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Partial confluence of maps onto graphs and inverse limits of single graphs

by

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Abstract. P(M) is the smallest integer such that if X is any continuum, f is any map from X onto M, and K is any subcontinuum of M, then there are P(M) or fewer continua in X the union of whose images under f is K. A formula is given for P(G) when G is a graph. In addition, an affirmative answer is given to a question of Hagopian who asked if an aposyndetic continuum that is the inverse limit of a single graph is locally connected.

1. Introduction. A general problem for a continuum M is to find the smallest integer P(M) such that if X is any continuum, f is any map from X onto M, and K is any subcontinuum of M, then there are P(M) or fewer continua in X the union of whose images under f is K. For example, class[M] is the set of all continua M for which P(M) = I, and if M is a simple closed curve or a simple triod, P(M) = 2. The first author has shown [7, Theorem II.2] if M is a continuum that, for some integer n, contains an n-od but no (n + 1)-od, n > 1, then P(M) < |n(n−1)|. One purpose of this paper is to show that if M is a graph, P(M) ≤ |n−1|. More precisely, P(M) = |n−1| where t(M) is the number of points in M of order 1.

A second purpose is to answer a question of Charles Hagopian who asked if an aposyndetic continuum that is the inverse limit of a single graph is locally connected. It is proved here that if X is semi-aposyndetic and is the inverse limit of continua for which there is an integer n such that no factor contains an n-od, then X is a graph.

2. Partial confluence of maps onto graphs. A continuum is a compact connected metric space (with metric g). A continuum M is an n-od, where n is an integer greater than 1, if M contains a subcontinuum K, called the core of the n-od, such that $M \setminus K$ has n components. If M is a continuum, let n(M) be the largest integer (if it exists) such that M contains an n(M)-od. A map is a continuous function. If f is a map from a continuum X onto a continuum Y, then a subcontinuum K of Y is an $w_f$-set if there is a continuum $K'$ in X such that $f(K') = K$. A subcontinuum J of a subcontinuum K of Y is a maximal $w_f$-set in K if J is a $w_f$-set, and J is not a proper subcontinuum of a $w_f$-set which is contained in K. The map f is $n$-partially confluent if every subcontinuum of Y is the union of n or fewer $w_f$-sets. For the continuum M let P(M) be the largest integer such
that there is a map $f$ from a continuum onto $M$ that is not $(P(M) - 1)$-partially confluent. Note that $P(M)$ is the smallest integer such that for every map of a continuum onto $M$, every subcontinuum of $M$ is the union of $P(M)$ or fewer $w_f$-sets.

A subcontinuum $A$ of a continuum $X$ is a free arc in $X$ if $A$ is an arc such that the boundary of $A$ is contained in the set of endpoints of $A$. A continuum $G$ is a graph if it is the union of a finite number of free arcs. For a graph $G$, let $e(G)$ be the number of edges of $G$, $v(G)$ the number of points of order one in $G$, called terminal points, and $v(G)$ the number of vertices of $G$ (here, a point of order one is not a vertex). Let $\beta(G)$ be the first Betti number for $G$. A spanning tree for $G$ is an acyclic subcontinuum of $G$ that contains all of the vertices and terminal points of $G$, and whose edges are edges of $G$. If $H$ is a spanning tree, $\beta(H) = \beta(G)/H = e(G) - e(H) = e(G) - ([G] + v(G)) + 1 [1, Theorem 1, p. 36]. If $K$ is a subcontinuum of $G$, a component of $G \setminus K$ whose closure is an arc with both endpoints in $K$ or a simple closed curve with one point in $K$ is a chord of $K$. Clearly, every spanning tree has $\beta(G)$ chords. The following lemma is probably well known, but the proof is short, and is included here for completeness.

**Lemma 1.** If $G$ is a graph, then every finite collection of points of order two in $G$ that does not separate $G$ is contained in a collection of $(p(G) + 1)$ points of order two in $G$ that does not separate $G$, and every collection of $\beta(G) + 1$ points of order two in $G$ separates $G$.

**Proof.** Let $x_1, \ldots, x_n$ be a finite collection of points of order two in $G$ that does not separate $G$, and such that the addition of any point to this collection yields a collection that does separate $G$. For each $j$, let $J_j$ be the interior of the edge of $G$ that contains $x_j$. Then $H = G \setminus \bigcup J_j$ is acyclic, since every arc of order two in $H$ separates $H$. Therefore, $H$ is a spanning tree for $G$, and $\beta(G) = e(G) - e(H) = \gamma$.

**Theorem 1.** If $G$ is a graph, then $\beta(G) = \beta(\gamma) + \gamma$.

**Proof.** Let $D$ be an $(\gamma)$-od in $G$ with core $K$. It follows that $K$ must contain each vertex of $G$ of order greater than two. For if it did not contain a vertex, then $K$ could be extended by an arc to a continuum $K'$ which contains that vertex, and is the core of an $(\gamma)$-od in $G$.

Each component of $G \setminus K$ is either a chord of $K$ or an arc, one of whose endpoints is a terminal point of $G$. Each of the latter type of component contains exactly one component of $D \setminus K$, and each chord of $K$ contains exactly two components of $D \setminus K$. A collection of points consisting of one element from each chord of $K$ does not separate $G$. So, by Lemma 1, the maximum number of chords of $K$ is $\beta(G)$. Thus $\gamma(G) \leq \beta(G) + \gamma(G)$.

On the other hand, $G$ must contain a spanning tree (see the proof of Lemma 1), and the spanning tree minus the interiors of the edges of $G$ that contain the terminal points of $G$ is the core of a $(\gamma)$-od. Thus, $\gamma(G) = \beta(\gamma) + \gamma(G)$.

**Lemma 2.** Let $f$ be a map of a continuum $X$ onto the continuum $M$. Let $K$ be a subcontinuum of $M$, and $C_1$ and $C_2$ be disjoint nonempty closed subsets of $K$ such that $Bd(K) = C_1 \cup C_2$. Then there exists a connected set $A$ that is either a $w_f$-set in $K$, or the union of two $w_f$-sets in $K$, such that $A \cap C_i \neq \emptyset$ and $A \not\subset C_i$. Moreover, if no $w_f$-set in $K$ intersects $C_1$ and $C_2$, then there is a component of $M \setminus K$ whose closure intersects $C_1$ and $C_2$.

**Proof.** For $i = 1, 2$, let $A_i$ be the set of all points $x$ in $K$ such that there is a continuum in $X$ whose image contains $x$, lies in $K$, and intersects $C_i$. Note that $A_1$ and $A_2$ are nonempty closed subsets whose union is $K$. Let $y$ be an element of $A_1 \cap A_2$. Then there exist $w_f$-sets $Y_1$ and $Y_2$ such that $y \in Y_1 \cap Y_2$, and $Y_1 \cap C_i \neq \emptyset \neq Y_2 \cap C_2$, so $A_1 = Y_1 \cup Y_2$ is the required set (it is possible that $Y_1$ or $Y_2$ might intersect both $C_1$ and $C_2$).

Suppose the closure of no component of $M \setminus K$ intersects $C_1$ and $C_2$. Then $M \setminus K = Q_1 \cup Q_2$, a separation, such that $\cl(Q_1) \cap K = C_1$ and $\cl(Q_2) \cap K = C_2$. Let $B$ be a subcontinuum of $X$ irreducible between $f^{-1}(C_1)$ and $f^{-1}(C_2)$. Then $B$ is a $w_f$-set in $K$ intersecting $C_1$ and $C_2$.

**Lemma 3.** If $K$ is a subgraph of a graph $G$, and $E_1, \ldots, E_n$ are arcs such that both endpoints of each arc in $K$ are in $G$, and rest of the arc is in $G \setminus K$, and no one of the arcs is contained in the union of the others, then there exist points $a_1, a_2, \ldots, a_n$ such that for $1 \leq i \leq n$, $a_i \in E_i$ and $\bigcup_{i=1}^{n} \{a_i\}$ does not separate $G$.

**Proof.** Let $E_i$ be a free open arc lying in $E_i$, such that $E_1 \cap \bigcup_{i=2}^{n} E_i = \emptyset$, and let $a_i$ be a point in $E_i$. Then $G \setminus \{a_i\}$ is connected. Suppose for $1 \leq i \leq n$, points $a_1, \ldots, a_i$ and arcs $E_1, \ldots, E_i$ have been selected so that $a_i \in E_i$, $1 \leq i \leq k$, $E_i \cap \bigcup_{j=1}^{i-1} E_j = \emptyset$ for $1 \leq i \leq k$ and $1 \leq j \leq n$, and $\bigcup_{i=1}^{n} \{a_i\}$ does not separate $G$. Let $E_{k+1}$ be a free open arc lying in $E_{k+1}$ such that $E_{k+1} \cap \bigcup_{i=1}^{k} a_i = \emptyset$, and let $a_{k+1} \in E_{k+1}$. Since $E_1 \cap \bigcup_{i=2}^{n} E_i = \emptyset$, $(G \setminus \{a_1, \ldots, a_k, a_{k+1}\}) = G \setminus \{a_1, \ldots, a_k\}$ is connected. By induction $G \setminus \{a_1, \ldots, a_k\}$ is connected, where each point $a_i \in E_i$ for $1 \leq i \leq n$.

**Theorem 2.** If $G$ is the graph then $\beta(G) = 3(\beta(G) + \gamma(G)) - 1$.

**Proof.** Suppose $f$ is a map from a continuum onto $G$. Let $K$ be a subcontinuum of $G$ such that $K$ is acyclic, each boundary point of $G$ is a point of order two in $G$, and $K$ does not contain a terminal point of $G$. In this case, $G$ is irreducible about its boundary $B$. Since $|B| \leq n(G)$, it follows from Theorem 1 that $|B| \leq 3(\gamma(G) + \gamma)$.

According to Lemma 2, if $B \neq \emptyset$, there is a point $b_{k+1}$ in $B \setminus \{a_i\}$ and $w_f$-sets $E_1$ and $E_2$ in $K$ such that $E_1 \cap \{a_1, \ldots, a_k, b_{k+1}\} = \emptyset$. Then $G \setminus \{a_1, \ldots, a_k, b_{k+1}\}$ is a maximal collection of points in $B$ that contains $b_{k+1}$, and is contained in the union of a collection $S = \{E_1, \ldots, E_{k+1}\}$ of $w_f$-sets in $K$ such that $\bigcup S$ is connected. Note that if no $w_f$-set in $K$ contains $b_{k+1}$ and another point of $B$, then $S$ may be empty. Also, note that no $w_f$-set in $K$ contains a point of $\{a_i\}$ and a point of $B \setminus \{a_i\}$.

According to Lemma 2, if $B \neq \emptyset$, there is a point $b_{k+1}$ in $B \setminus \{a_i\}$ and $w_f$-sets $E_1$ and $E_2$ in $K$ such that $E_1 \cap \{a_1, \ldots, a_k, b_{k+1}\} = \emptyset$. Then $G \setminus \{a_1, \ldots, a_k, b_{k+1}\}$ is a maximal collection of points in $B \setminus \{a_i\}$, that contains $b_{k+1}$, and is contained in the union of a collection $S = \{E_1, \ldots, E_{k+1}\}$ of $w_f$-sets in $K$ such that $\bigcup S$ is connected.

Suppose $a_i \notin \{a_i\}$, where $\{a_i\}$ is contained in $B$, and $a_i$ have been defined for $1 \leq i \leq k$, let $v = \sum_{i=1}^{k} x(i)$, and suppose $B \cup \{a_i\} \neq \emptyset$. By Lemma 2, there is a point $b_{k+1}$ in
Suppose $K$ is any subcontinuum of $G$. Then $K$ is the limit of a sequence $\{K_n\}_{n=1}^\infty$ of subcontinua of $G$ such that for each $i$, $K_i$ is acyclic, the boundary points of $K_i$ have order two, and $K_i$ is the union of $n = 3\beta(G) + (t(G) - 1)$ or fewer $w_j$-sets. So for each positive integer $i$, there exist $w_j$-sets $Q_1, \ldots, Q_i$ such that $K_i = \bigcup_{j=1}^i Q_j$. Choosing subsequences if necessary, assume that $\{Q_j\}_{j=1}^\infty$ converges to a continuum $Q_i$ for $1 \leq j \leq n$. Clearly $Q_j$ is a $w_j$-set for $1 \leq j \leq n$, and $\bigcup_{j=1}^i Q_j$ is contained in $K$. To see that $\bigcup_{j=1}^i Q_j = K$, let $x$ be an element of $K$. For every positive integer $i$, there exists $y_i \in K_i$ such that $\lim y_i = y$. For every positive integer $i$, there exists an integer $a(i)$, $1 \leq a(i) \leq n$, such that $y_i \in Q_{a(i)}$. There is an integer $\alpha$, $1 \leq \alpha \leq n$, such that $a(i) = \alpha$ for infinitely many $\alpha$. Without loss of generality assume that $a(i) = \alpha$ for all the $\alpha$. Then $y_i \in Q_{\alpha}$ for all $\alpha$, and $\lim y_i = \lim y_{\alpha} = y_{\alpha}$ which is contained in $\bigcup_{j=1}^i Q_j$. Hence $K$ is the union of $n = 3\beta(G) + (t(G) - 1)$ or fewer $w_j$-sets.

Let $K$ be an acyclic subcontinuum of $G$ such that $K$ contains all the vertices of $G$, the boundary points of $K$ have order two, and $K$ does not contain any of the terminal points of $G$. We will produce a map $f$ from a continuum onto $G$ such that $K$ is not the union of fewer than $3\beta(G) + (t(G) - 1)$ or fewer $w_j$-sets.

By an end arc of $K$ is meant an arc in $K$ which contains a terminal point of $K$ and is contained in a free arc of $K$. If $\beta(G) \neq 0$, there are $\beta(G)$ pairs of end arcs, $(a_i, b_i)$, $(a_i, b_i)$, $1 \leq i \leq \beta(G)$, where $a_i$ and $b_i$ are terminal points of $K$, and there is an arc $[a_i, b_i]$ in the closure of $G$. Note that if $\beta(G) = 0$, then there are an arc $a_i, b_i$ in the closure of $G$. For each end arc $(a_i, b_i)$ of $K$, let $x_i$ be a point in the interior of $[a_i, b_i]$. For each $x_i$ from 2 to $\beta(G)$, let $f(x_i)$ be an arc in $K \cup [a_i, b_i]$, which is irreducible from $x_i$ to $a_i$ and does not intersect $(a_i, b_i)$. Note that $[a_i, b_i]$ must contain $(a_i, b_i)$. Let $f(a_i)$ be an arc in $K \cup [a_i, b_i]$ which is irreducible from $a_i$ to $b_i$ and does not contain $b_i$.

If $(t(G) - 1)(t(G) - 1) = \beta(G) + (t(G) - 1)$ and $i \neq 1$, let $f(b_i)$ be an arc in $K \cup [a_i, b_i]$ which is irreducible from $b_i$ to $x_i$ and which does not intersect $(a_i, b_i)$.

Let $F$ be a simple fan which consists of $2\beta(G) + (t(G) - 1)$ or fewer $w_j$-sets. Define the map $f$ from $F$ onto $G$ as follows. For $1 \leq j \leq \beta(G) + (t(G) - 1)$, map one leg of $F$ one-to-one onto $K$, sending $x_i$ to $x_i$. For $1 \leq j \leq \beta(G) + (t(G) - 1)$, map one leg of $F$ onto $K$, sending $x_i$ to $x_i$. Note that each map $F$ onto $K$, $1 \leq j \leq \beta(G) + (t(G) - 1)$, contains exactly two $w_j$-sets which is maximal in $K$, and each $F$, $1 \leq j \leq \beta(G) + (t(G) - 1)$, contains exactly two $w_j$-sets which are maximal in $K$. Observe also that these $2\beta(G) + (t(G) - 1)$ or fewer $w_j$-sets are all necessary in order for their union to be $K$. Therefore, $K$ is the union of fewer than $3\beta(G) + (t(G) - 1)$ or fewer $w_j$-sets. So $P(G) = 3\beta(G) + (t(G) - 1)$.

Since $\beta(G) = \beta(G) - (\beta(G) + (t(G) - 1) - 1)\beta(G) = 3\beta(G) + (t(G) - 1)$, $P(G) = 3\beta(G) + (t(G) - 1)$ is a formula which makes $P(G)$ trivial to compute. Also, from Theorem 1 it follows that $\beta(G) = 3\beta(G) + (t(G) - 1)$. If $(G) = 0$, then $P(G) = \beta(G) - 1$, and in general, $P(G) = \beta(G) - 1$, which suggests the following question.

**QUESTION 1.** Is there a continuum $X$ such that $P(X) > 3\beta(X) - 1$?

The next theorem will allow us to consider $P(X)$ for a larger collection of continua.

**THEOREM 3.** Suppose $X$ is a positive integer, and the continuum $X = \lim X_{n=1}^\infty$, where each $\lim X_{n=1}^\infty$ is a continuum such that $P(X_{n=1}^\infty) \leq n$. Then $P(X) \leq n$.

**Proof.** For each positive integer $i$ there is a map $g_i$ from $X$ onto some $\mathbb{X}_i$, such that $\text{diam}(g_i^{-1}(g_i(x))) \leq 1/i$ for each $x$ in $X$. (4, Lemma 1.162, p. 167) Let $f$ be a map from the continuum $X$ onto $G$, and let $L$ be a subcontinuum of $X$. Since $g_i$ is $n$-partially confluent, for each positive integer $i$ there is a collection $K_i$, ..., $K_n$, $i \leq n$, in $X$ such that $\bigcup_{j=1}^i g_i^{-1}(K_j) = g_i(L)$. Let $L = f(K)$ for each $j$, $1 \leq j \leq n$. Choosing subsequences if necessary, assume that for each $j$, $1 \leq j \leq n$, the sequence $(g_i^{-1}(K_j))_{n=1}^\infty$ converges to a continuum $L_j$ in $X$, and the sequence $(K_j)_{n=1}^\infty$ converges to a continuum $K_j$ in $X$. It follows that $f(K_j) = L_j$ for each $j$, $1 \leq j \leq n$.

If $x$ is a point in $L$, there is a map $f$ from the positive integers into the integers from 1 to $n$, and a sequence of points $\{K_{n=1}^\infty\}_{n=1}^\infty$ such that $K_0 = \lim K_{n=1}^\infty$, and $g_i(\lim K_{n=1}^\infty) = g_i(x)$ for each positive integer $i$. There is a $j$, $1 \leq j \leq n$, such that $a_j = f$ for infinitely many $f$. Choosing subsequences if necessary, assume that $a_j = f$ for each positive integer $i$. Then $g_i^{-1}(K_j)_{n=1}^\infty$ converges to a point $p_j$ in $K_j$, and $f(p_j) = \lim f(\lim K_{n=1}^\infty) = x$ since $\text{diam}(g_i^{-1}(g_i(x))) \leq 1/i$ for each positive integer $i$. So $x \in L_j$. We have shown that $L = \bigcup_{j=1}^i L_j$, and since each $L_j$ is a $w_j$-set, $L$ is the union of $n$ $w_j$-sets.
If $X$ is the inverse limit of a single graph, define $P^*(X)$ to be the minimum of \{ $P(G)$ | $G$ is a graph and $X$ is the inverse limit of $G$ \}. According to Theorem 3, $P(X) \leq P^*(X)$. For example, if $M$ is the Ingram continuum [2, p. 100] then $M$ is the inverse limit of a simple triod, $X$ is not the inverse limit of an arc [2, Theorem 3, p. 106], and $M$ is in class $W$ [3, Theorem 1, p. 190]. So $P^*(M) = 2$ and $P(M) = 1$.

3. Inverse limits of a single graph. The purpose of this section is to answer a question of Charles Hagopian who asked if an aposyndetic continuum that is the inverse limit of a single graph is locally connected. This question is related to the more general problem of when a one-dimensional aposyndetic continuum is locally connected. For example, it is not known if a one-dimensional unicoherent and mutually aposyndetic continuum is locally connected [see the Problem Book, problem 48]. Since every one-dimensional continuum is the inverse limit of graphs, it is natural to view those continua that are the inverse limits of a single graph as an important subclass of one-dimensional continua.

A map $f : X \rightarrow Y$ is an $e$-map if $e$ is a positive number such that $f^{-1}(y)$ has diameter less than $e$ for each $y$ in $X$. A space $X$ is semi-aposyndetic if for each pair of points in $X$ there is a continuum in $X$ that contains one of the points in its interior and does not contain the other point.

**Theorem 4.** If a continuum $X$ contains an $n$-od, then there is a positive number $e$ such that if $f$ is an $e$-map from $X$ onto $Y$, then $Y$ contains an $n$-od.

**Proof.** Suppose $C$ is an $n$-od with core $K$ in $X$. Let $\{L_1, \ldots, L_n\}$ be the components of $C$. For each $i$, $1 \leq i \leq n$, let $x_i$ be an element of $L_i$. Let $\delta_i = \min \{d(x_i, K)\}$, and let $\delta_0 = \min_{\delta_i}(d(x_i, \delta_0))$. Let $\epsilon = \min(\delta_i, \delta_0)/2$.

Suppose $f$ is an $e$-map from $X$ onto $Y$. For each $i$, the set $f(L_i \cup K)$ is disjoint from $f(L_i \cup K)$. For each $x$, the union of the $x_i$ is a continuum in $X$. Therefore, the continua in $\{f(L_i \cup K)\}_{i=1}^n$ have a point in common and no one of them is contained in the union of the others. The union of this collection contains an $n$-od [6, Theorem 1].

If a continuum $X = \lim(X_n, f_n)$, then for each positive number $e$ there is an $e$-map into some $X_n$ [4, Lemma 1.162, p. 167]. So, if there is a positive integer $n$ such that each $X_n$ does not contain an $n$-od, then $X$ does not contain an $n$-od.

**Theorem 5.** A continuum is a graph if and only if it is semi-aposyndetic and does not contain an infinite-od.

**Proof.** Every hereditarily locally connected continuum that does not contain an infinite-od is a graph [5, Theorem III.1, p. 568]. Suppose the continuum $X$ is semi-aposyndetic and not hereditarily locally connected. Then there is a sequence $\{K_i\}_{i=1}^\infty$ of disjoint continua in $X$ that converges to a nondegenerate continuum $K$ in $X$. Let $x$ and $y$ be different points in $K$. Without loss of generality, it can be assumed that there is a continuum $J$ in $X$ that contains $x$ in its interior and does not contain $y$. There is an integer $N$ such that if $n \leq N$, $K_n \cap J \neq \emptyset$, and $K_n \cup J \neq \emptyset$. Then $J \cup K \cup (\bigcup_{n=N}^\infty K_n)$ is an infinite-od.

The next two theorems follow immediately from Theorems 4 and 5.

**Theorem 6.** If $X = \lim(X_n, f_n)$ and each $X_n$ is a continuum, $X$ is semi-aposyndetic, and if there is a positive integer $n$ such that each $X_n$ does not contain an $n$-od, and each $X_n$ is a continuum, then $X$ is a graph.

Since, for a positive integer $n$, there are only finitely many graphs that do not contain an $n$-od, if each $X_n$ in the statement of Theorem 6 is a graph that does not contain an $n$-od, then $X$ is the inverse limit of a single graph. Clearly, if $G$ is a graph, there is an integer $n$ such that $G$ does not contain an $n$-od.

**Theorem 7.** If $X$ is the inverse limit of a single graph $G$, and $X$ is semi-aposyndetic, then $X$ is a graph.

**References**


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