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DRUKARNIA UNIWERSYTETU JACIELLONSKIEGO W KRAKOWIE

On components of MANR-spaces

by
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Abstract. A metrizable space $X$ is a MANR-space [5] if and only if all components of $X$ are MANR-spaces open in $X$. For two MANR-spaces having the same shape in the sense of Fox [4] there exists one-to-one correspondence between components of these spaces such that corresponding components have the same shape.

The notion of MANR-space introduced by the author in [5] and studied in [6] and [7] is a generalization of the notion of FANR-space introduced and studied by K. Borsuk in [3]. In the case of compacta these notions coincide ([5], p. 62). In [3] K. Borsuk proved that components of a FANR-space $X$ are FANR-spaces open in $X$. In this paper we obtain analogous result for components of MANR-spaces. In [2] K. Borsuk has proved that for compacta having the same shape there exists one-to-one correspondence between components of these compacta such that corresponding components have the same shape. It is not known if it is true for arbitrary metrizable spaces when we consider shape in the sense of Fox [4]. In this paper we obtain the analogous result for MANR-spaces.

§ 1. Shape in the sense of Fox and connectivity. First, let us recall the basic notions of Fox shape theory [4].

Let $X$ be a closed subset of an ANR(90)-space $P$. The family $U(X, P)$ of all open neighborhoods of $X$ in $P$ is called the complete neighborhood system of $X$ in $P$.

Consider two arbitrary complete neighborhood systems $U(X, P)$ and $V(Y, Q)$. A mutation $f: U(X, P) \to V(Y, Q)$ is defined as a collection of maps $f: U \to V$, where $U \in U(X, P)$, $V \in V(Y, Q)$ such that:

\begin{enumerate}
  \item[(1.1)] if $ef: U \to U'$ and $f_1, f_2: U' \to V$, $f_1 \circ f_2$, $f_1 \circ f_2: U \to V'$, then there exists $U'' \in U(X, P)$ such that $U'' \subseteq U$ and $f_1(U'' \cap U') = f_2(U'' \cap U')$.
\end{enumerate}

It is easy to see that the condition (1.3) may be replaced by the following condition:

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Let $X$ be a closed subset of a metrizable space $X'$ considered as a closed subset of an ANR(9R)-space $P$. A mutation $r: U(X', P) \to U(X, P)$ is called a mutual retraction ([5], p. 52) if $r(x) = x$ for every $r \in r$ and every $x \in X$. A closed subset $X$ of a metrizable space $X'$ is called a mutual retraction ([5], p. 53) of $X'$ if there exists a mutual retraction $r: U(X', P) \to U(X, P)$. A metrizable space $X$ is called a mutual neighborhood retract ([5], p. 57) if for every metrizable space $X'$ containing $X$ as a closed subset, the set $X$ is a mutual neighborhood retract of $X'$. A closed subset $X$ of a metrizable space $X'$ is called a mutual neighborhood retract ([5], p. 56) of $X'$ if there exists a closed neighborhood $W$ of $X$ in $X'$ such that $X$ is a mutual neighborhood retract of $W$. A metrizable space $X$ is said to be a mutual absolute neighborhood retract (shortly MAR, [5], p. 57) for every metrizable space $X'$ containing $X$ as a closed subset, the set $X$ is a mutual neighborhood retract of $X'$. By the trivial shape we mean the shape of a space consisting of one point.

In [6] (Theorem 3.5), p. 90 we have proved the following

(2.1) Theorem. A metrizable space $X$ is a MAR-space if and only if the shape Sh$X$ is trivial.

From Corollary (1.8) and Theorem (2.1) we obtain the following

(2.2) COROLLARY. MAR-spaces are connected.

§ 3. Some properties of components of MANR-spaces. The notions of MAR and MANR-space are generalizations of the notions of FAR and PANR-space, respectively, introduced by K. Borsuk [3]. In the case of compacta these notions coincide ([5], (4.2), (4.4), (5.5), pp. 57, 62). K. Borsuk has proved ([2], p. 193) that components of a PANR-space $X$ are PANR-spaces open in $X$. So, it is natural to consider the following
(3.1) QUESTION. Is it true that every component of a MANR-space $X$ is a MANR-space open in $X$?

We are going to get a positive answer for Question (3.1). First let us prove the following

(3.2) THEOREM. If $X$ is a mutational retract of an ANR($\mathfrak{S}$,$\mathfrak{H}$)-space $P$, then every component of $X$ is a mutational retract of a component of $P$.

Proof. Take an arbitrary component $X_0$ of $X$. Let $P_0$ be a component of $P$ containing the component $X_0$. By the hypothesis there exists a mutation retraction $r: W(P_0, P) \to U(X, P)$. Since $P \in$ ANR($\mathfrak{S}$,$\mathfrak{H}$), then $P_0$ is open in $P$ and hence $P_0 \in$ ANR($\mathfrak{S}$,$\mathfrak{H}$). Since the set $X$ is closed in $P$, then $X \cap P_0$ is closed in $P_0$. Thus, we can consider complete neighborhood systems $W_0(P_0, P_0)$ and $U_0(X \cap P_0, P_0)$. Let us construct a mutation retraction $r_0: W_0(P_0, P_0) \to U_0(X \cap P_0, P_0)$.

Take an arbitrary $r \in r$: $P \to U$, $U \in U(X, P)$. Then $U \cap P_0 \in U_0(X \cap P_0, P_0)$. Let us show that $r(P_0) \subseteq U \cap P_0$. Since $r(P_0) \subseteq r(P) \subseteq U$, it suffices to show that $r(P_0) \subseteq r(U \cap P_0)$, and $r(P_0) \subseteq r(U \cap P_0)$ is connected, if sufficient to show that $r(P_0) \subseteq r(U \cap P_0)$. Thus, we have $r(P_0) \subseteq r(U \cap P_0)$. Let us define a map $r_0: r_0: U_0(P_0, P_0) \to U_0(X \cap P_0, P_0)$ a mutation retraction.

For an arbitrary point $x_0 \in X \cap P_0$ we have $r_0(x_0) = r(x_0) = x_0$, because $r \in r$. It remains to show that $r_0$ is a mutation. We are going to verify the conditions (1.1)-(1.3).

(3.3) THEOREM. MANR-spaces are the same as mutual retracts of ANR($\mathfrak{S}$,$\mathfrak{H}$)-spaces.

From Theorems (3.2) and (3.3) we obtain the following

(3.4) COROLLARY. Every component of a MANR-space is a MANR-space.

Proof. Take an arbitrary MANR-space $X$ and let $X_0$ be a component of $X$. By (3.3) $X_0$ is a mutational retract of an ANR($\mathfrak{S}$,$\mathfrak{H}$)-space $P$. By (3.3) $X_0$ is a mutational retract of a component $X_0$ of $P$, and since $P_0 \in$ ANR($\mathfrak{S}$,$\mathfrak{H}$), then by (3.3) $X_0 \in$ MANR.

(3.5) THEOREM. Components of MANR-space $X$ are open in $X$.

Proof. Let $X_0$ be a component of a MANR-space $X$. By Theorem (3.3) there exists an ANR($\mathfrak{S}$,$\mathfrak{H}$)-space $P$ such that $X_0$ is a mutational retract of $P$. By Theorem (3.2) the component $X_0$ is a mutational retract of a component $P_0$ of $P$. Then $P_0 \cap X_0 = P_0 \subseteq X_0$. Therefore, every component of a MANR-space $X$ is a retract of $X$.
Dr. J. Olędzki and Mr. W. Matuszewski independently obtained the following

(3.7) Theorem. If \( X \) is a mutual retraction of \( X' \) and \( X \in \text{MANR} \) then every component of \( X \) is a mutual retraction of a component of \( X' \).

Proof. Take an arbitrary component \( X_0 \) of \( X \). By (3.6) \( X_0 \) is a retract of \( X \) and since \( X \) is a mutual retraction of \( X' \), then \( X_0 \) is a mutual retraction of \( X' \), and hence \( X_0 \) is a mutual retraction of \( X'_0 \), where \( X'_0 \) is a component of \( X' \) containing \( X_0 \).

(3.8) Problem. Does Theorem (3.7) remain true without the hypothesis \( X \in \text{MANR} \)?

In [5] (Theorem (4.13), p. 59) we have proved the following

(3.9) Theorem. If \( X_t \) are MANR-spaces for every \( t \in I \), then \( \bigoplus_{t \in I} X_t \) is a MANR-space.

The converse is also true, because \( X_t \) is a retract of \( \bigoplus_{t \in I} X_t \) if and hence \( \bigoplus_{t \in I} X_t \in \text{MANR} \) then by Theorem (4.12) of [5] (p. 59) we have \( X_t \in \text{MANR} \). Therefore, we get the following

(3.10) Corollary. \( \bigoplus_{t \in I} X_t \) is a MANR-space if and only if \( X_t \) are MANR-spaces for every \( t \in I \).

From Theorem (3.5) and Corollary (3.10) we obtain the following

(3.11) Corollary. A metrizable space \( X \) is a MANR-space if and only if all components of \( X \) are MANR-spaces open in \( X \).

§ 4. Components of MANR-spaces having comparable shapes. Following K. Borsuk (21, p. 17) let us denote by \( \square(X) \) the set of all components of a space \( X \).

K. Borsuk has proved (21, pp. 215, 216) that

(4.1) If \( X \) and \( Y \) are metric compacts of the same shape then there exists one-to-one correspondence \( A: \square(X) \to \square(Y) \) such that for every component \( X_t \in \square(X) \) we have \( \text{Sh}_A(X_t) = \text{Sh}_Y(Y_t) \).

(4.2) If \( X \) and \( Y \) are metric compacts such that \( \text{Sh}_X \succeq \text{Sh}_Y \), then there exist functions \( A: \square(X) \to \square(Y) \) and \( A': \square(Y) \to \square(X) \) such that the composition \( A'A \) is the identity function on \( \square(Y) \) and then for every component \( Y_t \in \square(Y) \) we have \( \text{Sh}_X(Y_t) = \text{Sh}_X(Y_t) \).

(4.3) Problem. Do the results (4.1) and (4.2) remain true for arbitrary metrizable spaces \( X \) and \( Y \), where \( \text{Sh} \) denote the shape in the sense of Fox?

We are going to show that for MANR-spaces the answer for Problem (4.3) is "yes".

Let us observe that

(4.4) For every MANR-space \( X \) there exists an ANR(30)-space \( P \) such that \( X \) is a mutual retraction of \( P \) and every component of \( P \) contains exactly one component of \( X \).

Indeed, by Theorem (4.11) of [5] (p. 59) there exists an ANR(30)-space \( Q \) such that \( X \) is a mutual retraction of \( Q \). The set \( P \) being the union of all components of \( Q \) intersecting \( X \) satisfies the required conditions.

Suppose \( X_0 \subseteq X \subseteq P \); \( X_0 \subseteq P \); \( X \subseteq Y \subseteq Q \); \( Y_0 \subseteq Q \); \( P, Q, P_0 \)

\( \subseteq \text{ANR(30)} \); \( X \) and \( X_0 \) are closed in \( P \); \( Y \) and \( Y_0 \) are closed in \( Q \). Consider two mutations \( f: U(X, P) \to V(Y, Q) \) and \( f_0: U_0(X_0, P_0) \to V_0(Y_0, Q_0) \). We say that \( f_0 \) is a submutation of \( f \) if for every \( f_0 \in f_0 \) there exists \( f \) such that the domain of \( f_0 \) is a subset of the domain of \( f \), the range of \( f_0 \) is an intersection of the range of \( f \) with \( Q_0 \), and \( f_0(x) = f(x) \) for every \( x \) belonging to the domain of \( f_0 \).

(4.5) Lemma. Suppose \( P, Q \subseteq \text{ANR(30)} \), \( X \) and \( Y \) are closed subsets of \( P \) and \( Q \), respectively, and every component of \( P \) (of \( Q \)) contains exactly one component of \( X \) (of \( Y \)). Let \( f: U(X, P) \to V(Y, Q) \) be a mutation. Then there exists exactly one function \( A_f: \square(X) \to \square(Y) \) such that for every component \( X_t \subseteq X \) there exists a submutation \( f_0: U_0(X_0, P_0) \to V_0(A_0(X_0), Q_0) \), where \( P_0 \) and \( Q_0 \) are components of \( P \) and \( Q \) containing \( X_0 \) and \( A_0(X_0) \), respectively.

Proof. Take an arbitrary component \( X_0 \subseteq \square(X) \) and arbitrary \( f \) such that there exists exactly one component \( Q_0 \subseteq \square(Q) \) such that \( f(X_0) \subseteq Q_0 \). Since \( f \) is a mutation, then by (1.1) and (1.3) the choice of \( Q_0 \) does not depend of the choice of \( f \). By the hypothesis the component \( Q_0 \) contains exactly one component \( Y_0 \subseteq \square(Y) \). Let us set \( A_0(X_0) = Y_0 \). So, we have the definition of \( A_f: \square(X) \to \square(Y) \).

Let us define a mutation \( f_0: U_0(X_0, P_0) \to V_0(A_0(X_0), Q_0) \). Take an arbitrary \( f \neq f \): \( f: U \to V \). Then

\[ f(U \cap P_0) \subseteq U \cap Q_0, \quad U \cap P \subseteq U_0(X_0, P_0), \quad V \cap Q \subseteq V_0(A_0(X_0), Q_0). \]

Let us define a map \( f_0: U \cap P_0 \to V \cap Q_0 \) by the formula

\[ f_0(x) = f(x) \quad \text{for every } x \in U \cap P_0. \]

Let us denote by \( f_0 \) the collection of all maps \( f_0 \) which can be obtained in such a way.

Let us show that \( f_0: U_0(X_0, P_0) \to V_0(A_0(X_0), Q_0) \) is a mutation.

Let us verify the condition (1.1). Take an arbitrary \( f_0 \in f_0 \); \( U_0 \to V_0 \). Let \( U_0 \subseteq U_0(X_0, P_0), \quad V_0 \subseteq V_0(A_0(X_0), Q_0), \quad V_0 \subseteq U_0 \). We define \( f_0: U_0 \to V_0 \) by \( f_0(x) = f_0(x) \) for \( x \in U_0 \). By the definition of \( f_0 \) there exists a \( f_0 \) such that \( f_0 \neq f \). Let \( U \cap P_0 = U_0 \). Then \( f_0 \neq f_0 \). Let us set \( U' = U' \cap (U \cap (P_0 \cap P_0)) \) and \( V' = V' \cap U' \). Let us define a map \( f': U' \to V' \) by the formula \( f'(x) = f(x) \) for \( x \in U' \). Since \( f \) is a mutation, then \( f \neq f' \). Let us observe that \( U' \cap P_0 = U_0 \) and \( V' \cap Q_0 = V_0 \). Applying the definition of \( f_0 \) to the map \( f' \), we get the map \( f_0 \). Therefore, \( f_0 \neq f_0 \) and the condition (1.1) is satisfied.

Let us verify the condition (1.2). Take an arbitrary \( f_0 \in f_0 \). Let us set \( U \subseteq U \cap (P_0 \cap P_0) \). Then \( f_0 \subseteq f_0 \). Since \( f \) is a mutation, there exists a \( f_0 \) such that \( f_0 \subseteq f_0 \). Let us set \( U_0 = U \cap P_0 \) and let us observe that \( V_0 \cap Q_0 = V_0 \). Consider the map \( f_0: U_0 \to V_0 \) defined by \( f_0(x) = f(x) \) for \( x \in U_0 \). By the definition of \( f_0 \) we have \( f_0 \neq f_0 \). Therefore, the condition (1.2) is satisfied.

Let us verify the condition (1.3). Take arbitrary two maps \( f_0, f_0 \neq f_0 \) such
that \( f_{o1}, f_{o2} : U_o \to V_o \). Then there exist \( U_1, U_2 \in U(X, P) \), \( V_1, V_2 \in V(Y, Q) \) such that \( U_1 \cap U_2 = U_1 \cap V_o = U_o, \) \( V_1 \cap Q_o = V_o, \) and there exist maps \( f_1, f_2 \in f : U_i \to V_i \) for \( i = 1, 2, \) such that \( f(x) = f(x) \) for \( x \in U_o \). Since \( f \) is a mutation, then by (1.2) there exists \( U' \subseteq U(X, P) \) such that \( U' \subseteq U_1 \cap U_2 \) and \( f(U') = f(U') \) in \( V_1 \cup V_2 \). Let us set \( U'_o = U' \cap P_o \). Then \( U'_o \subseteq U_0(X_o, P_o) \) and \( f_{o1}(U'_o) = f_{o2}(U'_o) \) in \( V'_o \). Therefore, the condition (1.3) is satisfied.

Thus, \( f_o : U_0(X_o, P_o) \to V_o(A(X_o, X_o), Q_o) \) is a mutation. It is evident that \( f_o \) is a submutation of \( f \).

It is obvious that the function \( A_f : \Box(X) \to \Box(Y) \) satisfying the required condition is unique. Thus, the proof is finished.

(4.6) If \( f = g : U(X, P) \to V(Y, Q) \), then \( A_f = A_g : \Box(X) \to \Box(Y) \) and for every component \( X_o \in \Box(X) \) the submutations \( f_o \) and \( g_o \) are homotopic.

(4.7) If \( f : U(X, P) \to V(Y, Q) \) and \( g : V(Y, Q) \to W(Z, R) \), then \( A_{gf} = A_g A_f \).

(4.8) If \( u \) is the identity mutation for the system \( U(X, P) \) then \( A_u : \Box(X) \to \Box(X) \) is the identity function.

(4.9) Theorem. If \( X \in \text{MANR} \) and \( \text{Sh} X \approx \text{Sh} Y \), then there exist functions \( A : \Box(X) \to \Box(Y) \) and \( A' : \Box(Y) \to \Box(X) \) such that the composition \( AA' \) is the identity function on \( \Box(Y) \) and \( \text{Sh} A'(Y_o) \approx \text{Sh} Y_o \) for every component \( Y_o \in \Box(Y) \).

Proof. Since \( X \in \text{MANR} \) and \( \text{Sh} X \approx \text{Sh} Y \) then by Theorem (4.5) of [6] (p. 92) we have \( Y \in \text{MANR} \). By (4.4) there exist ANR(BR)-spaces \( P \) and \( Q \) containing \( X \) and \( Y \), respectively, and satisfying the hypotheses of Lemma (4.5). Since \( \text{Sh} X \approx \text{Sh} Y \), then there exist mutations \( f : U(X, P) \to V(Y, Q) \) and \( g : V(Y, Q) \to U(X, P) \) such that \( fg \approx \text{id} \). By Lemma (4.5) there exist functions \( A_f : \Box(X) \to \Box(Y) \) and \( A'_f : \Box(Y) \to \Box(X) \) satisfying the thesis of Lemma (4.5). Let us set \( A = A_f \) and \( A' = A'_f \). It follows by (4.6)-(4.8) that the functions \( A \) and \( A' \) satisfy the required conditions. Thus, the proof is finished.

Analogously we can prove the following

(4.10) Theorem. If \( X \in \text{MANR} \) and \( \text{Sh} X = \text{Sh} Y \), then there exists one-to-one function \( A : \Box(X) \to \Box(Y) \) such that \( \text{Sh} X_o = \text{Sh} A(Y_o) \) for every component \( X_o \in \Box(X) \).

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