Functionals on uniformly closed rings of continuous functions

by

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In this paper we are concerned with the following problem: Suppose $X$ is a completely regular space and let $E$ be a linear ring of continuous real-valued functions defined on $X$ which satisfies the following conditions:

1. All constant functions belong to $E$.

2. $E$ is closed with respect to the uniform convergence (i.e., if $(f_n)$ uniformly converges to $f$ and $f_n \in E$ ($n = 1, 2, \ldots$), then $f \in E$).

Under what conditions imposed on $X$ and $E$ each non-trivial linear multiplicative functional $\varphi$ defined on $E$ is of the form

$$(\ast) \quad \varphi(f) = f(p_0)$$

where $p_0$ is a fixed point of $X$?

We note some results related to this problem:

If $E$ is the ring of all bounded continuous functions on $X$, then the answer to our problem is positive if and only if $X$ is a compact space (Stone [4]).

If $E$ is the ring of all continuous functions on $X$ then the answer to the problem is positive if and only if $X$ is a $Q$-space (Hewitt [1], [2]).

The main role in our considerations is played by the evaluation mapping of $X$ into the Tihonov cube build up by means of all members $f$ of $E$ which satisfy the inequality $0 \leq f(p) \leq \bar{1}$ (i.e. denote by $E^*$ the set of all members $f$ of $B$ which satisfy the above inequality and agree that the coordinates of points of the Tihonov cube $I^n$ ($m = \bar{1}$) are enumerated by means of members of $E^*$). Then the evaluation mapping can be described as a mapping which carries a point $p \in X$ into the point $a \in I^n$ whose $f$th coordinate is equal to $f(p)$. We denote this evaluation mapping by $E_B$.

(1) A functional $\varphi$ is said to be non-trivial provided that $\varphi$ does not vanish identically.
If each member of a ring $R$ is bounded, then the answer to our problem is rather uninteresting; it is quite similar to that of the above-mentioned Stone result; namely, it is positive if and only if $F_R(X)$ is compact. The more interesting case is the case where $R$ contains possibly many unbounded functions, i.e., where $R$ contains the inverse of each member of $R$ whose each value is different from 0. The answer to our problem in this case is given in Theorem 2.

**I. Some properties of the mapping $F_R$.** In this section $X$ denotes a compactly regular space; $R$ denotes a fixed linear ring of real-valued continuous functions defined on $X$ which satisfies the conditions 1* and 2*; $F_R(X)$ denotes the closure of $F_R(X)$ with respect to the Tikhonov cube $I^X$.

(i) If $f$ is a bounded function in $R$, then there is a continuous real-valued function $h$ defined on $F_R(X)$, such that $f(p) = h[F_R(p)]$ for each $p$ in $X$.

Let $f^*(p) = a_f(p) + \beta$, where $a \neq 0$ and $\beta$ are real numbers chosen in such a way that $0 < f^*(p) < 1$ for each $p$ in $X$. Then $f^* \in R^*$. Let $h(a) = \frac{1}{a} [p_f(a) - \beta]$, for $a \in F_R(X)$, where $p_f(a)$ denotes the $j$th coordinate of $a$. Then $h$ is the required function.

(ii) If $h$ is a continuous real-valued function defined on $F_R(X)$, then the function $f$ defined on $X$ by the equality $f(p) = h[F_R(p)]$ belongs to $R$.

Let $C = (p_f)_{f \in R^*}$ be the family of all coordinate functions of points in $F_R(X)$. Since $F_R(X)$ is compact and $C$ distinguishes points of $F_R(X)$, by the Stone-Weierstrass approximation theorem for each positive $\epsilon$ there exists a polynomial $W(y_1, ..., y_n)$ of real variables $y_1, ..., y_n$ and members $t_1, ..., t_m$ of $R^*$ such that $|h(t) - W(t_1, ..., t_m)| < \epsilon$ for each $t \in F_R(X)$.

By the definition of $F_R$ and $f$, we obtain $|f(p) - W(t_1, ..., t_m)| < \epsilon$ for each $p$ in $X$. Since $W(f_1, ..., f_m) \in R$ is closed with respect to the uniform convergence, $f \in R$.

**II. Theorem 1.** If $R$ is a linear ring of bounded real-valued continuous functions on a completely regular space $X$ satisfying the conditions 1* and 2*, then each non-trivial linear multiplicative functional $\varphi$ defined on $R$ is of the form $\star$ if and only if $F_R(X)$ is compact.

Proof. Suppose that $F_R(X)$ is compact and let $\varphi$ be any non-trivial linear multiplicative functional defined on $R$. Denote by $E$ the ring of all continuous real-valued functions defined on $F_R(X)$. Let $h$ be any member of $E$, and let us set $\varphi(h) = \varphi(f)$, where $f$ is a function in $E$ satisfying the equality $f(p) = h[F_R(p)]$ (by (ii), $f \in R$). Then $\varphi$ is a non-trivial linear multiplicative functional on $R$, whence, by the Stone theorem, there is a point $a \in F_R(X)$ such that $\varphi(a) = h(a_p)$ for each $a \in R$.

Let $p_0$ be any point of $X$ with $F_R(p_0) = a_q$. If $f$ is any member of $R$, then there is an $h \in E$ such that $f(p) = h[F_R(p)]$ for each $p$ in $X$. We have $\varphi(f) = \varphi(h) = h(a_p) = h[F_R(p_0)] = f(p_0)$, whence $\varphi$ is of the form $\star$.

Conversely, suppose that $F_R(X)$ is not compact. Then $E = E_0 \neq F_R(X)$; let $a_q$ be any point of $E = F_R(X)$. Let us set $\varphi(f) = h(a_q)$ for any $f$ in $R$, where $h$ is a continuous function on $F_R(X)$ such that $f(p) = h[F_R(p)]$ for each $p$ in $X$. Then $\varphi$ is a non-trivial linear multiplicative functional on $R$. If $p_0$ is any point of $X$, then there is a continuous function $h$ on $F_R(X)$ with $h(a_p) = 0$ and $h(a_q) = 1$, where $a_q \in F_R(p_0)$. By (ii), there is a member $h$ in $R$ such that $f(p) = h[F_R(p)]$ for each $p$ in $X$. We have $\varphi(f) = 0$ and $f(p_0) = 1$ and it follows that $\varphi$ is not of the form $\star$.

**III.** In the sequel the following definition is needed: a subset $P$ of a topological space $S$ is said to be $Q$-closed (in $S$) provided that for each point $p \in S \\

S_P$ there is a $G_{\delta}$-set which contains $p$ and is disjoint from $P$.

If $S$ is a completely regular space, then we have the following:

(iii) A set $P \subseteq S$ is $Q$-closed in $S$ if and only if for each $p \in S \setminus P$ there is a continuous real-valued function $f$ on $S$ such that $f(p) = 0$ and $f(q) \neq 0$ for each $q \in P$.

The simple proof of (iii) can be left to the reader.

We are interested in the case where a space is $Q$-closed in a certain compactification of itself.

(iv) If $bS$ and $b'S$ are compactifications of a completely regular space $S$, $bS \subseteq b'S$ (i.e., there is a continuous mapping $F$ of $bS$ onto $b'S$ such that $F(p) = p$ for each $p$ in $S$) and $S$ is $Q$-closed in $bS$, then $S$ is $Q$-closed in $b'S$.

Indeed, if $p \in b'S \setminus S$, then $q = F(p) \in bS \setminus S$, whence there is a $G_{\delta}$-set $C \subseteq b'S$ which contains $q$ and is disjoint from $bS$. Then $F^{-1}(C)$ is a $G_{\delta}$-set in $b'S$ which contains $p$ and is disjoint from $S$.

(*) This definition was introduced in [3].

(1) This can be proved in the following way: suppose $q = F(p) \in S$. Then $p$ and $q$ are distinct points of $bS$, whence there exists a neighboured $U$ of $p$ with $q \not\in bS \cup U$ (the bar indicates the closure with respect to $S$). Since $S$ is dense in $bS$, $p \in S \cup U$ (the bar indicates the closure with respect to $S$). On the other hand, if $F(S \cap U) = S \cup U$, and since $F$ is continuous, $q = F(p) \in F(S \cap U) = S \cup U$ (the bar indicates the closure with respect to $S$) and this leads to a contradiction.

6*
The extreme case is explained by the following:
(v) A space $S$ is $Q$-closed in $\beta S$ if and only if $S$ is a $Q$-space.
This statement is given in [3].
(vi) A space $S$ is $Q$-closed in each of its compactifications if and only if $S$ is a Lindelöf space; i.e., each open covering of $S$ contains a countable subcovering.

Suppose that $S$ is a Lindelöf space. Let $bS$ be a compactification of $S$ and let $p_0 \in bS \backslash S$. For each $p \in S$ there is a neighbourhood $U_p$ of $p$ such that $p_0 \not\in \overline{U_p}$. Since $(U_p)_{p \in S}$ is an open covering of $S$, there is a countable covering $U_{p_0}, U_{p_1}, \ldots$ of $S$. Then $G = \bigcap_{n} (bS \backslash U_{p_n})$ is a $G_\delta$-set in $bS$ which contains $p_0$ and is disjoint from $S$.

Conversely, suppose that $S$ is not a Lindelöf space. Then there is a family $U = \{U_a\}_{a \in A}$ of open subsets of $\beta S$ which covers $S$ and such that no countable subfamily of $U$ covers $S$. Let $H = \beta S \backslash \bigcup\{U_a : a \in A\}$ and let $bS$ be the compactification of $S$ which is obtained from $\beta S$ by the identification of all points of $H$ to a single point; denote this point by $p_0$. Suppose that there is a $G_\delta$-set $G_i \subset bS$ which contains $p_0$ and is disjoint from $S$. Let $F = bS \backslash G$. Then $S \subset F \subset \bigcup\{U_a : a \in A\}$. But $F$, being an $F_\sigma$-set in a compact space, can be covered by a countable infinity of sets $U_a$. This leads to a contradiction, whence it is not $Q$-closed in $bS$.

An immediate consequence of (vi) is the following:
(vii) A locally compact space is $Q$-closed in its minimal one-point compactification if and only if it is a Lindelöf space.

IV. In this section we shall consider the cases of rings containing unbounded functions. We assume the following condition:
3° If $f \in R$ and $(p) \neq 0$ for each $p \in X$, then $1/f \in R$.

We shall prove some elementary properties of such rings (in (vii)-(viii)) $R$ is a fixed linear ring of continuous functions on a fixed space $X$ satisfying the conditions $1°$-$3°$.

(viii) If $f \in R$, then $|f| \in R$.

At first, suppose that $f$ is a bounded function. Then one can assume without loss of generality that $|f(p)| < \frac{1}{2}$ for each $p \in X$. We have $|f| = \sqrt{1 - (1 - |f|)^2}$, whence $|f|$ can be written as the sum of a uniformly convergent series of polynomials with respect to $f$. Thus, by $2°$, $|f| \in R$.

Now, suppose that $f$ is an arbitrary function in $R$. Let $f_1 = f / (1 + |f|)$.

Then $f_1$ is a bounded function, and, by $3°$, $f_1 \in R$. Consequently $|f_1| \in R$ and $|f| = |f_1| (1 + f_1) \in R$.

If $f, g \in R$, then $\max\{f, g\} \in R$ and $\min\{f, g\} \in R$.

This follows from (viii) and the formulas:
$$\max\{f, g\} = \frac{f + g + \sqrt{|f - g|^2}}{2}, \quad \min\{f, g\} = \frac{f + g - \sqrt{|f - g|^2}}{2}.$$

(x) Each member $f$ of $R$ can be written as the difference of two non-negative members of $R$.
In fact, $f = f^* - f^-$, where $f^* = \max\{f, 0\}$, $f^- = -\min\{f, 0\}$.

(xi) Each member $f$ of $R$ can be written in the form $f = 1/f_1 - 1/f_2$, where $f_1$ and $f_2$ are bounded positive functions in $R$.

Indeed, let
$$f_1 = \frac{1}{f + 1}; \quad f_2 = \frac{1}{f + 1},$$

where $f^*$ and $f^-$ have the same meaning as in the proof of (x).

(xii) For each $f$ in $R$ there is a continuous function $h$ defined on $\overline{F_R(X)}$ such that $f(p) = h(F_R(p))$ for each $p \in X$.

In view of (x), compact one can assume that $f$ is a non-negative function.

Then, by $3°$, $g = (f + 1)/f \in R^*$ (see the definition of the mapping $F_R$ given at the beginning of this paper). Let
$$h(a) = \frac{1 - p_0(a)}{p_0(a)},$$

where $p_0(a)$ denotes the $g$th coordinate of a point $a \in F_R(X)$. Then $h$ is the required function (this function is well-defined on $\overline{F_R(X)}$, since the $g$th coordinate of the point $a \in F_R(X)$ lies in the interval $0 < t \leq 1$).

Theorem 2. If $R$ is a linear ring of continuous real-valued functions defined on a topological space $X$ which satisfies the conditions $1°$-$3°$, then each non-trivial linear multiplicative functional $\varphi$ defined on $R$ is of the form (x) if and only if $\overline{F_R(X)}$ is $Q$-closed in $F_R(X)$.

Proof. Suppose that $F_R(X)$ is $Q$-closed in $F_R(X)$ and let $\varphi$ be a non-trivial linear multiplicative functional defined on $X$. Denote by $B_i$ the ring of all continuous functions defined on $F_R(X)$. In virtue of (i) and (ii) a one-to-one correspondence can be established between bounded members of $B_i$ and all members of $B_i$; corresponding functions $f, h \in R_i$, $h \in R_i$ satisfy the equality $f(p) = h(F_R(p))$ for each $p \in X$. Let us set $\varphi_i(h) = \varphi(h)$. Then $\varphi_i$ is a non-trivial linear multiplicative functional defined on $R_i$. Since $F_R(X)$ is compact, there is a point $a_\infty \in F_R(X)$ such that $\varphi_i(a_\infty) = h(a_\infty)$ for each $h \in R_i$. We shall show that $a_\infty \in F_R(X)$. 
Indeed, if \( a_0 \in \mathcal{F}_R(X) \setminus \mathcal{F}_b(X) \), then, by (iii), there is a continuous function \( h \) defined on \( \mathcal{F}_R(X) \) such that \( h(a_0) = 0 \) and \( h(x) \neq 0 \) for each \( x \) in \( \mathcal{F}_b(X) \). Let \( f \) be the function in \( R \) which corresponds to \( h \). Then \( f(p) \neq 0 \) for each \( p \) in \( X \), whence, by 3°, \( 1/f \in \mathcal{B}_R \), and it follows that \( \varphi(f) \neq 0 \). On the other hand, \( \varphi(f) = \varphi(h) = h(a_0) = 0 \), and this leads to a contradiction.

Now, let \( p_0 \) be any point in \( X \) with \( \mathcal{F}_b(p_0) = a_0 \). Using (i) and the definition of the functional \( \varphi_1 \), one can easily show that \( \varphi(f) = f(p_0) \) for each bounded function \( f \) in \( R \). Using (xi) we infer that the above equality holds true for each function \( f \) in \( R \). Thus the first part of our theorem is proved.

Conversely, suppose that \( \mathcal{F}_b(X) \) is not \( Q \)-closed in \( \mathcal{F}_R(X) \). Then, by (iii), there is a point \( a_0 \in \mathcal{F}_R(X) \setminus \mathcal{F}_b(X) \) such that for each continuous function \( h \) defined on \( \mathcal{F}_R(X) \) which is strictly positive on \( \mathcal{F}_b(X) \), we have \( h(a_0) > 0 \). Let \( f \) be any member of \( R \). By (xii), there is a continuous function \( h \) defined on \( \mathcal{F}_R(X) \) such that \( f(p) = h(\mathcal{F}_b(p)) \) for each \( p \) in \( X \). We shall show that \( h \) admits a continuous extension over \( \mathcal{F}_b(X) \cup \{a_0\} \). In fact, let

\[
f = \frac{1}{f_1} - \frac{1}{f_2},
\]

where \( f_1, f_2 \) are bounded positive functions in \( R \). By (i), there are continuous functions \( h_1, h_2 \) defined on \( \mathcal{F}_R(X) \) such that \( f_1(p) = h_1(\mathcal{F}_b(p)) \) for each \( p \) in \( X \) (\( i = 1, 2 \)). Since \( h_i \) (\( i = 1, 2 \)) are strictly positive on \( \mathcal{F}_b(X) \) and \( h(a_0) > 0 \) (\( i = 1, 2 \)), it follows that the function \( \frac{1}{f_1} - \frac{1}{f_2} \) is continuous on \( \mathcal{F}_b(X) \cup \{a_0\} \) and clearly it is an extension of \( h \).

Let us set \( \varphi(f) = h^*(a_0) \), where \( h^* \) is the continuous extension of \( h \) over \( \mathcal{F}_b(X) \cup \{a_0\} \). Then \( \varphi \) is a non-trivial linear multiplicative functional defined on \( R \). Using (iii), it is easy to show that \( \varphi \) is not of the form (ii).

V. Consequences of Theorem 2.

**Theorem 3.** If \( X \) is a Lindelöf space and \( R \) is any linear ring of continuous functions defined on \( X \) satisfying the conditions 1°-3°, then each non-trivial linear multiplicative functional \( \varphi \) defined on \( R \) is of the form (ii).

Conversely, if \( X \) is not a Lindelöf space, then there is a linear ring \( R \) of continuous functions on \( X \) satisfying the conditions 1°-3° and a non-trivial linear multiplicative functional \( \varphi \) defined on \( R \) which is not of the form (ii).

**Proof.** If \( X \) is a Lindelöf space, then each continuous image of \( X \) is a Lindelöf space and a Lindelöf space is \( Q \)-closed in each of its compactifications (clearly \( \mathcal{F}_R(X) \) is a compactification on \( \mathcal{F}_R(X) \)).

Conversely, if \( X \) is not a Lindelöf space, then there is a compactification \( kX \) of \( X \) such that \( X \) is not \( Q \)-closed in \( kX \). Let \( R \) be the least ring satisfying the conditions 1°-3° that contains all functions on \( X \) which admit a continuous extension over \( kX \). Then \( R \) is a homeomorphism and \( \mathcal{F}_R \) can be extended to a continuous mapping of \( kX \) onto \( \mathcal{F}_R(X) \). It follows that \( \mathcal{F}_R(X) \) is not \( Q \)-closed in \( \mathcal{F}_R(X) \).

**Theorem 4.** If \( R_1, R_2 \) are linear rings of continuous functions on \( X \) satisfying the conditions 1°-3°, \( R_1 \) distinguishes points and closed sets \(^4\) and \( R_1 \subset R_2 \), then if each non-trivial linear multiplicative functional \( \varphi \) on \( R_1 \) is of the form (ii), the same holds true for the ring \( R_2 \).

**Proof.** The mappings \( \mathcal{F}_R \) and \( \mathcal{F}_B \) are homeomorphisms, whence \( \mathcal{F}_R(X) \) and \( \mathcal{F}_B(X) \) can be regarded as compactifications of \( X \). One can easily verify that \( \mathcal{F}_R(X) \subset \mathcal{F}_B(X) \), whence the statement of the theorem follows directly from Theorem 2 and (iv).

References


\(^4\) I.e. for each closed set \( A \subset X \) and each point \( p \in X \setminus A \) there is a \( f \) in \( R \) with \( f(p) \neq f(A) \).

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