x) = a$, where $a \in K_\theta$ and $a \not\in [K]_\theta$.

From the penultimate paragraph we see that, as vector-spaces over $F_p$, $K = [\tau](\theta)$, where $\theta$ is the 1-dimensional space generated by $a$. Observing that $\tau(Abs(K)) \subseteq Abs(K)$, that $\tau^{-1}(Abs(K)) \subseteq Abs(K)$, and that $\tau(Abs(K))$, we conclude that

$Abs(K) = [\tau(Abs(K)) \otimes \theta]$.

So far in this analysis we have not used the assumption that $T_K(\theta)$ is totally transcendental, but now we do. We will assume also that $K$ is infinite.

By Lemma 6 and the Corollary to Lemma 8, $(K_\theta)^{\ast}$ is divisible for every finite algebraic extension $K_\theta$ of $K$. It follows easily that $(Abs(K_\theta))^{\ast}$ is divisible for every finite algebraic extension $K_\theta$ of $K$. Since the only finite divisible group is the trivial one-element group, it follows that $Abs(K_\theta)$ is infinite unless $(Abs(K_\theta))^{\ast}$ is.

Suppose $(Abs(K_\theta))^{\ast} = \{1\}$. Then $Abs(K_\theta) = F_p$ and $p = 2$. Let $\zeta$ be a root of the equation $\zeta^2 + \zeta + 1 = 0$.

Then $K(\zeta)$ is an extension of $K$ of degree 2, and $F_2 \subseteq K(\zeta)$. Let $K_\theta = K(\zeta)$. Then $F_2 \subseteq Abs(K_\theta)$, we claim $Abs(K_\theta) = F_2$. Suppose $u \in Abs(K_\theta)$, and let $f(\xi E_\zeta) = \frac{1}{2}$ be the minimum polynomial of $u$ over $F_2$. Since $u \in K(\zeta)$ there exist $a$, $b$ in $K$ such that $u = a + b\zeta$. If $b = 0$, $u \in K$, so $u \in Abs(K_\theta) = F_2$. Suppose $b \neq 0$. Consider the element $g$ of $K[x]$ defined by $g(x) = f(a + b\zeta)$. Since $f$ is not the zero polynomial, and $K$ is infinite, one verifies easily that $g$ is not the zero polynomial over $K$. By observing that $g(a) = g(b^2 - b^2\zeta^2)$ one proves easily that $g$ is irreducible over $K$. But $g(0) = 0$, and $g(b^2 + \xi^2 + 1 = 0$. It follows that for some constant $c$, $c \in K$, with $c \neq 0$, $g(x) = c(x^2 + x + 1)$, i.e. $f(a + b\zeta) = c(x^2 + x + 1)$. It follows that $f(a) = c_1 + c_2 a + c_3 a^3$ for some $c_1$, $c_2$ in $F_2$.

Therefore $c_1 + c_2 a + c_3 a^3 = 0$.

Therefore $c_1 + c_2 a + c_3 a^3 + b^2 a^2 = c x^2 + c x + c$.

Therefore $c_1 + c_2 a + c_3 a^3 = c$.

$c_2 b = c$.

$b^2 = c$.

Therefore $b^2 = c b$, and since $b \neq 0$,

$b = c b$, since $b \neq 0$,

$b = c b$.

In fact, $c_1$ must be 1. Therefore $b = 1$ and $c = 1$. Therefore $c_1 + c_2 a + c_3 a^3 = 1$.

If $c_1 = 0$, $1 + a = a^3 = 0$, and $a \in K$, although the equation $a^2 + a + 1 = 0$ has no root in $K$. Therefore $c_1 = 1$, so $a + a^3 = 0$, so $a \in F_2$. Since $a$, $b \in F_2$, $a + b^2 \in F_2(\zeta) = F_2$, so $u \in F_2$. We have proved that $Abs(K_\theta) = F_2$. But $(F_2)^{\ast}$ is not divisible. Therefore $(Abs(K))^{\ast}$ is not divisible, contrary to what was proved earlier. This rules out the case where $(Abs(K))^{\ast} = \{1\}$.

Therefore Abs(K) is infinite.

We now construct Abs(K) as an algebra over K. Since Abs(K) is infinite and K is finite, the dimension of Abs(K) over K is infinite. We select elements $b_n (n < 0)$ of Abs(K) which form a linearly independent set over K. Define $x \in K^{\ast}$ by

$s(0) = a$,

$s(n + 1) = b_n$ for $n \geq 0$.

We are going to show that $T_K(\theta)$ has $\infty$ complete extensions in $\infty(\theta)$, and $\infty(\theta)$ is totally transcendental. Suppose $K_\theta = F_\infty$. If $N$ is such that $K_\theta \subseteq K \cap F_N$, we define $T_N$ as the trace function from $K \cap F_N$ to $K_\theta$. Now $K \cap F_N$ is a cyclic extension $2^\infty$. 


of $K$, and it is well-known [11, 12] that the map $x \mapsto x^n$ generates the Galois group. We now apply the additive analogue of Hilbert’s “Zatz 90” [12], p. 77, to get immediately the following important fact:

**Fact 6.** If $y \neq K \cap F_{p^r}$, then $T_p(y) = 0$ if and only if $y = x^{p^m} - x$ for some $x \in K \cap F_{p^r}$.

Now we define a map $\delta : \text{Add}(K) \to \text{Add}(K)$ by:

$$\delta(x) = x^{p^m} - x.$$

Then $\delta$ is an endomorphism of $\text{Add}(K)$, and $\delta[\text{Add}(K)]$ is a subgroup of $\text{Add}(K)$.

We claim that $\alpha \in \delta[\text{Add}(K)]$. Since $\alpha \in \text{Add}(K)$, $\alpha \neq 0$. Since $\delta(x) = x^{p^m} = \delta(\text{Add}(K)) = \{0\}$, we see that the equation $x^{p^m} - x = 0$ has no root in $K$. If $y$ is any root of the above equation, and $\lambda \in F_{p^r}$, it is easily seen that $y^{p^m} - y = \delta(y)$ is also a root of the equation. It follows easily that the polynomial $x^{p^m} - x = \delta[\text{Add}(K)]$ is irreducible over $K$, and $K_{p^m} = K_{p^r}$ is an extension of $K_{p^r}$ degree $p^r$. Therefore $K_{p^m} = F_{p^{mr}}$. Therefore if $\alpha \in \delta[\text{Add}(K)]$, $\text{Add}(K) = K_{p^r}$, so $F_{p^r} \subseteq K$. But $F_{p^r}$ is the unique extension of $F_p$ of degree $p^r$, and since $\alpha \in \delta[\text{Add}(K)]$, $\alpha \notin \text{Add}(K)$. Since $\alpha \notin \text{Add}(K)$, $F_{p^r} \supseteq K$. Therefore $\alpha \notin \delta[\text{Add}(K)]$. Suppose $\alpha \in F_{p^r}$. We define a set $\text{Con}(\alpha)$ of conditions, involving the unknown $x_0$ as follows:

**Con(\alpha)** consists of the conditions $b_n x - (s(n) - \alpha) \in \delta[\text{Add}(K)]$, for $n \leq c_i$.

If $m \leq c$, we define $\text{Con}(\alpha)$ as the set of conditions $b_n x - (s(n) - \alpha) \in \delta[\text{Add}(K)]$, for $n \leq m$.

We claim that the set $\text{Con}(\alpha)$ is satisfiable in $K$, i.e. that there exists $x_{m_0} \in K$ such that $b_m x^{p^m} - (s(n) - \alpha) \in \delta[\text{Add}(K)]$, for $n \leq m$.

Since $b_n \in F_{p^r}$ and $\text{Add}(K)$ is finite, we can find $x_{m_0} \in K \cap F_{p^r}$ such that $b_m x^{p^m} - (s(n) - \alpha) \in \delta[\text{Add}(K)]$, for $n \leq m_0$. Then $x_{m_0}$ satisfies $\text{Con}(\alpha)$.

Therefore we want to solve the system of equations

$$T_\alpha(b_n x - (s(n) - \alpha)) = 0 \quad (n \leq m) \quad \text{in} \quad K \cap F_{p^r}.$$

This is equivalent to solving

$$T_\alpha(b_n x - s(n) - \alpha) = 0 \quad (n \leq m) \quad \text{in} \quad K \cap F_{p^r}.$$
Remark. Because of the lengthy analysis involved in Case B, it is worthwhile giving an example of an infinite field \( K \) such that:

(i) For every finite algebraic extension \( K_i \) of \( K \), \( (K_i)^* \) is divisible;

(ii) For every finite algebraic extension \( K_i \) of \( K \), Case B holds, i.e., there exists \( k \geq 0 \) such that

\[
K_i = \tau(K_i) = \cdots = \tau_{k+1}(K_i) = \tau_k(K_i) = \cdots
\]

Let \( F \) be a prime, and let \( K \) be the closure of \( F_p \) under algebraic extensions of degree prime to \( p \). In terms of the so-called supernatural numbers \((\Omega)(12)\), \( K \) is the unique extension of \( F_p \) of degree \( s \), where

\[
s = \prod_{\ell \in \text{prime}} q^{\ell(q)}
\]

and \( q(p) = \infty \) if \( q \neq p \), and \( n(p) = 1 \). In the terminology of Ax [3], \( K \) is \( (p) \)-pseudo-finite. (See Ax’s Proposition 9.)

Since \( K \) has a cyclic extension of degree \( p \), \( K \neq K \) by Fact 3. We claim \( \tau(K) = \tau(K) \). \( K \) has a unique extension of degree \( p \) generated by a root of the equation \( x^p - a = 0 \). By Fact 5, for every \( u \) in \( K \) there is an \( r \) in \( F_p \) such that \( u^r - a \in \tau(K) \). Therefore \( \tau(u^r) \in \tau(K) \). But \( \tau(r) = 0 \). Therefore \( \tau(u^r) \in \tau(K) \), whence \( \tau(K) = \tau(K) \).

Therefore Case B holds for \( K \). A similar argument will prove that Case B holds for each finite extension \( K_i \) of \( K \).

Now we prove that \( K^* \) is divisible. Clearly it suffices to prove that, for each prime \( q \), \( K^* = (K^*)^q \). Since \( K \) is perfect the result is clear for \( q = \).

Suppose \( q \) is a prime \( \neq p \), and that \( q \) divides \( p^{n-1} \) for some \( N \) which is relatively prime to \( p \). For such \( N \), \( F_p x \subset K \), so \( K \) contains a primitive \( (p^{n-1})^\text{th} \) root of unity. Since \( q \) divides \( p^{n-1} \), \( K \) contains a primitive \( q^\text{th} \) root of unity, so \( K \) contains a primitive \( q^\text{th} \) root of unity. Suppose now \( K^* = (K^*)^q \). Then for some \( a \in K^* \), \( a \not\in (K^*)^q \). The polynomial \( x^p - a \) is irreducible over \( K \), since \( K \) has \( q^\text{th} \) roots of unity. Therefore \( K^* = K^* \) if \( q \) divides \( p^{n-1} \) for some \( N \) which is relatively prime to \( p \) to \( \).

Finally suppose that \( q \) is a prime \( \neq p \), and for all \( N \) which are relatively prime to \( p \) which \( \neq p \) does not divide \( p^{n-1} \). Then \( q \) is relatively prime to \( p^{n-1} \), whenever \( N \) is relatively prime to \( p \). Now suppose \( a \in K^* \). Then \( a \in (F_p)^q \) for some \( N \) which is relatively prime to \( p \). Then \( a \in (K^*)^q \). Since \( q \) is relatively prime to \( p^{n-1} \), \( F_p \) has a root of unity \( t \), \( n \) such that \( \omega = i = (p^{n-1}) = 1 \). Then \( a = a^\omega = a^{n+1 - (a^{n+1})} = a^{n} = (a^q)^n \). Therefore \( a \in (K^*)^q \). Since \( a \) was arbitrary, \( K^* = (K^*)^q \).

This concludes our proof that \( K^* \) is divisible. A similar argument shows that \( (K_i)^* \) is divisible for all finite extensions \( K_i \) of \( K \).

Thus \( K \) has all the required properties.

9. \( \omega \)-categoricality. We now get, as a corollary of Theorem 1, the principal result of our paper.

**Theorem 2.** If \( \mathcal{K} \) is an \( \mathcal{E}_\omega \) class of fields such that \( \text{Th}(\mathcal{K}) \) is \( \omega \)-categorical, and \( \mathcal{K} \) has no finite members, then \( \mathcal{K} \) is one of the classes \( \mathcal{A}_\omega \).

**Proof.** Assume the hypothesis. Let \( K \) be a member of \( \mathcal{K} \). Then \( \text{Th}(K) \) is \( \omega \)-categorical, so by [14], 3.8, \( \text{Th}(K) \) is totally transcendental. \( K \) is infinite. We conclude by Theorem 1 that \( K \) is algebraically closed. Thus all members of \( \mathcal{K} \) are algebraically closed. By \( \omega \)-categoricity all members of \( \mathcal{K} \) have the same characteristic, so \( \mathcal{K} \subset \mathcal{A}_\omega \) for some \( n \). Since any two members of \( \mathcal{A}_\omega \) are elementarily equivalent [16], we conclude that \( \mathcal{K} = \mathcal{A}_\omega \).

10. Elimination of quantifiers. Tarski [17] proved that the theory of algebraically closed fields of specified characteristic admits elimination of quantifiers. We will prove a converse of this result.

**Definition.** Let \( \Sigma \) be an \( \mathcal{L} \)-theory. Then \( \Sigma \) admits elimination of quantifiers if and only if the following condition holds: if \( \varphi(v_0, \ldots, v_n) \) is an \( \mathcal{L} \)-formula with all its free variables in the list \( v_0, \ldots, v_n \), then there exists a quantifier-free \( \varphi(v_0, \ldots, v_n) \) such that

\[
\Sigma \models (\forall v_0 \cdots (\forall v_n) \varphi(v_0, \ldots, v_n) \rightarrow \varphi(v_0, \ldots, v_n)).
\]

(Here \( \leftrightarrow \) is material equivalence, definable from \( \exists \) and \( \land \).)

It is easily seen that if \( \Sigma \) admits elimination of quantifiers then \( \Sigma \) is model-complete.

We want to know which theories of fields admit elimination of quantifiers. We emphasize that we are talking about elimination of quantifiers in the basic logic for fields. It is well-known [17] that the theory of real-closed fields admits elimination of quantifiers when we use the auxiliary predicate \( \Sigma \) for order. The situation is analogous for valued fields [6].

Suppose \( K \) is a field such that \( \text{Th}(K) \) admits elimination of quantifiers. Then for each \( n \geq 1 \) there is a predicate \( p_a(x_0, \ldots, x_n) \) which is a Boolean combination of polynomial equations with integral coefficients such that for each \( K_i \subset K \), and \( x_0, \ldots, x_n \) in \( K_i \), \( p_a(x_0, \ldots, x_n) \) holds in \( K_i \) if and only if there is a \( y \) in \( K_i \) such that \( x_0 + x_1 y + x_2 y^2 + \cdots + x_n y^n = 0 \). The obvious example is when \( K \) is algebraically closed, and \( p_a(x_0, \ldots, x_n) \) is \( \varphi(x_0) \neq 0 \) \( \rightarrow \neg(|x_0| = x_0 = \cdots = x_n = 0) \).
Lemma 12. Suppose $K$ is a field such that $Th(K)$ admits elimination of quantifiers. Then $Th(K)$ is $\omega_1$-categorical.

Proof. Assume the hypothesis. Suppose $K_1$ and $K_2$ are models of $Th(K)$ of cardinality $\omega_1$. Then $K_1$ and $K_2$ have the same characteristic. Clearly $K_1$ and $K_2$ both have transcendence degree $\omega_1$ over their respective ground fields. Let $L_1$, $L_2$ be respectively the ground fields of $K_1$, $K_2$. Let $\{b_{1}\mid 1 < \omega_1\}$ and $\{b_{2}\mid 1 < \omega_1\}$ be respectively transcendence bases for $K_1$ over $L_1$ and $K_2$ over $L_2$. Let $J_1 = L_1(b_{1}, ..., b_{1}; ..., 1 < \omega_1)$, and $J_2 = L_2(b_{2}, ..., b_{2}; 1 < \omega_1)$, then $K_1$ is algebraic over $J_1$, and $K_2$ is algebraic over $J_2$. Also by the Steinitz theory there exists an isomorphism $\sigma: J_1 \cong J_2$. We claim $\sigma$ extends to an isomorphism $\tilde{\sigma}: K_1 \cong K_2$.

$\sigma$ extends to an isomorphism $\sigma^*: J_1[x] \cong J_2[x]$ such that $\sigma^*(x) = x$. We claim that for each $f \in J_1[x]$ $f$ has a root in $K_2$ if and only if $\sigma(f)$ has a root in $K_2$. Suppose $f = a_1x_1 + a_2x_2 + ... + a_nx^n$, where $a_1, a_2, ..., a_n \in K_1$. Then $f$ has a root in $K_1$ if and only if there exists $y \in K_2$ such that $a_1y + a_2y^2 + ... + a_ny^n = 0$. Now we use the predicate $P_2$ introduced earlier. $f$ has a root in $K_1$ if and only if $P_2(a_1, a_2, ..., a_n)$ holds in $K_2$. But $P_2$ is quantifier-free, so $P_2(a_1, a_2, ..., a_n)$ holds in $K_2$ if and only if $P_2(\sigma(a_1), \sigma(a_2), ..., \sigma(a_n))$ holds in $K_2$. But $P_2(\sigma(a_1), \sigma(a_2), ..., \sigma(a_n))$ holds in $K_2$ if and only if there exists $Z \in K_2$ such that $\sigma(z_1) + \sigma(z_2) + ... + \sigma(z_n) = 0$, i.e., if and only if $\sigma^*(f)$ has a root in $K_2$. This proves the claim that $f$ has a root in $K_2$ if and only if $\sigma^*(f)$ has a root in $K_2$.

Since $K_1$ and $K_2$ were arbitrary models of $Th(K)$ of cardinality $\omega_1$, we conclude that $Th(K)$ is $\omega_1$-categorical.

Theorem 3. Suppose $K$ is an infinite field such that $Th(K)$ admits elimination of quantifiers. Then $K$ is algebraically closed.

Proof. Assume the hypothesis. Then $Th(K)$ is $\omega_1$-categorical, and so by Theorem 2 $K$ is algebraically closed.

Theorem 3 is the promised converse to Tarski’s result.

11. Concluding remarks. It would be interesting to classify $\omega_1$-categorical theories of division rings. By the methods of this paper one can show that if $K$ is a division ring with $Th(K)$ totally transcendental then the centre of $K$ is either finite or algebraically closed, but this is all we know at the moment.

More generally one would like categoricity results for wider classes of rings. A specimen result is that if $K$ is an algebraically closed field and $M(K)$ is the ring of $n \times n$ matrices over $K$ then $Th(M(K))$ is $\omega_1$-categorical. Here our knowledge is fragmentary.

References


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