Semigroups on continua ruled by arcs
by
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Introduction. In our paper [4], we show that an acyclic Peano continuum \( P \) which has a compact set of endpoints \( I \) admits a semigroup structure with zero and unit. Later, we were able to prove that each acyclic Peano continuum \( P \) admits such a structure, i.e., there is a continuous mapping \( \sigma : P \times P \to P \) such that \( \sigma \) is associative and there are points 0 and 1 in \( P \) such that \( \sigma(\sigma(\sigma(0),0),0) = 0 \) and \( \sigma(\sigma(\sigma(1),1),1) = 1 \) for each \( z \in P \). We made the observation that the concept of being acyclic (i.e., contains no simple closed curve) was not essential to defining a semigroup structure on a Peano continuum \( P \) (i.e., locally connected, compact, and metric) provided that certain properties were possessed by \( P \). One of these is a natural kind of ruling of \( P \) by simple arcs. A disk, for example, can be ruled with arcs.

It is known [2] that a one dimensional compact connected semigroup with zero and unit is a generalized tree [3], i.e., a tree with the property that every compact connected semigroup with zero and unit is isomorphic to a generalized tree. Although a “ruled continuum” may be of large dimension, the one-dimensional ones include certain generalized trees, and therefore, admit the desired semigroup structure.

A special case of a ruled continuum: trees (acyclic Peano continuum). We take up this special case separately to provide motivation for the rather complicated description of a ruled continuum. The concepts were suggested by this special case and an example due to Professor Haskell Cohen. He has shown that a Cantorian swastika admits the required semigroup structure. It remains unpublished. However, Cohen’s example is a ruled continuum of a special kind which furnishes a technique for overcoming an obstacle which we encountered.

Suppose that \( S \) is an acyclic Peano continuum and that \( I \) is the set of all endpoints of \( S \) with the exception of one endpoint which we
denote by 0. There exists $S$ such that sup $\{d (0, x) : x \in S\} = d (0, u)$ where $d$ is a convex metric for $S$. Here, we refer to R. H. Bing's result [1] that each Peano continuum admits a convex metric. This is an extremely powerful tool and our techniques depend heavily upon it.

For each point $x$ in $S$, there is a unique arc $[0, x]$ in $S$. Furthermore, for each arc $[0, x]$ in $S$, there is an arc $[0, e]$ such that $e \in I$ and $[0, e] \supset [0, x]$. That is, $S$ is a union of a collection of arcs $[0, e]$ for the various points $e$ in $I$. If $[0, e]$ and $[0, f]$ are two elements of $S$, then $[0, e] \cup [0, f]$ is a proper subset of each if and only if $e \neq f$. Also, if $a_n \to a$, then the unique arcs $[0, a_n] \to [0, a]$. We can assume without loss of generality that $d (0, u) = 1$. Since $d$ is a convex metric for $S$, it can be shown that for each arc $[0, x]$ in $S$, the arc $[0, x]$ is isometric to a straight line interval. These are all properties which we shall use in our description of a "ruled continuum". We shall say that $S$ is radially convex since our arcs emanate from the point 0.

A linear ordering of $I$. Our next step is to linearly order the points of $I$. Note that $I$ may be dense in $S$.

Let $B$ denote the set of branch points of $S$, i.e., cutpoints of order $\geq 2$. It is known that if $b \in B$, then $S - b$ has at most countably many components. For $b \in B$ let $C_b$ denote those components of $S - b$ which do not contain 0, and let $I_b = I \cap C_b^*$ (the set of endpoints of the tree $C_b$), excluding 0. Note that the $C_b$ form a null collection, i.e., for $b > 0$, $\cap C_b \geq 0$ is countable. Now order the collection $C_b$ so that $\cap C_b \geq 0$ diam $C_b$ (and $b \not \geq \cap C_b$) if $0$ do not lie in the same component of $S - b$.

Define a relation $R (b)$ on $\cap I_b$ by: $(x, y) \in R (b)$ if and only if $x \in I_b$, $y \in I_b$ and $i < j$. We note that $R (b)$ is transitive and has the property that if $x \not \geq y$, $y \not \geq x$, then $x \not \geq z$. Let $E = \cap R (b) \cup A$, where $A$ is the diagonal of $I \times I$. The proof that $R$ is an ordering is given in [4]. We next coordinatize $S$ as follows: to each $x$ in $S$ is assigned $e_x \in I$ subject to the conditions $e_0 = [0, e_0] > [0, e_0]$, and $e_x = u$ if $x \not \geq u$. Now assign to each $x \in S$ two coordinates $(a_x, e_x)$, where $a_x = d (0, x)$ and $e_x$ is given above. Now, $(a_x, e_x)$ uniquely represents $x$ since $S$ is radially convex. Let $y = (a_y, e_y)$ and define

$$sy = \left( \min (a_x, a_y), \min (e_x, e_y) \right).$$

Multiplication in $S$ is easily seen to be well defined, associative, has the zero $(0, e_0)$, and unit $(1, u)$. It remains to be shown that multiplication is continuous.

Next, we state two lemmas which are not difficult to prove for trees. We shall prove them later for "ruled continua".

**Lemma 1.** Suppose that $a_n \to a$ with $a_n = (a_{n1}, e_{n1})$ and $a = (a_0, e_0)$. Let $b$ be a sequence of real numbers such that $0 < b_n < 1$ with $b_n \to 1$ and $b_n \leq e_{n1}$ then $(b_n e_{n1}) \to (b_0 e_0)$.

**Lemma 2.** Suppose that $a_n \to a$, $y_n \to y$, and $e_n \leq e_y$. If either $a \in (0, y)$ or $y \in (0, a)$, then $a_n y_n \to ay$.

We recall the following result ([4], Lemma 5): The function $f : S \times S \to S$ defined by: $f (x, y) = [0, p] \cap [0, q] = [0, p \cap y]$ is continuous. Let $(a_n) \to y$. By Lemma 2 we need only consider the case $a \in (0, y)$ and $y \in (0, a)$. For such $x, y$ let $f (x, y) = b$. We have that $(a_n, y) \to (a, y)$ and $(y_n, y) \to (y, y) = y$; hence we may choose subsequences $(a_n)$ and $(y_n) \to (a_n, y) \in C_b$, $(y_n, y) \in C_b$, and either $a_n \leq y_n$ or $a_n \not \leq a_n$ for some $i, j, m, n$. Note that $x$ and $y$ are distinct from $b$ since $x \not \geq y$, $y \not \geq x$. It follows from the definition of the ordering that $e_n \leq e_y$ all $n$.

If $a_n \leq e_y$ all $n$, then

$$a_n y_n = \left( a_{n1}, e_{n1} \right) \left( a_{n2}, e_{n2} \right) = (a_{n1}, e_{n2}) = a_n \to a y = ay.$$ 

If $a_n \leq a_n$ for each $n$, then

$$a_n y_n = \left( a_{n1}, e_{n1} \right) \left( a_{n2}, e_{n2} \right) = (a_{n1}, e_{n2}) \to (a, e_y) \quad (by \ Lemma 1)$$

and

$$a_n \leq (a_n, e_n) \to (a, e_y) \quad (by \ Lemma 1)$$

Thus, we have proved the following theorem.

**Theorem A.** Suppose that $P$ is an arcwise Peano continuum. Then $P$ admits the structure of a topological semimultiplicative with zero and unit.

**A description of ruled continua.** The facts that tree admits a semigroup structure with 0 and 1 depends upon a number of properties which we list below. It is easily seen that these are taken from trees. However, a large class of continua including discs as well as pathologically non-locally connected satisfy these conditions.

Let $S$ be a compact metric continuum, and let $0 \not \in S$. Suppose $I \subset S$, and suppose $\mathcal{H} = \{(0, e) : e \in I\}$ is a collection of arcs in $S$ satisfying the following conditions (1)-(8).

1. $S = \cup \mathcal{H}$
2. For each $e \in I$ there is a unique arc $[0, e] \subset \mathcal{H}$.
3. If $e, f \in I$ with $e \neq f$, then $[0, e] \cup [0, f]$ is a proper subset of each.
4. If $x \in S$ denote by $[0, x]$ the subarc with endpoints 0, $x$ of any member of $\mathcal{H}$ which contains $x$. This is said to be well defined by (5).

We say that a metric $d$ for $S$ is radially convex if for each $e \in I$ and $x, y \in [0, e]$ with $x \neq y$, $d (0, x) \neq d (0, y)$. Suppose further

4. If $a_n \to a$, then $[0, a_n] \to [0, a]$.
5. $S$ has a radially convex metric $d$. 


Using (5) and (1) it can be seen that there exists $u \in I$ such that $d(0, u)$ is maximal among $d(0, e) : e \in I$, and without loss of generality we may assume that $d(0, u) = 1$.

(6) For $x \in S$, let $e_x \in I$ be chosen with $x = (0, e_x)$ and satisfying $x \in I$ if $e_x = u$.

Suppose next that $I$ can be ordered, with maximal element $u$, subject to the restrictions.

(7) If $(x_k) \to x$ and $(y_k) \to y$ with $e_k < e_y < u$ and $x \neq (0, y)$, $y \neq (0, z)$, then there are subsequences $(x_{k_n})$ and $(y_{k_n})$ such that $e_{x_{k_n}} < e_{y_{k_n}}$ for each $k$.

Denote by $C$ the point 0 and all points $y$ in $[0, u]$ such that there exists $a \notin [0, u]$ and sequences $(x_k) \to y$, $(a_k) \to a$ such that (a) $e_{a_k} < e_y$, (b) $y \notin [0, a]$, and (c) $[0, a] \in [0, p] \subset [0, u]$ where $y \notin [0, p]$. Either $C$ is the set consisting of only 0 or $C = (0, k) (k \neq 0)$ for some $k \neq 0$ in $[0, u]$.

(8) If $C = (0, k)$ $(k \neq 0)$, then $S$ can be remetrized with a radially convex metric $d$ so that $d(0, u) = 1$, $d(0, k) = 1$, and if $e \in I$ with $[0, e] \subset [0, u]$, then $d(e, p) < 1$.

It follows from (8), that $u \notin C$.

The class $S$ of all ruled continua is that collection of arcwise connected continua satisfying conditions (1)-(8).

**Examples.** In view of the somewhat formidable description of $S$ it seems appropriate to give some illustrative examples.

**Example 1.** The Cantorian dual space $S$. This pathological continuum consists of four copies of $[0, 1] \times [0, 1]$ where $C$ is the Cantor set.

These are put together as illustrated (Fig. 1). Here, $I$ is the collection of all non weak-cutpoints and $S$ is given the metric inherited from the plane. It is clear that $S$ satisfies (1)-(8). The ordering of $I$ is given clockwise. The continuum $C = [0, k]$ describes a “discontinuity” in the ordering of $I$ in the sense that there are elements on low-indexed lines which are close to elements on the high indexed line $[0, u]$. This type of order discontinuity occurs only at the points of $O$ and occurs with a single multiplicity. Condition (6) can be satisfied by choosing the largest possible $e_k$ for points $a$ on the coordinate axes.

Condition (7) follows easily, and expresses the continuity of the ordering away from $[0, u]$. Condition (8) is clear.

**Example 2.** Two Cantor fans tangent along a segment (Fig. 2). Here, $I$ consists of the non weak-cutpoints. The ordering on $I$ is clockwise. The continuum $C$ is the point 0. We may describe this as the continuously ordered case. Conditions (1)-(8) are immediate.

**Example 3.** Two cones tangent along a line segment (Fig. 3). Here, $I$ consists of $(C_1 \cup C_2) - k$ together with $u$ where $C_1$ and $C_2$ are the boundaries of the cones (topological disks).

The ordering is as indicated, with $u$ the maximal element. This illustrates what we may call a discontinuity in the ordering of multiplicity two. An uncountable multiplicity may be obtained by filling in cones along a Cantor set.

**Example 4.** The two cell (Fig. 4). We may take $I$ to be a side opposite a vertex, and $C = 0$. This is a continuous ordering, as motivated by Example 1. Some typical members of $S$ are indicated.

Here (Fig. 5), $I = (u, v) \cup (e, k) \cup (d, v) \cup (f, k)$ and $C = [0, k]$. Some typical members of $S$ are indicated.

**Example 5** (Fig. 6). The “closed up” $\sin(1/2)$ curve, together with interior. Here $I$ is $u$ together with the graph of $y = \sin(1/2)$, $0 < x < 1$. 
This is another example of the continuously ordered case. Some typical members of \( \mathbb{R} \) are indicated.

**Definition of multiplication in a ruled continuum \( S \).** Let \( S \) be in \( \mathbb{R} \); if \( C = 0 \), then assign to each \( x \in S \) two coordinates \((x_a, x_b)\), where \( x_a = (0, x) \) and \( x_b \in \mathbb{R} \) is given by (6). If \( C \) is non-degenerate, we remetrize according to (8) and then assign coordinates as above. Note that \((x_a, x_b)\) uniquely represents \( x \) since \( S \) is radially convex. Let 
\[ y = (x_a, y_b) \] and define \( xy = (x_a, x_b)(x_a, y_b) = [(x_a, x_b), \min(x_a, y_b)] \), where \( x_a \geq y_a = \max(x_a + y_a - 1, 0) \). We note that \( \frac{1}{2}: \frac{1}{2} = 0 \), and \( \frac{1}{2} \left( \frac{1}{2} + p \right) = p \) for \( 0 < p \leq \frac{1}{2} \). Multiplication in \( S \) is easily seen to be well defined, associative, has the zero \((0, 0)\), and unit \((1, 0)\). It remains to be shown that multiplication is continuous. We proceed to show this fact.

**Proof of Lemma 1.** Note that \((t_0, e_{t_0}) \in [0, e_0) \rightarrow (0, y) \) by (4). Hence there is a subsequence \((t_n, e_{t_n})\) converging to an element of \([0, a] \subseteq [0, e_0)\), i.e., \((t_n, e_{t_n}) \rightarrow (t, e)\). But each subsequence of \((t_n, e_{t_n})\) has the cluster point \((t, e)\), and the result follows.

**Proof of Lemma 2.** Suppose \( x \neq y \) and \( x \in [0, y] \). We may assume, by choosing subsequences, that either \( e \leq e' \) for each \( n \), or that \( e_n < e' \) for each \( n \), and that \( a_n < a' \).

If \( e_n < e' \), then 
\[ a_n y_n = (a_n, e_n)(a_n, e_n) = (a_n, e_n) = xy \rightarrow xy \] if \( e_n < e' \), then 
\[ a_n y_n = (a_n, e_n)(a_n, e_n) = (a_n, e_n) = xy \rightarrow xy \] Since \( a_n < a' \), we conclude from Lemma 1 that \((a_n, e_n) \rightarrow (a, e)\). But \((x_a, y_b) = (x_a, y_b)\) since \( x \in [0, y] \subseteq [0, e_0] \); hence, \( x_n y_n = (x_n, y_n)(x_n, y_n) = (x_n, e_n) = (a_n, e_n) = xy \). A similar argument applies in case \( x \notin [0, y] \).

Now, suppose \( x = y \). Then, by choosing subsequences, we have either (1) \( e_n \leq e' \) for each \( n \) or (2) \( e_n < e' \) and \( e_n \geq e_0 \) for each \( n \). If (1) holds, then \( a_n y_n \rightarrow xy \) since each element of \( S \) is idempotent. If (2) holds, then 
\[ a_n y_n = (a_n, e_n) \rightarrow (a_n, e_n) = (a_n, e_n) = xy \] and \( a_n y_n \rightarrow xy \). The proof is complete.

Thus, we may assume that \( a_n \rightarrow a \), \( y_n \rightarrow y \), \( x \notin [0, y] \), and \( y \notin [0, x] \). It follows that \( e_n < e' \), for otherwise \( x \notin [0, y] \); and hence, either \( x \in [0, y] \)

**Case 1.** \( e_n < e' < u \). By (7) we may assume, by choosing subsequences, that \( e_n < e' \) for each \( n \).

Then 
\[ a_n y_n = (a_n, e_n)(a_n, e_n) = (a_n, e_n) = xy \] By Lemma 1, we conclude that \((a_n, e_n) \rightarrow (a_n + e_n, e_n) = xy \).
but these discontinuities are nice in the sense that $C = 0$ for these arcs.
It appears that our techniques would handle the cases of a finite number of arcs $A_n$ like $[0, n]$ with bad discontinuities, that is, the sets $C_n$ (like $C$) are nonempty and also $A_n - C_n$ is nonempty.
Finally, it is conjectured that $C$ contains $m$-cells, and that $C$ is
closed under the operation of taking cones.

**Question.** Suppose that a compact metric continuum $S$ contains
a subset $I$ such that (a) $I$ satisfies Conditions (1)-(6) for a ruled
continuum and (b) $I$ admits a topological semigroup structure with zero $z$
and unit $u$ where $I$ and $u$ have the same meaning as in (1)-(6). Does $S$
admit the structure of a topological semigroup with zero and unit $u$?
The continuum $S$ above is a more general type of ruled continuum
than that considered in Conditions (1)-(8).

References

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On the lexicographic dimension of linearly ordered sets

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1. Introduction. In earlier papers from the theory of representation
of linearly ordered sets chief interest was concentrated on finding
so-called universal sets. Under an $m$-universal linearly ordered set (where
$m$ is cardinality) we understand a linearly ordered set which contains
a subset isomorphic with every linearly ordered set of cardinality $\leq m$.
It was been shown that such universal sets are ordinal powers (in Birk-
hoff's sense) in which the base in any chain containing at least two ele-
ments and the exponent is a well-ordered set. Thus Hausdorff proved
[1, p. 131] that every linearly ordered set of cardinality $\leq \aleph_0$ where
$\aleph_0$ is a regular cardinal number is isomorphic with a certain set of se-
quences of type $\omega_1$ formed from three ciphers 0, 1, 2, and ordered lexi-
cographically. In other words, he proved that an ordinal power of type $\aleph_0$
is an $\aleph_0$-universal linearly ordered set if $\aleph_0$ is regular. Sierpiński
([2]) improved his result in the following way: An ordinal power of type $\aleph_2$
is an $\aleph_2$-universal linearly ordered set for every cardinal number $\aleph_k$.

Now it is clear that the type of base cannot be reduced. Hence
interest has been concentrated on the problem if it is possible to reduce
the type of the exponent. It has been shown, however, that in general
this type cannot be reduced. In some cases it is possible, however, to
map a given linearly ordered set of cardinality $\aleph_k$ isomorphically onto
a subset of a power with the exponent of a lower type than $\omega_1$. Thus
Novotný ([3]) proves that: Every $\aleph_2$-separable (1) linearly ordered set
can be isomorphically mapped onto a subset of ordinal power of type $\aleph_2$.
This survey makes clear the effort to find the most economical repre-
sentation, i.e. a representation in which both the base and the exponent
are of the smallest possible types.

Now it is possible to pose this problem: Let the type of the base be constant. What is the smallest possible type of exponent such that
the given linearly ordered set can be mapped isomorphically onto a sub-
set of the corresponding ordinal power? This problem was partially

(1) A linearly ordered set $G$ is called $m$-separable if it contains a dense subset $M$
of minimal possible cardinality $m$. 