INTEGRAL CLOSURES OF IDEALS
IN THE REES RING

by

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Introduction. The important ideas of reduction and integral closure of an ideal in a commutative Noetherian ring \( A \) (with identity) were introduced by Northcott and Rees [4]; a brief and direct approach to their theory is given in [6, (1.1)]. We begin by briefly summarizing some of the main aspects.

Let \( a \) be an ideal of \( A \). We say that \( a \) is a reduction of the ideal \( b \) of \( A \) if \( a \subseteq b \) and there exists \( s \in \mathbb{N} \) such that \( ab^s = b^{s+1} \). (We use \( \mathbb{N} \) (respectively \( \mathbb{N}_0 \)) to denote the set of positive (respectively non-negative) integers.) An element \( x \) of \( A \) is said to be integrally dependent on \( a \) if there exist \( n \in \mathbb{N} \) and elements \( c_1, \ldots, c_n \in A \) with \( c_i \in a^i \) for \( i = 1, \ldots, n \) such that

\[
x^n + c_1 x^{n-1} + \ldots + c_{n-1} x + c_n = 0.
\]

In fact, this is the case if and only if \( a \) is a reduction of \( a + Ax \); moreover,

\[
\overline{a} = \{ y \in A : y \text{ is integrally dependent on } a \}
\]

is an ideal of \( A \), called the classical integral closure of \( a \), and it is the largest ideal of \( A \) which has \( a \) as a reduction in the sense that \( a \) is a reduction of \( \overline{a} \) and any ideal of \( A \) which has \( a \) as a reduction must be contained in \( \overline{a} \).

In [8], Sharp, Tiras¸ and Yassi introduced concepts of reduction and integral closure of an ideal \( I \) of a commutative ring \( R \) (with identity) relative to a Noetherian module \( M \), and they showed that these concepts have properties which reflect those of the classical concepts outlined in the last paragraph. Again, we provide a brief review.

We say that \( I \) is a reduction of the ideal \( J \) of \( R \) relative to \( M \) if \( I \subseteq J \) and there exists \( s \in \mathbb{N} \) such that \( IJ^s M = J^{s+1} M \). An element \( x \) of \( R \) is said to be integrally dependent on \( I \) relative to \( M \) if there exists \( n \in \mathbb{N} \) such that

\[
x^n \cdot M \subseteq \left( \sum_{i=1}^{n} x^{n-i} I^i \right) \cdot M.
\]

In fact, this is the case if and only if \( I \) is a reduction of \( I + Rx \) relative to \( M \).
is an ideal of \( R \), called the integral closure of \( I \) relative to \( M \), and is the largest ideal of \( R \) which has \( I \) as a reduction relative to \( M \). In this paper, we indicate the dependence of \( I^{-} \) on the Noetherian \( R \)-module \( M \) by means of the extended notation \( I^{-}(M) \).

Now we give the definition of the Rees ring. The classical reference is [5, p. 33]. Let \( R \) be a commutative ring with identity.

Let \( t \) be an indeterminate. Let \( S = \{ t^i : i \in \mathbb{N}_0 \} \). Then \( S \) is a multiplicatively closed subset of \( R[t] \). So we get the ring \( S^{-1}(R[t]) \). The homomorphism \( \psi : R[t] \to S^{-1}(R[t]), f \mapsto f/1 \), is an injective ring homomorphism, and so we can consider \( R[t] \) as a subring of \( S^{-1}(R[t]) \). Next, suppose that \( I \) is a proper ideal of \( R \) generated by \( a_1, \ldots, a_s \) \((n \in \mathbb{N})\). Then \( \mathcal{R} = R[a_1, \ldots, a_s, t, t^{-1}] \) is a subring of \( R[t, t^{-1}] \). \( \mathcal{R} \) is called the Rees ring of \( R \) with respect to \( I \) (see [2, p. 120]). Note that each element of \( \mathcal{R} \) is of the form \( \sum_{i=m}^{n} b_i t^i \) where \( m, n \in \mathbb{Z} \) (the set of integers), and, for \( i > 0 \), \( b_i \in I^i \). Note also that for \( i \leq 0 \) we interpret \( I^i \) as \( R \).

Now we give another definition which will be helpful in this section.

**Definition.** Let \((R_n)_{n \in \mathbb{Z}}\) be a family of subgroups of \( R \). We say that \( R \) is a graded ring if the following conditions are satisfied.

(i) \( R \) is the direct sum of the subgroups \( R_n \), i.e. \( R = \bigoplus_{n=-\infty}^{\infty} R_n \).

(ii) \( R_q \cdot R_{q'} \subseteq R_{q+q'} \) for all \( q, q' \in \mathbb{Z} \). (Observe that \( R_q \cdot R_{q'} \) is the set of all elements \( x \) of \( R \) such that \( x \) is a sum of a finite number of elements of the form \( a \cdot b \) with \( a \in R_q, b \in R_{q'} \).)

The following proposition comes from [3].

**Proposition** [3, Proposition 28, p. 115]. Let \( A \) be a graded ring. If \( K \) is a submodule of the graded \( A \)-module \( E = \sum_{i \in \mathbb{Z}} E^i \), then the following statements are equivalent:

(a) \( K = \sum_{i \in \mathbb{Z}} (E^i \cap K) \);

(b) If \( y \in K \), then all the homogeneous components of \( y \) belong to \( K \);

(c) \( K \) can be generated by homogeneous elements.

Next we give the notations and terminology which we will need throughout this paper.

**Notations and terminology.** Let \( R \) be a commutative Noetherian ring and \( I \) be an ideal of \( R \) generated by \( a_1, \ldots, a_s \), \( I = (a_1, \ldots, a_s) \). Let
\( \mathcal{R} = R[t, t^{-1}] \) be the Rees ring of \( R \) with respect to \( I \). Let \( \mathcal{R} = \bigoplus_{n \in \mathbb{Z}} R_n \) where \( R_n \) denotes the subgroup of \( \mathcal{R} \) consisting of 0 and the homogeneous elements of \( \mathcal{R} \) of degree \( n \). For all \( k \in \mathbb{N} \), by (1.2)(a),

\[
\mathcal{R}t^{-k} = \bigoplus_{i \in \mathbb{Z}} (R_i \cap Rt^{-k}).
\]

Therefore

\[
R_i \cap Rt^{-k} = \begin{cases} R_i^{i+k} & \text{if } i > -k, \\ R_i^i & \text{if } i \leq -k. \end{cases}
\]

Let \( M \) be a finitely generated \( R \)-module. Then it is easy to see that

\[
M[t] = \sum_{i=1}^r R[t]u_i \quad \text{where } u_1, \ldots, u_r \text{ is a generating set for } M.
\]

Let \( S = \{ t^i : i \in \mathbb{N}_0 \} \) be a multiplicatively closed subset of \( R[t] \). Then \( M[t] \to S^{-1}(M[t]), \ f \mapsto f/1, \) is an injective module homomorphism. Now let

\[
\mathcal{R}(R, I) = \bigoplus_n I^n t^n = \left\{ \sum_{i=-q}^p a_i t^i \in R[t, t^{-1}] : a_i \in I^i \right\}.
\]

Then \( \mathcal{R}(R, I) \) is a subring of \( R[t, t^{-1}] \). Also let

\[
\mathcal{R}(M, I) = \left\{ \sum_{i=-r}^s m_i t^i \in M[t, t^{-1}] : m_i \in I^i M \right\}.
\]

We can regard \( \mathcal{R}(M, I) \) as an \( \mathcal{R}(R, I) \)-module with the following scalar multiplication:

\[
\mathcal{R}(R, I) \times \mathcal{R}(M, I) \to \mathcal{R}(M, I), \quad \left( \sum_{i=-n}^m c_i t^i, \sum_{j=-q}^p m_j t^j \right) \mapsto \sum_{i=-n}^m \sum_{j=-q}^p c_i m_j t^{i+j},
\]

where \( c_i m_j \in I^{i+j} M \).

Let \( X_1, \ldots, X_s, X_{s+1} \) be indeterminates over \( R \). Then \( R[X_1, \ldots, X_{s+1}] \) is a Noetherian ring. It is readily seen that \( M[X_1, \ldots, X_{s+1}] \) is a Noetherian \( R[X_1, \ldots, X_{s+1}] \)-module, and \( M[a_1 t, \ldots, a_s t, t^{-1}] \) is a Noetherian \( R[a_1 t, \ldots, a_s t, t^{-1}] \)-module.

2. Some related results. Throughout this section, unless otherwise stated, \( R \) will denote a commutative ring with identity. Essentially our aim is to investigate some interrelations between the Rees ring of \( R \) with respect to an ideal of \( R \) and the ground ring \( R \).

We begin with a well-known lemma which gives us the connection between the integral closure of an ideal in the Rees ring and the integral closure of the ideal in the ground ring \( R \).
(2.1) Lemma. Let \( R \) be a commutative Noetherian ring, \( I \) be an ideal of \( R \), and \( \mathcal{R} \) be the Rees ring of \( R \) with respect to \( I \). Let \( t \) be an indeterminate and \( u = t^{-1} \). Then
\[
(u^i \mathcal{R}) \cap R = (I^i)
\]
where the bar refers to classical integral closure.

From now on, let \( M = \mathcal{R}(M, I) = M[a_1 t, \ldots, a_s t, t^{-1}] \). Also, \( (\mathcal{R}t^{-k})^{-(M)} \), for \( k \in \mathbb{N} \), is the integral closure of \( \mathcal{R}t^{-k} \) relative to \( M \).

Now for all \( i > -k, k \in \mathbb{N}, \) define
\[
C_{i, k} = \{ x \in R : xt^i \in R_k \cap (\mathcal{R}t^{-k})^{-(M)} \}.
\]
It is clear that for all \( i > -k, C_{i, k} \) is an ideal of \( R \). In particular,
\[
C_{0, k} = R \cap (\mathcal{R}t^{-k})^{-(M)}.
\]

Now we give the relation between \( (I^k)^{-(M)} \) and \( (\mathcal{R}t^{-k})^{-(M)} \). The following theorem can be used to reduce problems about the integral closure of the powers of \( I \) relative to \( M \) to the corresponding problems for powers of the principal ideal \( \mathcal{R}t^{-k} \) in \( R \).

(2.2) Theorem. Let \( R \) be a commutative Noetherian ring and \( I \) be an ideal of \( R \). Let \( \mathcal{R} \) be the Rees ring of \( R \) with respect to \( I \). Let \( M \) be a Noetherian \( R \)-module and \( M = \mathcal{R}(M, I) \). Then for \( k \in \mathbb{N}, \)
\[
(\mathcal{R}t^{-k})^{-(M)} \cap R = (I^k)^{-(M)}.
\]

Proof. Let \( x \in (I^k)^{-(M)} \). Then there is an \( n \in \mathbb{N} \) such that
\[
x^n \cdot M \subseteq \left( \sum_{i=1}^{n} x^{n-i} I^i \right) \cdot M.
\]
It is enough to show that an element of the form \( x^m m_j \), where \( m_j \in I^j M \), is in \( (\sum_{i=1}^{n} x^{n-i} I^i)^{-(M)} \). Since
\[
x^m m_j \in x^n I^j M \subseteq \sum_{i=1}^{n} x^{n-i} I^i (I^k)^i M,
\]
we have \( x^m m_j = \sum_{i=1}^{n} x^{n-i} \beta_i \), where \( \beta_i \in I^j (I^k)^i M \), and the result follows.

For the converse, let us first give some useful ideas about the ideal \( C_{i, k} \).

Now to complete the proof we show that \( I^j \) is a reduction of \( C_{i-k, k} \) relative to \( M \). It is enough to show that each element of \( C_{i-k, k} \) is integrally dependent on \( I^j \) relative to \( M \) by the preceding paragraph and [8, (1.5)(v) ].
Let $x \in C_{i-k,k}$. Then $xt^{i-k} \in (\mathcal{R}t^{i-k})(M)$. Thus there exists an $n \in \mathbb{N}$ such that

\[(*) \quad \mathcal{R}(xt^{i-k})^n \cdot M \subseteq \left( \sum_{r=1}^{n} (\mathcal{R}t^{i-k})^r (\mathcal{R}xt^{i-k})^{n-r} \right) \cdot M.
\]

We claim that

\[x^n \cdot M \subseteq \left( \sum_{r=1}^{n} x^{n-r}(I^k)^r \right) \cdot M.
\]

Let $y \in x^n \cdot M$. Then $y = x^n \cdot m$ for some $m \in M$. Hence $(xt^{i-k})^n \cdot m \in \mathcal{R}(xt^{i-k})^n \cdot M$. By $(*)$,

\[(xt^{i-k})^n \cdot m \in \left( \sum_{r=1}^{n} (\mathcal{R}t^{i-k})^r (\mathcal{R}xt^{i-k})^{n-r} \right) \cdot M.
\]

Therefore

\[x^n t^{n(i-k)}m = \sum_{r=1}^{n} x^{n-r}t^{(i-k)(n-r)-kr} \gamma_r \quad \text{with} \quad \gamma_r \in M.
\]

By comparing components of degree $n(i-k)$, we get

\[x^n \cdot m \in \left( \sum_{r=1}^{n} x^{n-r}(I^k)^r \right) \cdot M.
\]

This means $x$ is integrally dependent on $I^i$ relative to $M$. Then by [8, (1.5)(v)], $I^i$ is a reduction of $C_{i-k,k}$ relative to $M$ for all $i \geq 1$. Now the result follows from [8, (1.5)(vii)].

One could naturally ask whether there exist any relations, as in (2.2), between $(\mathcal{R}t^{i-k})(M) \cdot M$ and $(I^k)^{n(M)} \cdot M$. It will be shown in (2.5) that the answer is yes, and to prove this we need to show first that the integral closure of a homogeneous ideal in a graded ring is homogeneous.

(2.3) **Proposition.** Let $R = \bigoplus_{n \in \mathbb{Z}} R_n$ be a graded Noetherian ring and let $I$ be a homogeneous ideal in $R$. Then $\overline{I}$, the integral closure of $I$ in $R$, is a homogeneous ideal of $R$.

**Proof.** Let $T = R[t, t^{-1}]$. Consider

\[T_n = \left\{ \sum_{i=-p}^{q} r_i t^i : r_i \in R_n \right\}, \quad n \in \mathbb{Z}.
\]

For $n \in \mathbb{Z}$, $T_n$ is an additive subgroup of $T$. Also $T_n \cdot T_m \subseteq T_{m+n}$. Let $\mathcal{R} = R[It, t^{-1}]$. Then $\mathcal{R} = \bigoplus_{n \in \mathbb{Z}} (R[It, t^{-1}])_n$ is a graded subring of $R[t, t^{-1}]$.

Now let $x = \sum_{i=-p}^{q} x_i \in \overline{I}$. Then by [6, (1.1)(ii)], $xt$ is integral over $R[It, t^{-1}]$. By [1, Proposition 20, p. 321] all homogeneous components of $xt$ are integral over $R[It, t^{-1}]$. This completes the proof. ■
Corollary. Let $R = \bigoplus_{n \in \mathbb{Z}} R_n$ be a graded ring and let $I$ be a homogeneous ideal of $R$. Suppose that $M$ is a Noetherian graded $R$-module. Then $I^{-(M)}$ is a homogeneous ideal of $R$.

Proof. Let the bar refer to the natural ring homomorphism $R \to R/0 : R M$. By [8, (1.6)], $\bar{I}^{-(M)} = (\bar{I})^{-(R)}$, the integral closure $\bar{I}$ in $\bar{R}$. By (2.3), $\bar{I}^{-(M)}$ is a homogeneous ideal. Now the result follows from the definition of the graded ring structure on the residue class ring. ■

Now we are able to give an answer to the question asked just after (2.2).

Theorem. Let $R$ and $M$ be as in (2.2). Then for all $k \in \mathbb{N},$

\[
(Rt^{-k})^{-(M)} \cdot M \cap M = (I_k^{-(M)}) \cdot M.
\]

Proof. By the result about $C_{i,k}$ given in (2.2), the zero component of $(Rt^{-k})^{-(M)} \cdot M$ is $C_{0,k} \cdot M$. This gives us $(I_k^{-(M)}) \cdot M = C_{0,k} \cdot M \subseteq (Rt^{-k})^{-(M)} \cdot M \cap M$.

Let $m \in (Rt^{-k})^{-(M)} \cdot M \cap M$. Since $m$ is a homogeneous element of $(Rt^{-k})^{-(M)} \cdot M$ of degree 0, it belongs to $C_{0,k} \cdot M$. This completes the proof. ■

We conclude this paper by giving the interrelation between the associated primes in $R$ and in $R$. To do this we need the following proposition.

Proposition [3, Proposition 20, p. 99]. Let $N$ be a $p$-primary submodule of an $R$-module $E$ and let $K$ be an arbitrary submodule of $E$. If $K \not\subseteq N$, then $(N : K)$ is a $p$-primary ideal. If $K \subseteq N$, then $(N : K) = R$. ■

Proposition. Let $R$ and $M$ be as in (2.5). Let

\[
p \in \operatorname{Ass}_R \left( \frac{M}{(I_k^{-(M)}) \cdot M} \right)
\]

for $k \in \mathbb{N}$. Then there exists

\[
\mathcal{P} \in \operatorname{Ass}_R \left( \frac{M}{(Rt^{-k})^{-(M)} \cdot M} \right)
\]

such that $\mathcal{P} \cap R = p$.

Proof. Let

\[
\mathcal{G} = \frac{M}{(Rt^{-k})^{-(M)} \cdot M} = \bigoplus_{n \in \mathbb{Z}} G_n.
\]

We have shown that

\[
G_0 = \frac{M}{(I_k^{-(M)}) \cdot M}.
\]

Let $p \in \operatorname{Ass}_R G_0$. Then there exists $g_0 \in G_0$ such that $(0 : R g_0) = p$. Now consider $\mathcal{R} g_0$, a homogeneous submodule of $\mathcal{G}$. Take a minimal primary decomposition for 0 in $\mathcal{R} g_0$ (because $\mathcal{R}$ is Noetherian). Then $0 = \bigcap_{i=1}^n \alpha_i$,
with $\alpha_i$ being $P_i$-primary homogeneous submodules of $Rg_0$ ($1 \leq i \leq n$). Then

$$p = (0 :_R g_0) = R \cap (0 :_R Rg_0) = \bigcap_{i=1}^{n} (R \cap (\alpha_i :_R Rg_0)).$$

Thus by (2.6), $(\alpha_i :_R Rg_0)$ is a $P_i$-primary ideal and by [7, (9.33)(ii)], $P_i \in \text{Ass}_{R} g_0 \subseteq \text{Ass}_{R} G$. Now the result follows by [7, 3.50].

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REFERENCES