Contents of Volume 132, Number 1

B. L. Brechner, J. C. Mayer, and E. D. Tymchatyn, Inaccessibility, essential maps, and shape theory ........................................ 1-23
H. Patkowski, On $1^-$-homogeneous ANR spaces .................. 25-48
W. Just, The space $(\omega^n)^{\omega^0}$ is not always a continuous image of $(\omega)^{\omega^0}$ ........................................ 59-72
R. Cauty, Sur le nombre de coéufs d'une sous-variété ......... 73-88

The FUNDAMENTA MATHEMATICAÆ publish papers devoted to Set Theory, Topology, Mathematical Logic and Foundations, Real Functions, Measure and Integration, Abstract Algebra

Each volume consists of three separate issues

Manuscripts and correspondence should be addressed to:
FUNDAMENTA MATHEMATICAÆ, Śniadeckich 8, 00-950 Warszawa, Poland

Paper for publication should be submitted in two typewritten (double spaced) copies and contain a short abstract. Special types (Greek, script, boldface) should be marked in the manuscript and a corresponding key should be enclosed. The authors will receive 75 reprints of their articles.

Orders for library exchange should be sent to:
INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, Exchanges, Śniadeckich 8, 00-950 Warszawa, Poland

The Fundamenta Mathematicae are available at your bookseller or at
ARS POLONA, Krakowskie Przedmieście 7, 00-068 Warszawa, Poland

© Copyright by Państwowe Wydawnictwo Naukowe, Warszawa 1989

ISBN 83-01-08646-3 ISSN 0016-2736

---

Inaccessibility, essential maps, and shape theory

by

B. L. Brechner (Gainesville, FL), J. C. Mayer * (Birmingham, AL)
and E. D. Tymchatyn (Saskatoon, SK)

Abstract. The accessibility of a point $p$ of a compactum $X \subset E^n$ from a complementary domain $U \subset E^n - X$ can be characterized in terms of the homotopy classes of certain maps. Let $p_U$ be the projection from $p$ of $E^n - \{p\}$ radially onto an $S^{n-1}$ with $p$ as center. Let $q \in U$. Then $p$ is accessible from $U$ iff $p_U \circ X - \{p\}$ is access $p_{U}$X - \{p\}$. In particular, if $U$ is the unbounded complementary domain of $X$, then $p$ is accessible from $U$ iff $p_U \circ X - \{p\}$ is inessential. As an application, suppose $X$ is a cellular plane continuum with an inaccessible point $p$ (for example the pseudo-arc). Then $X$ has trivial shape, but $X - \{p\}$ admits an essential map to $S^1$.

Nevertheless, $X - \{p\}$ is shape incompatible to $S^1$ in the weak and strong shape theories of Borsuk and the shape theory of Fox. It also follows that in the strong shape theory of Borsuk and in the shape theory of Fox, $X - \{p\}$ does not have trivial shape.

1. Introduction. We obtain a characterization, in terms of the homotopy classes of certain maps, of the accessibility of a point $p$ of a compactum $X \subset E^n$ from a complementary domain $U \subset E^n - X$ ($n \geq 2$). Theorem 3.1 essentially says the following: Let $U$ be a complementary domain of $X$ in $E^n$ and let $q$ be a point in $U$. Let $p_U$ denote the projection from $p$ of $E^n - \{p\}$ radially onto an $S^{n-1}$ with $p$ as center. Then $p$ is accessible from $U$ iff $p_U \circ X - \{p\}$ is access $p_{U}$X - \{p\}$. As a special case, Theorem 3.2 says that if $U$ is the unbounded complementary domain of $X$ in $E^n$, then $p$ is accessible from $U$ iff $p_U \circ X - \{p\}$ is inessential.

David Bellamy has asked us in conversation, the following two questions: (1) Does the pseudo-arc minus a point admit an essential map to $S^1$? (2) Is there a space with trivial shape which admits an essential map to $S^1$? An affirmative answer to (1) follows from the above results on accessibility.

It seemed likely that the pseudo-arc minus an (end) point had trivial shape, since it is the intersection of a tower of open 2-balls. However, in Section 5, where we consider several different extensions of shape theory to noncompact metric spaces, we show that the pseudo-arc minus a point does not have the shape of a point in any version we consider.

The answer to the second question may depend upon the version of shape theory one considers; in the version of shape theory due to Fox, that $X$ has trivial shape is equivalent to there being no essential map from $X$ to any ANR.

In Section 4, we determine sufficient conditions for a continuum $X$ minus a point
To admit an essential map to \( S^1 \), regardless of the embeddability of \( X \) in \( E^3 \), or the accessibility of point \( p \).

In Section 5, we obtain for each integer \( n \geq 1 \), an example of a cellular continuum \( X_n \) in \( E^{n+1} \) such that for a certain point \( p_n \in X_n \), \( X_n - \{p_n\} \) admits an essential map to \( S^n \), has nontrivial shape, and, for \( n = 1 \), is shape incomparable to any compact, noncontractible ANR, e.g. \( S^1 \). We also review the definitions and elementary properties of several versions of shape theory, including Borsuk's strong and weak shape theories and Fox's shape theory, all of which agree on compact metric spaces. Of course, \( X_n - \{p_n\} \) is not compact, which leads to some counter-intuitive results. We confine our attention to metric spaces throughout.

Understanding the construction of the pseudo-arc is not required for this paper, however, the interested reader is referred to [B] or [M].

We wish to thank David Bellamy, David Wilson, and Juan Toledo for helpful conversations. We also wish to thank the referee for extensive remarks which led to improvements in the paper, in particular, shortening the proof of Theorem 3.1, simplifying the arguments in Sections 5.4-5.4.4, and clarifying the proof of Lemma 5.4.6.

2. Preliminary definitions and theorems. A compactum is a compact (subset of \( E^n \)) metric space, and a continuum is a connected compactum. A domain is a connected open set in \( E^n \) (\( S^n \)). The double arrow in \( f : X \rightarrow Y \) means that \( f \) is an onto map. All maps are continuous. By \( f = q \) we mean that the map \( f \) is homotopic to the map \( q \). By \( f = 0 \) we mean that the map \( f \) is homotopic to a constant map. A map \( f : X \rightarrow S^n \), \( \geq 1 \), is called inessential iff \( f = q \); otherwise \( f \) is called essential. Unless stated otherwise, assume \( n \geq 2 \).

Let \( Y \subseteq E^n \) (\( S^n \)). By \( Cl(Y), Bd(Y), \) and \( Int(Y) \) we mean the closure, boundary, and interior of \( Y \), respectively, as a subset of \( E^n \) (\( S^n \)). If \( Cl(Y) \subseteq E^n \) is compact, we mean by \( Ext(Y) \) the unbounded complementary domain of \( Cl(Y) \) in \( E^n \).

Let \( X \) be a compactum in \( E^n \) (\( S^n \)), and let \( p \) be a point in \( X \). We say \( p \) is accessible iff there is an arc \( A \subseteq E^n \) such that \( A \cap X = \{q\} \). Otherwise \( p \) is called inaccessible. If \( X \) separates \( E^n \) (\( S^n \)), we say that \( p \) is accessible from a complementary domain \( U \subseteq X \) in \( E^n \) (\( S^n \)) provided that \( A = U \cup \{p\} \). If \( X \) separates \( E^n \) into exactly two complementary domains, we denote the unbounded domain by \( Ext(A) \) and the bounded domain by \( Int(A) \). Though we use "Ext" and "Int" in two distinct ways, context will make clear which is intended.

DEFINITION of the map \( \pi_n \). See p. 97 of [H–W]. Let \( S^n - 1 \) be the \((n-1)\)-sphere in \( E^n \) of radius 1 centered at the origin 0. For each point \( p \in E^n \), we define a map \( \pi_n : E^n - \{p\} \rightarrow S^{n-1} \) as follows: for each point \( x \in E^n - \{p\} \), \( \pi_n(x) \) is the projection of the point \( x - p \), in vector terminology, radially from 0 onto \( S^{n-1} \); that is:

\[
\pi_n(x) = \frac{x - p}{|x - p|}.
\]

2.1. Theorem (Theorem VI. 10 in [H–W]). If \( X \) is a compactum in \( E^n \), then points \( p \) and \( q \) in \( E^n - X \) are separated by \( X \) iff \( \pi_n(X) \not= \pi_n(X) \).

2.2. Corollary. If \( X \) is a compactum in \( E^n \), and point \( p \) lies in the unbounded complementary domain of \( X \) in \( E^n \), then \( \pi_n(X) \not= 0 \).

Proof. Let \( B \) be a closed polyhedral ball in \( E^n \) containing \( X \) in \( Int(B) \) and let \( q \in E^n - B \). Clearly, \( \pi_n(B) = 0 \). Since \( X \subseteq B \), \( \pi_n(X) = 0 \). Since both \( p \) and \( q \) lie in the same complementary domain of \( X \), \( \pi_n(X) \not= 0 \).

2.3. Theorem (Theorem IV. 5.1 of [W]). If \( X \) is a compactum in \( E^n \), \( U \) is a compact domain of \( X \), and there is a nondegenerate closed, connected subset \( K \subseteq Cl(U) \) meeting \( X \) only in \( p \), then \( p \) is accessible from \( U \).

2.4. Lemma, Let \( X \) be a compactum and \( U \) and \( V \) domains such that \( X \subseteq U \subseteq V \subseteq E^n \). Suppose \( q_0 \in V - Cl(U), p \in U - X \), and \( q_1 \in U - X \), with an arc \( A = [q_0, q_1] \subseteq V - X \). Then \( X \) separates \( q_0 \) from \( p \) in \( V \) iff \( X \) separates \( q_1 \) from \( p \) in \( U \).

2.5. Theorem. Let \( X \) be a compactum in \( E^n \), \( U \) a compact domain of \( X \), and \( \langle \rangle \) a component of \( X \). Suppose that no closed subset of \( X \) missing \( p \) separates \( p \) from a point of \( U \). Then \( p \) is accessible from \( U \).

Proof. Let \( q_0 \in U \). We show that there is an arc \( A \) from \( q_0 \) to \( p \) so that \( A - \{p\} \not= U \). Since \( A \) is a component of \( X \), there is a separation \( X = (X \cap U) \cup (X \cap V) \) of \( X \) by open sets \( U \) and \( V \) in \( E^n \) whose boundaries miss \( X \), with \( p \in U \) and \( diam(U) < \frac{1}{2} \). Since \( E^n \) is locally connected, the components of \( U \) are open. Hence we may suppose that \( U \) is connected, and also that \( q_0 \neq U \). Let \( X_1 = U \cap X \) and \( X_2 = V \cap X \). Note that \( X_1 \) is a domain and \( diam(U) < \frac{1}{2} \).

Since \( X_1 \) is a closed subset of \( X \), \( X_1 \) does not separate \( q_0 \) from \( p \) in \( E^n \). Let \( A_0 \) be an arc from \( q_0 \) to \( p \) in \( E^n - X_1 \). Give \( A_0 \) its natural order with initial point \( q_0 \). Observe that \( A_0 \) meets \( Bd(U) \). Let \( q_1 \) be the first point of \( A_0 \) in \( Bd(U) \). For sufficiently small \( s \), there is an \( e \)-ball \( W \) about \( q_1 \) so that \( W \cap X = \emptyset \). Let \( q_1 \) be a point of \( W \cap U \) and let \( A_1 \) be an arc from \( q_1 \) to \( q_1 \), \( W \cap U \). Let \( A_2 \) be an arc in \( A_0 \cup A_1 \) from \( q_0 \) to \( q_1 \). Then \( A_0 \cap X = \emptyset \).

Now \( X_1 \) is a compactum in \( U_1 \), \( q_1 \in U_1 - X_1 \), \( \langle \rangle \) is a component of \( X_1 \), so by Lemma 2.4, no closed subset of \( X_1 \) separates \( p \) from \( q_1 \) in \( U_1 \). Thus we can apply the above argument with \( X_1 \) in place of \( X \), \( q_1 \) in place of \( q_0 \), and \( U_1 \) in place of \( E^n \).

We can find a domain \( U_2 \subseteq U_1 \) and an open set \( V_2 \subseteq U_1 \), with \( p \in U_2 \), \( diam(U_2) < \frac{1}{4} \), \( V_2 \cap U_1 = \emptyset \), and \( X_2 = U_2 \cap V_2 \). Let \( X_2 = X_1 \cap U_2 \) and \( X_2 = X_1 \cap V_2 \). As above, we can find an arc \( A_3 \subseteq U_1 \) from \( q_1 \) to a point \( q_2 \in U_1 \), so that \( A_3 \) misses \( X \).

Proceeding in this way, we can construct a sequence of arcs \( A_1 \subseteq U_i \), from \( q_i \), to \( q_{i+1} \), so that \( A_i \) misses \( X \), \( diam(A_i) < \frac{1}{2} \), and \( p \in \lim_{i \to \infty} A_i \). Then \( \bigcup_{i=0}^{\infty} A_i \cup \{p\} \) contains an arc \( A \) from \( q_0 \) to \( p \) so that \( A - \{p\} \not= U \).

2.6. Corollary. Let \( U \) be a domain in \( E^n \), \( Bd(U) \) compact, and \( \langle \rangle \) a component of \( Bd(U) \). Then \( p \) is accessible from \( U \).

Proof. Since \( U \) is a component of \( E^n - Bd(U) \), no closed subset \( K \subseteq Bd(U) \) missing \( p \) separates \( p \) from some \( q \in U \).

\*
2.7. Theorem (a version of the Borsuk Homotopy Extension Theorem; see p. 86 of [H-W]). Let C be a closed subset of a normal space S. Let f \equiv g: C \to Y, where Y is an ANR (absolute neighborhood retract). Suppose that F is an extension of f to S and that G is an extension of g to S. Then there is a neighborhood U of C in S such that \( F|U = G|U \).

3. Accessibility and homotopic maps. In this section we characterize, for a point p of a compactum X in \( E^n \), the accessibility of p from a complementary domain U of X. Our main theorem of this section is Theorem 3.1 which asserts that p is accessible from U iff for any point q \( \in U \), \( \pi_q|X - \{p\} \approx \pi_q|X \).

We note that accessibility of a point p in X may depend on the embedding of X in \( E^n \). As a Corollary of the characterization, we answer Question (1) of the Introduction by showing that the pseudo arc minus a (any) point admits an essential map to \( S^1 \).

We also construct an example of a nonseparating continuum M in \( E^3 \) with an accessible point p, so that \( \pi_q|M - \{p\} \approx \pi_q|X \) is necessarily inessential, but p is not accessible by a polygonal arc.

3.1. Theorem. Let X be a compactum in \( E^n \), U a complementary domain of X, q a point in U, and p a point in X. Then p is accessible from U iff \( \pi_q|X - \{p\} \approx \pi_q|X \).

Proof. Suppose that p is accessible from U. Then there is an arc A from q to p such that \( A - \{p\} \subset U \). We parameterize \( A \) by a map \( f: [0, 1] \to A \) such that \( f(0) = q \) and \( f(1) = p \). Then we define a homotopy \( \{f_t\}_{0 \leq t \leq 1} \) between \( \pi_q|X - \{p\} \) and \( \pi_q|X \) by

\[ f_t = \pi_{f(t)}|X - \{p\} \]

Conversely, suppose that \( \pi_q|X - \{p\} \approx \pi_q|X \). By Theorem 2.7 applied to the closed subset \( X - \{p\} \) of the space \( E^n - \{p, q\} \), there is a bounded neighborhood \( V_0 \) of \( X - \{p\} \) in \( E^n - \{p, q\} \) such that \( \pi_q|V_0 \approx \pi_q|V \). By way of notation, if Y is a set in \( E^n - \{p\} \), let Y denote the closure of Y in \( E^n - \{p\} \).

Since q \( \neq X \), we may apply normality and conclude that there is a neighborhood V of \( X - \{p\} \) in \( E^n - \{p\} \) such that

1. \( X - \{p\} \subset V \subset V_0 \),
2. \( q \in E^n - \text{Cl}(V) \),
3. \( \text{Bd}(V) \cap X = \{p\} \), and
4. \( \pi_q|V \approx \pi_q|V \).

From (1) and (2) it also follows that there is a complementary domain W of Cl(V) in \( E^n \) such that

5. \( \forall \in W \subset U \subset E^n - X \), and
6. \( \text{Bd}(W) \subset \text{Bd}(V) \subset \text{Cl}(V) = V \cup \{p\} \).

Now p \( \notin W \). If p \( \notin \text{Bd}(W) \), then p lies in a complementary domain \( W_0 \) of the compactum \( \text{Bd}(W) \), with \( W_0 \) distinct from \( W \). Hence by Theorem 2.1, \( \pi_q|\text{Bd}(W) \) \( \neq \pi_q|\text{Bd}(W) \). Noting (6), this contradicts (4). Therefore, we have

7. \( p \in \text{Bd}(W) \).

We now have Cl(W), a closed, connected subset of Cl(U), meeting x only in p. It follows from Theorem 2.3 that p is accessible from U.

3.2. Corollary. Let X be a compactum in \( E^n \), p a point of X, and U the unbounded complementary domain of X. Then p is accessible from U iff \( \pi_q|X - \{p\} \) is inessential.

Proof. Apply Theorem 3.1 and Corollary 2.2.

3.3. Corollary. Let p be a point of a compactum X. If there is an embedding \( \alpha \) of X in \( E^n \) in such a way that p is inaccessible from the unbounded complementary domain of X, then there is an essential map of \( X - \{p\} \) onto \( S^{n-1} \).

3.4. Remark. It is by applying Corollary 3.3 that examples of continua-min us a-point can be produced which admit essential maps to a sphere.

For example, let P denote the pseudo-arc ([Mo], [B]) and let p be any point in P. By the indecomposability of P, there are inaccessible points in any planar embedding of P. By the homogeneity of P [B], there is an embedding of P in \( E^2 \) with p inaccessible. Hence, \( F - \{p\} \) admits an essential map to \( S^1 \), answering Question (1) of the Introduction in the affirmative.

The \( \sin x \) continuum C = Cl(\{(x, y) \in E^2 | \sin(1/x) = y \text{ and } x \in (0, 1)\}) minus any point in the limit segment (the interval from (0, 1) to (0, -1) on the y-axis) admits an essential map to \( S^1 \); see the embedding described in Section 5.3.

In [B-M] two of the authors show that there is an embedding of the Knaster U-continuum (bucket handle) in the plane with its unique endpoint inaccessible. Hence, the Knaster U-continuum minus its endpoint admits an essential map to \( S^1 \).

In [M1] or [M2] one of the authors shows that given an indecomposable chainable continuum X and a point p in X, there is an embedding of X in \( E^2 \) with p inaccessible. The techniques used to prove that theorem can be extended to prove the following:

3.4.1. Theorem. If X is a chainable compactum containing a continuum of convergence \( X_0 \), then there is a point p \( \in X_0 \) and an embedding \( e: X \to E^2 \) such that e(p) is inaccessible. Moreover, p may be taken to be any point in \( X_0 \), except for at most two (a pair of opposite endpoints of \( X_0 \)).

It follows from Theorem 3.4.1 that every chainable continuum except the arc admits an essential map to \( S^1 \) upon the removal of some point. In Section 4, we show that, for maps to \( S^1 \), inaccessibility, and even embeddability in \( E^2 \), are not necessary, however.

3.5. Corollary. Let X be a compactum in \( E^n \) such that \( \dim(X) \leq n-2 \). Then every point of X is accessible in every embedding of X in \( E^n \).
Proof. Suppose that $X$ is inaccessible from the unbounded complementary domain $U$ of $X$. Then $p, X \setminus \{p\}$ is an essential map onto $S^{n-1}$ by Corollary 3.2. But $\dim (X \setminus \{p\}) < n - 1$, so $p, X \setminus \{p\}$ must be essential by Theorem VI. 6 of [H-W], a contradiction.

3.6. Remark. Theorem VI. 13 of [H-W] asserts that a compact subset $C$ of $E^*$ separates $E^*$ if there is an essential map of $C$ onto $S^{n-1}$. Remark 3.4 and the examples in Section 5.3 show that the hypothesis of compact is required for the “if” part of the theorem.

3.7. Example. In $E^2$, let $S^2$ be the unit sphere, and let $p$ be the origin. Let $A$ be a Fox–Artin arc [F-A, p. 983] from some point $q$ in $S^2$ to $p$, with $p$ as the “bad” point. We can fattened $A$ into a tapering solid cone $T$, by fattening less and less as we approach $p$. Thus $T$ is a closed ball, shaped like a solid cone, knotted in Fox–Artin fashion, with $p$ the apex of $T$. Let $D$ be the closed disk containing $q$ in $S^2$ wherein $T$ intersects $S^2$, and let $\text{Int}(D)$ denote the open disk $D - \text{Bd}(D)$ in $S^2$. See Figure 1.

![Diagram](image)

Let $C = (S^2 \cup \text{Bd}(D)) - \text{Int}(D)$. Then $C$ is homeomorphic to $S^2$, though not ambienly so, since $C$ contains a wild arc lying in $\text{Bd}(D)$. Note that $C \cup (\text{Ext}(C) \cup \infty)$ is homeomorphic to $B^3$. However, $C \cap \text{Int}(C) = M$ is not homeomorphic to $B^3$, for if it were, then $C$ would be collared on each side, and would therefore be ambienly homeomorphic to a standard $S^2$ [Bro].

Now $p$ is accessible from both $\text{Ext}(C)$ and $\text{Int}(C)$; however, $p$ is accessible from $\text{Ext}(C)$ only by a nonpolygonal arc following the “Fox–Artin channel” of $\text{Int}(T)$. Nevertheless, both $\pi_j[M - \{p\}]$ and $\pi_j[C - \{p\}]$ are inessential.

4. Mappings to $S^1$. One might suspect that in Theorem 3.1 the condition that $X$ be embeddable in $E^*$ is much too strong. At least for the case of the maps to the circle, we are able to eliminate the condition of embeddability in the plane.

4.1. Theorem. Let $X$ be a compact subset which is not locally connected at point $p \in X$. Suppose that $U$ is a closed neighborhood of $p$ in $X$, $(K_j)$ is a sequence of distinct components of $U$, each $K_j$ isolated in the sequence, $(U_j)$ is a sequence of open sets such that $U_j \subset \text{Int}(U)$ and $\lim (U_j) = \{p\}$, and $(f_j)$ is a sequence of mappings $f_j: K_j \to S^1$ such that $f_j(K_j - \{p\}) = 1$, and $f_j$ is essential modulo the boundary of $U$. Then there exists an essential mapping $f: X - \{p\} \to S^1$.

Proof. Extend $f_j$ to a map $g_j: (X - U_j) \cup K_j \to S^1$ by

$$g_j(x) = \begin{cases} 1, & \text{if } x \notin U_j, \\ f_j(x), & \text{if } x \in K_j. \end{cases}$$

Since $S^1$ is an ANR, there exists a neighborhood $V_j$ of $K_j \cup (X - U_j)$ such that $g_j$ extends to a mapping $h_j: V_j \to S^1$. Let $W_j$ be an open and closed neighborhood of $K_j$ in $U$ such that $W_j \subset V_j - \{p\}$. We may suppose that $W_j \cap W_k = \emptyset$ for $k \neq j$.

If some $f_j$ is essential, let $f$ be the extension of $h_j$ to $X - \{p\}$ which is constant on $W_j$. We may suppose, therefore, that each $f_j$ is inessential. Define $f: X - \{p\} \to S^1$ by

$$f(x) = \begin{cases} 1, & \text{if } x \notin \bigcup_{j=1}^\infty W_j, \\ (h_j(x))^j, & \text{if } x \in W_j. \end{cases}$$

Since the sets $(W_j)$ are pairwise disjoint, $f$ is well-defined. Since $\lim (U_j) = \{p\}$ and $f$ is constant on $X - \bigcup_{j=1}^\infty U_j \cup \{p\}$, $f$ is continuous. Note that $(h_j(x))^j$ denotes the $j$th power of $h_j(x)$ as an element of the multiplicative group $S^1$.

Since $f_j$ is essential modulo the boundary of $U$, it follows that if $\phi_j: K_j \to R$ were a lifting of $g_j$ (i.e., $g_j(x) = \exp(2\pi i \phi_j(x))$, for each $x \in (X - U_j) \cup K_j$ and $\phi_j$ is continuous), then $\text{diam}(\phi_j(K_j \cap \text{Bd}(U))) \geq 1$. Then $\psi_j: K_j \to R$ defined by $\psi_j(x) = j(\phi_j(x))$ would be a lifting of $(f_j)^j$, and $\text{diam}(\psi_j(K_j \cap \text{Bd}(U))) \geq j$. Since $\text{Bd}(U)$ is compact, and any lifting $\psi: X - \{p\} \to R$ of $f$ would have $\text{diam}(\psi(\text{Bd}(U))) \geq \text{diam}(\psi_j(\text{Bd}(U) \cap K_j)) \geq j$, for each positive integer $j$, it follows that no such lifting of $f$ exists. Hence, $f$ is essential.

4.2. Remark. Theorem 3.1 guarantees that an essential map of $X - \{p\} \subset E^2$ onto $S^1$ exists provided that $p$ is inaccessible from some complementary domain of $X$ in $E^2$. The essential map of Theorem 4.1 does not depend upon the embeddability of $X$ in $E^2$ with $p$ inaccessible, but rather upon the structure of $X$ near $p$.

Thus, for example, the wedge $X$ of two sin $1/x$ continua at a point $p$ of the limit segment has no embedding in $E^2$ with $p$ inaccessible, but Theorem 4.1 guarantees an essential map of $X - \{p\}$ onto $S^1$, since it is easy to find the components $K_j$ and maps $f_j$ required in the hypothesis.

Of course, Theorem 4.1 also applies to continua that are not embeddable in $E^2$ at all. For example, let $X$ be a ray spiralling clockwise to a simple triod with a “sticker” attached to the junction point of the triod. Let $p$ be any point of the triod.

The reader should note that the non-local-connectivity of $X$ at $p$ is a necessary, but not sufficient, condition for satisfying the hypothesis of Theorem 4.1. For example, consider the comb space $X = (([0, 1) \cup (1/n, 1]) \times [0, 1]) \cup ([0, 1) \times [0, 1))$ in $E^2$. Let $p$ be the point $(0, 1)$ on the limit segment $(0) \times [0, 1)$ of $X$. 
5. Connections with shape theory. Using the results of either Section 3 or Section 4, there are one-dimensional nonseparating plane continua, which, upon the removal of a point, admit essential maps to $S^1$. Examples include the sit [1/ε constant and the pseudo-arc. In Section 3.6, we construct, for each $n \geq 1$, an $n$-dimensional continuum $X_n \subset S^{n+1}$, which is the intersection of a nested sequence of $(n+1)$-balls, so that for a certain point $p_n \in X_n$ (which $X_n$ is not locally connected), $X_n \setminus \{p_n\}$ admits an essential map to $S^1$. One might suppose that if $X_n \setminus \{p_n\}$ admits an essential map to $S^1$, then $X_n \setminus \{p_n\}$ shape dominates $S^1$. In Section 5.4, we show that $X_n \setminus \{p_n\}$ is shape incomparable to $S^1$, and we conjecture that for $n \geq 1$, $X_n \setminus \{p_n\}$ is shape incomparable to $S^1$.

5.1. Shape theories for metric spaces. Shape theory, developed by Borsuk [Bo1] for compact metric spaces (compacta), has several inequivalent extensions to wider classes of topological spaces, including, but not limited to, metric spaces, which nevertheless agree on compacta. Among these, the extensions due to Fox [F] and Mardešić and Segal [M–S], for arbitrary topological spaces, agree on metric spaces. Borsuk extended his theory to metric spaces in [Bo3]. Therefore, we shall confine our attention to the weak and strong shape theories of Borsuk, as presented in [Bo4], and the shape theory of Fox.

Godlewski and Nowak [G–N] show the interrelationship between the strong shape theory of Borsuk and that of Fox. In the process, they implicitly suggest a definition of shape intermediate to the strong shape theory of Borsuk and the shape theory of Fox. We make their definition explicit below.

5.1.1. Fundamental sequences and homotopies. Let $X$ and $Y$ be closed subsets of AR (absolute retract) spaces $P$ and $Q$, respectively. Let $\{f_n : P \to Q\}$ be a sequence of maps. Consider the following conditions:

$F$: For every neighborhood $V$ of $Y$ (in $Q$), there is a neighborhood $U$ of $X$ (in $P$), such that $f(U) \approx f_{k+1}(U)$ in $V$, for almost all $k$.

$FC$: For every compact $A \subset X$, there is a compact $B \subset Y$, such that for every neighborhood $V$ of $B$ (in $Q$), there is a neighborhood $U$ of $A$ (in $P$), such that $f(U) \approx f_{k+1}(U)$ in $V$, for almost all $k$.

The triple $f = \{f_n : P \to Q, X, Y\}$ is called a $G$ (weak) [strong] fundamental sequence if $f$ satisfies condition(s) $F$ ($FC$) [$F$ [PC]]. We abbreviate "$G$ (weak) [strong] fundamental sequence" by $G$-sequence ($W$-sequence) [$S$-sequence]. In the discussion to follow, we shall speak of $R$-sequence, where $R$ is any one of $G, W,$ or $S$.

Let $f = \{f_n : P \to Q, X, Y\}$ and $g = \{g_n : P \to Q, X, Y\}$ be $R$-sequences. Consider the following conditions:

$H$: For every neighborhood $V$ of $Y$ (in $Q$), there is a neighborhood $U$ of $X$ (in $P$), such that $f(U) \approx g(U)$ in $V$, for almost all $k$.

$HC$: For every compact $A \subset X$, there is a compact $B \subset Y$, such that for every neighborhood $V$ of $B$ (in $Q$), there is a neighborhood $U$ of $A$ (in $P$), such that $f(U) \approx g(U)$ in $V$, for almost all $k$.

We say that $f$ and $g$ are $W$-homotopic (strongly) homotopic, abbreviated $G$-homotopic ($W$-homotopic) [$S$-homotopic], denoted $f \approx_W g$ ($f \approx_S g$), if $f$ and $g$ satisfy condition(s) $H$ ($HC$) [H and HC].

Note that two $S$-sequences can be $W$- or $G$-homotopic. An $S$-sequence is clearly both a $W$-sequence and a $G$-sequence, though not conversely, as examples in [Bo4] show. Similarly, two $R$-sequences which are $H$-homotopic are both $W$- and $G$-homotopic. For compacta, the three types of sequence are equivalent, as are the three notions of homotopy. Composition of $R$-sequences is defined in the natural way, and can be shown to be an $R$-sequence.

The $S$-sequence $y = \{f_i : P \to Q, X, Y\}$, where $f_i = f_{i-1} : P \to P$, is defined as the identity sequence. For point $y \in Y$, the $S$-sequence $y = \{f_i : P \to Q, X, Y\}$, where $f_i$ is the constant map to the point $y$, for all $k$, is called a constant sequence. If we do not specify the point $y \in Y$, we denote a constant sequence by $0$. Note that it is not clear whether an $R$-sequence of constant maps is $R$-homotopic to a constant sequence, unless space $Y$ is connected.

If the $R$-sequence $f$ is $R$-homotopic to some constant sequence, then we say that $f$ is a trivial $R$-sequence. It is not generally true that two trivial $R$-sequences from $X$ to $Y$ are $R$-homotopic, for suppose that $Y$ has two components. The following elementary properties of $R$-sequences and $R$-homotopies may be easily established:

(1) $R$-homotopy is reflexive, symmetric, and transitive.

(2) If $g, h, j$ are $R$-sequences, $f \approx g, h \approx j$, and the compositions $fh$ and $gf$ are defined, then $fh \approx gf$.

(3) If $f$ is an $R$-sequence, then $f_0 \approx 0$.

(4) If $f$ is an $R$-sequence and $f_0$ is defined, then $f_0 \approx 0$.

(5) If $f$ is an $R$-sequence, $f_0$ is defined, and $Y$ is connected, then $f_0 \approx 0$.

5.1.2. Mutations and homotopies. (As presented in [G–N], but due to Fox [F]).

Let $X$ and $Y$ be closed subsets of ANR (absolute neighborhood retract) spaces $P$ and $Q$, respectively. The family $U(X, P)$ of all neighborhoods of $X$ in $P$ is called the complete neighborhood system of $X$. Let $U(Y, Q)$ be the complete neighborhood systems for $X$ in $P$ and $Y$ in $Q$, respectively. A mutation $f : U(X, P) \to V(Y, Q)$ is a collection of maps $f : U \to V, U \in U(X, P), V \in V(Y, Q)$, satisfying the conditions:

$M1$: If $f \in \tilde{f}$, $f : U \to V, U' \subset U, U' \in U(X, P), V' \subset V', V' \in V(Y, Q)$, and $f' : V' \to V''$ is defined by $f'(x) = f(x)$, then $f' \in \tilde{f}$.

$M2$: Every $V \in V(Y, Q)$ is the range of some $f$. (Note that $V$ need not be the image of $f$).
M3: If \( f_1, f_2, g \in \mathcal{F} \) and \( f_1, f_2 : U \rightarrow V \), then there is a \( U' \in U(X, P) \) such that \( U' \subset U \) and \( f_1[U'] \sim f_2[U'] \) (in \( V \)). Two mutations \( f, g : U(X, P) \rightarrow V(Y, Q) \) are homotopic, abbreviated \( F \)-homotopic, denoted \( f \sim F g \), iff \( f \) and \( g \) satisfy the condition.

HM: For every \( f \in \mathcal{F} \) and \( g \in \mathcal{G} \) such that \( f, g : U \rightarrow V \), there is a \( U' \in U(X, P) \) such that \( U' \subset U \) and \( f[U'] \sim g[U'] \) (in \( V \)).

Condition HM is equivalent to the condition that \( \mathcal{F} \cup \mathcal{G} \) be a mutation. Composition of mutations is defined in the natural way.

Gogolewski and Nowak [G-N] show that given a \( G \)-sequence

\[
\{ f_k : P \rightarrow Q, X, Y \},
\]

the collection of maps \( f : U(X, P) \rightarrow V(Y, Q) \) defined by \( f = f_k\{U \} \) in \( V \) for almost all \( k \), is a mutation, and say that \( f \) is associated to \( f_k \). They show that this association preserves homotopy and composition. That is, \( f \sim F g \) iff \( f_k \sim F g_k \), for \( f \) associated to \( f_k \) and \( g \) associated to \( g_k \).

The mutation \( \mathcal{F} \cup \mathcal{G} \): \( U(X, P) \rightarrow U(X, P) \) consisting of all inclusions \( i_k: U \rightarrow V \), where \( U, V \in U(X, P) \) and \( U \subset V \), is called the identity mutation. It is associated to the identity sequence \( I_k \). A mutation \( \mathcal{F} : U(X, P) \rightarrow V(Y, Q) \) consisting of all constant maps \( g : U \rightarrow V \) such that \( g(x) = y \), for all \( x \in U \), for a fixed \( y \in Y \), is called a constant mutation. It is associated to the constant sequence \( I_k \). We may denote a constant mutation without specifying the point \( y \) by \( \delta \). A mutation which is \( F \)-homotopic to a constant mutation is called a trivial mutation. Two trivial mutations are not necessarily \( F \)-homotopic if \( Y \) is not connected.

Now let \( R \) denote any member of \( \{ G, F, W, S \} \), and let "\( F \)-sequence" mean "mutation." With appropriate alterations in notation, properties 5.1.1 (1)-(3) apply to mutations and \( F \)-homotopies.

5.1.3. Shape domination and shape equivalence. Let \( X \) and \( Y \) be closed subsets of AR (ANR) spaces \( P \) and \( Q \), respectively. We say that \( X \) \( R \)-shape dominates \( Y \), denoted \( X \preceq_R Y \), iff there are two \( R \)-sequences \( f = \{ f_i : P \rightarrow Q, X, Y \} \) and \( g = \{ g_i : P \rightarrow Q, X, Y \} \) (mutations \( f : U(X, P) \rightarrow V(Y, Q) \) and \( g : V(Y, Q) \rightarrow U(X, P) \)) such that \( f \sim R g \). If also, \( f \sim R g \), we say that \( X \) and \( Y \) are \( R \)-shape equivalent, denoted \( X \equiv_R Y \). Thus we obtain \( R \)-shape theory for metric spaces. When we are referring to compacta, where the theories all agree, or when a statement is true in all four theories, we may drop the prefix letter.

\( W \)-shape and \( S \)-shape are, respectively, the weak and strong shape theories of Borsuk [Bo4]. \( G \)-shape is the theory implicit in [G-N], and furnishes therein the bridge to \( F \)-shape, which is the shape theory of Fox [F], restricted to metric spaces.

By the Kuratowski–Wojdyslawski Theorem [Bo2, p. 79], any metric space can be embedded in some normed linear space as a closed subset of its convex hull. Since a convex subset of a normed linear space is an AR, it follows that the above definitions apply to all metric spaces. Theorem III. 3.3.4 of [Bo4] and Theorem 3.2 of [F] show that the choice of AR's (ANR's) and closed embeddings does not alter the relationship (if any) of shape domination or equivalence between \( X \) and \( Y \); hence, the definitions of shape domination and equivalence are unambiguous.

5.1.4. Some relationships among shape theories. As before, let \( R \) denote any one of \( G, W, S \), or \( F \). Let \( 1 \) denote a one-point space, \( A \) denote an arc, \( C \) denote the unit interval, and \( W \) denote the Warsaw circle (obtained from \( C \) by identifying the points \((0, -1) \) and \((1, 1) \)). The following elementary properties are common to all four theories (\( X \) and \( Y \) are metric spaces):

(1) \( X \overset{r}{\sim} Y \) implies \( X \preceq_r Y \) and \( Y \preceq_r X \).
(2) \( X \preceq_r 1 \).
(3) \( X = 1 \) iff \( i \preceq_r 0 \) (\( i \preceq_r 0 \)).
(4) \( i \preceq_r 0 \) implies \( X = 1 \).
(5) \( 1 \overset{r}{\sim} A \equiv C \).
(6) \( S^1 \overset{r}{\sim} W \).

If \( X \overset{r}{\sim} 1 \), we say that \( X \) has trivial \( R \)-shape. By (4), contractibility of \( X \) implies that \( X \) has trivial shape; however, the converse is not true, as \( C \) is not contractible.

The following relationships among the four shape theories can be established from the preceding definitions and remarks:

(7) \( X \preceq_r Y \) implies \( X \preceq_w Y \),
(8) \( X \preceq_r Y \) implies \( X \preceq_0 Y \), and
(9) \( X \preceq_r Y \) implies \( X \preceq_F Y \); therefore
(10) \( X \preceq_r Y \) implies \( X \preceq_F Y \), and
(11) \( X \preceq_r Y \) implies \( X \preceq_F Y \) and \( Y \preceq_F Y \).

(7)-(11) As (7)-(11), with "\( = \)" replacing "\( \sim \)."

The converse of Property (1) is false. Let \( X \) be a sequence of concentric circles in \( \mathbb{E}^2 \) of decreasing diameter converging to a limit circle. Let \( Y \) be a sequence of concentric circles in \( \mathbb{E}^2 \) converging to a point. Then \( X \preceq_r Y \) and \( Y \preceq_r X \), but \( X \not\preceq_r Y \).

(Example appears in a preprint of Borsuk's "Lectures on the Shape of Theory" given at the University of California, Riverside in April 1974). Properties (7), (7'), (8), and (8') are obvious. Properties (9) and (9'), and consequently (10) and (10'), are established in [G-N] through the process of associating a \( G \)-sequence with a mutation, as described in Section 5.1.2. Gogolewski and Nowak also provide an example which they use to show that \((9), (9'), (10), \) and (10') are not reversible. Borsuk uses the same example to show that (7) and (7') are not reversible. The example follows.
Let \( X \) be a space. We say that \( X \) is \textit{continuum-wise connected} iff given any two points \( p \) and \( q \) in \( X \), there is a continuum \( X_0 \subset X \), such that \( p, q \in X_0 \). It follows directly from the definitions that

(6) \( X \) is NC iff \( X \) is connected.

(7) \( X \) is NCC iff \( X \) is continuum-wise connected.

\[ \text{5.2. Essential maps and trivial shape.} \]

Let \( X \) be a closed subset of an ANR space \( P \). We have observed that \( X \) is NC iff the inclusion map \( i_X : X \to P \) is such that \( i_X \circ \theta = 0 \) in every neighborhood \( V \) of \( X \) in \( P \). Suppose that \( f : X \to X \) is an essential map of \( X \) to an ANR space \( Q \). Then \( 0 \neq f|_X \). Hence \( f|_X \neq 0 \). On the other hand, if no essential map from \( X \) to any ANR exists, then the inclusion map is homotopic to a constant. Thus we have that the following are equivalent:

(1) \( X \) is NC.

(2) \( X \) admits no essential map to an ANR.

(3) \( X = \emptyset \).

(4) \( X = \emptyset \).

We therefore have a partial answer to the second question in the Introduction. By the above and 5.1.4(10), there is no metric space of trivial \( G \), \( F \), or \( S \)-shape which admits an essential map to \( S^1 \).

\[ \text{5.2.3. Questions.} \]

We raise several questions concerning the interrelationship of the notion of trivial shape in the four shape theories discussed.

(1) Is there a metric space (necessarily noncompact) such that \( \alpha \) and \( \beta \) are \( S \)-shape (so \( \alpha \) is NCC), but admits an essential map to \( S^0 \) (so \( \alpha \) is not NC and \( X \neq \emptyset \))?

(2) Is there a metric space \( \alpha \) (necessarily noncompact) which is NC (so \( \alpha \) is \( \emptyset \)), but not NCC (so \( \alpha \neq \emptyset \))?

(3) Do the four versions of shape theory discussed agree on the class of metric spaces with trivial shape?

\[ \text{5.2.4. One-dimensional examples.} \]

The sin1/\( x \) continuum minus a point in the limit segment and the pseudo-arc minus any point each map essentially to \( S^1 \); hence, by 5.2.2 and 5.1.4(10), each has nontrivial \( G \), \( F \), and \( S \)-shape. Each is continuum-wise disconnected by the removal of said point; hence, by 5.2.3(1) (and (7), each is of nontrivial \( W \)-shape.

In Lemma 5.4.6, we show that a cellular continuum in \( E^2 \) minus an inaccessible point is neither NC nor NCC.

\[ \text{5.3. n-dimensional examples.} \]

Let \( X_1 \) be the sin1/\( x \) continuum embedded in the interior of \( I^1 \subset \mathbb{R}^2 \) so that the midpoint \( p_1 \) of the limit segment is \((0, 0)\), the limit segment is the interval from \((0, \frac{1}{2})\) to \((0, \frac{1}{2})\) on the \( y \)-axis, and the ray "wraps around" the limit segment. See Figure 2. It is evident that \( p_1 \) is inaccessible (from \( E^2 - X_1 \)). Let \( I^2 = I_1 \times I_2 \), \( I_i = [-1, 1] \), for all \( i \), and \( I^1 = \prod_{i=1}^{2} I_i \).
Inaccessibility, essential maps, and shape theory

(3) \( X_e \) is not locally connected at \( p_e \).
(4) \( X_e - \{ p_e \} \) admits an essential map to \( S^n \).

Proof. In the embedding described for \( X_e \), we may assume that \( X_e \) is the intersection of a defining sequence \( \{ e_i \}_{i=1}^n \) of chains of rectangular 2-balls. The union of the links of any given chain, \( \cup e_i \), is a long, thin 2-ball. See Figure 2. For each \( e > 0 \), there is an \( i > 1 \), such that \( B^2 = \bigcup e_i \) is a 2-ball neighborhood of \( X_e \) contained in the \( e \)-neighborhood of \( X_e \). Then \( B^2 \times \mathbb{R}^{n-1} \) is an \( (n+1) \)-ball neighborhood of \( X_e \) in \( \mathbb{R}^n \) contained in the \( e \)-neighborhood of \( X_e \). Hence, \( X_e \) is the intersection of a decreasing sequence of \((n+1)\)-balls, and so is cellular. It can then be shown that \( X_e \) has trivial shape, using the fact that a point has a similar sequence of neighborhoods. (That is, \( X_e \) is NC.) Since \( p_e \) is inaccessible from \( E^{n+1} - X_e \) by Lemma 5.3.1, it follows by Corollary 5.2.1 that \( X_e - \{ p_e \} \) admits an essential map to \( S^n \).

Let \( L_\alpha \) denote the limit segment of \( X_e \). Let \( L_\alpha = L_{\alpha-1} \times \mathbb{R}^1 \), for all \( n > 1 \). Then \( L_\alpha \) is the limit hyperplane of \( X_e \). It is not hard to see that \( L_\alpha \) is a continuum of convergence of \( X_e \), and so \( X_e \) is not locally connected at any point of \( L_\alpha \), including \( p_e \).

5.3.3. Theorem. For \( n > 1 \), \( X_e - \{ p_e \} \) is neither NC, nor NCC. Thus, \( X_e - \{ p_e \} \) is of nontrivial R-shape, \( R \in \{ G, F, W, S \} \).

Proof. That \( X_e - \{ p_e \} \) is not NC follows from the existence of an essential map to \( S^n \) and Proposition 5.2.2. Therefore, by 5.2.2 and 5.1.4(10), \( X_e - \{ p_e \} \) has nontrivial G-, F-, and S-shape.

To show that \( X_e - \{ p_e \} \) is not NCC, and thus has nontrivial W-shape, we will show that for each \( n > 1 \), there is a compactum \( A_e \in X_e - \{ p_e \} \) for which there is no continuum \( B \) with \( A_e \subset B \subset X_e - \{ p_e \} \) satisfying condition NCC.

Let \( A_e \) be the two point set consisting of the endpoints of the limit segment \( L_1 \) of \( X_e \). For \( n > 1 \), let \( A_e = (A_{n-1} \times \{ 1 \}) \cup (A_{n-1} \times \{ 1, 2 \}) \). Observe that \( A_e \) is the \( S^{n-1} \) bounding the \( n \)-ball \( L_{n-1} \), where \( L_{n-1} = L_{n-1} \times \{ 1 \} \), is the limit hyperplane of \( X_e \).

Suppose that \( B \) is a continuum in \( X_e - \{ p_e \} \) containing \( A_e \). Since \( B \) is compact, there is an \( (n+1) \)-ball \( C \) about \( p_e \) missing \( B \). We may assume that there is a diameter \( D \) of \( C \) whose endpoints lie in \( E^{n+1} - X_e \). From the endpoints of \( D \) extend disjoint rays \( R_1 \) and \( R_2 \) in \( E^{n+1} - X_e \) to \( \infty \). Then \( L = R_1 \cup R_2 \) is a line in \( E^{n+1} - B \) such that \( A_e \) links \( L \). That is, \( A_e \) is not contractible in \( E^{n+1} - B \). Thus there is a neighborhood \( V = E^{n+1} - L \) of \( B \) such that no neighborhood \( U \) of \( A_e \) contracts in \( V \). Hence, \( X_e - \{ p_e \} \) is not NCC.

5.4. Shape incomparability to \( S^n \). Our main theorem of this section is Theorem 5.4.9, which implies that each of our one-dimensional examples \( (X_e - \{ p_e \}) \), the sine 1/x continuum minus any point in the limit segment, the pseudo-arc minus any point) is shape incomparable to \( S^1 \). We conjecture that for \( n > 1 \), example \( X_e - \{ p_e \} \) of Section 5.3 is shape incomparable to \( S^n \). The method of proof below, however, cannot be extended to higher dimensions.

It follows from Propositions 5.1.4(9) and (10) that we may restrict our attention

5.3.2. Theorem. For \( n > 1 \), \( X_e \) has the following properties:

(1) \( X_e \) is a cellular continuum in \( E^{n+1} \).
(2) \( X_e \) has trivial shape.
to $F$- and $W$-shape incompatibility. The proof in the case of $W$-shape is fairly direct and elementary, and is presented in Lemmas 5.4.6 and 5.4.7 and in Theorem 5.4.8. The proof in the case of $F$-shape is less trivial, requiring a lemma about neighborhoods of a cellular continuum minus an inaccessible point in the punctured plane, and is presented in Lemmas 5.4.3 and 5.4.4 and in Theorem 5.4.5.

For $x > 0$, let $S(A, a) = \{x|d(x, A) < \epsilon\}$ denote the open $\epsilon$-neighborhood of a subset $A$ of $E^2$. For $C$ a compactum in $E^2$, let $\hat{C}$ denote the union of $C$ and its bounded complementary domains, if any. We call $\hat{C}$ the *topological hull* of $C$. Note that we identify $E^2 \cup \{\infty\}$, the one-point compactification of $E^2$, with $S^2$.

The following two lemmas are required in the proof of Lemma 5.4.3. The proof of the first is well known.

5.4.1. Lemma. Let $A$ be a 0-dimensional compact subset of a domain $U$ contained in $S^2$. Then there is a 2-cell $D$ such that $A \subset \text{Int}(D) \subset D \subset U$.

5.4.2. Lemma. Let $P$ and $Q$ be compact subsets of $S^2$ such that $Q$ is 0-dimensional, $P \cap Q = \{p\}$ and $p \in \text{Cl}(Q \setminus \{p\})$. Let $q \in Q \setminus \{p\}$. Then there is a 2-cell $D$ and a sequence of mutually disjoint 2-cells $\{D_n\}_{n=1}^\infty$ such that

1. $P \cap (\bigcup_{n=1}^\infty D_n) = \text{Int}(D) \subset S^2 - \{q\}$,

2. $P \cap (\bigcup_{n=1}^\infty D_n) = \emptyset$,

3. the sequence $\{D_n\}_{n=1}^\infty$ converges to $p$ (denoted $D_n \to p$), and

4. $D \setminus (\bigcup_{n=1}^\infty \text{Int}(D_n)) \subset (S^2 - Q) \cup \{p\}$.

Proof. There is a 2-cell $D$ such that $P \subset \text{Int}(D) \subset D \subset S^2 - \{q\}$ and $\text{Bd}(D) \cap Q = \emptyset$. Let $\{G_n\}_{n=1}^\infty$ be a nested null sequence of open 2-cell neighborhoods of $p$ in $S^2$ such that $\cap_n \text{Bd}(G_n) = \emptyset$ for all $n \geq 1$. Let $U_1, U_2, \ldots$ be the components of $D \setminus (\bigcup_n \text{Bd}(D) \cup \bigcup_n \text{Bd}(G_n))$ such that $U_n \cap Q \neq \emptyset$. Then $Q = \bigcap_n U_n$ is a compact 0-dimensional subset of the domain $U_n$. By Lemma 5.4.1, there is a 2-cell $D_0 \subset U_n$ such that $Q \subset \text{Int}(D_0)$. Then $D_0$ converges to $p$, since $D_0$ does. The remaining properties of the 2-cells are evident from the construction.

5.4.3. Lemma. Let $X$ be a cellular continuum in $E^2$, $p$ an inaccessible point of $\text{Bd}(X)$, $V$ a bounded connected open neighborhood of $X \setminus \{p\}$ in $E^2 - \{p\}$, and $Y$ a compact subset of $V$. Then there is a 2-cell $D$ and a null sequence of mutually disjoint 2-cells $\{D_n\}_{n=1}^\infty$ such that

1. $X \cup Y \cap (\bigcup_{n=1}^\infty D_n) = \text{Int}(D)$,

2. $(X \cup Y) \cap (\bigcup_{n=1}^\infty D_n) = \emptyset$.

(3) $D_n \to p$, and

(4) $D \setminus (\bigcup_{n=1}^\infty \text{Int}(D_n)) \subset V \cup \{p\}$.

Proof. Let $\mathcal{F}$ be the upper-semicontinuous decomposition of $E^2$ into components of $E^2 - V$ and single points of $V$. Let $f: E^2 \to E^2/\mathcal{F}$ be the quotient map. By Moore’s Theorem [Mr], we may assume that $E^2/\mathcal{F} = S^2$. (Because $p$ is connected, no component of $E^2 - V$ separates $E^2$.) Let $A \subset E^2 - (X \cup Y)$ be a sequence of points converging to $p$. Then $P = f(X \cup Y)$ and $Q = f(A \cup (E^2 - V))$ are compact subsets of $S^2$ such that $Q = 0$-dimensional and $P \cap Q = \{p'\}$, where $p' = f(p)$.

Since $A$ converges to $p$ and $f(A) \subset Q \setminus \{p'\}$, we have $p' \in \text{Cl}(Q \setminus \{p'\})$. Note that $f^{-1}(p') = p$, since $p$ is inaccessible and $f^{-1}(p')$ is a continuum in $E^2 - V$. It follows that the set-valued mapping $f^{-1}: S^2 \to E^2$ is continuous at $p'$.

Since $P$ and $Q$ satisfy the hypotheses of Lemma 5.4.2, with $q = f(C)$ where $C$ is the unbounded component of $E^2 - V$, there exist a 2-cell $D'$ and a sequence of 2-cells $\{D'_n\}_{n=1}^\infty$ in $S^2$ satisfying the conclusion of Lemma 5.4.2. Let $D = f^{-1}(D')$ and $D_n = f^{-1}(D'_n)$. Using the continuity of $f^{-1}$ at $p'$, and the fact that (1)-(4) of Lemma 5.4.2 are satisfied by $D'$ and $\{D'_n\}_{n=1}^\infty$, it can be seen that $D$ and $\{D'_n\}_{n=1}^\infty$ are 2-cells satisfying conditions (1)-(4) above.

5.4.4. Lemma. Let $f: U(S^1, S^1) \to V(X - \{p\}, E^2 - \{p\})$ be a mutation, where $X \subset E^2$ is a cellular continuum and $p \in \text{Bd}(X)$ is inaccessible. Then $f \not\approx 0$.

Proof. Note that $U(S^1, S^1) = (S^1, S^1)$. Let $f \in f$, so that $f: S^1 \to V$ for some $V \in V(X - \{p\}, E^2 - \{p\})$. Since $p$ is inaccessible from $E^2 - X$

(1) $X - \{p\}$ is connected.

For, if $X - \{p\}$ were not connected, then the closure of a separator of $E^2 - \{p\}$ which separated $X - \{p\}$ would contain a nondegenerate component meeting $X$ only in $p$, contradicting Theorem 2.3.

Let $V_1$ be the component of $V$ containing $X - \{p\}$. We may assume that $V_1$ is bounded and open, and that

(2) $f(S^1) \subset V_1$.

For, if $f(S^1)$ is not contained in $V_1$, then, since $f(S^1)$ is connected, $f(S^1)$ is contained in some component of $V$ distinct from $V_1$. Because $V_1 \subset V$ and $V_1 \cap V(X - \{p\}, E^2 - \{p\}) = \emptyset$, there is an $f_0 \in f$, $f_0: S^1 \to V_1$ such that $f_0 \cong f$ in $V$, which is a contradiction since $f(S^1)$ and $f_0(S^1)$ lie in different components of $V$.

Since each neighborhood of $X - \{p\}$ in $E^2 - \{p\}$ contains a path-connected neighborhood of $X - \{p\}$, it suffices to show that $f \not\approx 0$ in $V_1$.

By Lemma 5.4.3, there is a 2-cell $D$ and a null sequence $\{D_n\}_{n=1}^\infty$ of mutually disjoint 2-cells such that

3 — Fundamenta Mathematicae 132:1
Inaccessibility, essential maps, and shape theory

B. L. Brechner, J. C. Mayer, and E. D. Tymchatyn

(3) \( D_0 \subseteq \text{Int}(D) - (X \cup f(S^1)) \),

(4) \( D_n \to p \), and

(5) \( X \cup f(S^1) \subseteq \text{Int}(D) - \bigcup_{n=1}^\infty D_n \subseteq V_1 \cup \{p\} \).

Let \( M_\omega = \bigcup \{ D_i \mid i = 1, 2, \ldots, n \} \) and \( N_\omega = \bigcup \{ D_i \mid i = n, n+1, \ldots, \infty \} \). Note that \( N_\omega \) is the disjoint union of \( M_{\omega-1} \) and \( N_\omega \).

By (5), it suffices to show that \( f \equiv 0 \) in \( D - (N_\omega \cup \{p\}) \). As a first step, we claim that

(6) \( f \equiv 0 \) in \( D - \{p\} \).

For, suppose \( f \not\equiv 0 \) in \( D - \{p\} \). For any nonseparating continuum \( B \subseteq E^2 \) and any map \( g : S^1 \to E^2 \setminus B \), let \( \text{WN}(g, B) \) denote the winding number of \( g \) about \( B \). Then \( D_0 \to p \) by (4), it follows from the continuity of \( \text{WN}(f, D_0) \neq 0 \), for almost all \( n \). Let \( f \) be such that \( \text{WN}(f, D_0) \neq 0 \). Hence, \( f \not\equiv 0 \) in \( D - D_0 \).

Since \( D_0 \subseteq \text{Int}(D) - (X \cup f(S^1)) \), and \( D_0 \subseteq D - D_0 \), there is a 2-cell \( D' \) such that \( X \subseteq D' \subseteq D - D_0 \). Since \( f \not\equiv 0 \) in \( D - D_0 \), there is a map \( g : S^1 \to D' \) such that \( f \not\equiv 0 \) in \( D - D_0 \). Since \( D' \) is contractible, \( f \not\equiv 0 \) in \( D - D_0 \), a contradiction.

Since the homotopy of \( 6 \) miss a neighborhood of \( p \) in \( E^2 \), there is an integer \( k \geq 1 \) such that

(7) \( f \equiv 0 \) in \( D - (N_k \cup \{p\}) \).

If \( k = 1 \), we are done: so assume that \( k > 1 \). As in the proof of Lemma 5.4.3, there is a 2-cell \( D_0 \) such that

(8) \( (X \cup N_k) \subseteq \text{Int}(D_0) \subseteq D_0 \subseteq \text{Int}(D) - M_{k-1} \).

Since \( D_0 - (N_k \cup \{p\}) \subseteq V(X - \{p\}, E^2 - \{p\}) \), there is a map \( f_1 : S^1 \to D_0 - (N_k \cup \{p\}) \). By (8), \( D_0 - (N_k \cup \{p\}) \subseteq D_0 - (N_1 \cup \{p\}) \subseteq D - (N_1 \cup \{p\}) \).

Therefore, there is a deformation retraction of the 2-cell \( D_0 \) onto the 2-cell \( D_0 - (N_k \cup \{p\}) \), which is the inclusion. It follows from (7) and (9) that

(10) \( f_1 \equiv 0 \) in \( D - (N_k \cup \{p\}) \).

Thus, we have

(9) \( f_1 \equiv 0 \) in \( D - (N_k \cup \{p\}) \).

Consequently, by (9), (11), and (12), we have that

(13) \( f \equiv 0 \) in \( D - (N_k \cup \{p\}) \).

5.4.5. Theorem. Let \( X \subseteq E^2 \) be a cellular continuum and let \( p \in \text{Bd}(X) \) be inaccessible. Then \( X \setminus \{p\} \) is \( F \)-shape incomparable.

Proof. Suppose \( S^1 \not\subseteq X - \{p\} \). Then there are mutations \( f : U(S^1, S^1) \to U(X - \{p\}, E^2 - \{p\}) \) and \( g : V(X - \{p\}, E^2 - \{p\}) \to U(S^1, S^2) \) such that \( f \not\equiv g \), where \( f \) is the identity mutation on \( X - \{p\} \). By Lemma 5.4.4, \( f \not\equiv 0 \). By Proposition 5.1.1(4) (and the remark at the end of Section 5.1.2), \( f \not\equiv g \). So by Proposition 5.2.1(2), \( X - \{p\} \) is NC. Therefore, by Proposition 5.2.2, \( X - \{p\} \) admits no essential map to \( S^1 \). However, since \( p \) is inaccessible, Corollary 3.3 implies that \( X - \{p\} \) does admit an essential map to \( S^1 \). In view of this contradiction, \( S^1 \not\subseteq X - \{p\} \).

Now suppose \( X - \{p\} \not\subseteq S^1 \). Then for mutations as defined above, we have \( g \not\equiv f \), where \( f \) is the mutation whose only element is the identity \( \text{id}_X \) on \( S^1 \). Let \( g \equiv g \) for which \( g \not\equiv 0 \). By Lemma 5.4.4, \( g \equiv 0 \). Hence, \( g \equiv 0 \equiv 0 \), a contradiction. Thus, \( X - \{p\} \not\subseteq S^1 \).

5.4.6. Lemma. Let \( X \subseteq E^2 \) be a cellular continuum and \( p \in \text{Bd}(X) \) an inaccessible point. Then each subcontinuum of \( X - \{p\} \) is contained in a cellular subcontinuum of \( X - \{p\} \). However, there is a compactum in \( X - \{p\} \) not contained in any subcontinuum of \( X - \{p\} \); hence, \( X - \{p\} \) is not NC.

Proof. Let \( K \) be a subcontinuum of \( X - \{p\} \). Suppose \( K \) is not cellular. Then \( K \) separates \( E^2 \), and such continua coincide with nonseparating continua in \( E^2 \). All bounded complemented domains of \( K \) in \( E^2 \) are contained in \( \text{Int}(X) \), since \( X \) does not separate \( E^2 \). Since \( p \not\in \text{Int}(X), K \), the topological hull of \( K \), is a nonseparating, cellular, subcontinuum of \( X - \{p\} \) containing \( K \).

To show that \( X - \{p\} \) is not NC, it suffices to establish the following claim:

(1) There is an \( \epsilon > 0 \) such that \( A = X \cap (E^2 - S^1(0, \epsilon)) \) is not contained in any continuum \( K \subseteq X - \{p\} \).

Suppose the claim is false. Then for each \( n \), there is a real number \( a_n > 0 \), an open disk \( D_n = S(p, a_n) \), a compactum \( A_n = X \cap (E^2 - D_n) \), and a continuum \( K_n \) such that

(2) \( A - \{p\} \subseteq \cdots \subseteq K_n \subseteq A_{n+1} \subseteq \cdots \subseteq A_1 \).

(3) \( K_n \cap \text{Cl}(D_{n+1}) = \emptyset \), and

(4) \( (\epsilon_{n+1} - \epsilon_n) \) decreases to \( 0 \).

By the first paragraph of the proof, we may assume that \( K_n = K_{n+1} \). Since \( K_n \) is nonseparating, there is an arc \( B_n \) from\( \infty \) to \( p \) missing \( K_n \). Let \( B_n \subseteq B_n' \) be an arc.
Irreducible from $\infty$ to $\text{Bd}(D_3)$ missing $X$. We now inductively construct a locally connected continuum $M$ in $E^3$ meeting $X$ only in $p$, thus contradicting the inaccessibility of $p$.

Let $M_2 = B_2$. Let $R$ be the component of $M_2 \cap D_1$ meeting $\text{Bd}(D_3)$ and let $S$ be the component of $B_3 \cap D_1$ meeting $\text{Bd}(D_3)$. We claim there is an arc $F$ (possibly degenerate, if $\text{Cl}(R) \cap \text{Cl}(S) = \emptyset$) in $D_3 - X$ from $\text{Cl}(R)$ to $\text{Cl}(S)$. For if not, then some component $L$ of $\text{Cl}(D_3) \cap S$ separates $\text{Cl}(R)$ from $\text{Cl}(S)$. Now $L$ must meet $D_3$ and both components of $\text{Bd}(D_3) - (\text{Cl}(R) \cup \text{Cl}(S))$ in points of $X$. See Figure 3. But $K_1$ does not meet $D_3$. Hence, $B_1 \cup \text{Cl}(D_3) \cup M_2$ separates $E^3$ between points of $L \cap K_1 \cap \text{Bd}(D_3)$ without meeting the continuum $K_1$, a contradiction.

![Figure 3](image)

Let $M_3$ be an arc in $M_3 \cup F \cup B_1$ from $\infty$ to $\text{Bd}(D_3)$ which differs from $M_3$ only within $D_1$. We may repeat the above process with $D_1$, $D_2$, $D_3$, $B_4$, and $M_3$ replacing $D_1$, $D_3$, $B_2$, and $M_2$, respectively, to obtain an arc $M_4$ from $\infty$ to $\text{Bd}(D_3)$ missing $X$ and differing from $M_3$ only within $D_2$.

Inductively, for $n = 1, 2, \ldots$, there exists an arc $M_n$ from $\infty$ to $\text{Bd}(D_3)$ missing $X$ and such that $M_{n+1}$ differs from $M_n$ only inside $D_{n}$. It is clear that $M = (\bigcup_{n=1}^{\infty} M_n) \cup \{p\}$ is a locally connected continuum meeting $X$ only at $p$.

5.4.7. **Lemma.** Let $f = \{f_n: E^3 \to S^3 - \{p\}, S^1, X - \{p\}\}$ be a $W$-sequence, where $X \subset S^3$ is a cellular continuum and $p \in \text{Bd}(X)$ is inaccessible. Then $f \not\equiv \emptyset$.

**Proof.** It follows from the definition of a $W$-sequence (5.1.1(FC)) that with $A = S^3$ there is a continuum $B \subset X - \{p\}$ such that for every neighborhood $V$ of $B$, there is a neighborhood $U$ of $A$, such that for almost all $k, f_k[U] \approx f_k[U]$ in $V$. That $B$ may be taken to be a continuum follows from the fact that $S^3$ is connected. By the first part of Lemma 5.4.6, we may assume that $B$ is cellular. Thus, any neighborhood $V$ of $B$ contains a 2-cell neighborhood $V'$ of $B$. There is a neighborhood $U$ of $V'$ such that $f_k(U) \approx V'$, for almost all $k$. By the contractibility of $V'$, $f_k[U] \approx 0$ in $V$, for almost all $k$. Therefore, since $B$ is a continuum, $f \not\equiv 0$.

5.4.8. **Theorem.** Let $X \subset E^3$ be a cellular continuum and $p \in \text{Bd}(X)$ inaccessible. Then $X - \{p\}$ is $W$-shape incomparable to $S^1$. 

**Proof.** Suppose that $S^1 \not\subset X - \{p\}$. Recall that the choice of AR containing $X - \{p\}$ is immaterial. Thus we may assume that there are $W$-sequences

$$f = \{f_n: E^3 \to S^3 - \{p\}, S^1, X - \{p\}\}$$

and

$$g = \{g_n: S^3 - \{p\} \to E^3, X - \{p\}, S^1\}$$

such that $f_k \not\equiv g_k \not\equiv \emptyset$, the identity sequence on $X - \{p\}$. By Lemma 5.4.7, $f \not\equiv 0$; hence, by Proposition 5.1.1(4), $I_{x - \{p\}} \not\equiv f \not\equiv 0$. By Proposition 5.2.1(3), $X - \{p\}$ is NCC, which contradicts the second part of Lemma 5.4.6. Therefore, $S^1 \not\subset X - \{p\}$. Now suppose that $X - \{p\} \not\subset S^1$. Then there are $W$-sequences as defined above for which $g_k \not\equiv f_k$, the identity sequence on $S^1$. Since $S^1$ is connected, it follows from Proposition 5.1.1(5) and Lemma 5.4.7, that $I_{S^1} \not\equiv g_k \not\equiv 0$. But clearly, $I_{S^1} \not\equiv 0$. In view of this contradiction, $X - \{p\} \not\subset S^1$.

Combining Theorems 5.4.5 and 5.4.8, and by virtue of Propositions 5.1.4(9)-(10), we obtain the main theorem of this section.

5.4.9. **Theorem.** If $X \subset E^3$ is a cellular continuum and $p \in \text{Bd}(X)$ is inaccessible, then $X - \{p\}$ is shape incomparable to $S^1$.

5.4.10. **Corollary.** Each of our one-dimensional examples $(X - \{p\})$, the $\sin(1/x)$ function minus any point in the limit segment, the pseudo-circle minus any point) is shape incomparable to $S^1$.

5.5. **Shape Incomparability to compact, noncontractible ANRs.** In this section we indicate how to extend Theorem 5.4.9 to the case where $S^1$ is replaced by any compact, noncontractible ANR. We obtain as our main theorem the following:

5.5.1. **Theorem.** Let $X \subset E^3$ be a cellular continuum, $p \in \text{Bd}(X)$ an inaccessible point, and $Z$ any compact, noncontractible ANR. Then $X - \{p\}$ is shape incomparable to $Z$.

**Proof.** We carry out the proof modulo two lemmas to be proved subsequently. The theorem follows immediately provided that Theorems 5.4.5 and 5.4.8 can be strengthened by replacing $S^1$ with $Z$. We first assume that $Z$ is connected.

Lemma 5.4.7 can easily be extended by replacing $S^1$ with a connected $Z$ and $E^3$ with an AR $M$ containing $Z$. The strengthened version of Theorem 5.4.8 immediately follows. Thus, $X - \{p\}$ is not $W$-shape comparable to a connected $Z$. 


It follows from Lemma 5.5.2, proved below, that Lemma 5.4.4 can be strengthened by replacing $S^1$ with a connected $Z$. We use Lemma 5.5.2 to find a simple closed curve in $Z$ to which the argument of the proof of Lemma 5.4.4 is then applied. The strengthened version of Theorem 5.4.5 immediately follows. Thus, $X-\{p\}$ is $F$-shape incomparable to a connected $Z$.

Now suppose that $Z$ is not connected. By Lemma 5.5.3, proved below, if $F$ shape dominates $X-\{p\}$, then some component $Z_0$ of $Z$ does. Thus, this direction reduces to the first case. Since $X-\{p\}$ is connected, $X-\{p\}$ cannot shape dominate $Z$, which has more than one component.

**5.5.2. Lemma.** If $f: Y \to G$ is an essential mapping of a Peano continuum $Y$ onto a graph $G$, then there is a simple closed curve $S$ in $Y$ such that $f$ is essential on $S$.

**Proof.** Let $T$ be a maximal tree in $G$ [H, p. 117]. Let $E_1, \ldots, E_n$ be a listing of the edges of $G$ not in $T$. Barycentrically subdivide each $E_i$ into $E_{i1}$ and $E_{i2}$, for $i = 1, 2$. Let $T = \cup_{i=1}^n T_i$, where each $T_i$ be a tree. By Theorem 7 on p. 452 of [K], there exist Peano continua $B_1$ and $B_2$ in $G$ such that $f^{-1}(T) \subseteq B_1$ and $f$ is essential on $B_1$. Let $x \in B_1 \cap B_2$. Let $\varphi_i: B_i \to J$ be liftings of $f|_{B_i}$ to the universal covering space $J$ of $G$ such that $\varphi_i(x) = \varphi_i(x)$. Since $f$ is essential and does not lift to $J$, there exists $y \in B_1 \cap B_2$ such that $\varphi_i(y) \neq \varphi_i(y)$.

For $i = 1, 2$, let $K_i$ be an arc in $B_i$ from $x$ to $y$. Let $K_i$ have its natural order with initial point $x$. Let $z$ be the first point of $K_i$ in $K_i \cap B_2$ such that $\varphi_i(z) \neq \varphi_i(z)$. Let $L_i$ be the arc in $K_i$ from $x$ to $z$. Note that $z$ is an isolated point in $L_i \cap K_i$. Let $w$ be the last point in $L_i - \{z\}$ such that $w \in L_i$. Let $M_i$ be the arc in $K_i$ from $w$ to $z$. Then $S = M_1 \cup M_2$ is the required simple closed curve $S$.

**5.5.3. Lemma.** If the compactum $X$ $R$-shape dominates the connected set $Y$, then some component of $X$ $R$-shape dominates $Y$.

**Proof.** In order that the maps comprising an $R$-sequence from $Y$ to $X$ be (almost) homotopic, the fact that $Y$ is connected implies that (almost) all the maps take $Y$ into a neighborhood of some single component of $X$, no matter how small the neighborhood of that component.

References


UNIVERSITY OF FLORIDA, Gainesville, FL 32611
UNIVERSITY OF ALABAMA AT BIRMINGHAM, Birmingham, AL 35294
UNIVERSITY OF SASKATCHEWAN, Saskatoon, SK, Canada S7N 0W0

Received 25 April 1985; in revised form 3 September 1985 and 5 October 1987