ON HYPERGRAPHS OF MAXIMAL SIMPLE PATHS
OF A CLASS OF HAMILTONIAN GRAPHS

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Let $G = (V, E, \phi)$ be an arbitrary simple graph. A hypergraph of maximal simple paths of $G$ is a hypergraph $H = (X, \mathcal{E})$, where $\mathcal{E} = \{E_i\}_{i=1}^n$, $E_i = E_j \iff i = j$ is a family of subsets of $X$ corresponding to subsets of edges of an arbitrary maximal path of $G$. In [1] some fundamental problems in structural hypergraph theory have been formulated. Some solutions to these problems related to hypergraphs of maximal simple paths of a graph are given in [2]. In this paper necessary and sufficient conditions for a hypergraph $H$ to be a hypergraph of maximal simple paths of $G$ in certain subclass $\mathcal{G}$ of Hamiltonian graphs are established. Also, the unicity of reconstruction of $G \in \mathcal{G}$ based on corresponding hypergraph $H$ will be proved. The class $\mathcal{G}$ contains, as a proper subclass, all Hamiltonian graphs for which $r(e) \geq 3$.

A graph $G \in \mathcal{G}$ iff there exists a Hamiltonian cycle $C$ of $G$ such that if a vertex $v \in C$ is not incident with any chord of cycle $C$, then there is a chord $d$ linking both neighbours of $v$ in $C$. The class of hypergraphs of maximal paths of elements in $\mathcal{G}$ is denoted by $\mathcal{H}$. The following properties of a hypergraph $H = (X, \mathcal{E}) \in \mathcal{H}$ are evident:

1. $E_i \in \mathcal{E}, A \subseteq E_i, A \neq E_i \Rightarrow A \notin \mathcal{E}$.
2. There exists a set $C \subseteq X$ such that $A \subseteq C$, $|A| = |C|-1 \Rightarrow A \in \mathcal{E}$.

The set $C$ called a Hamiltonian cycle of hypergraph $H$. The set $X \setminus C$ will be called a set of chords of cycle $C$ and its elements—chords of cycle $C$.

Let $d$ be an arbitrary chord of a Hamiltonian cycle of $G$. There exist in $D = C \cup \{d\}$ exactly two maximal paths of $G$ with the length $|D|-2$ containing edge $d$. Hence, hypergraph $H$ has to satisfy:

3. $d \in X \setminus C \Rightarrow$ there exist exactly two sets $E_i, E_j \in \mathcal{E}; E_i, E_j \subseteq C \cup \{d\}, d \in E_i, d \in E_j, |E_i| = |E_j| = |C \cup \{d\}| - 2$.

Let $E_i$ and $E_j$ be maximal paths in $G$ determined by chord $d$ in condition (3). Then, it is easy to notice that

4. For every $x \in C \setminus E_i$ there exists exactly one $y \in C \setminus E_j$ such that $(x, y, d) \notin E_i$ for every $E_i \in \mathcal{E}$. 

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Let us denote \( \{x_1, x_2\} = C \setminus E_1 \) and \( \{y_1, y_2\} = C \setminus E_2 \). By condition (4) it follows that for a fixed chord \( d \) there are three cases possible:

\( (a) \) \( \{x_1, y_1, d\} \) and \( \{x_2, y_2, d\} \) are not contained in any set \( E_k \in \mathcal{E} \); for the remaining two sets: \( \{x_1, y_2, d\} \in E_k \), \( \{x_2, y_1, d\} \in E_k \); \( p, q \in I \),

\( (b) \) only \( \{x_1, y_1, d\} \) is contained in \( E_k \); \( k \in I \),

\( (c) \) none of the sets \( \{x_1, y_1, d\} \) is contained in \( E_k \in \mathcal{E} \).

Let us form a family of subsets of \( X \) denoted by \( \mathcal{F} \). Let \( d \) be an arbitrary chord of cycle \( C \). In case (a) we include sets \( \{x_1, x_2, d\} \) and \( \{x_2, y_2, d\} \) into family \( \mathcal{F} \), in case (b) the sets \( \{x_1, y_1, d\}, \{x_2, y_2, d\}, \{x_1, x_2, d\} \), \( \{y_1, y_2, d\} \) or \( \{x_1, y_2, d\}, \{x_2, y_1, d\}, \{x_1, y_1, d\}\).

Now, let \( \mathcal{F} \) be an arbitrary family of subsets of \( X \) and \( C \) an arbitrary non-empty subset of \( X \). Let us denote by \( \mathcal{F}^{(2)} \) the least family of a subset in \( X \) such that

\( (i) \) \( \mathcal{F} \subseteq \mathcal{F}^{(2)} \),

\( (ii) \) \( F_1, F_2 \in \mathcal{F}^{(2)}, |F_1 \cap F_2| \geq 1 \Rightarrow F_1 \cup F_2 \in \mathcal{F}^{(2)} \).

Let us denote by \( \mathcal{F}^{(2)}_{\text{max}} \) the family of all maximal sets of family \( \mathcal{F}^{(2)} \). We say that family \( \mathcal{F} \) determines a Hamiltonian structure in \( X \) with respect to a set \( C \) if the following conditions are satisfied:

\( (iii) \) \( F_1 \in \mathcal{F}^{(2)}_{\text{max}} \Rightarrow |F_1 \cap C| \geq 2 \),

\( (iv) \) \( x \in X \Rightarrow \) there exist exactly two sets \( F_1, F_2 \in \mathcal{F}^{(2)}_{\text{max}} \) such that \( x \in F_1, x \in F_2 \),

\( (v) \) \( \forall y \in |X|; F_1, F_2 \in \mathcal{F}^{(2)}_{\text{max}}, F_1 \neq F_2 \Rightarrow \{x \in X, y \neq x, y \in F_1 \Rightarrow y \notin F_2 \} \).

Let \( \mathcal{F} \) determine in \( X \) a Hamiltonian structure with respect to \( C \). A subset \( S \) of \( X \) is called elementary if for every \( F_1 \in \mathcal{F}^{(2)}_{\text{max}} \), we have \(|F_1 \cap S| \leq 2 \). In particular, \( C \) is an elementary set. \( n(S) \) denotes the number of those \( F_1 \in \mathcal{F}^{(2)}_{\text{max}} \), for which \(|F_1 \cap S| = 1 \). The number \( n(S) \) is an index of an elementary set \( S \). \( \mathcal{F} \) denotes a family of subsets of \( X \), all elementary maximal sets of a given Hamiltonian structure with index 2.

It is easy to see that if \( \mathcal{F} \) is a family of subsets of \( X \) determined by conditions (a), (b), (c), then for a hypergraph \( H \) of maximal paths of a Hamiltonian graph \( G \) the following condition should be satisfied:

\( (5) \) the family \( \mathcal{F} \) determines a Hamiltonian structure in \( X \) with respect to cycle \( C \) and \( E_k \in \mathcal{E} \).

Now, we shall prove the following

**Theorem.** Hypergraph \( H = \langle X, \mathcal{E} \rangle \) is a hypergraph of maximal simple paths for a graph \( G = \langle V, X, \phi \rangle \in \mathcal{G} \) iff for conditions (1)-(5) are satisfied. There is a one-to-one correspondence (isomorphism) between \( H \in \mathcal{H} \) and \( G \in \mathcal{G} \).

**Proof.** The necessity of conditions (1)-(5) is evident.

Let \( H = \langle X, \mathcal{E} \rangle \) be a hypergraph satisfying (1)-(5) and \( C \) its Hamiltonian cycle. We form graph \( G \) in the following way: let \( G = \langle V, X, \phi \rangle \) where \( V = \mathcal{F}^{(2)}_{\text{max}} \), function \( \phi : X \rightarrow V \) is defined according to condition (iv): \( x \in F_1 \Rightarrow \phi(x) = (F_1, F_2) \).

By condition (5), and taking into account (iii), (iv), and (v), we obtain the fact that \( G \) is a simple graph and for every vertex \( F_1 \in \mathcal{F}^{(2)}_{\text{max}} \), (iii) it follows that there exist exactly two edges \( x \in C \) of the graph which are incident with \( F_1 \) and hence \( C \) is a Hamiltonian cycle of \( G \). By conditions (2)-(4) the unicity of construction for family \( \mathcal{F} \) follows, and so it follows consequently for \( \mathcal{F}^{(2)}_{\text{max}} \). Further, by condition (5) and (iv) the unicity of construction for \( G \) is obtained. Only in case (c) a fictitious ambiguity occurs for family \( \mathcal{F} \) when for a chord \( d \) none of the sets \( \{x_1, y_1, d\} \) is contained in \( E_k \in \mathcal{E} \). By conditions (1)-(iv) it follows that if \( G \) has a chord with this property, then its Hamiltonian cycle contains exactly four edges and one or two diagonals. Here, the unicity (isomorphism) of \( G \) is evident. The vertices of \( G \), as it follows by construction of \( \mathcal{F} \), are determined either by a chord incident with a vertex or by a pair of edges belonging to a Hamiltonian cycle and incident with the same chord \( d \). Hence, it follows that \( G \in \mathcal{G} \).

By the construction of sets in family \( \mathcal{F} \) it follows that each \( S \in \mathcal{F} \) is a set of edges of \( G \) belonging to an arbitrary maximal path of \( G \). A set of edges of \( G \) belonging to an arbitrary maximal simple path of the graph is an elementary set with index 2 and it belongs to \( \mathcal{G} \). Hence, according to condition (5) hypergraph \( H = \langle X, \mathcal{E} \rangle \) is! a hypergraph of maximal simple paths of \( G \).

The set of conditions (1)-(5) allows to formulate a simple algorithm for verification whether \( H \in \mathcal{H} \). It seems that analysis of independence and reduction of the set of conditions should be interesting.

**References**


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