Table des matières du tome CVI, fascicule I

P. Minc, Fixed points and locally connected cyclic continua in \( E^3 \) ................. 1-20
H. P. Tuschik, On the decidability of the theory of linear orderings with generalized quantifiers ......................... 21-32
L. Bican, P. Jambor, T. Kepka and P. Náleš, Prime and coprime modules ................. 33-44
W. Szmielew, Concerning the order and the semi-order of \( n \)-dimensional Euclidean space .......................................................... 47-56
J. Dyda, Pointed and unpointed shape and pro-homotopy .................................................. 57-69
H. R. Bennett and D. J. Lutzer, Certain hereditarily properties and metrizability in generalized ordered spaces .................. 71-84

Les FUNDAMENTA MATHÉMATICA ET publient, en langues des congrès internationaux, des travaux consacrés à la Théorie des Ensembles, Topologie, Fonctions Réelles, Théorie Desccriptive des Ensembles, Algèbre Abstraite

Chaque volume paraît en 3 fascicules

Adresse de la Rédaction:
FUNDAMENTA MATHÉMATICA ET, Śniadeckich 8, 00-950 Warszawa (Pologne)

Adresse de l’Échange:
INSTITUT MATHÉMATIQUE, ACADEMIE POLONAISE DES SCIENCES
Śniadeckich 8, 00-950 Warszawa (Pologne)

Tous les volumes sont à obtenir par l’intermédiaire de
ARS POLONA, Krakowskie Przedmieście 7, 00-068 Warszawa (Pologne)

Correspondence concerning editorial work and manuscripts should be addressed to:
FUNDAMENTA MATHÉMATICA ET, Śniadeckich 8, 00-950 Warszawa (Poland)

Correspondence concerning exchange should be addressed to:
INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, Exchange
Śniadeckich 8, 00-950 Warszawa (Poland)

The Fundamenta Mathematicae are available at your bookseller or at
ARS POLONA, Krakowskie Przedmieście 7, 00-068 Warszawa (Poland)

© Copyright by Państwowe Wydawnictwo Naukowe, Warszawa 1980

ISBN 83-01-04102-4 ISSN 0016-2736

02735

Fixed points and locally connected cyclic continua in \( E^3 \)

by

Piotr Minc (Warszawa)

Abstract. In this paper is given an example of a locally connected continuum \( Y \subset E^3 \) such that \( Y \) separates \( E^3 \) and has the fixed point property.

It is well known that a planar locally connected acyclic continuum has the fixed point property. On the other hand, each locally connected continuum separating the plane admits a fixed point free mapping. The case differs in \( E^3 \). There is an example of a locally connected acyclic continuum contained in \( E^3 \) without the fixed point property (see [1]). Borsuk posed the problem whether there exists a locally connected continuum \( Y \subset E^3 \) which separates \( E^3 \) and has the fixed point property (see also [5] Problem 7, p. 66). The aim of this paper is to give such an example. The construction gives also an example of a locally connected and acyclic continuum lying in \( E^3 \) and containing a simple closed curve which is not contractible in it.

The author acknowledges his gratitude to Professors K. Borsuk, K. Sieklucki, and H. Toruńczyk for valuable conversations during preparation of this note.

1. Combinatorial preliminaries. In this section we introduce some notions which are needed later.

Let \( G \) be a free group with two generators, \( a \) and \( b \). Let \( e \) denote the neutral element of \( G \). Denote also \( a_1 = bab^{-1} \) and \( b_1 = aba^{-1} \). Let \( T \) be the set \( \{ a, a^{-1}, a_1, a_1^{-1}, b, b^{-1}, b_1, b_1^{-1} \} \). Define a function \( i: T^2 \to \{-1, 0, 1\} \) by the formula

\[
 i(c, d) = \begin{cases} 
 1 & \text{if } c = d, \\
 -1 & \text{if } c = d^{-1}, \\
 0 & \text{if } c \neq d \neq c^{-1}. 
\end{cases}
\]

Denote \( i_0(c) = i(a, c) + i(a_1, c) \) and \( i_1(c) = i(b, c) + i(b_1, c) \) for \( c \in T \).

Let \( x \) be a function from \( [1, 2, \ldots, n] \) into \( T \). Define an integer \( t(x) \) as follows:

\[
 t(x) = \sum_{k=1}^{n-1} (i(x(f))) \sum_{j=f+1}^{n} i(x(j)).
\]

\[ t \] — Fundamenta Mathematicae 1, CVI (1980)
1.1. **Lemma.** Let \( x \) be a function from \( \{1, 2, \ldots, n\} \) into \( T \), such that for some \( r \) (\( 2 \leq r \leq n-1 \)) one of the following conditions holds:

(i) \( x(r-1) = a, x(r) = b \) and \( x(r+1) = a^{-1} \),
(ii) \( x(r-1) = a, x(r) = b^{-1} \) and \( x(r+1) = a^{-1} \),
(iii) \( x(r-1) = b, x(r) = a \) and \( x(r+1) = b^{-1} \),
(iv) \( x(r-1) = b, x(r) = a^{-1} \) and \( x(r+1) = b^{-1} \).

Let \( \gamma \) be a function from \( \{1, 2, \ldots, n-2\} \) into \( T \) defined as follows:

\[
\gamma(j) = \begin{cases} 
  x(j) & \text{for } j = 1, 2, \ldots, r-2, \\
  x(j+2) & \text{for } j = r, r+1, \ldots, n-2
\end{cases}
\]

and

\[
\gamma(r-1) = \begin{cases} 
  b_1 & \text{in case (i)}, \\
  b_1^{-1} & \text{in case (ii)}, \\
  a_1 & \text{in case (iii)}, \\
  a_1^{-1} & \text{in case (iv)}.
\end{cases}
\]

Then

\[
\tau(\gamma) - \tau(x) = \begin{cases} 
  1 & \text{in case (i)}, \\
  -1 & \text{in case (ii)}, \\
  -1 & \text{in case (iii)}, \\
  1 & \text{in case (iv)}.
\end{cases}
\]

**Proof.** Case (i).

\[
\tau(x) = \sum_{j=1}^{n-1} \sum_{l=1}^{r-1} i_l(x(j)) \sum_{j=r+1}^{n-1} i_l(x(l)) + \sum_{j=r+1}^{n-1} i_l(x(r)) + \sum_{j=r+1}^{n-1} i_l(x(l)) = \tau(\gamma) - 1.
\]

1.2. **Lemma.** Let \( x \) be a function from \( \{1, 2, \ldots, n\} \) into \( T \). Suppose that there are integers \( r \) and \( s \) such that \( 1 \leq r < s \leq n \) and either

(i) \( x(j) \in \{a, a^{-1}, a_1, a_1^{-1}\} \) for \( r \leq j \leq s \) and \( \sum_{j=r}^{s} i_{x(j)} = 0 \), or

(ii) \( x(j) \in \{b, b^{-1}, b_1, b_1^{-1}\} \) for \( r \leq j \leq s \) and \( \sum_{j=r}^{s} i_{x(j)} = 0 \).

Let \( \gamma \) be a function from \( \{1, 2, \ldots, n-s-r-1\} \) into \( T \) defined by the formula

\[
\gamma(j) = \begin{cases} 
  x(j) & \text{for } 1 \leq j < r, \\
  x(j+s+r-1) & \text{for } r \leq j \leq n-s-r-1.
\end{cases}
\]

Then \( \tau(\gamma) = \tau(x) \).

**Proof.** Case (i).

\[
\tau(\gamma) = \sum_{j=1}^{r-1} i_l(x(j)) \sum_{l=r+1}^{r+s-1} i_l(x(l)) + \sum_{j=r+1}^{r+s-1} i_l(x(r)) + \sum_{j=r+1}^{r+s-1} i_l(x(l)) = \tau(x).
\]

Case (ii).

\[
\tau(\gamma) = \sum_{j=1}^{r-1} i_l(x(j)) \sum_{l=r+1}^{r+s-1} i_l(x(l)) + \sum_{j=r+1}^{r+s-1} i_l(x(r)) + \sum_{j=r+1}^{r+s-1} i_l(x(l)) = \tau(x).
\]

1.3. **Lemma.** Let \( x \) be a function from \( \{1, 2, \ldots, n\} \) into \( T \) such that

\[
e = x(1) \times (2) \ldots \times (n).
\]

Then

\[
\tau(\gamma) = \sum_{j=1}^{n} i_l(b_1, x(j)) - \sum_{j=1}^{n} i_l(a_1, x(j)).
\]

**Proof.** In the case where \( x(j) \in \{a, a^{-1}, b, b^{-1}\} \) for all \( j = 1, \ldots, n \), one can prove using 1.2 that \( \tau(\gamma) = 0 \). Thus the lemma follows from 1.1.

1.4. **Lemma.** Let \( x \) be a function from \( \{1, 2, \ldots, n\} \) into \( T \) such that \( \tau(\gamma) \neq 0 \).

Suppose that \( N_1, N_2, \ldots, N_s \) are mutually disjoint subsets of \( \{1, 2, \ldots, n\} \) such that \( \bigcup_{i=1}^{s} \bigcup_{j=1}^{s} N_i = \{1, 2, \ldots, n\} \) and there is a number \( l_0 \) (\( 1 \leq l_0 \leq l \)) such that

(i) \( x(j) \in \{a, a^{-1}, a_1, a_1^{-1}\} \) for \( j \in N_1 \) and \( \sum_{j \in N_1} i_{x(j)} = 0 \) for \( 1 \leq j \leq l_0 \), and

(ii) \( x(j) \in \{b, b^{-1}, b_1, b_1^{-1}\} \) for \( j \in N_2 \) and \( \sum_{j \in N_2} i_{x(j)} = 0 \) for \( l_0 < j < l \).
Then there are natural numbers $s_1, s_2, j_1, j_2, j_3$ and $j_4$ such that $1 \leq s_1 \leq s_2$, $1 \leq j_1 < j_2 < j_3 < j_4 \leq n$, and either $j_1, j_3 \in N_{s_2}$ and $j_1, j_2 \in N_{s_1}$, or $j_1, j_3 \in N_{s_1}$ and $j_1, j_4 \in N_{s_2}$.

Now suppose that the smallest natural number satisfying the assumption but not satisfying the conclusion of the lemma.

There are $f_1, f_2, \ldots, f_r$ such that, either $1 \leq s_1 \leq s_2$ for all $s = 1, 2, \ldots, r$ or $s_1 = s_{r+1}$ for all $s = 1, 2, \ldots, r$, and there are $j_1$ and $j_2$ such that $1 \leq j_1, j_2 \leq n$ and $
abla_j N_{s_2} = \{j_1, j_1 + 1, \ldots, j_2\}$. Let $\gamma$ be a function from $\{1, 2, \ldots, n - j_1 - 1\}$ into $T$ defined by the formula

$$
\gamma(j) = \begin{cases} 
\psi(j) & \text{for } 1 \leq j < j_1, \\
\nu(j + j_2 - j_1 + 1) & \text{for } j_1 \leq j \leq n - j_2 + j_1 - 1.
\end{cases}
$$

By 1.2 $\tau(x) = \tau(y)$. Now from our supposition the lemma follows.

2. Perforated discs.

Let $C$ be a continuum lying in the plane $E^2$. By $C$ we denote the union of $C$ and all bounded components of $E^2 - C$. By $C$ we denote the boundary of $C$ in $E^2$.

Let $C \subseteq E^2$ be a simple closed curve. By the orientation $+1$ of $C$ we mean the clockwise orientation. By the orientation $-1$ we mean the opposite one. Note that the orientation of $C$ determines the orientation on any subarc of $C$.

2.1. Proposition. Let $C_1$ and $C_2$ be two simple closed curves with the orientation $+1$. If $C_1 \cap C_2$ is an arc, then $C_1 \cap C_2$ is an arc determined by $C_1$ and $C_2$ agree.

If $C_1 \cap C_2 = C_1 \cap C_2$, then the orientations of $I$ determined by $C_1$ and $C_2$ are opposite.

By a perforated disc with $n$-holes ($n = 0, 1, 2, \ldots$) we mean a two-dimensional continuum $F \subseteq E^2$ such that $F$ is the union of $n+1$ mutually disjoint simple closed curves.

If $C \subseteq F$ is a simple closed curve, then we say that $F$ determines the orientation $+1$ on $C$ if $F \cap C$, and the orientation $-1$ in the opposite case.

2.2. Proposition. Let $F_1$ and $F_2$ be two perforated discs. Let $F_1$ be an arc contained in $F_3$, then the orientations on $F_1$ and $F_3$ are identical. If $F_1 \cap F_2 = F_1 \cap F_2$, then the orientations on $F_1$ determined by $F_1$ and $F_2$ are opposite.

Let $L$ be a one-dimensional polyhedron, and let $K \subseteq L$ be an oriented simple closed curve such that $K \cap L - K$ is void or consists of a single point $p$. Denote by $r$ the retraction of $L$ onto $K$ with $r(L - K) = \{p\}$. Let $C$ be another oriented simple closed curve and let $f$ be a mapping of $C$ into $L$. By $w(f, C, K)$ we denote the integer $j$ such that $w(f, C - K) = k^j$, where $c$ and $k$ are generators of the fundamental groups of $C$ and $K$, respectively, determining the chosen orientation of $C$ and $K$.

**Note:** The number $w(f, C, K)$ is equal to the number of oriented components of $f^{-1}(K - \{p\})$ which $f$ maps onto $K - \{p\}$ and preserves the orientations minus the number of the remaining components of $f^{-1}(K - \{p\})$ which are mapped onto $K - \{p\}$.

If $F$ is a union of a finite number of mutually disjoint perforated discs and $f : F \to L$, then by $w(f, F, K)$ we denote the sum of $w(f, C, K)$, where $C$ runs over all components of $F$ with orientations determined by the component of $F$ which contains $C$ ($f_C$ denotes $f$ restricted to $C$).

2.3. Proposition. Let $F$ be a perforated disc and let $f_0$ be a mapping of $F$ into an oriented simple closed curve $K$. Then $w(f_0, F, K) = 0$ if and only if there exists a mapping $f : F \to K$ such that $f_0 = f_0$.

2.4. Proposition. Let $F_1$ and $F_2$ be two plane sets such that

(i) each of the sets $F_1$, $F_2$ and $F_1 \cup F_2$ is a union of a finite number of mutually disjoint perforated discs, and

(ii) $F_1 \cap F_2 = F_1 \cap F_2$.

Let $f$ be a mapping of $F_1 \cup \hat{F}_2$ into a one-dimensional polyhedron $L$ containing an oriented simple closed curve $K$ as in the preceding definition. Then $w(f, F_1, K) + w(f, F_2, K) = w(f, F_1 \cup F_2, K)$.

We finish the section by the following

2.5. Lemma. Let $F_0$ be a disc and let $F$ be a perforated disc. Suppose $f : F \to F_0$ is a mapping such that $f(E) \subseteq F_0$ and $w(f, F, F_0) = 0$. Then there is a homotopy $H$:

$$H : F \times [0, 1] \to F_0$$

such that

(i) $H(u, 0) = f(u)$ for $u \in F$,

(ii) $H(u, 1) = f(u)$ for $u \in F$ and $t \in [0, 1]$ and

(iii) $H(u, 1) \subseteq F_0$ for $u \in F$.

**Proof.** By 2.3 there is a mapping $g : F \to \hat{F}_0$ such that $g|_0 = f_0$. Define $g_1 : (F \times [0, 1]) \cup (\hat{F} \times (0, 1) \cup (\hat{F} \times [1])) \to F_0$ by the formula

$$g_1(v, t) = \begin{cases} f(v) & \text{for } 0 < t < 1, \\
g(v) & \text{for } t = 1.
\end{cases}$$

An extension of $g_1$ over the whole of $F \times [0, 1]$ has the desired properties.

3. Basic constructions.

In this section we construct a locally connected acyclic continuum $X \subseteq E^3$. In Section 4 we shall show that $X$ contains a simple closed curve $A_D$ which is not contractible in $X$. The continuum $Y$ is the union of $X$ and a disc $D$ meeting $X$ at $A_D$. In Section 5 we shall prove that $Y$ has the fixed point property. In this section we shall only show that $X$ has the fixed point property and $Y$ satisfies the Lefschetz fixed point theorem.

The notation established in this section will be used freely throughout the rest of this paper.

Let $S_1$ and $S_2$ be two mutually disjoint circles. Fix two points, $x_1 \in S_1$ and $x_2 \in S_2$. Let $Z_i = \{x_1, 2, -1\}$ denote $S_i \times [-2, -1]$ with points $(z_i, -2)$ and $(z_i, -1)$ identified. Attach $Z_1$ to $Z_2$ by homeomorphisms sending $\{z_i\} \times [-2, -1]$ onto
$S_1 \times \{-2\}$ and $S_1 \times \{-1\}$ onto $z_1 \times [-2, -1]$. Denote the resulting space by $U$.

One can assume that $Z_1 \subset U$ and $Z_2 \subset U$. There is an embedding of $U$ into the Cartesian three-space $E^3$ such that

\[ U \cap \{(x, y, z) \in E^3 ; \, \, z = 0, \, x \geq 0, \, y \geq 0\} = S_1 \times \{-1\}, \]
\[ U \cap \{(x, y, z) \in E^3 ; \, \, z = 0, \, x \geq 0, \, y 

(\text{comp. Fig. 1).} \text{ Observe that } U \text{ has the same homotopy type as one-point union of two circles, namely } S_1 \times \{-1\} \text{ and } S_2 \times \{-2\}.}

Note that any orientation on $S_n$ ($n = 1, 2$) determines, by the natural projection, orientations on $S_1 \times \{-1\}$ and $S_2 \times \{-2\}$. There are orientations on $S_1$ and $S_2$ such that if $a, a_1, b$ and $b_1$ are elements of $\pi_1(U)$ corresponding, respectively, to the orientations on $S_1 \times \{-1\}, S_1 \times \{-2\}, S_2 \times \{-2\}$ and $S_2 \times \{-1\}$, then $a_1 = b a b^{-1}$ and $b_1 = a b a^{-1}$ (see Fig. 1). Fix these orientations on $S_1$ and $S_2$.

Let $N$ denote the set natural numbers (including $0$). In the set $(S_1 \times [0, 1] \times N) \cup \{(S_2 \times [0, 1] \times N) \cup U \times N, \text{ for each } n \in N \text{ and } z \in S_1 \cup S_2 \text{ identify points } (z, -2, n) \text{ and } (z, 1, n), (z, -1, n) \text{ and } (z, 0, n + 1). \text{ Observe that the resulting space is a tangle of two canals, say } C^1 \text{ and } C^2. \text{ Denote this space by } C^1 \cup C^2. \text{ Points of } C^1 \cup C^2 \text{ we shall denote as points of } (S_1 \times [0, 1] \times N) \cup (S_2 \times [0, 1] \times N) \cup (U \times N).

For each $n \in N$ and $\gamma = 1, 2$ let us adopt the following notation:

\[ C^1_n = U \times [n], \]
\[ Z^* = Z_2 \times [n], \]
\[ C^*_n = S_1 \times [0, 1] \times [n], \]
\[ A^*_n = S_1 \times [0] \times [n], \]
\[ B^*_n = S_1 \times [1] \times [n], \]

\[ \begin{align*}
L^*_1 &= (S_1 \times [0, 1] \times [n]), \\
P^*_1 &= (S_1 \times [0, 1] \times [n]).
\end{align*}

In this notation $C^* = \bigcup_{n \in N} (C^*_n \cup Z^*_n)$.

Note that the fixed orientation on $S_4$ determines by the natural projection the orientations on $A^*_n$ and $B^*_n$.

Let $f_1 : [0, \infty) \to (-\infty, \infty)$ be defined by the formula

\[ f_1(t) = \begin{cases}
\frac{t + 2}{t + 1} (-1 + \sin t) & \text{for } 2\pi \leq t \leq (2n + 1)\pi, \, n \in N, \\
\frac{t + 2}{t + 1} (-1 - 2\sin t) & \text{for } (2n + 1)\pi \leq t \leq (2n + 2)\pi, \, n \in N.
\end{cases}

Define the function $\phi_1 : [0, \infty) \to E^3$ by the formula

\[ \phi_1(t) = \left( f_1(t), \frac{\sin t}{t + 1}, \frac{1}{t + 1} \right). \]

Figure 2 shows the result of the projection of $\phi_1([0, \infty))$ into the plane $\{(x, y, z) \in E^3 ; \, z = 0\}$.

Let $f_2 : [0, \infty) \to (-\infty, \infty)$ be defined by the formula

\[ f_2(t) = (\sin(t + n) + 1). \]

Define $\phi_2 : [0, \infty) \to E^3$ as follows:

\[ \phi_2(t) = \left( f_2(t), \frac{1}{t + 1}, \frac{1}{t + 1} \right). \]
Figure 3 shows the image of the projection of $\varphi_1([0, \infty)) \cup \varphi_2([0, \infty))$ into the plane $(x, y, z) \in E^3; z = 0$.

$J = J(t)$ (v = 1, 2) denote the straight segment between points $(r, t, 0, 0)$ and $(r, t')$. If $(r, t) \neq (r', t')$, then the intersection of $J_r(t)$ and $J_r(t')$ is either void or consists of a single point $(r, t, 0, 0) = (r, t', 0, 0)$.

![Figure 3](image)

Let us adopt the following notation: $P_v = \bigcup \{J_r(t); t \in [0, \infty]\}$ for $v = 1, 2$, and $J = \{(x, y, z) \in E^3; -1 \leq x \leq 1, y = z = 0\}$. There is an embedding of $C^1 \cup C^2$ into $E^3$ such that:

1. $L_n^v = \varphi_v([1 + 2(n-1)] \pi, \frac{1}{2} + 2n\pi])$ for $v = 1, 2$, and $n = 1, 2, \ldots$

2. $(P_n \cup P_0) \cap (C^1 \cup C^2) = \varphi_1([0, \infty)) \cup \varphi_2([0, \infty))$.

3. $\det S \times \{x\} \times \{2^a\}$ for $v = 1, 2$, and $a = 1, 2, \ldots$ and $t \in [-2, -1) \cup [0, 1]$.

4. $\{x, y, z\} \in C_1^\infty \times \{1\}$ and $\{x, y, z\} \in C_2^\infty \times \{3\}$.

5. The sets $\{(x, y, z) \in C_1^\infty; x = -1\}$ and $\{(x, y, z) \in C_2^\infty; x = -1\}$ are both unions of two disjoint circles, for $n \in \mathbb{N}$.

Denote $P_1 \cup P_2 \cup C^1 \cup C^2$ by $X$. A schema of $X$ is illustrated in Figure 4. Denote also by $C^2_n \subset (C^1 \cup C^2) \cap \{(x, y, z) \in X; -1 \leq x \leq 1\}$ containing $C^2_n$. Note that $C^2_n$ is of the homotopy type of $C_1^\infty$, so $C^2_n$ is of the homotopy type of a one-point union of two circles.

Let $D$ be a disc in $E^3$ such that $A_2$ is its boundary and $D \cap X = A_2$. Denote $X \cup D$ by $Y$. Observe that $X$ and $Y$ are both locally connected, $X$ is acyclic, and $Y$ separates $E^3$ into two components.

3.1. Lemma. $X$ and $Y$ are QANR-spaces (see [4], comp. [6]).

Proof. Since $\varphi_1([0, \infty))$ is homeomorphic to the $(\sin(1/x))$-curve (the closure of $(x, y) \in E^3; y = \sin(1/x), 0 < x < 1)$, $\varphi_2([0, \infty)) \equiv \text{QANR}$ (see [6]). If we identify points $(1-x, 0)$ and $(1+x, 0)$ for $0 \leq x \leq 1$, in the $(\sin(1/x))$-curve we obtain a continuum homeomorphic to $\varphi_1([0, \infty))$. Hence, by [4, 3.4], $\varphi_2([0, \infty)) \equiv \text{QANR}$.

Let $F_v (v = 1, 2)$ be a continuum obtained from $\varphi_v([0, \infty)) \times S_1$ by the identification of each $(x, y) \times S_1$ with a point, where $x \in E^3$.

![Figure 4](image)

By [4, 4.1 and 4.3] we get $F_v \equiv \text{QANR}$.

Let $D'$ be a disc in $E^3$ such that $A_2$ is its boundary and $D' \cap X = A_2$. Let $Y$ be the union of $Y \cup D'$ and of the two bounded components of $E^3 - (Y \cup D')$. It is easy to see that $Y$ is an ANR-space. There is a neighbourhood $V$ of $X - J$ in $\bar{Y}$-J such that $X$ is a retract of $V$ and the boundary of V in $\bar{Y}$-J consists of two components, $K_1$ and $K_2$, such that $K_1$ is homeomorphic to $F_v$, $v = 1, 2$, $K_2$, and $K_2 \cup K_2 \cup J \equiv \text{QANR}$. By [4, 2.4], there are neighbourhoods $K_1$ of $K_1 \cup K_2 \cup J$ in $\bar{Y}$ and a quasi-deformation $H_{1, 2}$ of $P$, onto $K_1 \cup K_2 \cup J$ in $\bar{Y}$. It is easy to see that $H_2: (V_1 \cup V) \times [0, \infty) \to \bar{Y}$ defined by the formula

$$H(v, t) = \begin{cases} v & \text{for } v \in V_1 \\ H(v, t) & \text{for } v \in V_1 - V \end{cases}$$

is a quasi-deformation of $V_1$ to $V$ in $\bar{Y}$. Hence $\bar{Y} \equiv \text{QANR}$. Since $X$ is a retract of $V$, $X \equiv \text{QANR}$ (see [4, 4.7]). Again by [4, 4.3] we infer that $Y \equiv \text{QANR}$.
By [4, 3.1] we get the following

3.2. Corollary. X has the fixed point property.

For a mapping \( f: Y \to Y \), \( A(f) \) denote the Lefschetz number of \( f \), i.e. \( A(f) = \sum (-1)^i \text{Tr}( f_*) \), where \( \text{Tr}( f_*) \) is the trace of the homomorphism \( f_* \) induced by \( f \) on the \( i \)-th (Vietoris or Čech) homology group of \( Y \), over the rationals. Again by [4, 3.1] we get the following

3.3. Corollary. \( Y \) satisfies the Lefschetz fixed point theorem, i.e. each mapping \( f: Y \to Y \) with \( A(f) \neq 0 \) has a fixed point.

4. Auxiliary lemmas. In this section we prove the most significant Lemma 4.4 of this paper. As a corollary we conclude that \( A^0_0 \) is not contractible in \( X \). The following three lemmas are needed in the proof of 4.4.

4.1. Lemma. Let \( \delta \) be a positive real number. For \( v = 1, 2 \) and for \( n = 1, 2, \ldots \), let \( Q_1^n \) be an arc with endpoints \( c_1^n \) and \( d_1^n \), contained in the plane. Suppose that the following conditions are fulfilled for all \( n \geq 1 \):

(i) \( Q_1^n \cap Q_2^n \cap \bigcup_{j=n}^{\infty} Q_j = \emptyset \),

(ii) \( \text{dist}(c_1^n, Q_2^n) > \delta \), \( \text{dist}(d_1^n, Q_2^n) > \delta \), \( \text{dist}(c_2^n, Q_2^n) > \delta \) and \( \text{dist}(d_2^n, Q_2^n) > \delta \),

(iii) \( Q_1^n \cap Q_2^n \neq \emptyset \) and

(iv) there is a simple closed curve \( K \subset E^2 \) such that

(a) \( K \cap Q_1^n \) consists of two points, \( e_1^n \) and \( e_1^n,2^n \), for \( v = 1, 2 \),

(b) points \( e_1^n \) and \( e_1^n \) separate \( K \) between \( c_1^n \) and \( d_1^n \), and

(c) sets \( c_1^n \) and \( d_1^n \) and \( Q_1^n \cap Q_2^n \) are contained in three distinct components of \( Q_2^n - \{ e_1^n, e_2^n \} \) for \( v = 1, 2 \).

Then the set \( \bigcup_{n=1}^\infty Q_1^n \) is unbounded.

Proof. We claim that if \( C \) is a continuum such that \( c_1^n, d_1^n \subset C \) and \( Q_1^n \cap C = \emptyset \) for some \( n \), then \( C \cap Q_2^n \) separates the plane between \( c_1^n \) and \( d_1^n \). Suppose that this is not true. Let \( L = Q_2^n - \{ c_1^n, d_1^n \} \) be an arc containing \( Q_1^n \cap Q_2^n \). Since \( C \cap Q_2^n \) and \( Q_1^n \cap Q_2^n \cup L \) do not separate the plane between \( c_1^n \) and \( d_1^n \), and \( C \cap Q_2^n \cap Q_2^n \cap Q_2^n \) separates the plane between \( c_1^n \) and \( d_1^n \) (see [3, § 61, I, Th. 7]). Let \( g \) be the canonical mapping of \( E^2 \) onto the quotient space of \( E^2 \) decomposed onto \( L \) and single points. By [3, § 61, IV Th 8], \( g(E^2) \) is homeomorphic to the plane.

Let \( Q_1^n \subset g(Q_1^n) \) be an arc with endpoints \( g(c_1^n) \) and \( g(d_1^n) \). Observe that \( g(Q_2^n) \cap Q = g(L) \) is a single point and, by (iv), \( Q \) cuts sufficiently small neighbourhoods of \( g(L) \) onto two components intersecting \( g(Q_1^n) \). Thus points \( g(c_1^n) \) and \( g(d_1^n) \) belong to two different components of \( g(E^2) - (Q \cup g(C)) \), but this is impossible, because \( Q_2^n \cup C \cup L \) does not separate the plane between these points. This contradiction proves the claim.

Now, suppose that the lemma fails. Choosing a convergent subsequence of \( (Q_1^n) \) in the hyperspace of a disc containing \( \bigcup_{n=1}^\infty Q_1^n \), one can assume without loss of generality that \( Q_1^n \subset B(Q_2^n, 1/2) \) for all \( n \) and \( j \), where \( B(Q_2^n, 1/2) \) denotes a 1/2-ball around \( Q_2^n \). By (ii) we get

1. \( B(Q_2^n, 1/2) \cap B(Q_2^n, 1/2) \cap \bigcup_{j=1}^\infty Q_j = \emptyset \)

Similarly, choosing convergent subsequences of \( (c_1^n) \) and \( (d_1^n) \), one can assume that there are two \( 1/2 \)-balls \( B_1^n \) and \( B_2^n \) such that \( c_1^n \in B_1^n \) and \( d_1^n \in B_2^n \) for all \( n \). By (ii) we infer

2. \( Q_1^n \cap (B_1^n \cup B_2^n) = \emptyset \)

Let \( G_1 \) be a component of \( E^2 - (B_1^n \cup B_2^n \cup Q_1^n \cup Q_2^n) \) which contains infinitely many \( Q_2^n \)'s (see 2 and (i)). By the claim, 2 and (i) there is another component, \( G_1' \), of \( E^2 - (B_1^n \cup B_2^n \cup Q_1^n \cup Q_2^n) \) which contains \( c_1^n \) or \( d_1^n \), By 1, \( G_1' \) contains a \( 1/2 \)-ball.

Now, take \( n_1 \) such that \( Q_1^n \subset G_1 \). Let \( G_2 \) be a component of \( G_1 - Q_1^n \), which contains infinitely many \( Q_2^n \)'s. By the claim, (i) and (ii) there is another component, \( G_2' \), of \( G_1 - Q_1^n \), which contains \( c_1^n \) or \( d_1^n \). By 2, \( G_2' \) contains a \( 1/2 \)-ball. Repeating the argument, we construct an infinite sequence of mutually disjoint \( 1/2 \)-balls lying in a bounded plane region. This contradiction completes the proof.

Let us prove the following

4.2. Lemma. Let \( F \) be a perforated disc with \( k \) holes, and let \( j: F \to X \) be a mapping. Then there is an \( n \in N \) such that for each \( n \geq n \) and for each simple closed curve \( S \) contained in \( f^{-1}(C^n) \), the mapping \( f_{j,n} \) is homotopic to \( C^n \) to a constant map.

Proof. By the compactness of \( F \) there is a \( \delta > 0 \) such that if \( g(f(x_1), f(x_2)) < \delta \) then \( g(f(x_1), f(x_2)) < \delta \) for each \( x_1, x_2 \in F \). Let \( m \) be a natural number such that there is no family of \( m \) mutually disjoint \( 1/2 \)-balls contained in \( F \). Suppose that there are \( k + m \) naturals \( n_1, \ldots, n_{k+m} \) such that \( f^{-1}(C^n) \) contains a simple closed curve \( R_1 \) with \( f_{j,n} \) not homotopic in \( C^n \) to a constant map for \( j = 1, 2, \ldots, k + m \). The curves \( R_1, \ldots, R_{k+m} \) are mutually disjoint. One can easily see that there are least \( m \) components \( G_1, \ldots, G_m \) of \( F \) such that the boundary (with respect to the plane) of each of them is contained in \( \bigcup_{j=1}^{k+m} R_j \). Observe that \( f(G_j) - ((x, y, z) \in E^3; |z| < 1/2) \) is nonvoid. Hence each set \( G_j \) contains a \( 1/2 \)-ball. This contradiction completes the proof.

4.3. Lemma. Let \( F \) be a perforated disc and let \( f: F \to X \) be a mapping such that \( f^{-1}(C^n) \subset \bigcup_{j=1}^{k+m} R_j \), Then there exist a mapping \( f \) and a sequence \( F_0, F_1, \ldots \) of subsets of \( F \) such that

(i) \( f_{j,n} = f_{j,n} \),

(ii) \( f \) is homotopic to \( X \) in \( f \) relatively to \( F \),

(iii) each \( F_j (j = 0, 1, \ldots) \) is the union of a finite number of mutually disjoint perforated discs,
(v) \( F_0 \cap F_j \) is the union of a finite number of mutually disjoint arcs and simple closed curves for \( n \neq j \).

(vi) \( f(F_0) = C_1 \), \( f(F_{j+1}) = C_1 \) and \( f(F_{j+1}) = C_1 \) for \( j = 0, 1, \ldots \).

(vii) \( f^{-1}(C_2 \cup C_3) = \bigcup_{n=0}^{\infty} F_n \).

Proof. Let us adopt the following notation: \( C_j = T_{3j+1} \) for \( j = 0, 1, \ldots \) and \( v = 1, 2, 3 \).

There is a sequence \( U_0, U_1, \ldots \) of neighbourhoods of (respectively) \( T_0, T_1, \ldots \) in \( X \) such that

1. \( U_n = U_0 \) for \( |n-k| \geq 5 \) and
2. \( U_n \cap A_n = \emptyset \) for \( n \geq 1 \).

Observe that for each \( n = 0, 1, \ldots \), there are a neighbourhood \( V_n \) of \( T_n \) in \( U_n \) and a homotopy \( h_n: X \times [0, 1] \to X \) such that

3. \( h_n(x, 1) \) is a retract of \( V_n \) onto \( T_n \),
4. \( h_n(x, t) = x \) for \( x \in T_n \cup (X - U_n) \) and \( t \in [0, 1] \),
5. \( h_n(x, 0) = x \) for \( x \in X \),
6. \( h_n(x, t) = T_k \) for \( k = 0, 1, \ldots \), \( t \in [0, 1] \) and \( x \in T_k \),
7. \( h_n(x, t) = X - \bigcup_{k=0}^n T_k \) for \( x \in X - \bigcup_{k=0}^n T_k \) and \( t \in [0, 1] \),
8. \( h_n(x, 1) = T_n \cup (X - \bigcup_{k=0}^n T_k) \) for \( x \in X - \bigcup_{k=0}^n T_k \) and
9. \( g(h_n(x, t), x) < 2^{-t} \) for \( x \in X \) and \( t \in [0, 1] \).

The proof of existence of such a homotopy is omitted here, but can easily be carried out by using the fact that \( T_n \) is topologically a polyhedron. A polyhedron having nice polyhedral neighbourhoods.

There is a set \( F_0 = f^{-1}(V_0) \) such that

10. both \( F_0 \) and \( F_n - F_0 \) are the unions of a finite number of mutually disjoint perforated discs and
11. \( f^{-1}(T_n) \subseteq F_0 \).

Define \( g_0: X \times [\frac{1}{2}, 1] \to X \) by the formula

\[ g_0(x, t) = h_0(f(x), 2(1-t)|d(x, F_0)|^{-1}) \]

where \( d(x, F_0) \) denotes the distance between \( x \) and \( F_0 \). Observe that by 5

12. \( g_0(x, 1) = f(x) \) for \( x \in F \), by 4
13. \( g_0(x, t) = f(x) \) for \( x \in A_0 \) and \( t \in [\frac{1}{2}, 1] \), and by 3, 7 and 8
14. \( \{ x \in F; g_0(x, 1) \notin F_0 \} = F_0. \)

We shall construct a sequence of mappings \( g_0, g_1, \ldots \) and a sequence \( F_0, F_1, \ldots \) of subsets of \( F \) satisfying the following conditions:

15. \( g_0: F \times [\frac{1}{n+2}, 1] \to X \) is continuous for \( n = 0, 1, \ldots \),
16. \( g_0(x, t) = g_0(x, t) \) for \( x \in F, k, n = 0, 1, \ldots \) and \( t \in [\frac{1}{n+2}, 1] \cap [\frac{1}{k+2}, 1] \),
17. \( \text{diam} \left( g_0(x \times [\frac{1}{n+2}, 1] \cup [\frac{1}{n+2}, 1] \right) < 2^{-t} \) for \( n = 0, 1, \ldots \),
18. each set \( F_n \) and \( F_n - \bigcup_{j=0}^{n-1} F_j \) is the union of a finite number of mutually disjoint perforated discs and \( F_n \cap \bigcup_{j=0}^{n-1} F_j \) is the union of a finite number of mutually disjoint arcs and simple closed curves,
19. \( F_n = F_n - \bigcup_{j=0}^{n-1} F_j \) for \( n = 1, 2, \ldots \),
20. \( \{ x \in F; g_0(x \times [\frac{1}{n+2}, 1] \cup [\frac{1}{n+2}, 1] \} = F_n \) for \( n = 0, 1, \ldots \),
21. \( g_0(x \times [t]) = T_k \) for \( k, n = 0, 1, \ldots \), \( k \leq n \) and \( t \in [\frac{1}{n+2}, 1] \cap [\frac{1}{k+2}, 1] \),
22. if \( n > k+5, t \in [\frac{1}{n+2}, \frac{1}{n+1}] \) and \( g_0(x \times [t]) : U_n \), then \( g_0(x, t) = g_0(x \times [t] \cup [\frac{1}{n+2}, 1]) \),
23. \( g_0(x, t) = f(x) \) for \( n = 0, 1, \ldots, x \in F \) and \( t \in [\frac{1}{n+2}, 1] \).

By conditions 10–14, the mapping \( g_0 \) and the set \( F_0 \) satisfy 15–23. Now suppose that \( g_0, \ldots, g_{n-1} \) and \( F_0, \ldots, F_{n-1} \) have been constructed. To finish the construction it remains to construct \( g_n \) and \( F_n \). By 18 and 20 for \( n-1 \), there is a set \( F_n \) contained in

\[ F_n \cap \bigcup_{j=0}^{n-1} F_j \cap \{ x \in F; g_{n-1}(x \times [\frac{1}{n+1}, 1] \notin F_n \} \]

which satisfies the condition 18 for \( n \) and is such that

24. \( \{ x \in F; g_{n-1}(x \times [\frac{1}{n+1}, 1] \notin F_n \} \cap \bigcup_{j=0}^{n-1} F_j \} \subseteq F_n. \)
Define $g_n: \mathbb{R} \times \left[ \frac{1}{n+2}, 1 \right] \rightarrow \mathbb{R}$ by the following formula:

$$g_n(x, t) = \begin{cases} 
 g_{n-1}(x, t) & \text{for } t \in \left[ \frac{1}{n+1}, 1 \right], \\
 h \left( g_{n-1} \left( x, \frac{1}{n+1} \right) \right) \cdot (n+1)(1-(n+1)) \left( 1 + d(x, F_n) \right)^{-1} & \text{for } t \in \left[ \frac{1}{n+2}, \frac{1}{n+1} \right],
\end{cases}$$

where $d(x, F_n)$ denotes the distance between $x$ and $F_n$.

By 5 $g_n$ is a well-defined continuous mapping. Condition 16 immediately follows from the formula. By 9 we get 17. Equality 20 follows from 3, 6, 7, 8 and 24. By 6 we get 21, 22 follows from 1 and 4 and finally by 2 and 4 we get 23. The construction is completed.

The mapping $g: \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$, such that $g(x, t) = g_n(x, t)$ for $n = 0, 1, \ldots$ and $t \in \left[ \frac{1}{n+2}, \frac{1}{n+1} \right]$, is continuous (see 15, 16 and 17). Define $f(x) = g(x, 0)$. The mapping $f$ is homotopic to $g$ with homotopy $g$ not moving points of $\mathbb{R}$ (comp. 23). By 21 $f(F_n) \subset T_k$. The arbitrary $x \in g$ such that $f(x) \in C^1 \cup C^2$. There is an integer $k$ such that $f(x) \in T_k$. There is an $n \geq k + 5$ such that $g_n(x, 1/n+1) \in U_k$. By 22 we get $g(x, t) = g_n(x, t) = g_n(x, 1/n+1)$ for $t \in \left[ \frac{1}{n+2}, \frac{1}{n+1} \right]$. By 16 and again by 22 we have $g_t(x, 1/n+2) = g_n(x, 1/n+1)$ for $m \geq n$ and $t \in \left[ \frac{1}{n+2}, \frac{1}{n+1} \right]$.

Thus $g(x, 1/n+2) = g(x, 0) = f(x) \in T_k$. Therefore by 20 we get $x \notin \bigcup_{j=0}^{\infty} F_j$. Hence $f^{-1}(C^1 \cup C^2) = \bigcup_{n=0}^{\infty} F_n$, which completes the proof of the lemma.

4.4. **Lemma.** Let $F$ be a perforated disc and let $f_0$ be a mapping of $F$ into $A^1_0$ such that $w(f_0, F, A^1_0) \neq 0$. Then there is no continuous mapping $f: F \rightarrow X$ such that $f|_{\bar{B}} = f_0$.

**Proof.** Suppose that the lemma fails. Then there is a mapping $f: F \rightarrow X$ and a sequence $F_0, F_1, \ldots$ of subsets of $F$ such that $f|_{\bar{B}} = f_0$ and conditions (ii)-(vi) of Lemma 4.3 are fulfilled.

Observe that

1. $f(F_0) \subseteq A^1_0 \cup B^1 \cup L^1$,
2. $f(F_{j+1}) \subseteq A^1_0 \cup B^1 \cup L^1$ and
3. $f(F_{j+2}) \subseteq B^1 \cup B^1 \cup A^1_{j+1} \cup A^1_{j+2}$.

Since for each $x = 1, 2$ there is a mapping from $C^1_0$ onto an oriented circle which maps $L^1_0$ onto a single point and which is an orientation preserving homeomorphism onto $A^1_0$ and $B^1_0$, by 1, 2 and Proposition 2.3 we infer that

4. $w(f, F_{j+1}, A^1_0) + w(f, F_{j+1}, B^1_0) = 0$ and
5. $w(f, F_{j+1}, A^1_0) + w(f, F_{j+1}, B^1_0) = 0$ for $j = 0, 1, \ldots$

Similarly, since for each $x = 1, 2$ there is a mapping from $C^1_0$ onto an oriented circle which maps $B^1_0 \cup A^1_{j+1}, (m = 1, 2, m \neq n)$ onto a single point and which is an orientation preserving homeomorphism on $B^1_0$ and $A^1_{j+1}$, by 2 and 23 we infer that

6. $w(f, F_{j+2}, B^1_0) + w(f, F_{j+2}, A^1_{j+1}) = 0$ for $x = 1, 2$ and $j = 0, 1, \ldots$

Observe that

7. $f^{-1}(B^1_0 - \{p_j\}) \cap F_{j+1} = f^{-1}(B^1_0 - \{p_j\}) \cap F_{j+2}$
8. $f^{-1}(A^1_0 - \{p_j\}) \cap F_{j+2} = f^{-1}(A^1_0 - \{p_j\}) \cap F_{j+2}$ for $x = 1, 2$ and $j = 0, 1, \ldots$

By 7, condition (iv) of 4.3 and 2.4 we get

9. $w(f, F_{j+1}, B^1_0) = w(f, F_{j+2}, B^1_0)$ for $x = 1, 2$ and $j = 0, 1, \ldots$

By 8, condition (iv) of 4.3 and 2.4 we get

10. $w(f, F_{j+1}, A^1_0) = w(f, F_{j+2}, A^1_0)$ for $x = 1, 2$ and $j = 0, 1, \ldots$

Since $f^{-1}(A^1_0 - L^1_0) \cap F_j = f - f^{-1}(L^1_0) (f(F)) = f_0(F) = A^1_0$, and by 2.2 we infer

11. $w(f, F_0, A^1_0) = w(f_0, F, A^1_0)$.

Since $f^{-1}(A^1_0 - L^1_0) \cap F_j = f^{-1}(A^1_0 - L^1_0)$

12. $w(f, F_0, A^1_0) = 0$.

Combining 4, 5, 6, 9, 10, 11 and 12, one can easily get

13. $w(f, F_{j+2}, B^1_0) + w(f_0, F, A^1_0) = 0$.

14. $w(f, F_{j+2}, A^1_0) = w(f_0, F, A^1_0)$.

15. $w(f, F_{j+2}, B^1_0) + w(f, F_{j+2}, A^1_0) = 0$ for $j = 0, 1, \ldots$

By 4.2 there is an integer $n_0 \geq 2$ such that, for each $n \geq n_0$ and for each simple closed curve $C \subseteq f^{-1}(C^1_0)$, $f|_{\bar{B}}$ is homotopic in $C^1_0$ to a constant map.

Let $d$ be a positive real such that

16. if $g(x_1, x_2) \leq d$, then $g(x_1, x_2)f(x_1, x_2) < \frac{1}{4}$ for $x_1, x_2 \in F$.

For each $n = 0, 1, \ldots$, let $H_n$ be the union of $F_{n+2}$ and all components of $F_{n+3}, F_{n+4}, F_{n+5}$ and $F_{n+6}$ contained in $f^{-1}(C^1_0)$. By (iii) and (iv) of 4.3, $H_n$ is the union of a finite number of mutually disjoint perforated discs.

We claim that

17. there is an $n_1 \geq n_0$ such that for each $n \geq n_1$ there is a component $K_0$ of
Fixed points and locally connected cyclic continua in $E^3$

Let $H$ be the union of $H_0$ and all components of $F_{3n}$, $F_{3n+1}$, $F_{3n+2}$ and $F_{3n+3}$ contained in $f^{-1}(S^3) - K$. Let $H' = S^3 - H$ be the union of $F_{3n+2}$ and of all components of $F_{3n+3}$ contained in $H$. By (iii) and (iv) of 4.3, each of the sets $H$, $H'$ and $H''$ is the union of a finite number of mutually disjoint perforated discs. Observe that $H = H' \cup H''$, $f(H') \subseteq B_1 \cup A_{3n+1} \cup L_1 \cup L_2$ and $f(H'') \subseteq \{p_0\} = S^3 - (B_2 \cup A_{3n+1} \cup L_1 \cup L_2)$. Note also that each component of $f^{-1}(B_1 \cup A_{3n+1} \cup L_1 \cup L_2)$ which is mapped by $f$ onto $B_2 - \{p_0\}$, $B_2 - \{p_0\}$, $A_{3n+1} - \{p_0\}$ or $A_{3n+1} - \{p_0\}$ is contained in $K$. By 13, 14, 15, the choice of $n (n \approx n_0)$, 2.3 and 2.4, we get

$w(f, H, B_2) = -w(f, H, A_{3n+2}) \neq 0$ and $w(f, H, B_2) = w(f, H, A_{3n+2}) = 0$. Therefore, there is a component $K$ of $H$ such that

$20. w(f, K, A_{3n+2}) \neq w(f, K, B_2)$.

Observe that $H''$ is a component of $H$ which contains $K$. Let $I^2(S)$ be a collection of all components $I$ of $f^{-1}(L_1 \cup L_2 \cup A_{3n+1} \cup L_2)$ such that $I(f) = S - \{p_0\}$, where $S = B_2 \cup B_2 \cup A_{3n+1}$. Denote by $f^{-1}(S)$ a subcollection of those elements of $I(S)$ which are mapped by $f$ onto $S - \{p_0\}$ so that their orientations determined by $H'$ and $S$ are preserved. Denote also $I(S) = I(S) - f^{-1}(S)$. Observe that each element of $I(S)$ is mapped onto $I(H') \cup I(A_{3n+1}) \cup I(L_2)$. Arrange the elements of the collection $I(S)$ in a sequence $I_1, I_2, I_3, \ldots, I_k$ such that the elements occur cyclically on $K$ according to the orientation of $K$. Observe that, for each $j = 1, \ldots, k$, if $I_j \in I(L_2)$, then there is a component $K_j$ of $F_{3m}$ such that $F_{3m+2} \cup f^{-1}(S) - \{x, y, z\} \subseteq X$. Let $M_1, M_2, \ldots, M_n$ be components of $H'$ which meet $\{ I_j \in I(R_0) \} \cup I(A_{3n+1})$ and let $M_{3n+1}, M_{3n+2}, \ldots, M_{3n}$ be components of $H'$ which meet $\{ I_j \in I(R_2) \} \cup I(A_{3n+1})$. Since $f_0$ is homotopic to a constant map $(n \approx n_1 \approx n_2)$, we get

$21. w(f, M_1, B_2) + w(f, M_1, A_{3n+1}) = 0$ for $r = 1, 2$ and $l = 1, 2, \ldots, l_1$.

Denote $N_I = \{ I_j \in M_I \}$ for $I = 1, \ldots, d_1$. By 18, the sets $N_1, N_2, \ldots, N_l$ form a decomposition of $(1, 2, \ldots, k)$ into mutually disjoint subsets. Let $G$ denote the fundamental group of $C^2_k$. $G$ is a free group with generators $a$ and $b$ corresponding to the oriented simple closed curves $A_{3n+2}$ and $B_2$, respectively. The elements $a_i = bab^{-1} - 1$ and $b_i = bab^{-1}$ correspond to the oriented simple closed curves $A_{3n+2}$ and $A_{3n+1}$, respectively. Let $x$ be a function from $(1, 2, \ldots, k)$ into $X = \{a, a^{-1}, a_i, a_i^{-1}, b, b^{-1}, b_i, b_i^{-1}\}$ defined as follows:

$x(1) = \{a, a^{-1}, a_i, a_i^{-1}, b, b^{-1}, b_i, b_i^{-1}\}$ for $I_j \in I(L_2)$,

$x(2) = \{a, a^{-1}, a_i, a_i^{-1}, b, b^{-1}, b_i, b_i^{-1}\}$ for $I_j \in I(A_{3n+1})$,
Let \( Q_1 \) be the arc \( e_1^{A_1} \cup e_1^{A_1} \cup e_1^{A_2} \) and let \( Q_2 \) be the arc \( e_2^{A_2} \cup e_2^{A_2} \cup e_2^{A_2} \). Observe that \( Q_1 \) and \( Q_2 \) satisfy conditions (ii), (iii) and (iv) of 4.1 (for \( \delta \) see 16). Note also that \( Q_1' \), \( Q_2' \), \( Q_1'' \) and \( Q_2'' \) constructed in that manner satisfy the condition

\[
(Q_1' \cup Q_2') \cap (Q_1'' \cup Q_2'') = \emptyset \quad \text{for} \quad |r - r'| > 2.0
\]

Hence 4.1 give a contradiction, which completes the proof of the lemma.

4.4. Corollary. \( A_4 \) is not contractible in \( X \).

5. The main theorem. First let us prove the following

5.1. Lemma. Each continuous mapping \( f \) from the two-dimensional sphere \( S^2 \) in \( Y \) induces the trivial morphism on the (Cech or Vietoris) homology groups.

Proof. Denote by \( S \) the equator of \( S^2 \). \( S \) decomposes \( S^2 \) into two discs \( S_1 \) and \( S_2 \). Let \( g \) be a mapping from \( Y / S \) onto \( S^2 \) such that \( g(D) = S_1 \), \( g(X) = S_2 \) and \( \theta(A) \) is a homeomorphism onto \( S \). Observe that \( g \) induces an isomorphism on the homology groups. Since \( D \) has a polyhedral neighbourhood in \( Y \), one can assume that \( f^{-1}(D) \) is the union of a finite number of mutually disjoint perforated discs. Denote by \( \eta \) the closure of \( S^2 - f^{-1}(D) \). Note that \( \eta \) is also the union of a finite number of mutually disjoint perforated discs. The boundary of \( \eta \) is mapped by \( f \) onto \( A_4 \).

Let \( F \) be an arbitrary component of \( \eta \). From 4.4 we infer that \( w(f, F, A_4) = 0 \). Observe that \( w(f, F, S) = w(f, F, A_4) \) (\( S \) is oriented by the homomorphism \( p(A_4) \)). Hence \( w(f, F, S) = 0 \). Thus by 2.5, \( g f \) is homotopic in \( S^2 \) to a mapping with values in \( S \) with a homotopy not moving points of \( F \). Consequently \( g f \) is homotopic to a constant map. But \( g \) induces an isomorphism on homology groups of \( Y \) and \( S^2 \); therefore \( f \) induces the trivial morphism.

5.2. Theorem. There is a locally connected continuum contained in \( E^3 \) which separates \( E^3 \) and has the fixed point property.

Proof. To prove the theorem, it suffices to show that \( Y \) has the fixed point property.

Let \( f \) be a mapping from \( Y \) into itself. If \( f(Y) = Y \) there is a fixed point. Now, suppose that there is a \( y_0 \) \( \neq \) \( f \) such that \( f(y_0) \neq f \). Since \( Y \) is locally contractible at each point of the set \( Y - f \), consequently \( Y \) is locally contractible at \( f(Y) = Y \). Let \( W \) be a neighbourhood of \( f(y_0) \) contractible in \( Y \). There is a neighbourhood \( V \) of \( y_0 \) in \( Y \) such that \( f(V) \subseteq W \).

Now, we construct an auxiliary continuum \( Q \). Let \( F \) be a disc with the boundary \( S \). Fix a point \( x \) \( \in \) \( S \). Suppose that \( F \times [0, \infty) \) is embedded in \( E^3 - f \) such that \( f(t, x) \) is equal \( \eta(x) \) for \( t \in [0, \infty) \) and \( \text{diam}(F \times [t]) \) converges to zero as \( t \) approaches infinity. Denote by \( Q \) the continuum \( (F \times [0]) \cup (S \times [0, \infty)) \cup J \). There is a mapping \( g : Q \rightarrow Y \) inducing an isomorphism of the Vietoris homology groups of \( Q \) and \( Y \) and such that

(i) \( g(F \times [0]) = D \),

(ii) \( g(S \times [0, \infty)) = C^1 \),

(iii) \( g(J) = J \) for \( J \) \( \in \) \( \) \( J \) \( \),

(iv) \( \text{diam}(g(F \times [t])) \) converges to zero as \( t \) approaches infinity.

There is a strictly monotone sequence of positive reals \( t_1, t_2, \ldots \) converging to infinity and such that \( g(S \times [t_j]) \subseteq W \) for \( j = 1, 2, \ldots \) (Notice that this is the unique reason for \( \eta(x) \) being so complicated). Denote \( Q_j = Q \cup \bigcup_{j=1}^{m_j} F \times [t_j], m_j = 0 \) and \( K_j = (F \times [t_j]) \cup (S \times [t_j, t_{j+1}]) \cup (F \times [t_{j+1}]) \) for \( j = 0, 1, \ldots \).

Also, denote by \( p \) the inclusion of \( Q \) into \( Q_j \). Note that \( p \) (the homomorphism induced by \( p \)) is a monomorphism of the homology groups of \( Q \) into the homology groups of \( Q_j \).

Since \( \gamma_j(S \times [t_j]) \subseteq W \) for \( j = 1, 2, \ldots \), there is a mapping \( h : Q_1 \rightarrow Y \) such that \( h|_{Q_0} = f|_{Q_0} \), i.e. the diagram

\[
\begin{array}{ccc}
Q & \xrightarrow{e} & Y \\
\downarrow & & \downarrow \\
Q_1 & \xrightarrow{f} & Y
\end{array}
\]

is commutative.

For each \( j = 0, 1, \ldots \), let \( \gamma_j = \{ \gamma_j \} \) be a two-dimensional true cycle (see 2, p. 40) with a carrier \( K_j \) and a majorant \( 2^{-3} \) such that

1. \( \gamma_j \) represents a generator of the two-dimensional homology group of \( K_j \) over the rationals, and

2. \( \sum_{j=1}^{\infty} \gamma_j \) represents a generator of the two-dimensional homology group of \( (\sum_{j=1}^{\infty} F \times [t_j]) \cup (S \times [t_1, t_{j+1}]) \cup (F \times [t_{j+1}]) \) over the rationals, for \( n = 0, 1, \ldots \).

By 5.1, the true cycle \( h(\gamma_j) = (h(\gamma_j)) \) is homologous in \( Y \) to zero. Thus there is an infinite chain \( \gamma_j = (\gamma_j) \) with a majorant \( (\gamma_j) \) and a carrier \( Y \), such that \( \delta h(\gamma_j) = h(\gamma_j) \). Let \( \epsilon_1, \epsilon_2, \ldots \) be a sequence of natural numbers such that the sequence \( e_0, e_1, e_2, \ldots \), where \( e_n = \max(e_n, e_n, e_n, \ldots, e_n) \), converges to zero.

Put \( \beta_n = \sum_{j=1}^{n} \gamma_j \). Note that \( \beta_n = \{ \beta_n \} \) is an infinite chain with a majorant \( \{ e_n \} \) and a carrier \( Y \).

Form another infinite chain \( \gamma = \{ \gamma_x \} \) putting \( x = \sum_{j=1}^{\infty} \gamma_j \). Observe that \( x \) is a true cycle which represents a generator of \( p_d(H_3(Q)) \). Since

\[
\delta(\beta_n) = \sum_{j=1}^{n} \delta \gamma_j = \sum_{j=1}^{n} h(\gamma_j) = h(\gamma_x),
\]
we have $\delta^2 = h(v)$; in other words $h(v)$ is homologous to zero. Thus $h^*(\pi_k(H_*(Q))) = 0$. Since the diagram

$$
\begin{array}{ccc}
H_*(Q) & \xrightarrow{k_*} & H_*(Y) \\
\downarrow \gamma_* & & \downarrow f_* \\
H_*(Q_0) & \xrightarrow{h} & H_*(Y) \\
\end{array}
$$

is commutative

and $\gamma_*$ is an isomorphism, $f$ induces the trivial morphism on the two-dimensional homology group of $Y$. Since $Y$ has the same homologies as the two-dimensional sphere, the Lefschetz number $\Lambda(f) = 1$, then by 3.3, $f$ has a fixed point, which completes the proof.

References


WARSAW UNIVERSITY, INSTITUTE OF MATHEMATICS

Accepté par la Rédaction le 5. 4. 1977

On the decidability of the theory of linear orderings with generalized quantifiers

by

H. P. Tuschik (Berlin)

Abstract. LO($Q_0, Q_1, ..., Q_n$) be the theory of linear orderings with the additional quantifiers $Q_0, Q_1, ..., Q_n$. Under various hypotheses on set theory it is proved that LO($Q_0, ..., Q_n$) is always decidable. This generalizes the result of the author for LO($Q$). The proof uses methods from Leonhard and Läuchli. The theorems can be generalized to arbitrary finite sets of regular cardinality quantifiers.

A. Ehrenfeucht proved in [1] that the elementary theory LO of linear orderings is decidable. In [4] H. Läuchli and J. Leonard established the same result using games. Let us extend the elementary language of linear order by adding the generalized quantifiers $Q_0, Q_1, ..., Q_n$ to it.

We interpret the quantifier $Q_0$ as: "there exist at least $\omega_0$ many". Generalized quantifiers were introduced by A. Mostowski [6].

Let $LO(Q_0, ..., Q_n)$ be the theory of linear orderings with these additional quantifiers. Then we will prove that $LO(Q_0, ..., Q_n)$ is decidable. This generalizes the result of H. P. Tuschik [9] for $LO(Q)$. As a corollary we infer that $LO(Q; \text{I}<\omega)$ is decidable.

§ 1. Let $L$ be the first order language with identity and one binary predicate $\prec$. $L_0^0(Q)$ arises from $L$ by adding the quantifiers $Q_0, ..., Q_n$. LO is the following theory:

1. $\forall x \prec x$,
2. $x \prec y \land y \prec z \rightarrow x \prec z$,
3. $x = y \Leftrightarrow x \prec y \Leftrightarrow y \prec x$.

We use some definitions from [4] and [9]: $x \prec y (\text{mod} A)$ denotes the order relation of an ordered set $A, [A]$ denoted the field of $A$. $B$ is said to be a segment of $A$ if $B$ is a substructure of $A$ and if $x \prec y (\text{mod} B)$ and $x \prec z \prec y (\text{mod} A)$ implies $z \in B$.

Some special segments are the open interval $(x, y) = \{ z \in [A] : x \prec z \prec y (\text{mod} A) \}$, the left-open and right-open interval $(x, y) = \{ z \in [A] : x \prec z \prec y (\text{mod} A) \}$, the left-closed and right-open interval $[x, y) = \{ z \in [A] : x \leq z \prec y (\text{mod} A) \}$ and the closed interval $[x, y] = \{ z \in [A] : x \leq z \leq y (\text{mod} A) \}$. A map $f : A \rightarrow B$ of an ordered set $A$