those homeomorphisms which also send $P_n$ to $P_{n}$, then it is possible to construct all isotopies used in the proof of Theorem 6 of [9] so that $P_n$ is send to $P_{n}$ by these isotopies. Thus by a slight modification of the proof of Theorem 6 of [9] we have that $X$ is an isomorphism from $H(Y_n, P_n)$ to $H(X_{m})$.

Remark. If $m$ and $n$ are such that $m + n = 3$, then the presentation obtained for $H(X_{m})$ yields the group $S_m \times S_n \times Z_2$. This follows since when $F$ consists of three elements, the twist homeomorphisms $a_{1,2}$ and $a_{2,3}$ are isotopic (red $F$) to the identity and each dial homeomorphism is its own inverse.

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Hyperspaces where convergence to a calm limit implies eventual shape equivalence

by

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Abstract. We introduce the calm fundamental metric as a means of topologizing the collection $2^X$ of nonempty subcompacta of a compactum $X$. The calm fundamental metric $d_\mathcal{H}$ induces a topology stronger than that of Borsuk's fundamental metric and has the following property: if $A_n$ is calm and $\lim_{n \to \infty} d_\mathcal{H}(A_n, A) = 0$, then $\text{Sh}(A_n) = \text{Sh}(A)$ for almost all $n$. The relationship between $d_\mathcal{H}$ and other hyperspace metrics is explored for certain subsets of $2^X$.

§ 1. Introduction. For a metric space $X$, let $2^X$ denote the collection of nonempty compact subsets of $X$. There have been several methods developed for imposing a metric topology on $2^X$. The best-known is by use of the Hausdorff metric $d_h$. The Hausdorff metric has interesting properties, but is displeasing from the following standpoint: for fixed $A \in 2^X$, we may have $\lim_{n \to \infty} d_h(A_n, A) = 0$ and yet for all $n$, $A_n$ and $A$ may be very different topologically. For example, every member of $2^X$ is a limit of finite sets in the topology of $d_h$.

Metrics for $2^X$ that induce stronger topologies than that induced by $d_h$ were introduced by Borsuk in [B1] and [B2]. The fundamental metric $d_\mathcal{H}$ defined in the latter paper was shown in [Co-Sc] to have the following property: if $\lim_{n \to \infty} d_\mathcal{H}(A_n, A) = 0$ and $A$ is a calm compactum (see § 3 for the definition of calm) then $\text{Sh}(A_n) > \text{Sh}(A)$ for almost all $n$.

In this paper, we assume that $X$ is a nonempty compactum. Our main results include the introduction of the calm fundamental metric $d_\mathcal{H}$, which induces on $2^X$ a topology stronger than that of $d_\mathcal{H}$ and has the following property: if $\lim_{n \to \infty} d_\mathcal{H}(A_n, A) = 0$ and $A$ is calm, then $\text{Sh}(A_n) = \text{Sh}(A)$ for almost all $n$.

After submitting the first draft of this paper, the author received a preprint of [Ce2]. We show the notion of calmly regular convergence introduced there is essentially equivalent to convergence in the topology of $d_\mathcal{H}$ and we answer a question raised in [Ce2].

We assume the reader is familiar with shape theory [B3] and the topology of the Hilbert cube [Ch].
The author wishes to thank B.J. Ball for suggesting the problem of finding a nondiscrete metric related to $d_V$ to whose topology nearness would imply shape equivalence, R. B. Sher for suggestions that improved several proofs, and the referee for suggestions on the organization of this paper.

§ 2. Preliminaries. We let $Q$ denote the Hilbert cube. For $A \in 2^Q$, $v>0$, we let $N_v(A) = \{ x \in Q, d(x, A) < v \}.$

By map we will always mean a continuous function. An e-map is a map $E$ whose domain and range lie in a metric space $(E, d)$ and that satisfies $d(f(y), f(y')) < e$ for all $y, y'$ in the domain of $f$.

If $A$ and $B$ are compact subsets of an AR-space $M$, we say a fundamental sequence $f = \{ f_n, A, B \}_{n=0}^\infty$ is an $e$-fundamental sequence if it satisfies: for some neighborhood $U$ of $A$ in $M$, there is a $B_0$ such that $k \geq B_0$ implies $f(U)$ is an e-map.

The metric of continuity $d_f$ is defined [B1] as follows: for $A, B \in 2^Q$, $d_f(A, B) = \inf \{ e > 0 \} \{ E : E \subset A \subset B \}.$ The space obtained by topologizing $2^Q$ by $d_f$ is denoted $\mathcal{C}(Q).$

The fundamental metric is defined [B2] as follows: for $A, B \in 2^Q$, $d_f(A, B) = \inf \{ e > 0 \} \{ E : E \subset A \subset B \}.$

where $Q$ is the class of all topological spaces, we abbreviate the above by $h$-Comp$(V, A)$.

For $A \in 2^Q$, we say $A$ is $\mathcal{C}$-calm if for every neighborhood $U$ of $A$ in $Q$, there is a neighborhood $V$ of $A$ in $Q$ such that $h$-Comp$(V, A)$.

We say $A$ is calm if $A$ is $\mathcal{C}$-calm and $\mathcal{C}$ is the class of all topological spaces.

In $\mathcal{C}(Q)$ it is shown that $\mathcal{C}$-calmness is a hereditary shape property. The relation of calmness to more familiar shape properties is illustrated by the following facts: Solenoids are calm [Cel]. If $Y \in 2^Q$, then $Y \in \text{FANR}$ if and only if $Y$ is calm and movable [Ce-So].

The next three results have easy proofs that are left to the reader:

3.1 Lemma. Suppose $A \in 2^Q$, $\mathcal{C}$ is a class of topological spaces, and $V$ is a neighborhood of $A$ in $Q$ such that $h$-Comp$(V, A)$. Let $V'$ be a neighborhood of $A$ in $V$, then $h$-Comp$(V', A)$.

3.2 Corollary. Let $A \in 2^Q$. Then $A$ is $\mathcal{C}$-calm if and only if there is a $\mathcal{C}$-calm $V$ in $Q$ such that $h$-Comp$(V, A)$.

3.3 Lemma. Let $A \in 2^Q$ and let $f: Q \rightarrow Q$ be a homeomorphism. Let $B = f(A)$.

If $V$ is a neighborhood of $A$ in $Q$ such that $h$-Comp$(V, A)$ for a class $\mathcal{C}$ of topological spaces, then $h$-Comp$(f(V), B)$.

We define for each $A \in 2^Q$ an index of calmness $i(A)$ as follows:

$$i(A) = \sup \{ 0 \} \cup \{ e > 0, N_v(A) \neq Q \} \text{ and } h$-Comp$(N_v(A), A)$.

Observe that $i(A) > 0$, and by (3.2) we have $i(A) = 0$ if and only if $A$ is calm.

According to [Co-So], if $A$ and $B$ are compacta lying in AR-spaces $M$ and $N$, respectively, then fundamental sequences $f = \{ f_n, A, B \}_{n=0}^\infty$ and $g = \{ g_n, A, B \}_{n=0}^\infty$ are e-close if there is a neighborhood $U$ of $A$ in $M$ such that for some integer $m$, $m \geq 0$ implies $d(f_n(x), g_n(x)) < e$ for all $x \in U$.

3.4 Theorem [Co-So, (4.1)]. If $Y \in 2^Q$ is topologically calm, then there is an $e > 0$ such that for every compactum $X$ lying in an AR-space $M$, every pair of e-closed fundamental sequences $f = \{ f_n, X, Y \}_{n=0}^\infty$ and $g = \{ g_n, X, Y \}_{n=0}^\infty$ are homotopic.

3.5 Lemma. Suppose $\{ A_n \}_{n=0}^\infty \subset 2^Q$, $\lim d(A_n, A_0) = 0$, and $A_0$ is calm. If $\delta < i(A_n)$ then for almost all $n, \delta < i(A_n)$.

Proof. Otherwise there would be a sequence $\{ A_n \}_{n=0}^\infty \subset 2^Q$ and a $\delta > 0$ such that $\lim d(A_n, A_0) = 0$ and $i(A_n) < \delta < i(A_n)$ for $n = 1, 2, ...$

Let $0 < \delta < i(A_0) < \delta$. Since convergence in the fundamental metric implies convergence in the Hausdorff metric, there is an integer $m_1$ such that $n \geq m_1$ implies $A_n \subset N_{\delta}(A_0) \subset N_{\delta}(A_0).

Let $\{ e_n \}_{n=0}^\infty$ be a sequence of positive numbers converging to 0 such that there are e-fundamental sequences $f = \{ f_n, A, A \}_{n=0}^\infty$ and $g = \{ g_n, A, A \}_{n=0}^\infty$. By (3.4), there is a positive integer $m_2$ such that $n \geq m_2$ implies $\{ e_n \}$-closed fundamental sequences to $A_n$ are homotopic.

There is a compact ANR neighborhood $A_0 \subset A_0 \subset 2^Q$ such that $A \subset N(A_0).$ There is a positive integer $m_3$ such that $n \geq m_3$ implies $A_n \subset A_0$. There is a positive integer $m_4$ such that $n \geq m_4$ implies $A_n \subset A_0.

Let $n \geq \max \{ m_1, m_2, m_3, m_4 \}$ be fixed. Let $U$ be a neighborhood of $A_0$ in $Q$. There is a neighborhood $V$ of $A_0$ in $Q$ and a positive integer $k(n)$ such that $k(n)$ implies $f(U) = U$ and $g(U) = U$ is an e-map.

Since $\delta < i(A_0)$, it follows from (3.1) that $h$-Comp$(N_{\delta}(A_0), A_0).$ Thus there is a neighborhood $W$ of $A_0$ with (by choice of $m_3$) $W \subset A$ such that $f(W)$ and $g(W)$ are maps of a topological space into $W$ with $f(W) \subset N_{\delta}(A_0)$, then $f(W) \subset W$.}
There is a neighborhood $T$ of $A_0$ in $Q$ and a positive integer $k_2(n)$ such that $k \geq k_2(n)$ implies
\[ g_k^n[T] \text{ is an } e_\rho \text{-map,} \]
\[ g_k^n[T] \subseteq W, \]
and (by choice of $m_2$), since $f^n \equiv f^n \text{ is } (2e_\rho)\text{-close to } A_0$,
\[ f^n_k \equiv g^n_k \equiv 1 \text{ in } U. \]

Let $P$ be any topological space. Let $f, g : P \rightarrow T$ with $f \equiv g$ in $N(A_0)$. Fix $k \geq \max\{k_1(n), k_2(n)\}$. Then $g^n_k \equiv f^n_k \equiv g^n_k \equiv g$ are maps from $P$ into $W \subseteq A = N(A_0)$, with $g^n_k \equiv f^n_k \equiv g$ in $A$ by choice of $m_2$. By choice of $A_0$, $f^n_k \equiv g^n_k \equiv g$ in $N(A_0)$. By choice of $W$, $g^n_k \equiv f^n_k \equiv g$ in $V$. Our last condition on our choice of $T$ implies
\[ f^n_k \equiv g^n_k \equiv f^n_k \equiv g^n_k \equiv g \text{ in } U. \]

Thus $f \equiv g$ in $U$. It follows that $h^{\text{-Comp}}(N(A_0), A_0)$. This is impossible, since $n > (A_0)$. The assertion follows.

(3.6) Lemma. Suppose $\{A_0\}_{n=0}^{\infty} \subseteq Z^H, \lim d_p(A_0, A_0) = 0$, $\lim \lambda_k(n) = 0$, and $V$ is a neighborhood of $A_0$ in $Q$ such that $h^{\text{-Comp}}(V, A_0)$. Then for almost all $n$, $h^{\text{-Comp}}(V, A_0)$.

Proof. Let $x = i(A_0)/2$. Let $P$ be a compact ANR neighborhood of $A_0$ in $Q$ such that $P \supseteq V \cap N(A_0)$. There is an integer $n_1$ such that $n \geq n_1$ implies $A_0 \subseteq P$. There is a sequence of positive numbers $\{\varepsilon_n\}_{n=1}^{\infty}$, $\lim \varepsilon_n = 0$ and there are $e_\rho$-fundamental sequences $f^n_k \equiv g^n_k \equiv g^n_k \equiv g$ in $P$. Since $P$ is an ANR, there is a positive integer $n_2$ such that $n \geq n_2$ implies maps into $P$ that are $(2e_\rho)$-close are homotopic.

Let $\delta$ satisfy $\lambda_k(2\delta) < \varepsilon < \lambda_k(A_0)$. Since convergence in $d_p$ implies convergence in $d_\rho$, our choice of $P$ implies $P \subseteq N(A_0)$ for almost all $n$. Hence $\lim d_p(A_0) = 0$, our choice of $\delta$ and (3.1) imply $h^{\text{-Comp}}(N(A_0), A_0)$ for almost all $n$. Thus there is, by (3.1), a positive integer $n_3$ such that $n \geq n_3$ implies $h^{\text{-Comp}}(P, A_0)$.

Let $n \geq \max\{n_1, n_2, n_3\}$ be fixed. Let $U$ be a neighborhood of $A_0$ in $Q$. There is a neighborhood $W$ of $A_0$ in $Q$ such that for some positive integer $k_1(n)$, $k \geq k_1(n)$ implies
\[ g_k^n[W] \subseteq P \]
and $g_k^n[W]$ is an $e_\rho$-map.

There is a neighborhood $S$ of $A_0$ in $Q$ such that $S \cap P \subseteq W$ and maps into $S$ that are homotopic in $V$ are homotopic in $W$. There is a neighborhood $T$ of $A_0$ in $Q$ and a positive integer $k_2(n)$ such that $k \geq k_2(n)$ implies
\[ f^n_k[T] \subseteq S, \]
\[ f^n_k[T] \text{ is an } e_\rho \text{-map.} \]

and (by choice of $n_2$) maps into $T$ that are homotopic in $P$ are homotopic in $U$.

Let $k \geq \max\{k_1(n), k_2(n)\}$ be fixed. Let $Y$ be a topological space and let $f, g : Y \rightarrow U$ be maps such that $f \equiv g$ in $V$.

Then $f^n_k \equiv f^n_k \equiv g^n_k \equiv g^k \equiv g$ in $Y$. Hence $f^n_k \equiv f^n_k \equiv g^n_k \equiv g$ in $V$. By choice of $S$, $f^n_k \equiv f^n_k \equiv g^n_k \equiv g$ in $P$. By choice of $W$, $g^n_k \equiv f^n_k \equiv g^n_k \equiv g$ in $P$. Our choice of $n_2$ implies $f^n_k \equiv g^n_k \equiv f^n_k \equiv g^n_k \equiv g$ in $P$. Hence $f \equiv g$ in $P$.

By choice of $T$ we have $f \equiv g$ in $U$. Therefore $h^{\text{-Comp}}(V, A_0)$.

(3.7) Lemma. Let $\{A_0\}_{n=0}^{\infty} \subseteq Z^H, \lim f : Q \rightarrow Q$ be a homeomorphism, and let $B_n = f(A_n)$ for all $n$. Suppose $\lim \lambda_k(A_0, A_0) = 0$. Then $\lim i(A_\infty) = i(A_0)$ if and only if $\lim i(B_\infty) = i(B_0)$.

Proof. By (B2), it suffices to show that $\lim i(A_\infty) = i(A_0)$ implies $\lim i(B_\infty) = i(B_0)$.

Let $\varepsilon > 0$. By (3.5) we have $i(B_\infty) < i(B_0) + \varepsilon$ for almost all $n$. Since $\varepsilon$ is arbitrary, we are done if we can show $i(B_\infty) = i(B_0)$ for almost all $n$. Clearly we may assume $0 < i(B_\infty) - i(B_0)$.

Let $\delta$ satisfy $i(B_\infty) - \varepsilon < i(B_0) + \varepsilon$ (2. Let $V = N(i(B_\infty), i(B_0))$. Since (3.1) implies $h^{\text{-Comp}}(V, B_0)$, we have $h^{\text{-Comp}}(f^{-1}(V), A_0)$ by (3.3). It follows from (3.6), since we assume $\lim i(A_\infty) = i(A_0)$, that there is an integer $n_1$ such that $n \geq n_1$ implies $h^{\text{-Comp}}(f^{-1}(V), A_0)$.

Thus $h^{\text{-Comp}}(f^{-1}(V), A_0)$, by (3.3), $h^{\text{-Comp}}(V, B_\infty)$, for $n \geq n_1$.

Our choice of $\delta$ implies there is an integer $n_2$ such that $n \geq n_2$ implies $N(B_\infty) \subseteq Q$. It follows from (3.1) that for $n \geq \max\{n_1, n_2\}$, $h^{\text{-Comp}}(N(B_\infty), B_\infty)$. Hence for $n \geq \max\{n_1, n_2\}, i(B_\infty) > \delta > i(B_0) - \varepsilon$, and the proof is done.

Chapter 4. The calm fundamental metric. Let $X$ be a compactum. The index of calmness allows us to compare quantitatively members of $Z^2$ as follows: let $h: X \rightarrow Q$ be an embedding such that $h(X)$ is a $Z$-set in $Q$. Since closed subsets of $Z$-sets in $Q$ are also $Z$-sets in $Q$, we define for all $A, B \subseteq Z$, $\lambda_k(A, B) = \lambda(h^{-1}(A), (h^{-1}(B))$.

We define the calm fundamental metric on $Z^2$ for (the embedding $h$) by $d_p(A, B) = d_p(h(A), B) = d_p(A, h(B))$ for all $A, B \subseteq Z$. It is easily seen that this formula defines a metric, since $\lambda_k$ is symmetric in $A$ and $B$, nonnegative, and satisfies the triangle inequality. Let us denote by $C_k^\infty$ the space obtained by topologizing $Z^2$ by $d_p$.

An embedding of a compactum as a $Z$-set of $Q$ will be called a $Z$-embedding. The following shows that $2^d$ is a topological invariant of $X$ and is topologically independent of the $Z$-embedding $h$ chosen.

(4.1) Theorem. Let $g: X \rightarrow X'$ be a homeomorphism. Let $h: X \rightarrow Q$ and $h': X' \rightarrow Q$ be $Z$-embeddings. Let $G: 2^d_{Z}(X) = 2^d_{Z}(X')$ be the function defined by $G(A) = g(A)$ for all $A \subseteq Z$. Then $G$ is a homeomorphism.
Proof. By [Ch. 11.1, p. 14] there is a homeomorphism $H$ of $Q$ extending $h \circ g \circ h^{-1}: h(X) \to h(X')$. (We remark that this is the reason we have insisted on working with $Z$-sets.)

It is clear that $G$ is a bijection. It follows from (B2) that $d_p(A_n, A_0) = 0$ in $2^X$ if and only if
\[
\lim_{n \to \infty} d_p(A_n, A_0) = 0 \quad \text{in} \quad 2^X.
\]
Further, it follows from (3.7) that $\lim_{n \to \infty} \lambda_2(A_n, A_0) = 0$ if and only if
\[
0 = \lim_{n \to \infty} \lambda_2(G(A_n), G(A_0)) = H(G(A_n)), \quad \text{since for} \ n = 0, 1, 2, \ldots, \ \text{we have} \ h(G(A_n)) = h(G(A_1)) = H(h(x)). \quad \text{Thus} \ G \quad \text{and} \ G^{-1} \quad \text{are continuous, and the proof is complete.}
\]

In view of (4.1), we will drop "h" from the notation, writing $2^X$ for the hyper-space of $X$ metrized by the calm fundamental metric $d_{2^X}$ for any (fixed) $Z$-embedding of $X$ into $Q$. Alternatively, when it suits our purpose, we may simply consider $X$ as a $Z$-set of $Q$, using the inclusion map for the $Z$-embedding.

Let us see that if $A, B \in 2^X$ and there are $\epsilon$-fundamental sequences $f = \langle f_n, A, B \rangle_{n=0}^\infty$ and $g = \langle g_n, A, B \rangle_{n=0}^\infty$ where $f_n \neq g_n \Rightarrow k_n$ the identity fundamental sequence on $B$, then $A \epsilon$-dominates $B$. If we also have $g_n \neq f_n \Rightarrow l_n$, we say $A$ and $B$ are $\epsilon$-shape equivalent.

The metric of calmly regular convergence $d_\infty$ for the collection $ca(X)$ of members of $2^X$ (X not necessarily compact), satisfies for $M$ an ANR containing $X$:

\[(4.2) \quad \text{Theorem \ [Ce2 (2.2) and (4.6)]}. \quad \text{Let} \ \{A_n\}_{n=0}^\infty \in ca(X). \quad \text{Then}
\]
\[
\lim_{n \to \infty} d_\infty(A_n, A_0) = 0
\]
if and only if
\[\quad \text{a) \ \lim_{n \to \infty} d_p(A_n, A_0) = 0 \ \text{and}
\]
\[\quad \text{b) \ \therefore \ \text{there is a neighborhood} \ V \ \text{of} \ A_0 \ \text{in} \ M \ \text{such that} \ h-\text{Comp}(V, A_0) \ \text{for almost all} \ n.
\]

In the following theorem, the requirement that $X$ be compact in using $d_{2^X}$ is avoided by observing that $\lim d_p(A_n, A_0) = 0$ implies that $A = \bigcup_{n=0}^\infty A_n$ is compact; then we consider $d_{2^X}$ on the hyper-space of $A$. We remark that the equivalence of a) and b) below motivated this paper, while the equivalence of b) and c) improves [Ce2, (4.9)].

\[(4.3) \quad \text{Theorem \ [Let} \ \{A_n\}_{n=0}^\infty \in ca(X). \quad \text{The following are equivalent}
\]
\[\quad \text{a) \ \lim_{n \to \infty} d_p(A_n, A_0) = 0.
\]
\[\quad \text{b) \ \lim_{n \to \infty} d_\infty(A_n, A_0) = 0.
\]
\[\quad \text{c) \ \text{Given} \ \varepsilon > 0, \text{there is an integer} \ m \text{such that} \ n \geq m \text{implies} \ A_n \text{and} \ A_0 \text{are} \epsilon-\text{shape equivalent}.
\]

Proof. That a) implies b) follows from (3.6) and (4.2).

To show b) implies c): Assume b). Then there is a compact ANR neighborhood $P$ of $A_0$ in $Q$ such that $h$-Comp($P, A_0$) for $n = 0$ and $n \geq n_1$, where $n_1$ is some positive integer. Let $\varepsilon > 0$ be such that any two $\varepsilon$-close maps into $P$ are homotopic. There is a positive integer $n_2$ such that if $n \geq n_2$, $A_n \subset P$ and $d_P(A_n, A_0) < \varepsilon$. Fix $n_3 > \max(n_1, n_2)$.

By choice of $n_2$, there are $\epsilon$-fundamental sequences $f = \langle f_n, A_n, A_0 \rangle_{n=0}^\infty$ and $g = \langle g_n, A_n, A_0 \rangle_{n=0}^\infty$. By the proof of [Ce2, (4.20)], we have $f = g \Rightarrow \lambda_2$. Let $W$ be a neighborhood of $A_0$ in $Q$. By choice of $P$, there is a neighborhood $T$ of $A_n$ in $Q$, $T \cap P$, such that maps into $T$ that are homotopic in $P$ are homotopic in $W$. There is a neighborhood $Y$ of $A_n$ in $Q$, $Y \cap T$, and a positive integer $m$ such that $k_m \geq m$ implies $g = f(x(T), y(x(T))) \Rightarrow 2 \varepsilon$ for all $x \in A_n$ and $g = f(x(T), y(x(T))) \Rightarrow \varepsilon$.

Our choice of $\gamma_0$ implies $\gamma_0 = f(x(T), y(x(T))) \Rightarrow \varepsilon$ in $P$, hence in $W$ by choice of $T$. Therefore $\gamma_0 = f(x(T), y(x(T))) \Rightarrow \varepsilon$, and the assertion follows.

To show c) implies a): suppose there is a sequence of positive numbers $\{s_n\}_{n=1}^\infty$ whose limit is 0 such that $A_n$ and $A_0$ are $\epsilon$-shape equivalent. Clearly we have $\lim_{n \to \infty} d_p(A_n, A_0) = 0$. Thus we must show $\lim d_\infty(A_n, A_0) = 0$.

Suppose $\delta < \varepsilon < \langle i(A_n) \rangle$. By (3.5), it suffices to show that there is some $\gamma_0$ such that $0 < \gamma_0 < \langle i(A_n) \rangle$.

For each $n$, there are $\epsilon$-fundamental sequences $f = \langle f_n, A_n, A_0 \rangle_{n=0}^\infty$ and $g = \langle g_n, A_0, A_0 \rangle_{n=0}^\infty$ such that $f_n = g_n \Rightarrow \lambda_2$ and $f_n = g_n \Rightarrow \lambda_2$. Let $n \geq n_1$ be fixed and let $U$ be a neighborhood of $A_0$ in $Q$. There is a neighborhood $V$ of $A_0$ in $Q$ and a positive integer $k_n$ of $n$ such that $k_n \geq k_n(0) \Rightarrow g = f(x(T), y(x(T))) \Rightarrow U$. Let $W$ be a neighborhood of $A_0$ in $Q$ such that

$W \cap P$ and maps into $W$ that are homotopic in $N_d(A_0)$ are homotopic in $W$. Our choice of $V$ implies there is a neighborhood $T$ of $A_n$ with $T \cap P$ and there is a positive integer $k_n$ such that $k_n \geq k_n(0)$ implies $f = g \Rightarrow f = g \Rightarrow U$. Hence $f = g \Rightarrow f \Rightarrow U$. It follows that $h$-Comp($N_d(A_0)$, $A_0$).
Since convergence in $d_0$ implies convergence in $d_0$ and $\tau=\delta$, we have, for some positive integer $n_0$, $N(A_0)=N(A_0)$ for $n>n_0$. It follows from (3.1) that for $n=\max\{n_0, n_2\}$ we have $h$-Comp$(N(A_0), A_0)$. This completes the proof.

For $A \in (c)(X)$, let $X[A] = \{B \in 2^X \mid Sh(B) = Sh(A)\}$. An immediate consequence of (4.3) is:

(4.4) COROLLARY. a) For $X$ compact, $X[A]$ is open in $2^X$.
   b) $X[A]$ is open and closed in $(c)(X)$.

The following question is raised in [Co2]: If $X$ is separable, is $(c)(X), d_0)$ separable? A negative answer is given in:

(4.5) EXAMPLE. Let $E^3$ denote euclidean $3$-space. Then $(c)(E^3), d_0)$ is not separable.

Proof. By [G], there is an uncountable family $\{S_x, x \in A\}$ of solenoids in $E^3$ such that $x \neq \beta$ implies $Sh(S_x) \neq Sh(S_\beta)$. Since solenoids are calm [Cl, 4.11], it follows from (4.4) that $\{E^3[S_x] \mid x \in A\}$ is an uncountable family of nonempty, pairwise disjoint open sets in $(c)(E^3)$. Hence $(c)(E^3), d_0)$ is not separable.

We remark that in (4.3), c) implies a) even if $A_0$ is not calm: For c) clearly implies

$$\lim_{n \to \infty} d_b(A_n, A_0) = 0,$$

and $d_f$ and $d_\infty$ coincide on pairs of non-calm compacta. However, if $A_0$ is not calm, then a) does not imply c), as the following shows:

Suppose $A_0$ is the usual "middle-third" Cantor set of real numbers. For $n = 1, 2, \ldots$, let $A_n$ be the set of endpoints of the $2^n$ intervals remaining after the nth step in the construction of $A_0$, i.e., $A_n = \{x \in A_0 \mid x = m(3)^n \}$ for some integer $m$.

(4.6) EXAMPLE. If $\{A_n\}_{n=0}^\infty$ is as described above, then $\lim_{n \to \infty} d_b(A_n, A_0) = 0$ in $2^X$.

Proof. In [Bx-Sh] it was shown that $\lim_{n \to \infty} d_b(A_n, A_0) = 0$. Since $A_0$ has infinitely many components, it is not calm [Co1, 4.6]. Thus $i(A_n) = 0$ (we are assuming $A_0$ is $Z$-embedded in $Q_0$). Therefore we must show $\lim_{n \to \infty} i(A_n) = 0$. But since $n \geq 1$ implies $A_n$ is discrete, the fact that

$$\lim_{n \to \infty} \min\{d(x, y) \mid x, y \text{ are distinct points of } A_n\} = 0$$

and the easily-shown fact that $h$-Comp$(V, A)$ implies no component of $V$ contains distinct points of $A_n$, imply $\lim_{n \to \infty} i(A_n) = 0$.

§ 5. On restricting $d_{\text{AF}}$ to certain subsets of $2^X$. We have seen that for non-calm members of $2^X$, $d_{\text{AF}} = d_f$. In this section, we examine $d_{\text{AF}}$ for the following subsets of $2^X$:

$\text{FAR}^X = \{Y \in 2^X \mid Y \in \text{FAR} \}$;

$\text{ANR}^X = \{Y \in 2^X \mid Y \in \text{ANR} \}$ (the latter only in the case where $\dim X = \infty$).

(5.1) LEMMA. Let $A \in Z^0, A \in \text{FAR}$. Then $h$-Comp$(Q, A)$.

The following theorem characterizes the topology of $2^X$.

(5.3) THEOREM [B1]. Let $\{A_n\}_{n=0}^\infty \subset 2^X$. Then $\lim_{n \to \infty} d_b(A_n, A_0) = 0$ if and only if

a) $\lim_{n \to \infty} d_b(A_n, A_0) = 0$, and

b) given $\varepsilon > 0$, there is a $\delta > 0$ such that for all $n$, every subset of $A_n$ with diameter less than $\delta$ contracts to a point within a subset of $A_0$ of diameter less than $\varepsilon$.

The following is a weak version of [B1, Lemma on p. 188 and Theorem on p. 196].

(5.4) THEOREM. Suppose $\lim_{n \to \infty} d_b(A_n, A_0) = 0$ in $2^X$. Then there is a neighborhood $U$ of $A_0$ in $2^X$ and a positive integer $p$ such that $n \geq p$ implies $A_0$ is a retract of $U$.

We have:

(5.5) THEOREM. Let $\{A_n\}_{n=0}^\infty \subset \text{ANR}^X$. If $\lim_{n \to \infty} d_b(A_n, A_0) = 0$, then

$$\lim_{n \to \infty} d_b(A_n, A_0) = 0$$

Proof. Since $X$ is a finite-dimensional compactum, it embeds in $2^X$ for some positive integer $n$. We may regard $2^X$ as a subspace of $2^X$ [B1, Corollary 5, p. 198]. We may regard $2^X$ as a $Z$-set in $Q_0$ by identifying $2^X$ with a neighborhood $U$ of $X$ in $2^X$ and a positive integer $p$ such that $n \geq p$ implies there is a retraction $r_p: U \to A_n$. There is a compact $\text{ANR}$ neighborhood $P$ of $A_0 \cup \bigcup_{n=1}^\infty A_n$ in $Q_0$ such that $P \cap U = Q_0_1$.

Since $\lim_{n \to \infty} d_b(A_n, A_0) = 0$ implies $\lim_{n \to \infty} d_b(A_n, A_0) = 0$ [B1, (79), p. 190], there is a sequence of positive numbers $\{\varepsilon_n\}_{n=1}^\infty$ whose limit is 0, and for $n \geq 1$ there are $\varepsilon_n$-maps $f_n: A_n \to A_0$ and $g_n: A_0 \to A_n$. Since $\varepsilon_n$-maps induce $\varepsilon_n$-fundamental sequences, it follows from (4.2) that it suffices to show that $f_1$ and $g_0$ are homotopy inverses, for almost all $n$. By [Bx-Sh, 3.14, p. 852], there is a positive integer $n_2$ such that $n \geq n_2$ implies $f_n$ or $g_n$ is a homotopy inverse.
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There is a positive integer $n_3$ such that $n \geq n_3$ implies $(2n)$-close maps into $P$ are homotopic. Let $n = \max\{n_1, n_2, n_3\}$ be fixed. Let $r: P \to A_n$ be the retraction defined by

$$r(x, g_{a+1}, g_{a+2}, \ldots) = r_a(x), \quad \text{for} \quad x \in U.$$ 

Since $g_a \circ f_a: A_a \to A_a \subset P$ is a $(2n)$-map, our choice of $n_3$ implies there is a homotopy $F: A_a \times I \to P$ with

$$F(x, 0) = g_a \circ f_a(x) \quad \text{and} \quad F(x, 1) = x \quad \text{for all} \quad x \in A_a.$$ 

Thus $r \circ F: A_a \times I \to A_a$ is a homotopy with

$$r \circ F(x, 0) = g_a \circ f_a(x) \quad \text{and} \quad r \circ F(x, 1) = x.$$ 

This completes the proof.

The converse of (5.5) is not true: Let $(A_n)_{n=0}^{\infty}$ be the sequence of [Bx, (4.9)], in which it was shown that $A_0 \neq \lim_{\to n} A_n$ in the topology of $d_{\alpha}$. However, $A_0 = \lim_{\to n} A_n$ in the topology of $d_{\alpha}$, hence in the topology of $d_{\alpha}$, hence (by (5.2)) in the topology of $d_{\alpha}$.

Thus $d_{\alpha}$ induces a stronger topology on $ANR^\alpha$ than does $d_{\alpha}$.

References


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On non compact FANR's and MANR's

by

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Abstract. It is proved that a finite dimensional metrizable space $X$ is a FANR if and only if $X$ is a MANR and the set of points at which $X$ is not locally contractible has the compact closure. As an application, for finite dimensional metrizable spaces $X$ and $Y$, a necessary and sufficient condition under which $X \times Y$ be a FANR is obtained in terms of $X$ and $Y$.

1. Introduction. The notion of FANR is introduced by K. Borsuk [2]. According to [2, p. 94] a metrizable space $X$ is a FANR if for every metrizable space $X'$ containing $X$ as a closed subset, $X'$ is a fundamental neighborhood retract of $X'$. S. Godlewski [4] has introduced the concept of MANR. From the definition it is obvious that every FANR is a MANR. By [4] and [6] the properties "to be a FANR" are not generally shape invariants in the sense of Fox [3]. In this paper we shall show that a finite dimensional metrizable space $X$ is a FANR if and only if $X$ is a MANR and the set of points at which $X$ is not locally contractible has the compact closure. Obviously the second condition is not a shape invariant.

All spaces under considerations are metrizable and are continuous. AR and ANR mean those for metrizable spaces.

2. Theorems. Let $X$ be a space and let $x \in X$. If for every neighborhood $U$ of $x$ in $X$ there exists a neighborhood $V$ of $x$ such that $V$ is contractible in $U$, then $X$ is said to be locally contractible at $x$. Put $L(X) = \{x : x \in X$ and $X$ is locally contractible at $x\}$ and $L(X) = \{x : x \in X$ and $X$ is locally contractible at $x\}$ and $L(X)$ be the closure of $X$ in $X$.

THEOREM 1. A finite dimensional space $X$ is a FANR if and only if $X$ is a MANR and $L(X)$ is compact.

Proof. If "part". Let $M$ be an AR containing $X$ as a closed set. It is assumed by [7] that $M$ is finite dimensional and $X$ is unstable in $M$ in the sense of Sher [9, p. 346]. Since $X$ is a MANR, there is a closed neighborhood $W$ of $X$ in $M$ and a mutational retraction $r: U(W, M) \to U(X, M)$. Here $U(A, M)$ means the family of all open neighborhoods of $A$ in $M$. See [3] and [5] for notations and definitions. Let $d$ be a metric in $M$. Choose an open cover $\mathcal{U}$ of the set $M - L(X)$ such that if $d(x, y) > 0 (t \to \infty$ for $x \in M - L(X)$ then diameter $\mathcal{U}(x, \mathcal{U}) = 0 (t \to \infty)$, where $\mathcal{U}(x, \mathcal{U}) = \bigcup \{U : x \in U \in \mathcal{U}\}$. Since $X$ is locally contractible at each point of the set $X - L(X)$ and $M$ is finite dimensional, by [1, Theorem (9.1), p. 80]