Ramified analysis and the minimal $\beta$-models of higher order arithmetics

by

Zygmunt Vetulani (Poznań)

Abstract. We discuss the minimal $\beta$-models of $A_n$ in terms of ramified analysis. A $\beta$-interpretation of $A_n$ in $A_0$ is obtained, which leads from a $\beta$-model to the minimal $\beta$-model of $A_0$. We propose an alternative notion of constructibility for $A_0$. We generalize the notion of ramified analysis and we prove (by Jensen's method) a generalization of the theorem on "correspondence in levels" between the hierarchy of constructible sets $L$ and the ramified analysis $RA$. All proofs and lemmas are formulated for $n = 3$. However, they work for $n \geq 3$.

Chapter I

Introduction. The present investigation was inspired by the following question, formulated by K. Apt and W. Marek ([1], p. 226): "Can the smallest $\beta$-model of $A_n$ ($n > 2$) be characterized "from below" similarly to the "ramified analytical" characterization in the case of $A_2"? The answer we give is positive.

Higher order arithmetics, and especially the second order arithmetic $A_2$, have been thoroughly investigated since the fifties: particularly in Warsaw by Professor Mostowski and his colleagues and students. It was Mostowski's idea to classify models of $A_2$ with respect to the notion of well-foundedness: $\beta$-models (i.e., models with respect to which the notion of well-ordering is absolute) were carefully investigated.

The class of all $\beta$-models of $A_2$ has a nice property: there exists a smallest $\beta$-model of $A_2$. This model was investigated by R.O. Gandy [4] and others. Gandy proved that this is exactly what we call ramified analysis. The proof of this fact was obtained by recursion-theoretic methods by using the properties of the notion of hyperjump; unfortunately, it does not generalize to the higher order cases. (This is not very surprising: let us recall here some very important differences between $A_2$ and $A_3$, such as for example different positions of the notion of well-ordering and also of constructibility. See [1]). A strong connection between constructibility and...
ramified analysis was first established by Boolos [2], who proved that $R_{A_n} = P(\omega) \cap L_n$. We shall generalize R. Jensen's proof of this result. Let us recall also one important result obtained by Zbierski [10]: it shows the equivalence (in the sense of interpretability) of the arithmetic $A_n$ and the set theory ZFC$^+$ $+ \Pi^1_2(\omega)$ exists. Zbierski discusses the formula $\forall A$ is a well-founded tree (Tree ($A$)) and shows that for all axioms $\varphi$ of ZFC$^+$ $+ \Pi^1_2(\omega)$ exists we have $A_{\varphi}$ $\equiv$ $A_{\varphi}^{true}$. The model theoretic counterpart of this theorem establishes correspondence between $\beta$-models of arithmetic and standard models of set theory.

Theorem (Zbierski [10]). 1. If $M$ is a standard transitive model of ZFC$^+$ $+ \Pi^1_2(\omega)$ exists, then $A_\varphi(M)$ is a $\beta$-model of $A_n$.

2. If $N$ is a $\beta$-model of $A_n$, then the trees of $N$ with the relation of isomorphism of trees and the relation of being a maximal proper subtree form a model which is isomorphic to a standard transitive model $M$ of ZFC$^+$ $+ \Pi^1_2(\omega)$ exists, so that $N = A_\varphi(M)$.

We shall give below a very natural generalization of ramified analysis for $n \geq 3$ and we shall show its properties. The main feature of our proofs is the absence of any recursion-theoretic notions and methods. This constitutes the main difference between our results and those mentioned above ([1], [2], [4]). A possible generalization of recursion-theoretic methods of Gandy, Boolos, Putnam and others would require an essential development of the recursion-theoretic technique to higher type objects. A general question arises in which scale would it be possible?

To solve the problem of Apt and Marek we introduce generalized ramified analysis in the following way:

**Definition 1.1.**

$RA_{\varphi}^n \equiv \langle \omega, RA_{\varphi}^{n+1}, \ldots, RA_{\varphi}^{n+1} \rangle \in \langle i, i, i \rangle, \in, +, \cdot, 0, 1 \rangle$,

$RA_{\varphi}^n \equiv \langle \omega, RA_{\varphi}^{n+1}, \ldots, RA_{\varphi}^{n+1} \rangle \in \langle i, i, i \rangle, \in, +, \cdot, 0, 1 \rangle$,

where

$RA_{\varphi}^{n+1} \equiv P(\omega) \cap HF$,

$RA_{\varphi}^{n+1} \equiv \{ D(\langle \omega, RA_{\varphi}^{n+1} \rangle) \}$

subsets of $RA_{\varphi}^{n+1}$ definable over $RA_{\varphi}^n$ by $L(A_n)$-formulas with parameters from $RA_{\varphi}^n$ (here $RA_{\varphi}^{n+1} \equiv \omega), RA_{\varphi}^{n+1} \equiv \bigcup RA_{\varphi}^{n+1}$ for $i \in \omega$.

We call this structure ramified analysis because for $n = 2$ it is just the usual ramified analysis. In the Chapter IV we prove the main Theorem claiming the existence of the $\beta$-interpretation of $A_n$ in $A_n$ which defines in every $\beta$-model the ramified analysis $RA_{\varphi}^n$.

**Notations. 1. Variables.** We shall use letters $m, n, i, k, t$ to denote natural numbers; $\psi, \psi$ denote formulas or their Gödel numbers, capitails will usually denote the highest order objects (classes) and small letters the lower order objects (sets or numbers).

2. Pairing functions. In 1st-order Peano arithmetic we can define a function $J$ being a "one-one" mapping defined on pairs of natural numbers onto all natural numbers. It is possible to write a formula $\langle (x, y) = \{z \rangle$ defining the pair $(x, y)$ of two objects $x$ and $y$ (of orders $i$ and $j$ respectively) as an object $z$ of the order $max(i, j)$. One can do this by using the function $J$. Also one can code $n$-tuples of objects of order smaller than or equal to $i$ as an object of order $i$.

3. Coding. We shall use this term in two different meanings.

a. $x \in \forall \exists (Ea)(b \in X \leftrightarrow \langle a, b \rangle \in Y) \iff \exists (Ea)(X = \{a \rangle (y, y \in Y))$.

where $X = \{y \} \in \{b \} \in X \leftrightarrow \langle a, b \rangle \in Y$. We then say that $Y$ is a code of the family $\{X : x \in Y\}$. We define similarly $a_\in X$ for $x$ and $Y$ of different order. Dom$(Y) \equiv \{ a : \langle b \rangle \in Y \}$. We can imagine that the family coded by $Y$ is numerated by the elements of Dom$(Y)$.

b. We also say that $X$ codes the relation $A \iff \langle a, b \rangle \in X \leftrightarrow \langle a, b \rangle \in A$.

In particular, the class $X$ codes a tree iff

- $X$ codes a partial ordering with no loops,
- there exists the unique maximal element in the partial ordering coded by $X$,
- every linear subordering of the ordering coded by $X$ is finite.

When $X$ codes a tree, we write Tree$(X)$ or $X \in Trees$. It is easy to write the formula of $L(A_n)$ defining trees.

**Chapter II**

**Definability of RA.** The purpose of this chapter is to formalize the notion of ramified analysis in the language of arithmetic, which yields a formula which is absolute with respect to $\beta$-models. This will be accomplished in Theorem 1. The method of proof of this theorem was presented by Mostowski [7] for the $\beta$-models of KM and can be directly translated to our case. This permits us to omit all details and to restrict ourselves to some general remarks concerning the proof.

**Theorem 1 (formulated here for $n = 3$).** Let $M = \langle F_1, F_2, \ldots \rangle$ be a $\beta$-model of $A_3$, $\eta$ being the height of $M$. Then, for $i = 1, 2$

1. there exist formulas $r_\eta(\cdot, \cdot)$ and $r_\eta(\cdot, \cdot)$ such that

   $a \in RA_{\varphi}^{n+1} \iff \langle (a, F_1, F_2, \ldots) \rangle \equiv r_\eta(T, a)$

   and

   $a \in RA_{\varphi}^{n+1} \iff a \in RA_{\varphi}^{n+1} \iff \langle (a, F_1, F_2, \ldots) \rangle \equiv r_\eta(a)$,

   where $T$ is a code of a well-ordering of order-type $\alpha$ and $T \in F_3$.

2. $RA_{\varphi}^{n+1} \subseteq \{ F_1 \}$

3. $RA_{\varphi}^{n+1} (RA_{\varphi}^{n+1} - 1$ in the general formulation) does not contain any well-ordering of order-type $\eta$. 

Our aim is to write down in an appropriate way the formulas $r_{\alpha}^*(\cdot, \cdot)$ and $r_{\alpha}^*(\cdot)$, which are the arithmetical reconstruction of the set-theoretical definition of ramified analysis. The difficulties which we have to overcome are typical for the problems of formalization of mathematics in arithmetic.

One of them is the nonexistence of objects which are collections of elements of different ranks. It is also impossible to discuss explicitly collections of classes, even as small as pairs. (We can not apply here Kuratowski's notion of pair.) We overcome this difficulty by employing pairs $\langle \langle \cdot, \cdot \rangle \rangle$ and codes. In particular, we can write down formula $\text{Bord}(X)$, which says that the class $X$ is a well-ordering by using the same formula as in the set theory but with $\langle \langle \cdot, \cdot \rangle \rangle$ instead of Kuratowski's pairing function.

Another important problem is connected with inductive definitions. The main difficulty here is due to the absence of ordinal numbers, and so we are reduced to working with well-orderings. The problem is that, generally, there is no distinguished well-ordering of a given order type. We have to prove additionally that a construction where a well-ordering of a given type is to be used does not depend on the particular choice of the well-ordering. In our proof we define ramified analysis by using sequence-constructors whose domains are well-orderings. The totality of these sequence-constructors determines objects of ramified hierarchy. It is important to show that any two sequence-constructors based on isomorphism domains (well-orderings) produce the same objects.

To be able to reconstruct the definition of $RA$ in arithmetic we first have to reconstruct the notion of satisfaction. We write a formula $\text{Sat}(X, Y, \varphi, \bar{\alpha}, \bar{\beta})$, where $X, Y$ code some families of objects and $\bar{\alpha}, \bar{\beta}$ code some finite sequences of objects, with the following property:

$$M \models \text{Sat}(X, Y, \varphi, \bar{\alpha}, \bar{\beta}) \iff \langle \langle \alpha, [a : n X], \langle A : \text{Ap}^{-1} \rangle, \varepsilon, \ldots \rangle \rangle = \varphi[n, X^{(0)}, Y^{(0)}]$$

for any $\omega$-model $M$. This formula has to represent the inductive definition of satisfaction. For example, one of its inductive conditions is:

$$\text{Sat}(X, Y, \varepsilon^{(1)} \in X^{(1)}, a, b) \iff X^{(1)} \in Y^{(0)}, \text{Sat}(\ldots) i \in 2^\varepsilon \text{ in all } \omega \text{-models of } A_3 \text{ (generally } \text{Sat}(... i) \equiv \text{Sat}(\ldots) i \text{ in all } \omega \text{-models of } A_3). \text{ Using the formula } \text{Sat}(\ldots) i \text{, we can define the class } \text{Sat}(X, Y) \text{ coding the family of classes which are definable over the structure coded by } X, Y \text{, i.e., } (\text{Sat}(X, Y))^i \text{ is a code (in any } \omega \text{-model } M \text{ of the family } \text{Def}^{i}(\langle [a : n X], \langle A : \text{Ap}^{-1} \rangle, \varepsilon, \ldots \rangle). \text{ Now we are able to write the formulas } r_{\alpha}(T, a) \text{ which, informally speaking, express what follows: } \alpha \text{ is an object of rank } i \text{ which is produced by a suitable sequence-constructors where their domain is the well-ordering } T \text{.} \text{ It follows that } (\exists T) \text{ is a well-ordering of order type } \alpha, \text{ then } M \models \text{Sat}(T, \alpha) \text{ if and only if } r_{\alpha}(T, a) \text{ for any } \beta \text{-model } M. \text{ This fact may be proved by induction with respect to } \alpha. \text{ As a by-product of his reasoning Mostowski got the existence of a definable well-ordering of } RA \text{ (17). This is also true in our case.}
use a Gödel style inductive definition with the aid of the pairing function $\langle \cdot, \cdot \rangle$. For example, $\langle 5, \varphi, \psi \rangle$ will be a formula (intuitively $\varphi \land \psi$) if $\varphi$ and $\psi$ are. Then we define in the usual way all fundamental syntactic notions connected with $RL_\alpha$, in particular $TL_{\alpha}$ — the class of terms, $FL_\alpha$ — the class of formulas and $FL_\alpha^v$ — the class of variable-free formulas. We define satisfaction $\models_{\alpha}$ for $RL_\alpha$ formulas and realization $\lbrack \cdot \rbrack_{\alpha}$ for $RL_\alpha$ terms. The following property holds:

$$L_\alpha = \lbrack [x]_\alpha, t \in T_{\alpha} = \lbrack \sigma(x = x) \rbrack \]$$

for $S$ such that $S = \alpha$.

$a$ being the last element in $R$ and $\sigma$ being the abstraction operator in $RL_\alpha$.

We shall also use the following lemma. Its straightforward but tedious proof will be omitted.

**Lemma 3.6.** Let $\alpha \in \text{Lim}$. 1. The following relations are definable over $Ar^3(L_\alpha)$: $t \in T_{\alpha}^\prime$, $\varphi \in FL_\alpha^v$, $\psi \in FL_\alpha^v$.

2. There is a formula $\Theta$ such that, for each $D, S \in P^2(\alpha) \cap L_\alpha$

$$D = \{ \varphi \in FL_\alpha^v, \models_\alpha \varphi \} \leftrightarrow Ar^3(L_\alpha) \models \Theta(D, S).$$

3. If $S, R \in P^2(\alpha) \cap L_\alpha$, then $\langle \varphi \in FL_\alpha^v, \models_\alpha \varphi \rangle \in P^2(\alpha) \cap L_\alpha$.

4. Any order homomorphism (isomorphism) $H$ of $S$ and $R$ belonging to $P^2(\alpha) \cap L_\alpha$ may be extended in a natural way to the homomorphism (isomorphism) $H'$ of $T_\alpha \cup FL_\alpha$ and $T_\alpha \cup FL_\alpha$, which also belongs to $P^2(\alpha) \cap L_\alpha$.

To finish the proof of the definability of $W_\alpha$ over $Ar^3(L_\alpha)$ we analyse two cases. The first of them is: there is no $R \in P^2(\alpha) \cap L_\alpha$, $R$ being a linear ordering of type $\alpha + \tau$ and such that $R_\alpha \subseteq W_\alpha$, $\gamma < \alpha$, $\tau$ arbitrary order type, empty or not. (If $S$ is a code of a well-ordering, then $S_\alpha$ is its initial segment of type $\nu$.)

Our Lemma 3.3 then follows from the fact that $R \in W_\alpha$ is $Ar^3(L_\alpha)$ is $\text{Bord}(\alpha)$ (because $Ar^3(L_\alpha)$ has the $\beta$-property). In the opposite case, i.e., if a well-ordering $R$ with the above property exists, it is enough to show that the set $[R \alpha : \gamma < \alpha]$ is definable over $Ar^3(L_\alpha)$ (because $W_\alpha = \{ S : (E(V_\chi \in Ar^3(L_\alpha))(E(V_\chi \in H) : H \vdash \tau \equiv \tau) \}$.)

**Fact 1.** $\varphi(\beta, (ED_\alpha)(\alpha(\gamma) \in L_\alpha) \models Ar^3(L_\alpha) = \Theta(D, R^D \cap R^\alpha)$.

**Fact 2.** For each $e \in Fd(R)$ such that $R \cap \alpha$ has the order type $\alpha + \tau$ and each $D \in P^2(\alpha) \cap L_\alpha$ we have $Ar^3(L_\alpha) = \Theta(D, R \cap \alpha) (\Theta$ is the formula from Lemma 3.6).

This completes the proof of Lemma 3.3.

**Lemma 3.7.** Let $\varphi$ be a formula of $L(EF)$ Then there exists a formula $\delta$ of $L(A_\alpha)$ such that

$$L_\alpha \models [x_1, \ldots, x_\alpha] \leftrightarrow Ar^3(L_\alpha) \models \delta(R_1, \ldots, R_\alpha, t_1, \ldots, t_\alpha)$$

where $R_\alpha \in W_\alpha, t_\alpha \in T_\alpha, z_\alpha = \lbrack \alpha \rbrack_{\alpha}$.
The proof of this lemma is by induction. Let \( \varphi \) be a bounded formula from \( L(EZ) \) and \( R \subseteq W^n_1 \subseteq T^n \), \( r \equiv \| R \| \).

\[
L_n \models \varphi[z_1, ..., z_k] \iff (E_{p_n})_n L_n \models \varphi[z_1, ..., z_k] \iff (E_{p_n})_n (E_{S_n})_n (E_{H_n})_n (E_{T_n})_n (E_{R_n})_n \text{ ("S", and initial segments of } S \text{ isomorphic to } R \text{ by isomorphisms } H \text{ and } T \text{ is a common extension of isomorphisms } H \text{ to terms and formulas of } R \text{ such that } H(t_i) = t_i \rightarrow H(t_i)[z_1, ..., z_k]).
\]

To verify those equalities we use the well-known properties of \( \models _{S} \) and \( \equiv _{S} \). From the definability of syntactical notions of RL and from the definability of \( W_n \) it follows that the last sentence can be relativized to \( A^2(L_n) \).

The inductive step: we use the following equivalence.

\[
L_n \models (E_{x}) \varphi(x)[z_1, ..., z_k] \iff A^2(L_n) \models (E_{x}) \varphi(x)[z_1, ..., z_k] \iff (E_{p_n})_n (E_{S_n})_n (E_{H_n})_n (E_{T_n})_n (E_{R_n})_n \text{ ("S", and initial segments of } S \text{ isomorphic to } R \text{ by isomorphisms } H \text{ and } T \text{ is a common extension of isomorphisms } H \text{ to terms and formulas of } R \text{ such that } H(t_i) = t_i \rightarrow H(t_i)[z_1, ..., z_k]).
\]

**Corollary.** There is a formula \( \equiv \) such that, for \( n \in \omega \),

\[
\forall a \in A^2(L_n) \models \equiv[R_n, R, b, t, \{A]\] \text{ where } R_n \text{ is a code of a well-ordering of type } n, \text{ and } \equiv \text{ is a term of the ramified language which denotes } n.
\]

Now we can easily obtain the inclusions we need to complete the proof of Theorem 2. For example: let \( a \in p^2(x) \cap L_n \). Then

\[
a = [b \in L_n : L_n \models \varphi[b, ..., z_k]] \text{ for a certain } \varphi.
\]

We show that

\[
b \equiv a \iff L_n \models (E_{x}) \varphi(x)[z_1, ..., z_k] \iff (E_{p_n})_n (E_{S_n})_n (E_{H_n})_n (E_{T_n})_n (E_{R_n})_n \text{ ("S", and initial segments of } S \text{ isomorphic to } R \text{ by isomorphisms } H \text{ and } T \text{ is a common extension of isomorphisms } H \text{ to terms and formulas of } R \text{ such that } H(t_i) = t_i \rightarrow H(t_i)[z_1, ..., z_k]).
\]

It follows that \( a \in Def^2(A^2(L_n)) \).

**Chapter IV**

The Main Theorem: \( ra(\cdot) \) is a standard \( \beta \)-interpretation of \( A_3 \) in \( A_3 \).

We are ready now to prove the main theorem about the ramified analysis \( ra(\cdot) \) (formulated below for \( A_3 \)).

**Theorem 3** (\( ra \)-theorem). 1. Formulas \( ra_3(\cdot) \) and \( ra_3(\cdot) \) from Theorem 2 (Chapter II) give a standard (\( \beta \)) interpretation of \( A_3 \) in \( A_3 \), i.e., \( A_3 \models \varphi^{ra} \) for all axioms \( \varphi \) of \( A_3 \). For an arbitrary \( \beta \)-model \( M \models A_3 \) the equalities \( R^{A_3}(\cdot) = ra(\cdot) \) hold.

2. \( M \models \varphi^{A_3}(\cdot) \) holds for all axioms \( \varphi \) of \( A_3 \).

This equivalence is easily provable by induction with respect to \( \psi \). The right-hand side of this formula will be abbreviated to \( \psi^{A_3}(\cdot) \) and the left-hand side to \( (\psi^{A_3})^{Tram} \). We then have the following lemma:

**Lemma 2.4.** \( A_3 \models \psi^{A_3}(\cdot) \) for all axioms \( \varphi \) of \( A_3 \).
Formulas \((RA^{(3)})_{\text{Thm}}^e\) with \(+_{\text{Thm}}, \cdot_{\text{Thm}}\), etc., give a nonstandard interpretation of \(A_3\) in \(A_3\). Now we shall find out how to pass from this interpretation to a standard one.

Let \(M\) be a \(\beta\)-model of \(A_3\). From Zbierski’s theorem it follows that trees from \(M\) form a model of set theory uniquely isomorphic to a standard transitive one. The collapsing isomorphism is called “realization”. It follows that a fragment of this realization is definable in \(A_3\). This definable fragment is just the restriction of the isomorphism in question to trees coding arithmetical objects. Informally speaking, we define in \(A_3\) functions \(g\) and \(f\) such that:

1. \(g = (\text{tree coding arithmetical object }\mapsto \text{its realization}).\)
2. \(f = (\text{object } a \mapsto \text{tree coding object } a).\)
3. \(g\) is a surjection and \(f\) is defined everywhere.
4. \(g \circ f\) is an identity.

We shall limit ourselves to giving only the definition of the function \(g\).

Let \(t_n\) be a natural number which codes (via the pairing function \(J\)) the canonical tree for \(n\). Such coding can be done uniformly. Moreover, we may assume that the relations “\(m \in \text{Fild}(t_n)\)” and “\(m < r, r^*\) as well as the function \((n = t_n)\) are recursive (\(m < r, r^*\) means that \(r\) is less than \(r^*\) in the sense of the tree coded by \(t_n\)).

We define \(g\) by means of formulas \(g_0, g_1, g_2, \ldots\):

\[
g_0(T, n) = \text{TeTrees} \& (Ex) ((X) \text{ is bijection of } \text{Fild}(T) \text{ and } \text{Fild}(t_n) \& \& (m)(r)(d)(\langle c, m', e \in X \& \langle d, r' \rangle \in X \Rightarrow (m < r, r^* \leftrightarrow \langle c, d', e' \rangle \in T)))
\]

\[
g_0(T, n) \text{ says that } T \text{ is isomorphic to the canonical tree for } n.
\]

\[
g_{i+1}(T, A) = \text{TeTrees} \& (T)(T \in \text{Eps}(T) \Rightarrow (Ex)g_0(T, n) \& \& (a)(a \in A \Rightarrow (ET)(T \in \text{Eps}(T) \& g_0(T, n))))
\]

\[
g_{k+1}(T, A) \text{ says that } A \text{ is a set of objects which are coded by the maximal proper subtrees of } T. \text{(Let us recall that} \)

\[
X \in \text{Eps} Y \Rightarrow (Ex)\text{Amax}(T)(X \Rightarrow Y)
\]

and

\[
X \in \text{Eps} Y \Rightarrow (Ex)\text{max}(T)(X \Rightarrow Y)
\]

see also [1], p. 204.

**Lemma 4.4** (homomorphism lemma).

1. \(A_3 \vdash (T_1)_{x, y, z} (T_2)_{x, y, z} (T_1, Eps T_2 \mapsto g(T_1) \mapsto g(T_2))\).
2. \(A_3 \vdash (T)_{x, y, z} (x_1) \in \text{Eps}(T) \mapsto g(T_1) \mapsto g(T_2))\).
3. \(A_3 \vdash (T_1)_{x, y, z} (x_1) \in \text{Eps}(T_1) \mapsto g(T_1) \mapsto g(T_2))\).
4. \(A_3 \vdash g_0(RA^{(3)}_{\text{Thm}}) = \omega\).

\[
5. A_3 \vdash (T_1)_{x, y, z} (T_2)_{x, y, z} (x_1) \in \text{Eps} \rightarrow (T_1 \cdot_{\text{Thm}} T_2) \mapsto g(T_1) \mapsto g(T_2) = g(T_2))
\]

\[
6. A_3 \vdash (T_1)_{x, y, z} (T_2)_{x, y, z} (x_1) \in \text{Eps} \rightarrow (T_1 \cdot_{\text{Thm}} T_2) \mapsto g(T_1) \mapsto g(T_2) = g(T_2))
\]

\[
7. A_3 \vdash (T_1)_{x, y, z} (T_2)_{x, y, z} (x_1) \in \text{Eps} \rightarrow (T_1 \cdot_{\text{Thm}} T_2) \mapsto g(T_1) \mapsto g(T_2) = g(T_2))
\]

\[
8. g_0(RA^{(3)}_{\text{Thm}}) = 0, g_0(1_{\text{Thm}}) = 1.
\]

The homomorphism lemma shows that \(g\) is a definable homomorphism of a nonstandard model \(\langle RA^{(3)}_{\text{Thm}}, \text{Eps}, Eq, +_{\text{Thm}}, \cdot_{\text{Thm}}, \ldots \rangle\) onto \(\langle g'RA^{(3)}_{\text{Thm}}, e, \ldots \rangle\), i.e., onto a standard one: taking the superposition of the interpretation of Zbierski with the function \(g\), we obtain the required standard interpretation of \(A_3\) in \(A_3\). We have proved that \(A_3 \vdash g'RA^{(3)}_{\text{Thm}}\) for all axioms \(\varphi\) of \(A_3\).

All that remains to be proved is **Lemma 4.5**.

**Lemma 4.5.** \(A_3 \vdash g'(g(RA^{(3)}_{\text{Thm}}) = \rho'a))\).

The proof is by induction. First we introduce the relation \(A(T)\) (for any given \(T\)), which is defined as follows:

\[
\langle a, b \rangle \in A(T) \Rightarrow (On(T))_{\text{Thm}} \& (ET_1)_{x, y, z} (T_1, Eps T_2 \mapsto (T_2 \in \text{Eps} T \& \& a = \max(T_1) \& \& b = \max(T_2) \& T_1 T_2 Eps T))
\]

If \(T\) is a tree which codes an ordinal number, then \(A(T)\) is a well-ordering of its almost maximal elements. We have \(A: On^{(3)}_{\text{Thm}} \text{Bord}.\) Conversely: for every \(X\) such that Bord \((X)\) there is a tree \(T\) such that \(A(T)\) is isomorphic to \(X\). This may be shown by induction (in \(A_3\)); namely, we show by induction that

\[
(\partial_{x, y, z} ET_{x, y, z}) (On^{(3)}_{\text{Thm}} (T) \& A(T) = X \uparrow a))
\]

Employing \(A(\cdot)\) we can reformulate the lemma to get a form more convenient for induction in \(A_3\):

\[
(X)_{\text{Bord}} (T)_{x, y, z} (On^{(3)}_{\text{Thm}} (T) \& A(T) = X \uparrow a)
\]

We leave out the tedious details of this induction.

The remaining part of 1 is contained in Theorem 1 in Chapter II. To complete the proof of Theorem 3 it remains to prove 5.

We use essentially the same methods as in the proof of 1. Let \(Ar^{(3)}(V)\) be a formula defining the “full arithmetic” \(Ar^{(3)}(V)\) and let Bord \((T)\) be
the arithmetical definition of the class of well-orderings, i.e., $\operatorname{Bord}(T) \equiv \operatorname{Bord}\omega^0(T)$. Let us recall that the $\beta$-property of $RA$ was established in $\text{ZFC}^- + \text{"P}(\omega) \text{ exists}“$, i.e.,

$$\text{ZFC}^- + \text{"P}(\omega) \text{ exists}“ \vdash (T \text{ is RA } \Rightarrow (\operatorname{Bord}^a(T) \equiv \operatorname{Bord}(T))).$$

Now we can employ Zbierski’s theorem as in 1.

$$A_3 \vdash (T)_{\text{r.e.}} (\operatorname{Bord}^a)_{\text{r.e.}} (T) \equiv \operatorname{Bord}^a(T).$$

$$A_3 \vdash (T)_{\text{r.e.}} (\operatorname{Bord}^\omega)_{\text{r.e.}} (T) \equiv \operatorname{Bord}^\omega(T).$$

This means that $A_3 \vdash "\text{RA has \beta-property}"$.

**Chapter V**

**Other nice properties of ra. Reflection.** In Chapter IV it was proved that $ra$ is an inner model of $A_3$. Zbierski’s theorem shows that it is 1

## Definition 5.1

1. "ra becomes stabilized on $X$" $\iff$ $\operatorname{Bord}(X)$ & $(ra(X, \cdot) = ra(X + 1, \cdot))$ & $(Y) (Y \leq X \Rightarrow (ra(Y, \cdot) \neq ra(Y + 1, \cdot))).$
2. "ra does not stabilize" $\iff (X)$ (ra becomes stabilized on $X$).

(Y $\leq X$ means that $Y$ is isomorphic to an initial segment of $X$).

Now we are ready to formulate and to prove the reflection principle.

**Theorem 4 (Principle of reflection).**

1. $A_3 \vdash ["ra(\cdot)" \text{ becomes stabilized on } X \mapsto (Y)_{\text{r.e.}} (\varphi) (Y \leq X \mapsto (ra(X, \cdot) = ra(X + 1, \cdot))$ & $(Y) (Y \leq X \Rightarrow (ra(Y, \cdot) \neq ra(Y + 1, \cdot))).$
2. $A_3 \vdash ["ra(\cdot)" \text{ does not stabilize} \mapsto (Y)_{\text{r.e.}} (\varphi) (Y \leq X \mapsto (ra(X, \cdot) = ra(X + 1, \cdot))$ & $(Y) (Y \leq X \Rightarrow (ra(Y, \cdot) \neq ra(Y + 1, \cdot))).$

**Proof.** Let us now introduce two abbreviations (in $L(ZF)$):

a. "RA becomes stabilized on $\alpha" \mapsto (\operatorname{RA}_{\alpha+1} = \operatorname{RA}_{\alpha} & (\beta_{\alpha+1} (\operatorname{RA} \neq \operatorname{RA}_{\alpha+1})).$

b. "RA does not stabilize" $\iff (\alpha) ("\text{RA becomes stabilized on } \alpha")$.

It is well known (W. Marek, M. Srebrny [3]) that $L_{\omega_3}$ is pointwise definable and satisfies the principle of reflection ($\eta_3$ is the first 3-gap). From this it follows that

$$\text{ZFC}^- + \text{"P}(\omega) \text{ exists“} \vdash "\text{RA satisfies the principle of reflection}, “$$

i.e.,

$$\text{ZFC}^- + \text{"P}(\omega) \text{ exists“} \vdash (\alpha) ("\text{RA becomes stabilized on } \alpha")$$

and

$$\text{ZFC}^- + \text{"P}(\omega) \text{ exists“} \vdash (\alpha) ("\text{RA does not stabilize} \mapsto (\beta_{\alpha+1} (\operatorname{RA} \neq \varphi (p)) \mapsto (\beta_{\alpha+1} (\operatorname{RA} \neq \varphi (p)))).$$

for all formulas $\varphi$

We can apply, as usual, Zbierski’s theorem to the following formula:

$$(\ast) \quad A_3 \vdash (T)_{\text{r.e.}} (\text{"RA becomes stabilized on } T^\alpha)_{\text{r.e.}}$$

$$\mapsto (T) (T^\alpha (E \mapsto (E^\alpha)^p) \mapsto (T^\alpha (E \mapsto (E^\alpha)^p)) \mapsto (T)_{\text{r.e.}} (\text{"RA becomes stabilized on } T^\alpha)_{\text{r.e.}}.$$

The formula ("RA becomes stabilized on $T^\alpha)_{\text{r.e.}}$, i.e., the formula

$$\text{"RA becomes stabilized on } T^\alpha)_{\text{r.e.}} = \operatorname{RA}_{\alpha+1} \text{ and } (T)_{\text{r.e.}} (T^\alpha (E \mapsto (E^\alpha)^p))$$

is equivalent in $A_3$ to the formula

$$\mapsto (T) (E \mapsto (E^\alpha)^p) \mapsto (T)_{\text{r.e.}} (\text{"RA becomes stabilized on } T^\alpha)_{\text{r.e.}}.$$

and by virtue of Lemma 4.5 is $A_3$-equivalent to "ra(\cdot) becomes stabilized", namely to the formula

$$ra (A (T + 1), \cdot) = ra (A (T), \cdot) \& (X)_{\text{r.e.}} (X \leq A (T) \Rightarrow ra (X, \cdot) \neq ra (X + 1, \cdot)).$$

(here $g' A$ is the image of $A$ by $g$). So in $A_3$ the formula "ra(\cdot) becomes stabilized on $T^\alpha)_{\text{r.e.}} is equivalent to the formula ("RA becomes stabilized on $T^\alpha)_{\text{r.e.}}.

We make use of a lemma stating that

$$A_3 \vdash [\text{SAT} (N_1, N_2, f (\varphi), f (\varphi)) \mapsto (Y \mapsto f (\varphi)) \mapsto (Y \mapsto f (\varphi))].$$

where Sat is arithmetical but SAT is a set-theoretical formula of satisfaction, $N_1$ is such that

$$A_3 \vdash [X \mapsto (N_1, N_2, f (\varphi), f (\varphi))]$$

for $i = 1, 2,$

$g_i$ are functions defined in Chapter IV and $Y = \{X : X \in N_i\}$. (Incidentally this fact is useful in the proof of Lemma 4.5 in Chapter IV).
Using this and (∗) together with Lemma 4.5, we obtain the principle of reflection.

Another nice property of the inner model \( r_a \) is established in the following theorem. It corresponds to the absoluteness (with respect to the class of constructible sets) of the notion of constructibility.

**Theorem 5.** \( A_\nu \models r_a^{\lambda} = r_a \).

Before beginning the proof we introduce some auxiliary notions.

**Definition 5.2.** We say that \( (\psi^1, \ldots, \psi^\nu) \) is a transitive system of formulas of \( L(A_\nu) \) if and only if the only free variable in \( \psi^i \) is of order \( i \) and

1. \( A_\nu \models [\psi^i(a^0) \& \psi^{i+1} \in a^0 \rightarrow \psi^{i+1}(b^{i+1})] \) for \( i = 2, 3, \ldots, \nu - 1 \),
2. \( A_\nu \models [\psi^i(a, b) \rightarrow \psi^i(a) \& \psi^i(b)] \).

**Example.** \( (r_a^1, r_a^2, \ldots, r_a^\nu) \) is a transitive system of formulas of \( L(A_\nu) \).

**Definition 5.3.** \( TC, T_0, T, \) are \( A_\nu \)-definable operations such that

\[
T_0(x) = \{ a : a \in x \} \cup \{ \langle a, b \rangle : (E \circ (a, b)^2 = c) \}
\]

and

\[
T_0(T_0(x)) = T_0(x),
\]

\[
T_{n+1}(x) = T_n(x) \cup T_{n+1}(x),
\]

\[
TC(x) = \bigcup T_n(x).
\]

(\( TC(x) \) is a kind of transitive closure of \( x \)).

The following easy lemma is a counterpart to the lemma on absoluteness of \( A_\nu \)-formulas with respect to standard transitive models of ZF.

**Lemma 5.1.1.** Formulas of the form \((EX^{\nu-1})(\psi(X^{\nu-1}, \ldots, X^1))^{TC(X^{\nu-1})}\) are upward absolute with respect to transitive systems of formulas.

2. Formulas of the form \((X^{\nu-1})(\psi(X^{\nu-1}, \ldots, X^1))^{TC(X^{\nu-1})}\) are downward absolute with respect to transitive systems of formulas.

By a careful analysis of \( \Pi^1_{\nu-1} \) and \( \Sigma^1_{\nu-1} \) definitions of ramified analysis we obtain:

**Lemma 5.2.** There exist formulas \( P_\nu \) and \( Q_\nu \) such that:

\[
A_\nu \models [r_a(\check{\cdot}, \check{\cdot}) \leftrightarrow (EX^{\nu-1})P_\nu(X^{\nu-1}, \check{\cdot}, \check{\cdot})^{TC(X^{\nu-1})}]
\]

and

\[
A_\nu \models [r_a(\check{\cdot}, \check{\cdot}) \leftrightarrow (X^{\nu-1})Q_\nu(X^{\nu-1}, \check{\cdot}, \check{\cdot})^{TC(X^{\nu-1})}].
\]

Theorem 5 follows easily from the above lemmas and from the fact that \( (r_a, r_a) \) is a transitive system of formulas of \( L(A_\nu) \).

Notice. For \( n = 2 \) the theorem immediately follows from the absoluteness of \( (A_\nu)^{AL} \)-formulas with respect to \( r_a(\check{\cdot}) \).

I dedicate this paper to my friend R. Z. Kufner.

**References**


Adam Mickiewicz University
Poznań, Poland

Received 28 May 1979
in revised form 18 January 1982