ON MANIFOLDS ADMITTING METRICS WHICH ARE
LOCAL CONFORMAL TO COSYMPLECTIC METRICS:
THEIR CANONICAL FOLIATIONS, BOOTHBY–WANG
FIBERINGS, AND REAL HOMOLOGY TYPE

BY

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1. Introduction. The present paper builds on work by Z. Olszak [16]. There, locally conformal cosymplectic (l.c.c.) manifolds are defined to be almost contact metric (a.ct.m.) manifolds whose almost contact and fundamental forms $\eta$, $\Theta$ are subject to $d\eta = \frac{1}{2} \omega \wedge \eta$, $d\Theta = \omega \wedge \Theta$ for some closed 1-form $\omega$ and with a $(1,1)$-structure tensor $\varphi$ integrable. The reason for which such manifolds are termed l.c.c. is that the metric of the underlying a.ct.m. structure appears to be conformal to a (local) cosymplectic metric in some neighborhood of each point of the manifold. Our results are organized as follows. Totally geodesic orientable real hypersurfaces $M^{2n+1}$ of a locally conformal Kaehler (l.c.K.) manifold $M^{2n+2}$ are shown to carry a naturally induced l.c.c. structure, provided the Lee field $B_0$ of $M^{2n+2}$ is tangent to $M^{2n+1}$. The same conclusion occurs if $M^{2n+1}$ is totally umbilical and its mean curvature vector is given by $H = -\frac{1}{2} \text{nor}(B_0)$ (cf. our Theorem 7). In Section 3 we show that odd-dimensional real Hopf manifolds $\mathbb{R}H^{2n+1} \cong S^{2n} \times S^1$, $n \geq 2$, thought of as local similarity (l.s.) manifolds carrying the metric discovered by C. Reischer and I. Vaisman [19] turn out to be l.c.c. manifolds in a natural way, yet admit no globally defined cosymplectic metrics, by a result of D. E. Blair and S. Goldberg [3]. Leaving definitions momentarily aside, we may also state

Theorem 1. Each leaf of the canonical foliation $\Sigma$ of a strongly non-cosymplectic l.c.c. manifold $M^{2n+1}$ carries an induced $(f,g,u,v,\lambda)$-structure whose 1-form $v$ is closed. If the characteristic 1-form $\omega$ of $M^{2n+1}$ is parallel, then $\Sigma$ has totally geodesic leaves. If moreover the local cosymplectic metrics $g_i$, $i \in I$, of $M^{2n+1}$ are flat then the leaves of $\Sigma$ are Riemannian manifolds of constant sectional curvature. If additionally $M^{2n+1}$ is normal, then each complete leaf of $\Sigma$ is holomorphically isometric to $\mathbb{C}P^n(c^2)$, $c = \frac{1}{2} \|\omega\|$. 


Theorem 2. Let $M^{2n+1}$ be a compact normal l.c.c. manifold. If the structure vector $\xi$ is regular then:

(i) $M^{2n+1}$ is a principal $S^1$-bundle over $M^{2n} = M^{2n+1}/\xi$,
(ii) the almost contact 1-form $\eta$ yields a flat connection 1-form on $M^{2n+1}$,
(iii) the base manifold $M^{2n}$ has a natural structure of Kaehlerian manifold.

Theorem 3. Let $M^{2n+1}$ be a connected compact orientable (strongly non-cosymplectic) l.c.c. manifold with a parallel characteristic 1-form $\omega$ and flat Weyl connection. Then the Betti numbers of $M^{2n+1}$ are given by:

$$b_0(M^{2n+1}) = b_{2n+1}(M^{2n+1}) = 1, \quad b_1(M^{2n+1}) = b_{2n}(M^{2n+1}) = 1,$$

$$b_p(M^{2n+1}) = 0, \quad 2 \leq p \leq 2n - 1,$$

i.e. $M^{2n+1}$ is a real homology real Hopf manifold.

In addition to (odd-dimensional) real Hopf manifolds, several examples of l.c.c. manifolds (such as real hypersurfaces of a complex Inoue surface endowed with the l.c.K. metric discovered by F. Tricerri [23]) are discussed in Section 7.

2. Conformal changes of almost contact metric structures. Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost contact metric (a.ct.m.) manifold of (real) dimension $2n + 1$ (cf. D. E. Blair [2], pp. 19–20). It is said to be normal if $N^1 = 0$, where $N^1 = [\varphi, \varphi] + 2d\eta \otimes \xi$. An a.ct.m. manifold is cosymplectic if it is normal and both the almost contact and fundamental forms are closed. See D. E. Blair [1], Z. Olszak [15], S. Tanno [22] for general properties of cosymplectic manifolds.

Let $M^{2n+1}$ be an a.ct.m. manifold. Then $M^{2n+1}$ is said to be locally conformal cosymplectic (l.c.c.) if there exists an open covering $\{U_i\}_{i \in I}$ of $M^{2n+1}$ and a family $\{f_i\}_{i \in I}, f_i \in C^\infty(U_i)$, of real-valued smooth functions such that $(U_i, \varphi_i, \xi_i, \eta_i, g_i)$ is a cosymplectic manifold, where $\varphi_i = \varphi|_{U_i}, \xi_i = \exp(f_i/2)\xi|_{U_i}, \eta_i = \exp(-f_i/2)\eta|_{U_i}, g_i = \exp(-f_i)g|_{U_i}, i \in I$. Clearly, if $M^{2n+1}$ is l.c.c. then $\varphi$ is integrable.

Let $M^{2n+1}$ be an a.ct.m. manifold and $f \in C^\infty(M^{2n+1})$ a smooth real-valued function on $M^{2n+1}$. A conformal change of the a.ct.m. structure (cf. I. Vaisman [25]) is a transformation of the form

$$\varphi_f = \varphi, \quad \xi_f = \exp\left(\frac{f}{2}\right)\xi, \quad \eta_f = \exp\left(-\frac{f}{2}\right)\eta, \quad g_f = \exp(-f)g.$$

The Riemannian connections of $g, g_f$ are related by

$$\nabla_X Y = \nabla_X Y - \frac{1}{2}[X(f)Y + Y(f)X - g(X,Y)\text{grad}(f)],$$

where $\text{grad}(f) = (df)^\sharp$ and $\sharp$ denotes raising of indices with respect to $g$. 

Clearly \((M^{2n+1}, \varphi, \xi, \eta, g)\) is an a.c.t.m. manifold and is cosymplectic iff
\[d\eta = \frac{1}{2}df \wedge \eta, \quad d\Theta = df \wedge \Theta, \quad [\varphi, \varphi] = 0, \quad \text{where} \quad \Theta(X, Y) = g(X, \varphi Y).\]
We may establish the following:

**Lemma 4.** Let \((M^{2n+1}, \varphi, \xi, \eta, g)\) be a cosymplectic manifold, \(n \geq 1\). If the cosymplectic property is invariant by the transformation \((1)\) then \(df \equiv 0\) on \(M^{2n+1}\).

**Proof.** Note that \((2)\) yields
\[(3) \quad \left(\nabla_X \varphi\right)Y = (\nabla_X \varphi)Y + \frac{1}{2}[Y(f)\varphi X - (\varphi Y)(f)X + \Theta(X, Y)\operatorname{grad}(f) - g(X, Y)\varphi(\operatorname{grad}(f))].\]
Since \(M^{2n+1}\) is cosymplectic it is normal, so that \(N^1 = 0\). This yields \(N^2 = 0\), where \(N^2 = (L_{\varphi X}\eta)Y - (L_{\varphi Y}\eta)X\) (cf. [2], p. 50). Here \(L\) denotes the Lie derivative. Then \(\nabla \varphi = 0\), by [2], p. 53. Now, by \((3)\) we obtain
\[(4) \quad Y(f)\varphi X + \Theta(X, Y)\operatorname{grad}(f) = (\varphi Y)(f)X + g(X, Y)\varphi(\operatorname{grad}(f)).\]
Let \(X = Y = \xi\) in \((4)\). Then \(\varphi(\operatorname{grad}(f)) = 0\). Use this to modify \((4)\) and apply \(\varphi\) to the resulting equation. This yields \(Y(f)\varphi^2 X = (\varphi Y)(f)\varphi X\).
Take the inner product with \(\varphi^2 X\) to get \(Y(f)||\varphi^2 X||^2 = 0\). Finally, replace \(X\) by \(\varphi X\); as \(\varphi\) is an \(f\)-structure (in the sense of [26], p. 379), \(\operatorname{rank}(\varphi) = 2n,\ n \geq 1\), so that \(Y(f) = 0\) for any \(Y\).

**Theorem 5.** Let \((M^{2n+1}, \varphi, \xi, \eta, g)\) be a l.c.c. manifold. Then for any \(i, j \in I,\ i \neq j\), with \(U_i \cap U_j \neq \emptyset\), one has \(df_i = df_j\) on \(U_i \cap U_j\); therefore the (local) 1-forms \(df_i\) glue up to a globally defined (closed) 1-form \(\omega\). Also the Riemannian connections \(\nabla^f\) of \((U_i, g_i)\), \(i \in I\), glue up to a globally defined torsion-free linear connection \(D\) on \(M^{2n+1}\) expressed by
\[(5) \quad D_X Y = \nabla_X Y - \frac{1}{2}[\omega(X)Y + \omega(Y)X - g(X, Y)B],\]
where \(B = \omega^2\) and \(\nabla\) is the Levi-Civita connection of \((M^{2n+1}, g)\).

**Proof.** Let \(U_{ij} = U_i \cap U_j,\ i \neq j,\ i, j \in I,\ U_{ij} \neq \emptyset\). Then both \((\varphi, \xi_i, \eta_i, g_i)\), \((\varphi, \xi_j, \eta_j, g_j)\) are cosymplectic structures on \(U_{ij}\) and are related by a conformal transformation \((1)\) with \(f = f_j - f_i\); thus one may apply Lemma 4.

The 1-form \(\omega\) furnished by Theorem 5 is referred to as the characteristic 1-form of \(M^{2n+1}\); also \(B\) is the characteristic field and \(D\) the Weyl connection. Since \(d\eta = 0\), \(d\Theta = 0,\ i \in I\), where \(\Theta_i\) denotes the fundamental 2-form of \((\varphi, \xi_i, \eta_i, g_i)\), it follows that
\[(6) \quad d\eta = \frac{1}{2}\omega \wedge \eta, \quad d\Theta = \omega \wedge \Theta.\]
Also, for any l.c.c. manifold, \([\varphi, \varphi] = 0\). Conversely, any a.c.t.m. manifold \(M^{2n+1}\) satisfying \((6)\) for some closed 1-form \(\omega\) and with \(\varphi\) integrable is l.c.c.
If $\omega \equiv 0$ then $M^{2n+1}$ is a cosymplectic manifold. If $\omega$ has no singular points, $M^{2n+1}$ is termed strongly non-cosymplectic.

3. Odd-dimensional real Hopf manifolds. A similarity transformation of $\mathbb{R}^n$ is given by

$$x^i = ga^i_j x^j + b^i,$$

where $g > 0$ and $[a^i_j] \in O(n)$. A manifold $M^n$ is a local similarity (l.s.) manifold if it possesses a smooth atlas whose transition functions have the form (7) (see [19]). Let $0 < \lambda < 1$ be fixed. Let $\Delta_\lambda$ be the cyclic group generated by the transformation $x^i = \lambda x^i$ of $\mathbb{R}^n - \{0\}$. Then $\mathbb{R}H^n = (\mathbb{R}^n - \{0\})/\Delta_\lambda$ is the real Hopf manifold. Define a diffeomorphism $f : \mathbb{R}H^n \to S^{n-1} \times S^1$ by setting:

$$f([x]) = \left(\frac{x^1}{|x|}, \ldots, \frac{x^n}{|x|}, \exp\left(\frac{-2\pi \log |x|}{\log \lambda}\right)\right)$$

for any $[x] \in \mathbb{R}H^n$. Here $[x] = \pi(x)$, $x = (x^1, \ldots, x^n)$, $x \in \mathbb{R}^n - \{0\}$, $|x|^2 = \sum_{i=1}^n (x^i)^2$ and $\pi : \mathbb{R}^n - \{0\} \to \mathbb{R}H^n$ denotes the natural projection. Then $\mathbb{R}H^n$, $n > 1$, is a compact connected l.s. manifold (with transition functions $x^i = \lambda x^i$). Let us endow $\mathbb{R}^{2n+1} - \{0\}$ with the metric

$$ds^2 = (|x|^2 + t^2)^{-1}\left\{\delta_{ij} dx^i \otimes dx^j + dt^2\right\}$$

where $(x^i, t)$, $1 \leq i \leq 2n$, are the natural coordinates (cf. (4.4) in [19], p. 287). As (8) is invariant under any transformation

$$x^i = \lambda^m x^i, \quad m \in \mathbb{Z},$$

it gives a globally defined metric $g_0$ on $\mathbb{R}H^{2n+1}$. We organize $\mathbb{R}H^{2n+1}$ into a l.c.c. manifold as follows. Let $\sigma = \log(|x|^2 + t^2)$. One may endow $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^1$ with a cosymplectic structure (cf. Z. Olszak [15], p. 241). Namely, let $g = \delta_{ij} dx^i \otimes dx^j + dt^2$ be the product metric on $\mathbb{R}^{2n+1}$. Let $\varphi(X + f \partial/\partial t) = JX$, where $X$ is tangent to $\mathbb{R}^2n$ and $f \in C^\infty(\mathbb{R}^{2n+1})$. Here $J$ denotes the canonical complex structure of $\mathbb{R}^n \approx \mathbb{C}^n$. Also set $\eta(X + f \partial/\partial t) = f$. Then $(\varphi, \xi, \eta, g)$, $\xi = \partial/\partial t$, is a cosymplectic structure on $\mathbb{R}^{2n+1}$. Note that $e^{\sigma/2} \xi$, $-e^{-\sigma/2} \eta$ and (as noticed above) $e^{-\sigma} g$ are invariant under any transformation (9). Therefore $\mathbb{R}H^{2n+1}$ inherits a l.c.c. structure ($\varphi_0, \xi_0, \eta_0, g_0$). Furthermore, by Proposition 3.5 in [19], p. 286, any orientable compact l.s. manifold of dimension $m \geq 3$ is a real homology real Hopf manifold, i.e. it has the Betti numbers $b_0 = b_1 = b_{m-1} = b_m = 1$ and $b_p = 0$ for $2 \leq p \leq m - 2$. By a theorem of D. E. Blair and S. Goldberg (Th. 2.4, in [3], p. 351), the Betti numbers of a compact cosymplectic manifold are non-zero. Combining the above statements one obtains in particular
Theorem 6. Any odd-dimensional real Hopf manifold $\mathbb{R}H^{2n+1}$, $n \geq 2$, has a natural structure of $\text{l.c.c.}$ manifold but admits no globally defined cosymplectic metrics. The Weyl connection of $\mathbb{R}H^{2n+1}$ is flat and its characteristic form $\omega = d\sigma$ is parallel with respect to the Levi-Civita connection of $(\mathbb{R}H^{2n+1}, g_0)$.

4. Real hypersurfaces of a locally conformal Kaehler manifold.

Let $(M^{2n+2}, g_0, J)$ be a locally conformal Kaehler (l.c.K.) manifold, with the complex structure $J$ and the Hermitian metric $g_0$ (cf. e.g. P. Libermann [14]). Let $M^{2n+1}$ be an orientable real hypersurface of $M^{2n+2}$. Given a unit normal field $N$ on $M^{2n+1}$, we put as usual $\xi = -JN$. Set $\varphi X = \tan(JX), \; FX = \text{nor}(JX)$, for any tangent vector field $X$ on $M^{2n+1}$. Here $\tan_x, \text{nor}_x$ denote the natural projections associated with the direct sum decomposition $T_x(M^{2n+2}) = T_x(M^{2n+1}) \oplus E_x$, $x \in M^{2n+1}$. Also $E \rightarrow M^{2n+1}$ is the normal bundle of $\iota : M^{2n+1} \subset M^{2n+2}$. Let $\eta(X) = g_0(FX, N)$. Let $g = \iota^*g_0$ be the induced metric. By a result of [2], p. 30, $(\varphi, \xi, \eta, g)$ is an a.c.t.m. structure on $M^{2n+1}$. Let $\omega_0 = (1/n)i(\Omega)d\Omega$. Here $i(\Omega)$ denotes the adjoint (with respect to $g_0$) of $e(\Omega)$, where $e(\Omega)\lambda = \Omega \wedge \lambda$, for any differential form $\lambda$ on $M^{2n+2}$, while $\Omega$ is the Kaehler 2-form of $M^{2n+2}$. Then $d\omega_0 = 0$, $d\Omega = \omega_0 \wedge \Omega$ (see e.g. [24]). Let $\omega = \iota^*\omega_0$. Let $\Theta$ be the fundamental form of the a.c.t.m. structure $(\varphi, \xi, \eta, g)$. Clearly $\Theta = \iota^*\Omega$. Thus

$$d\Theta = \omega \wedge \Theta, \quad d\omega = 0. \quad (10)$$

We recall the Gauss–Weingarten formulae:

$$\nabla^0_X Y = \nabla_X Y + g(AX, Y)N, \quad \nabla^0_X N = -AX, \quad (11)$$

where $A$ denotes the shape operator of $\iota$, while $\nabla$ is the induced connection. Then (11) leads to

$$\nabla_X \varphi Y = \eta(Y)AX - g(AX, Y)\xi \quad (12)$$

$$+ \frac{1}{2} \{ \omega(\varphi Y)X - \omega(Y)\varphi X + g(X, Y)\varphi B - \Theta(X, Y)B$$

$$+ \omega_0(N)[\eta(Y)X - g(X, Y)\xi] \}.$$

Here $B = \tan(B_0), \; B_0 = \omega_0^0$ (indices being raised with respect to $g_0$). Moreover,

$$\nabla_X \eta Y = -\Theta(AX, Y) \quad (13)$$

$$+ \frac{1}{2} [g(X, Y)\omega(\xi) - \Theta(X, Y)\omega_0(N) - \eta(X)\omega(Y)].$$

As $\nabla$ is torsion free, (13) leads to

$$2(d\eta)(X, Y) = (\omega \wedge \eta)(X, Y) - \Theta(AX, Y) \quad (14)$$

$$- \Theta(X, AY) - \Theta(X, Y)\omega_0(N).$$
Also (12) gives
\begin{equation}
[\varphi, \psi](X, Y) = \eta(Y)[A, \varphi]X - \eta(X)[A, \varphi]Y
- \left\{ g((A\varphi + \varphi A)X, Y) - \Theta(X, Y)\omega(N) \right\} \xi.
\end{equation}
As an application of (14)--(15) one obtains

**Theorem 7.** Let $M^{2n+1}$ be a real hypersurface of the l.c.K. manifold $M^{2n+2}$, and assume that either $M^{2n+1}$ is totally umbilical and its mean curvature vector satisfies $H = -\frac{1}{2}B^\perp$, $B^\perp = \text{nor}(B_0)$, or $M^{2n+1}$ is totally geodesic and tangent to the Lee field $B_0$ of $M^{2n+2}$. Then $(\varphi, \xi, \eta, g)$ is a l.c.c. structure on $M^{2n+1}$.

Let $CH^{2n+1} \approx S^{2n+1} \times S^1$ be the complex Hopf manifold ([13], Vol. II, p. 137) carrying the l.c.K. metric $g_0$ induced by the ($G_d$-invariant) metric $ds^2 = |x|^{-2}\delta_{ij}dx^i \otimes dx^j$, where $(x^1, \ldots, x^{2n+2})$ are the natural (real-analytic) coordinates on $\mathbb{C}^{n+1}$. Here $G_d = \{d^m : m \in \mathbb{Z}\}$, $d \in \mathbb{C} - \{0\}$, $|d| \neq 1$, while $I$ is the identical transformation of $\mathbb{C}^{n+1} - \{0\}$. Let $\pi : \mathbb{C}^{n+1} - \{0\} \to CH^{n+1}$ be the natural map (a local diffeomorphism). Let $\iota : M^{2n+1} \to (\mathbb{C}^{n+1} - \{0\}, \delta_{ij})$ be an orientable totally geodesic real hypersurface. Then $\psi : M^{2n+1} \to CH^{n+1}$, $\psi = \pi \circ \iota$, is totally umbilical. Indeed, let $h, h'$ be the second fundamental forms of $M^{2n+1}$ in $(\mathbb{C}^{n+1}, |x|^{-2}\delta_{ij})$ and $(\mathbb{C}^{n+1}, \delta_{ij})$, respectively. Let $g$ be the metric induced on $M^{2n+1}$ by $|x|^{-2}\delta_{ij}$. Then $\psi$ is an isometric immersion of $(M^{2n+1}, g)$ (in $(CH^{n+1}, g_0)$). Let $B^\perp$ be the normal component of $-2x^i\partial/\partial x^i$ (with respect to $M^{2n+1}$). Then
\begin{equation}
2h' = 2h + g \otimes B^\perp.
\end{equation}
Now (16) and $h' = 0$ give $h = g \otimes H$, $2H = -B^\perp$, i.e. $M^{2n+1} \to (\mathbb{C}^{n+1} - \{0\}, |x|^{-2}\delta_{ij})$ is totally umbilical. Let $\nabla$ be the Riemannian connection of $|x|^{-2}\delta_{ij}$. For any tangent vector fields $X, Y$ on $\mathbb{C}^{n+1}$ one has $\nabla^0_{\pi_* X} \pi_* Y = \pi_* \nabla_X Y$ (cf. [13], Vol. I, p. 161). Thus $h_\psi = \pi_* h$, where $h_\psi$ is the second fundamental form of $\psi$. Also (16) yields
\begin{equation}
H' = \exp(f)\{H + \frac{1}{2}B^\perp\},
\end{equation}
where $f$ is the restriction to $M^{2n+1}$ of $\log|x|^{-2}$. Thus (17) gives $h_\psi = g \otimes H_\psi$, i.e. $\psi$ is totally umbilical. We may apply Theorem 7 to $M^{2n+1} \to CH^{n+1}$ to conclude that $M^{2n+1}$ inherits a l.c.c. structure.

**5. The canonical foliation of a locally conformal symplectic manifold.** Let $M^{2n}$ be a real $2n$-dimensional differentiable manifold. An $(f, g, u, v, \lambda)$-structure on $M^{2n}$ consists of a $(1, 1)$-tensor field $F$, a Riemannian metric $G$, two 1-forms $u$, $v$ and a smooth real-valued function
\( \lambda \in C^\infty(M^{2n}) \) subject to:

\[
\begin{align*}
  f^2 &= -I + u \otimes U + v \otimes V, \\
  u \circ f &= \lambda v, \\
  v \circ f &= -\lambda u, \\
  fU &= -\lambda V, \\
  fV &= \lambda U,
\end{align*}
\]

(18)

\( u(V) = v(U) = 0, \quad u(U) = v(V) = 1 - \lambda^2, \)

\( g(fX, fY) = g(X, Y) - u(X)u(Y) - v(X)v(Y), \)

where \( U = u^2, V = v^2 \) (raising of indices is performed with respect to \( g \)) (see [26], p. 386).

Let \((M^{2n+1}, \varphi, \xi, \eta, g_0)\) be a strongly non-cosymplectic manifold with characteristic 1-form \( \omega \). Then \( M^{2n+1} \) admits a canonical foliation \( \Sigma \) whose leaves are the maximal connected integral manifolds of the Pfaffian equation \( \omega = 0 \).

Now we may prove Theorem 1. To this end, let \( M^{2n} \) be a leaf of \( \Sigma \). Let \( B_0 = \omega^2 \) be the characteristic field of \( M^{2n+1} \). Then \( C = \|\omega\|^{-1} B_0 \) is a unit normal vector field on \( M^{2n} \). Let \( X \) be tangential and set \( fX = \tan(\varphi X) \), \( u(X) = g_0(\varphi X, C), v(X) = \eta(X), \lambda = \eta(C) \). Then \( M^{2n} \) inherits an obvious \((f, g, u, v, \lambda)\)-structure, where \( g \) is the induced metric, while \( V = \tan(\xi), U = -\varphi C \). Since \( \omega = 0 \) on \( T(M^{2n}) \) by (6) one has \( dv = 0 \).

Let \( D^0 \) be the Weyl connection of \( M^{2n+1} \) and \( K_0 \) its curvature tensor field. As a consequence of (5) one has

\[
\begin{align*}
  K_0(X, Y)Z &= R_0(X, Y)Z - \frac{1}{2}\|\omega\|^2(X \wedge Y)Z \\
  &\quad - \frac{1}{2} \{L(X, Z)Y - L(Y, Z)X + g_0(X, Z)L(Y, \cdot)^2 - g_0(Y, Z)L(X, \cdot)^2\}.
\end{align*}
\]

Here \( R_0 \) denotes the curvature of \((M^{2n+1}, g_0)\) and

\[
\begin{align*}
  L(X, Y) &= (\nabla^0_X\omega)Y + \frac{1}{2}\omega(X)\omega(Y), \\
  (X \wedge Y)Z &= g_0(Y, Z)X - g_0(X, Z)Y.
\end{align*}
\]

Let \( K_0 = 0 \); apply (19) and the Gauss equation of \( M^{2n} \to M^{2n+1} \) to obtain

\[
\begin{align*}
  R(X, Y)Z &= \frac{1}{2}\|\omega\|^2(X \wedge Y)Z + (AX \wedge AY)Z \\
  &\quad + \frac{1}{2}\{\omega(h(Y, Z))X - \omega(h(X, Z))Y\} \\
  &\quad + \frac{1}{2}\|\omega\|\{g(Y, Z)AX - g(X, Z)AY\}.
\end{align*}
\]

As \( \Sigma \) has codimension 1 and \( \omega \) is parallel, \( h = 0 \) and (20) gives \( R(X, Y) = c^2 X \wedge Y, c = \frac{1}{2}\|\omega\| \), i.e. \( M^{2n} \) is an elliptic space-form. To prove the last statement in Theorem 1, assume \( M^{2n+1} \) is normal. Then \( \omega = 2\lambda \eta \gamma \); as \( \eta(C) = \lambda \), this yields \( \lambda^2 = 1 \). Then (18) gives \( u = 0, v = 0, f^2 = -I \) and \( M^{2n} \) turns out to be an almost Hermitian manifold. Moreover, \( [\varphi, \varphi] = 0, u = 0 \) lead to \( [f, f] = 0 \). Let \( \Omega \) be the Kähler 2-form of \( M^{2n} \). By (6), \( d\Omega = 0 \), i.e. \( M^{2n} \) is Kählerian. Suppose \( M^{2n} \) is complete. Then \( \pi_1(M^{2n}) = 0 \),
by a classical result in [20] and one may apply Th. 7.9 in [13], Vol. II, p. 170.

6. Regular locally conformal cosymplectic manifolds. A l.c.c. manifold $M^{2n+1}$ with the characteristic 1-form $\omega$ is normal iff

$$\omega = \omega(\xi) \eta.$$  

(21)

The structure vector $\xi$ is regular if it defines a regular foliation (i.e. each point of $M^{2n+1}$ admits a flat coordinate neighborhood, say $(U,x',t)$, $1 \leq i \leq 2n$, which intersects the orbits of $\xi$ in at most one slice $x^i = \text{const.}$, cf. [18]). By (21), if $M^{2n+1}$ is strongly non-cosymplectic, then $\xi$ is regular iff $B = \omega^2$ is regular.

Let $M^{2n+1}$ be compact; then $\xi$ is complete (cf. [13], Vol. I, p. 14). Let $P(\xi)$ be the period function of $\xi$, $P(\xi)_x \neq 0$, $x \in M^{2n+1}$ (see [5], pp. 722–723). The global 1-parameter transformation group of $P(\xi)^{-1}\xi$, $P(P(\xi)^{-1}\xi) = 1$, induces a free action of $S^1$ on $M^{2n+1}$. By standard arguments (cf. [5], p. 725, [4], p. 178, and [2], p. 15), $M^{2n+1}(M^{2n},\pi,S^1)$ is a principal $S^1$-bundle over the space of orbits $M^{2n} = M^{2n+1}/\xi$. By a result in [21], p. 236, as $\eta(\xi) = 1$ and $L_\xi \eta = 0$ it follows that $P(\xi) = \text{const.}$ Thus $L_{P(\xi)^{-1}\xi} \eta = 0$ and therefore $\eta$ is invariant under the action of $S^1$. Now we may prove Theorem 2. Clearly $\xi$ is vertical, i.e. tangent to the fibres of $\pi$. Let $A \in L(S^1)$ be the unique left invariant vector field on $S^1$ with $A^* = \xi$. (Here $A^*$ denotes the fundamental vector field on $M^{2n+1}$ associated with $A$, cf. [13], Vol. I, p. 51). Let $\overline{\eta} = \eta \otimes A$. Then $\overline{\eta}$ is a connection 1-form on $M^{2n+1}$. Let $H = \text{Ker}(\overline{\eta})$. By normality $N^3 = 0$, where $N^3 = L_\xi \varphi$ (see [2], p. 50). Thus $\varphi$ commutes with right translations. Consequently, $J_p Z_p = (d_\pi \varphi) Z_p^{H^+}$, $x \in \pi^{-1}(p)$, $p \in M^{2n}$, $Z \in T_p(M^{2n})$, is a well defined complex structure on $M^{2n}$. (Here $Z^H$ denotes the horizontal lift of $Z$ (with respect to $\overline{\eta}$).) Let $\overline{\eta}(Z,W) = g(Z^{H^+},W^{H^+})$. By (21), $\omega = 0$ on $H$ and thus $(M^{2n},\overline{\eta},J)$ is Kaehlerian.

Remark. $M^{2n}$ carries the Riemannian metric $g$, so it is paracompact.

By [13], Vol. I, p. 92, as $\overline{\eta}$ is flat, if $\pi_1(M^{2n}) = 0$ then $M^{2n+1} \approx M^{2n} \times S^1$ (i.e. $M^{2n+1}$ is the trivial $S^1$-bundle).

7. Submanifolds of complex Inoue surfaces. Let $C^+ = \{z \in C : \text{Im}(z) > 0\}$ be the upper half of the complex plane. Let $(z,w)$ be the natural complex coordonates on $C^+ \times C$. We endow $C^+ \times C$ with the Hermitian metric

$$ds^2 = y^2 dz \otimes d\overline{z} + y dw \otimes d\overline{w},$$

where $z = x + iy$, $i = \sqrt{-1}$. Then (22) makes $C^+ \times C$ into a globally conformal Kaehlerian manifold with the Lee form $\omega = y^{-1}dy$. Let $A \in \text{SL}(3,\mathbb{Z})$ with a real eigenvalue $\alpha > 0$ and two complex eigenvalues
\( \beta \neq \bar{\beta} \). Let \((a_1, a_2, a_3), (b_1, b_2, b_3)\) be respectively a real eigenvector and an eigenvector corresponding to \( \alpha, \beta \). Let \( G_A \) be the discrete group generated by the transformations \( f_\alpha, \alpha = 0, 1, 2, 3 \), where \( f_\alpha(z, w) = (\alpha z, \beta w), \) \( f_i(z, w) = (z + a_i, w + b_i), \) \( i = 1, 2, 3 \). Then \( G_A \) acts freely and properly discontinuously on \( \mathbb{C}^+ \times \mathbb{C} \) so that \( CI^2 = (\mathbb{C}^+ \times \mathbb{C})/G_A \) becomes a (compact) complex surface. This is the Inoue surface (cf. [12]). It was observed in [23], p. 84, that (22) is \( CI^2 \) complex surface. This is the manifold with a non-parallel Lee form (see Prop. 2.4 of [23], p. 85). Let \( \pi : \mathbb{C}^+ \times \mathbb{C} \to CI^2 \) be the natural projection. Let \( \iota : M \subset \mathbb{C}^+ \times \mathbb{C} \) be a submanifold and \( g \) the metric induced by (22). Then \( \psi : M \to CI^2, \psi = \pi \circ \iota, \) is an isometric immersion of \((M, g)\) into \( CI^2 \).

It is our purpose to build examples of (immersed) submanifolds of \( CI^2 \) (and motivate the results in Section 4). Let \( w = a + ib; \) we set \( X = \partial/\partial x, \) \( Y = \partial/\partial y, A = \partial/\partial a, B = \partial/\partial b. \) The real components of (22) are:

\[
g_0 = \begin{pmatrix}
y -2 & 0 & 0 & 0 \\
0 & y & 0 & 0 \\
0 & 0 & y -2 & 0 \\
0 & 0 & 0 & y \\
\end{pmatrix}
\]

Thus the non-zero Christoffel symbols of the Levi-Civita connection \( \nabla^0 \) of \( CI^2 \) are

\[
(23) \quad \begin{array}{cc}
\Gamma^1_{13} = \Gamma^3_{13} = -\Gamma^3_{11} = -y^{-1}, \\
\Gamma^2_{23} = \Gamma^3_{23} = \frac{1}{2} y^{-1}, \quad \Gamma^2_{22} = \Gamma^3_{44} = -\frac{1}{2} y^2.
\end{array}
\]

The Lee field of \( CI^2 \) is (locally) given by \( \mathcal{L} = yY. \) Let \( L^h = \{z \in \mathbb{C}^+ : \text{Im}(z) = 1\} \) and \( \iota : L^h \times \mathbb{C} \to \mathbb{C}^+ \times \mathbb{C} \) the natural inclusion. The tangent space at a point of \( L^h \times \mathbb{C} \) is spanned by \( X, A \) and \( B. \) Then \( N = yY \) is a unit normal vector field on \( L^h \times \mathbb{C}. \) By (23) one obtains

\[
(24) \quad \nabla^h_A N = -X, \quad \nabla^h_A N = \frac{1}{2} A, \quad \nabla^h_B N = \frac{1}{2} B.
\]

Let \( a_N \) be the shape operator of \( \psi : L^h \times \mathbb{C} \to CI^2, \psi = \pi \circ \iota. \) Then \( \text{Trace}(a_N) = 0, \) i.e. \( \psi \) is minimal. Clearly \( L^h \times \mathbb{C} \) is a maximal connected integral manifold of the Pfaff equation \( y^{-1}dy = 0, \) i.e. a leaf of the canonical foliation of the (strongly non-Kaehler) l.c.K. manifold \( CI^2, \) and therefore normal to \( \mathcal{L}. \)

Let \( L^v = \{z \in \mathbb{C}^+ : \text{Re}(z) = 0\} \) and \( \iota : L^v \times \mathbb{C} \to \mathbb{C}^+ \times \mathbb{C} \) the inclusion. Tangent spaces at points of \( L^v \times \mathbb{C} \) are spanned by \( A, Y, B, \) and \( N = yX \) is a unit normal. By (23),

\[
(25) \quad \nabla^0_A A = -\frac{1}{2} y^2 Y, \quad \nabla^0_A Y = \frac{1}{2} y^{-1} A, \quad \nabla^0_A B = 0, \\
\nabla^0_B Y = -y^{-1} Y, \quad \nabla^0_B A = \frac{1}{2} y^{-1} B, \quad \nabla^0_B B = -\frac{1}{2} y^2 Y.
\]

Consequently, \( \psi : L^v \times \mathbb{C} \to CI^2, \psi = \pi \circ \iota, \) is a totally geodesic immersion.
Clearly $L^v \times \mathbb{C}$ is tangent to $\mathcal{L}$ and inherits a l.c.c. structure (via our Theorem 7). Both $L^h \times \mathbb{C}$ and $L^v \times \mathbb{C}$ are generic, as real hypersurfaces of $\mathbb{C}^2$.

8. Betti numbers of locally conformal cosymplectic manifolds.

Let $M^{2n+1}$ be a l.c.c. manifold with $\nabla \omega = 0$, $K = 0$ (i.e. having a flat Weyl connection). Set $||\omega|| = 2c, c > 0$. By (19) the curvature of $M^{2n+1}$ has the expression

\begin{equation}
R_{ij} = c^2 \{ g_{jk} \delta_i^m - g_{ik} \delta_j^m \} + \frac{1}{4} \{ (\omega_j^m \omega_i^m - \omega_i^m \omega_j^m) \omega_k + (g_{ik} \omega_j - g_{jk} \omega_i) B^m \}.
\end{equation}

Suitable contraction of indices in (26) gives the Ricci curvature

\begin{equation}
R_{jk} = (2n-1) \{ c^2 g_{jk} - \frac{1}{4} \omega_j \omega_k \}.
\end{equation}

If $\alpha = (1/p!) \alpha_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$ is a differential $p$-form on $M^{2n+1}$, we consider the quadratic form

\begin{equation}
F_p(\alpha) = R_{ijk} \alpha^{ji_2 \ldots i_p} \alpha_i^j - \frac{1}{2} (p-1) R_{ijkm} \alpha^{ji_3 \ldots i_p} \alpha^{km}_{i_3 \ldots i_p}
\end{equation}

(cf. [10], p. 88). Then (26)–(27) lead to

\begin{equation}
R_{ijk} \alpha^{ji_2 \ldots i_p} \alpha_i^j = (2n-1) \{ c^2 p! ||\alpha||^2 - \frac{1}{4} (p-1)! ||B \alpha||^2 \},
\end{equation}

\begin{equation}
R_{ijkm} \alpha^{ji_3 \ldots i_p} \alpha^m_{i_3 \ldots i_p} = 2c^2 p! ||\alpha||^2 - (p-1)! ||B \alpha||^2,
\end{equation}

where $B$ denotes interior product with $B$.

Now we may prove our Theorem 3. Let $\alpha$ be a harmonic $p$-form on $M^{2n+1}$. By (3.2.9) in [10], p. 88, it follows that

\begin{equation}
\int_M \{ p F_p(\alpha) + \nabla_i \alpha_{i_1 \ldots i_p} \nabla^i \alpha^{i_1 \ldots i_p} \} \wedge 1 = 0.
\end{equation}

On the other hand, by (28)–(29),

\begin{equation}
F_p(\alpha) = c^2 \{ p! (2n-p) ||\alpha||^2 + (p-1)! (2p-2n-1) ||U \alpha||^2 \},
\end{equation}

where $U = ||\omega||^{-1} B$. Hence, if $n+1 \leq p \leq 2n-1$, then $b_p(M^{2n+1}) = 0$ (cf. our (30)–(31)). By Poincaré duality one also has $b_p(M^{2n+1}) = 0$ when $2 \leq p \leq n$. Since $\omega$ is parallel, it is harmonic. Thus $b_1(M^{2n+1}) = b_{2n}(M^{2n+1}) \geq 1$ (as $c \neq 0$). To compute the first Betti number of $M^{2n+1}$, let $\sigma$ be a harmonic 1-form. Then $\sigma$ is a harmonic $2n$-form, where $\star$ denotes the Hodge operator. Then (31) leads to

\begin{equation}
F_{2n}(\star \sigma) = c^2 (2n-1)!(2n-1) ||U(\star \sigma)||^2
\end{equation}

and thus $t_{2n}(\star \sigma) = 0$, by (30). By applying once more the Hodge operator one has $u \wedge \sigma = 0$ or $\sigma = f u$ for some nowhere vanishing $f \in C^\infty(M^{2n+1})$. Here $u = ||\omega||^{-1} \omega$. As $\sigma$ is harmonic, it is closed, so that $df \wedge u = 0$ or $df = \lambda v$ for some $\lambda \in C^\infty(M^{2n+1})$. But $\sigma$ is also coclosed, so that
\((df, \sigma) = (f, \delta \sigma) = 0\) (by (2.9.3) in [10], p. 74), i.e. \(df\) and \(\sigma\) are orthogonal. Thus \(0 = (df, \sigma) = \lambda f \text{vol}(M^{2n+1})\) yields \(\lambda = 0\). As \(M^{2n+1}\) is connected one obtains \(f = \text{const.}\), i.e. \(b_1(M^{2n+1}) = 1\).

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