all codes $k$ of those pairs $(i, j)$ for which $|i| = m$. We put $m(k) = |i|$ and $m(k) = 0$ for $k \notin \bigcup K_m$. (Of course $m(k) = 0$ also if $|i| = 0$.) So there is a natural bijection between $S_n$ and the set $B$ defined prior to Theorem 3 with $\mathcal{M} = \mathcal{M}_0$, $L = \mathcal{M}_0$ by this bijection. So we can write $\mathcal{M}_0 = \langle B, R_0, R_1, \ldots \rangle$.

Now we have to show that all the relations $R_1$ have the form prescribed prior to Theorem 3. Let $\langle (k(i, j, 1), \ldots, k(i, j, n(i)): j = 0, 1, \ldots) \rangle$ be an enumeration of all $n$-tuples of integers. Let

$$R_0 = R_1 \cap \{ (k(i, j, 1)) \times A^{[0,1]} \cup \cdots \cup \{ (k(i, j, n(i)): j = 0, 1, \ldots) \} \times A^{[0,1]} \}$$

Then, if we look again at the meaning of the codes $k \in \bigcup K_m$ and use the fact that $\mathcal{M}_0$ satisfies the axioms (a), by Lemma (i) it is easy to see that $R_0$ is of the form required prior to Theorem 3, with a formula $\varphi_0$ in the language of $\mathcal{M}_0$ with $\sum_{i=0}^m m(k(i, j, r))$ variables.

This concludes our proof that $\mathcal{M}_0$ is an $\mathcal{M}_0$-model of $T$. The inequality $K_i \neq \emptyset$ follows from the fact that $K_i$ must contain a code of the pair $(x_1, 1)$.

Now let $\mathcal{U}$ be an arbitrary dense linear order without endpoints. The same functions $k(i, j, r), m(k)$ and formulas $\varphi_0$ which we found for $\mathcal{M}_0$ yield a certain $\mathcal{M}_0$-model $\mathcal{M}$ with $K_i \neq \emptyset$. It remains to check that $\mathcal{M}$ satisfies $T$. But it is clear that every finite part of $\mathcal{M}$ is isomorphic to some finite part of $\mathcal{M}_0$. Since $\mathcal{M}_0 \models T^*$ and the axioms of $T^*$ are universal, $\mathcal{M} \models T^*$. Since $T^* \models T$, the proof is complete.

Note added in July 1989. After this paper was written the authors learned that the problem of existence of Borel models was independently posed and solved by H. Friedman, see [3] C. C. Steinhorn, Borel structures for first order and extended logics, in: Harvey Friedman Research in the Foundations of Mathematics, L. A. Harrington et al. (eds), Elsevier Science Publishers B. V. (North-Holland), 1983, 161–178.


The proofs of the existence of Borel models presented in those papers are closely related to ours, but we decided to keep Theorem 2 and its proof because it gives a sharper estimate of the Borel classes of the relations and because the concrete structure of the model described in Theorem 3 may be of independent interest. The papers [2] and [3] discuss additional aspects and extensions of Theorem 2 but its proofs given there are not as detailed as ours.

References


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The shrinking property of products of cardinals

by

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Abstract. It is known that for cardinals $\kappa > \omega$ and $\lambda > 1$, $\kappa^\lambda$ is normal if and only if $\kappa$ is regular and $\kappa < \lambda$. We show that normality can be replaced by the shrinking property in this result.

Ordinals and cardinals are considered as sets of smaller ordinals. In particular, $\kappa = \{0, 1, \ldots, n-1\}$ for each $n \in \omega$. Let $\{x_e : e \in \kappa\}$ be a collection of spaces. $\prod_{e \in \kappa} X_e$ denotes the usual Tikhonov product space of $X_e$. Each element $f$ of $\prod_{e \in \kappa} X_e$ is considered as a function whose domain is $\lambda$ and $f(e) \in X_e$ for each $e \in \kappa$. Whenver $X_e$ is a single space $X$ for each $e \in \kappa$, $\prod_{e \in \kappa} X_e$ is denoted by $X^\kappa$.

Let $X$ be a space and let $\kappa$ be a cardinal. Assume $\kappa$ is an open cover of $X$. A cover $\mathcal{V} = \{V(U) : U \in \mathcal{U}\}$ is said to be a shrinking of $\mathcal{U}$ if $\forall V(U) \subseteq U$ for each $U \in \mathcal{U}$. In particular, $\mathcal{V}$ is said to be an open (closed) shrinking of $\mathcal{U}$ if each member of $\mathcal{V}$ is open (closed, respectively). $X$ is said to have the $\kappa$-shrinking property if every open cover of size $\leq \kappa$ has an open shrinking. It has the shrinking property if it has the $\kappa$-shrinking property for every infinite cardinal $\kappa$. Note that $\kappa$-shrinking property is normality and that $\omega_1$-shrinking property is countable paracompactness plus normality. It is easy to show that a normal space which has the property that every open cover of size $\leq \kappa$ has a closed shrinking has the $\kappa$-shrinking property. Note that paracompact spaces, in particular compact Hausdorff and regular Lindelöf spaces, have the shrinking property. On the other hand, $\omega_1$ with the order topology has the shrinking property but is not paracompact. In general, ordered spaces have the shrinking property, see [Ke]. But the product space $\omega_1 \times (\omega_1 + 1)$ does not have the shrinking property, in fact it is not normal, see [Pr, 2.2]. But note that it is countably paracompact so it is a perfect preimage of the countably paracompact space $\omega_1$. Note that $\kappa$-shrinking property implies normality if $\kappa \geq 2$. It is strangely difficult to find an example of a normal space without the $\kappa$-shrinking property for $\kappa \geq \omega$. For each $\kappa \geq \omega$, we know of essentially one real such example, namely the $\kappa$-Dowker space, see [Ru1], [Ru2].

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For \( a \) and \( a' \in S(g) \) with \( g \in G \), define \( a \equiv a' \) by \( \gamma(a, g) = \gamma(a', g) \). Then clearly \( \equiv \) is an equivalence relation on \( S(g) \). Let \( \overline{S}(g) \) be the quotient of \( S(g) \) by \( \equiv \). And for each \( E \in \overline{S}(g) \), define \( \gamma(E) = \gamma(a, g) \) for some (in fact every) \( a \in E \). Then clearly \( \gamma(E) \neq \gamma(E') \) whenever \( E \neq E' \in \overline{S}(g) \) holds and \( \{ \gamma(a, g) : a \in E \} \) is an open cover of \( \{ \gamma(a', g) : a' \in E \} \). Define \( \overline{E} \) and \( \overline{E'} \) for each \( E \in \overline{S}(g) \). Then for each \( g \in G \), \( S_g = \{ \overline{E} : E \in \overline{S}(g) \} \) is a collection of closed sets in \( Y \) such that \( \overline{E} \subseteq U_{\overline{E}} \) for each \( E \in \overline{S}(g) \). For each \( g \in G \), define

\[
F_{\overline{E}} = \bigcup_{\overline{E} \subseteq U_{\overline{E}}} \{ \gamma(a, g) : a \equiv a' \} \quad \text{and} \quad \gamma(E) = \gamma(E). 
\]

Then each \( F_{\overline{E}} \) is closed in \( Y \) and contained in \( U_{\overline{E}} \). Therefore, since \( S_g \) covers \( \{ \overline{E} \} \times \{ V \} \), \( \gamma(E) \equiv \gamma(E') \) for each \( E \in \overline{S}(g) \). Then for each \( g \in G \), \( S_g \) covers a collection of closed sets in \( Y \). Since \( \overline{E} \subseteq U_{\overline{E}} \) for each \( E \in \overline{S}(g) \), we have \( \gamma(E) \equiv \gamma(E') \) for each \( E \in \overline{S}(g) \). Thus the proof of Fact 1 is complete.

To show \( \overline{S}(g) \) has the k-shrinking property, assume \( U = \{ U_{\overline{E}} : \gamma(E) \equiv \gamma(E') \} \) be an open cover of \( \overline{S}(g) \). As above, for each \( a \in S(g) \), define \( \delta(a) \equiv (a) \equiv \gamma(a, g) \) for each \( a \in S(g) \). For each \( \gamma(a, g) \in \overline{S}(g) \), fix a \( \gamma(a, g) \in \overline{S}(g) \) such that \( \beta(\gamma(a, g), \delta(a)) \subseteq \gamma(a, g) \). Furthermore, since \( \{ h \in [0, \delta^-1] : h < \delta \} \subseteq \kappa \), there is an \( h \) such that a stationary set \( S(h) \subseteq S(g) \) such that \( \gamma(h, g) = \beta(\gamma(a, g)) \) for each \( a \in S(g) \). This means

\[
\{ \beta(\gamma(a, g)) : a \in S(g) \} \subseteq \overline{U_{\overline{E}}}.
\]

For each \( a \in S(g) \). Note that for each \( g \in [0, \delta^-1] \), \( V_g = \bigcap_{h \in [0, \delta^-1]} \{ \gamma(h, g) \} \) is a clopen set of \( S(g) \). Put \( \delta = \max \{ \beta(\gamma(a, g)) : a \in S(g) \} \). Then \( S(g) = (S(g) \setminus [0, \delta^-1]) \) is stationary for each \( g \in G \). Then by (5),

\[
\beta(a, g) \subseteq U_{\overline{E}} \quad \text{for each} \quad a \in S(g) \quad \text{with} \quad g \in G.
\]
and \( \delta(\theta; \theta \in \kappa) \) in \( \kappa \) such that \( \delta(\theta) \in S_{\delta(\theta)} \) for each \( \theta \in \kappa \). First define
\[
\gamma(0) = \min \{ \gamma : \delta(\gamma) \in S_0, \delta(\gamma) = 0 \}
\]
and fix \( \delta(\theta) \in S_{\delta(\theta)} \) for each \( \theta \in \kappa \). Next assume that \( \gamma(\theta) \) and \( \delta(\theta) \) have already been defined for every \( \theta \in \kappa \). Put \( \gamma_\theta = \sup \{ \gamma(\theta) : \theta \in \kappa \} \). Since each \( S_\theta \) is
bounded in \( \kappa \), sup \( \{ \sup \{ S_\gamma \} : \gamma \in \kappa \} \in \kappa \). Thus there is a \( \gamma(\theta) \in \kappa \) such that \( S_{\delta(\theta)} \subseteq S_{\gamma_\theta} \). Note that \( \gamma(\theta) \in \kappa \) for every \( \theta \in \kappa \).
By fixing \( \delta(\theta) \in S_{\delta(\theta)} \) for each \( \theta \in \kappa \), we have constructed the desired sequences. For each \( \theta \in \kappa \), put \( H_\theta = \{ \delta(\theta); \theta \in \kappa \} \). Then since \( \delta(\theta) \in S_{\delta(\theta)} \) (thus \( \delta(\theta) = \gamma(\theta) \)) we have \( H_\theta = 0 \). Since \( \delta(\theta) \in \kappa \) is unbounded in \( \kappa \),
\[
\bigcup_{\theta \in \kappa} H_\theta = \bigcup_{\theta \in \kappa} H_\theta = \bigcup_{\theta \in \kappa} \{ \delta(\theta); \theta \in \kappa \} = 0.
\]
Thus \( \{ H_\gamma : \gamma \in \kappa \} \) is the desired collection of closed sets. This completes the proof of Fact 2.

Finally, for each \( \theta \in \kappa \), define \( Z_\theta = k^* \times [0, \beta] \times k^* \). Then each \( Z_\theta \) is clopen in \( k^* \) and \( k^* \setminus (0, \beta) \times k^* \). Moreover, since each \( Z_\theta \) is homeomorphic to \( \gamma \times [0, \beta] \times k^* \), each \( Z_\theta \) has the \( \kappa \)-shrinking property by Fact 1. Therefore for each \( \theta \in \kappa \), there is a collection \( \{ F_\gamma : \gamma \in \kappa \} \) of closed sets in \( k^* \) such that \( F_\gamma \subseteq \gamma \times [0, \beta] \times k^* \), \( \bigcup_{\theta \in \kappa} F_\gamma = Z_\theta \). Then it is easy to show that \( \{ \gamma_\theta \cup \{ \gamma_{\delta(\theta)} ;\gamma \in \kappa \} \) is a closed shrinking property of \( \kappa \). This completes the proof of the theorem.

Remark. By putting \( U_0 \cup U_0 = 0 \) for \( \theta \in \kappa \), the above proof shows the normality of \( k^* \). But in this case, Case 2 of Fact 2 cannot happen.

**Lemma.** A normal product \( \prod_{n \in \omega} X_n \) has the \( \kappa \)-shrinking property if and only if \( \prod_{n \in \omega} X_n \) has the \( \kappa \)-shrinking property for every finite \( S \in \kappa \).

Using this lemma, we can show:

**Corollary.** For cardinals \( \kappa > \omega \) and \( \lambda > 1 \), the following are equivalent:

(i) \( k^* \) has the \( \kappa \)-shrinking property,

(ii) \( k^* \) is regular and \( \lambda < \kappa \).

**Proof.** (i) \( \Rightarrow \) (ii). If \( k^* \) has the \( \kappa \)-shrinking property, then \( k^* \) is normal. Thus this follows from [Pr, 6.7].

(ii) \( \Rightarrow \) (i). Assume that \( \kappa \) is regular and \( \lambda < \kappa \). Then by [Pr, 6.7], \( k^* \) is normal. Furthermore, applying the above theorem and lemma, we can show that \( k^* \) has the shrinking property. The proof is complete.

The shrinking version of [Pr, 6.9] is also valid.

**Corollary.** Let \( \kappa \) be an arbitrary infinite cardinal. The space \( k^* \) has the \( \kappa \)-shrinking property if and only if \( \kappa \) is regular.

To end this paper, we note that we can remove the condition \( \lambda < \kappa \) from 6.8 of [Pr].

**Proposition.** Let \( \kappa, \lambda, \tau \) be cardinals with \( \omega < \kappa \) and \( \tau < \kappa \). Every cover of \( k^* \) by \( \tau \) open sets has a finite subcover. In particular, \( k^* \) is \( \tau \)-paracompact for every \( \tau < \kappa \).

**Proof.** Here \( \kappa \) denotes the cofinality of \( \kappa \). Let \( \tau \) be the first cardinal for which this proposition fails and let \( \Psi = \{ \gamma \} \) be an open cover of \( k^* \) which does not have a finite subcover. Put \( V_\gamma = \bigcup_{\gamma \in \Psi} U_\gamma \) for each \( \gamma \in \Psi \). Then \( \{ V_{\gamma} : \gamma \in \Psi \} \) is an increasing open cover, and by the definition of \( \tau \), \( k^* \setminus \bigcup_{\gamma \in \Psi} U_\gamma \) is non-empty. So fix \( \delta(\tau) \in \tau \) for each \( \gamma \in \Psi \). Define for each \( \alpha \in \lambda \), \( \delta(\alpha) = \sup \{ \delta(\alpha) : \alpha \in \kappa \} \). Then \( Z = \prod_{\alpha \in \lambda} [0, \delta(\alpha)] \) is compact, thus there is a \( \gamma \in \kappa \) such that \( \gamma = \delta(\alpha) \). But this yields a contradiction, since \( \delta(\tau) \in \tau \).

**Corollary.** Let \( \kappa, \lambda, \tau \) be cardinals with \( \omega < \kappa \) and \( \tau < \kappa \). Then every increasing open cover \( \{ U_\gamma : \gamma \in \kappa \} \) of \( k^* \) has an increasing open shrinking.

**Proof.** For such a cover \( \{ U_\gamma : \gamma \in \kappa \} \), there is a \( \gamma \in \kappa \) such that \( k^* \subseteq U_\gamma \) by the above proposition. Put \( V_\gamma = 0 \) for \( \gamma \in \gamma \) and \( V_\gamma = k^* \) for \( \gamma \in \gamma \). Then \( \{ V_{\gamma} : \gamma \in \kappa \} \) is the desired shrinking.

**References**


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